Parametric manifolds. II. Intrinsic approach

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A *parametric manifold* is a manifold on which all tensor fields depend on an additional parameter, such as time, together with a *parametric structure*, namely a given (parametric) one-form field. Such a manifold admits natural generalizations of Lie differentiation, exterior differentiation, and covariant differentiation, all based on a nonstandard action of vector fields on functions. There is a new geometric object, called the *deficiency*, which behaves much like torsion, and which measures whether a parametric manifold can be viewed as a one-parameter family of orthogonal hypersurfaces. © *1995 American Institute of Physics.*

I. INTRODUCTION

It is often useful to project the geometric structure of a manifold onto an embedded hypersurface. This leads to the well-known Gauss–Codazzi formalism, which relates the projected geometry of the hypersurface to the original manifold. Initial value problems are often posed in this setting, with a one-parameter family of embedded hypersurfaces being used to describe the evolution. Identifying these hypersurfaces leads to the interpretation of tensor fields in the original manifold as one-parameter families of tensor fields on a given hypersurface. This is the beginnings of a theory of *parametric manifolds*.

We recently generalized the Gauss–Codazzi formalism from the setting just described to the case where the manifold is foliated by the integral curves of a (suitably regular) vector field, but where these curves are *not* assumed to be hypersurface orthogonal.[1] (The recent work of Gowdy[2] also introduces a generalized notion of the Gauss–Codazzi formalism.) We will refer to this as the *extrinsic* approach to parametric manifolds. This results in a picture of a parametric manifold which is now the *manifold of orbits* of the given curves, on which there are one-parameter families of tensor fields.

However, there are implicit properties which such parametric manifolds inherit from the original manifold. Notable among these is the behavior under *reparameterizations*, which consist of relabeling the parameter along the given curves, and which are hence a special class of coordinate transformations in the original manifold.

We show here that parametric manifolds can be be defined completely *intrinsically*, without reference to an "original manifold." The key idea is to generalize the action of vector fields on functions in a way reminiscent of the notion of horizontal lift in a fiber bundle. This naturally leads to generalizations of Lie differentiation, exterior differentiation, and covariant differentiation. These derivative operators reproduce intrinsically the corresponding projected operators obtained in our earlier extrinsic approach.

The geometry of parametric manifolds is "almost a fiber bundle," and as such may provide the groundwork for a generalization of Yang–Mills theory.

We start by defining parametric manifolds in Sec. II. We then introduce parametric exterior differentiation in Sec. III, which allows us to define the all-important notion of *deficiency*, which measures whether a parametric manifold can be viewed as a one-parameter family of orthogonal hypersurfaces. In Sec. IV, we then use the deficiency to define a parametric bracket, and hence a

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parametric Lie derivative. Parametric connections are discussed in Sec. V, including their associated (generalized) torsion and curvature. Finally, in Sec. VI, we discuss our results.

II. PARAMETRIC FUNCTIONS AND VECTOR FIELDS

Consider a smooth manifold Σ. We wish to consider one-parameter families of tensor fields on Σ, parameterized by a parameter t. Since the particular choice of parameter should not be important, we first need to describe how to change the parameterization.

Definition 1: A reparameterization of Σ is an assignment

$$s=t+F(p)$$

for $p\in\Sigma, s,t\in\mathscr{R}$, and $F:\Sigma\rightarrow\mathscr{R}$.

A *parametric structure* on Σ is a preferred one-parameter family of one-forms $\omega(t)$ on Σ which behaves as follows under a reparameterization:

$$\hat{\omega}(s)=\omega(t)-dF, \tag{1}$$

i.e., $\omega(t)$ transforms to $\hat{\omega}(s)$ under a reparameterization.

We can now start to consider parametric objects on Σ.

Definition 2: A parametric function on Σ is a mapping $f:\Sigma\times\mathscr{R}\rightarrow\mathscr{R}$. Denote the collection of such mappings by $\mathscr{F}_*(\Sigma)$.

Given a parametric function $f\in\mathscr{F}_*(\Sigma)$, for a fixed $t\in\mathscr{R}$ f can be considered as a function from Σ to $\mathscr{R}$. Denote this function by f_t. Thus $f_t\in\mathscr{F}(\Sigma)$, the ring of functions on Σ, and can be acted on by tangent vectors of Σ.

Proposition 3: The action of ∂_t *on parametric functions is a covariant operation.*

Proof: Under a coordinate transformation of Σ, the operator ∂_t remains unaffected. This is because the parameter t is not a coordinate and, hence, any coordinate transformation of Σ must be independent of t. Therefore, $\partial f/\partial t|_{(p,t_0)}$ does not depend on the choice of coordinates for $p\in\Sigma$. Furthermore, under a reparameterization $s=t+F(p)$, $\partial f/\partial s=\partial f/\partial t$. ■

While tangent vector fields do not act uniquely on parametric functions, one-parameter families of tangent vector fields do. These one-parameter families of vector fields, called parametric vector fields, will act on parametric functions in a way reminiscent of the action of projected vector fields.

Definition 4: A parametric vector field is a smooth mapping $X:\Sigma\times\mathscr{R}\rightarrow T\Sigma$ such that for each $p\in\Sigma$, $X(p,t)\in T_p\Sigma$ for all $t\in\mathscr{R}$. Let $\chi_*(\Sigma)$ represent the collection of smooth parametric vector fields defined on Σ.

For a fixed t, let $X_t:\Sigma\rightarrow T\Sigma$ denote the obvious tangent vector field. We define the action of a parametric vector field on a parametric function as follows:

$$Xf(p,t)=X_tf_t(p)+\omega(t)(X_t)\,\frac{\partial f}{\partial t}\,(p).$$

Suppressing the point p, we can write this action as

$$X(f)=X_tf_t+\omega(X_t)\,\frac{\partial f}{\partial t}. \tag{2}$$

Theorem 5: *$X(f)$ is invariant under reparameterizations and coordinate transformations.*

Proof: Consider coordinates $\{x^i\}$ and a parameter t. Writing $\omega=:M_i\,dx^i$, we have that

$$X(f)=X^i\left(\frac{\partial f_t}{\partial x^i}+M_i\,\frac{\partial f}{\partial t}\right).$$

Under a reparameterization $s=t+F(p)$, the components of ω transform according to Eq. (1). Denote the parametric structure ω under this new parameterization by $\hat{\omega}$. Thus

$$\hat{\omega}=\hat{M}_i\, dx^i=\left(M_i-\frac{\partial F}{\partial x^i}\right)dx^i=\omega-dF.$$

Although $\partial f/\partial s=\partial f/\partial t$, we must be careful computing $\partial f/\partial x^i$. Using the notation introduced above, let $f_t:\Sigma\rightarrow\mathscr{R}$ and let $\hat{f}_s(p)=f(p,s)=f(p,t+F(p))$ denote its reparameterization. Then

$$\frac{\partial\hat{f}_s}{\partial x^i}=\frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial x^i}+\frac{\partial f}{\partial s}\frac{\partial s}{\partial x^i}=\frac{\partial f}{\partial x^i}+\frac{\partial f}{\partial t}\frac{\partial t}{\partial s}\frac{\partial F}{\partial x^i}=\frac{\partial f_t}{\partial x^i}+\frac{\partial F}{\partial x^i}\frac{\partial f}{\partial t}.$$

Therefore

$$\begin{aligned}X(f)&=X^i\left(\frac{\partial f_t}{\partial x^i}+M_i\frac{\partial f}{\partial t}\right)\\&=X^i\left(\frac{\partial f_t}{\partial x^i}+\left(\hat{M}_i+\frac{\partial F}{\partial x^i}\right)\frac{\partial f}{\partial t}\right)\\&=X^i\left(\left(\frac{\partial\hat{f}_s}{\partial x^i}-\frac{\partial F}{\partial x^i}\frac{\partial f}{\partial t}\right)+\left(\hat{M}_i+\frac{\partial F}{\partial x^i}\right)\frac{\partial f}{\partial t}\right)\\&=X^i\left(\frac{\partial\hat{f}_s}{\partial x^i}+\hat{M}_i\frac{\partial f}{\partial s}\right),\end{aligned}$$

which is the expression for $X(f)$ with respect to the parameter s, showing that $X(f)$ is invariant under a reparameterization. If we consider a coordinate transformation of Σ, X_t and $\partial/\partial x^i$ will transform as usual, guaranteeing that $X_t(f_t)$ is independent of the choice of coordinates. Since ω and ∂_t are unaffected, $X(f)$ remains invariant under a coordinate transformation of Σ. ■

Theorem 6: *Parametric vector fields are derivations on the ring* $\mathscr{F}_*(\Sigma)$. That is,

(i) $X(rf+sg)=rX(f)+sX(g)$ *and*

(ii) $X(fg)=fX(g)+gX(f)$ *for all* $r,s\in\mathscr{R}$ *and* $f,g\in\mathscr{F}_*(\Sigma)$.

Proof: This follows directly from Eq. (2) since X_t and ∂_t are derivations. ■

Parametric vector fields have a very nice representation in terms of a local coordinate system, $\{x^i\}$. Since a parametric vector field is just a family of tangent vector fields, we may write

$$X=X^i\frac{\partial}{\partial x^i}=X^i\partial_i$$

as usual, where we let the functions X^i depend on the parameter. That is, the X^i are parametric functions on Σ. In terms of this representation we may write out the action of parametric vector fields on parametric functions

$$X(f)=X_t(f_t)+\omega(t)(X_t)\dot{f}=X^i(f_{,i}+M_i\dot{f})=:X^i(f_{*i}),$$

where we have introduced the use of $\dot{f}$ for $\partial f/\partial t$.

The action of parametric vector fields on parametric functions mimics the action of vector fields which are orthogonal to ∂_t in some bigger manifold, typically $\Sigma\times\mathscr{R}$, which can be thought of as a fiber bundle over Σ. In this interpretation, the action of X on f is given by taking the horizontal lift, as specified by ω.

We can similarly define parametric tensors of higher rank.

Definition 7: A parametric (p,q)-tensor, $T\in\mathscr{T}^p_q(\Sigma)$, on Σ is a one-parameter family of (p,q)-tensors on Σ. That is,

$$T{:}T\Sigma\times\cdots\times T\Sigma\times T^*\Sigma\times\cdots\times T^*\Sigma\times\mathscr{R}\to\mathscr{R}$$

such that $T(\ldots,t)\in\mathscr{T}^p_q(\Sigma)$.

As with parametric vector fields, parametric tensors can easily be expressed in a coordinate basis

$$T^{i_1\cdots i_p}{}_{j_1\cdots j_q}\frac{\partial}{\partial x^{i_1}}\cdots\frac{\partial}{\partial x^{i_p}}\,dx^{j_1}\cdots dx^{j_q},$$

where the $T^{i_1\cdots i_p}{}_{j_1\cdots j_q}$ are parametric functions. We can also talk about one-parameter families of metrics on Σ, that is, *parametric metrics*.

The Lie bracket of two vector fields orthogonal to a given family of curves need not be a vector field orthogonal to the curves. This "deficiency" is carried over to the parametric theory, as can be seen explicitly by calculating the action of the commutator $(XY-YX)$ on a parametric function.

$$X(Y(f))=X^i(Y^jf_{*j})_{*i}=X^i(Y^j{}_{*i}f_{*j}+Y^jf_{*ji})$$

so

$$(XY-YX)(f)=(X^iY^j{}_{*i}-Y^iX^j{}_{*i})f_{*j}+X^iY^j(f_{*ji}-f_{*ij}),\tag{3}$$

where, in general, $f_{*ji}-f_{*ij}\neq 0$.

The first term on the right-hand side can indeed be written as the (parametric) action of some vector field on f, but the second turns out to involve (only) differentiation of f with respect to the parameter, and hence cannot be so written. In terms of horizontal lifts, the first term of Eq. (3) is again horizontal, and can thus be identified with (the action of) a parametric vector field, while the second term involves differentiation in the vertical direction, which does not correspond to a parametric vector field.

We would nevertheless like to define a notion of the "bracket" of parametric vector fields. The noncommutativity of the mixed parametric derivative makes this nontrivial. Without the use of a projection operator, or equivalently of a horizontal lift, it is difficult to isolate the first term, which is the one we want. However, there is an intrinsic calculation that yields the second term, which is the deficiency. In order to define the deficiency intrinsically we will now turn our attention to exterior differentiation of parametric forms.

III. PARAMETRIC EXTERIOR DIFFERENTIATION

Perjés[3] introduced a notion of exterior differentiation of parametric functions, namely,

$$d_*f=df+\omega\dot{f},$$

where d is the usual exterior differentiation on differential forms. Parametric functions may be considered as parametric differential zero-forms. Parametric differential p-forms are just one-parameter families of differential p-forms defined on Σ. Thus, in a coordinate basis, a parametric differential p-form may be written as

$$\theta=\theta_{i_1\cdots i_p}\,dx^{i_1}\wedge\cdots\wedge dx^{i_p},$$

where the $\theta_{i_1\cdots i_p}$ are functions of x^i and t.

There are four axioms needed to completely determine the exterior derivative d (see Ref. 4), namely,

(i) $df(X)=X(f)$ for functions f and vector fields X,

(ii) wedge-product rule: $d(\theta\wedge\tau)=d\theta\wedge\tau+(-1)^p\theta\wedge d\tau$ where θ is a p-form,

(iii) $d(df)=0$, and

(iv) d is linear: $d(\theta+\tau)=d\theta+d\tau$.

We already have that $d_*f(X)=X(f)$ for parametric vector fields X and parametric functions f. Properties (ii) and (iv) also carry over easily. However, it is not clear that we wish $d_*(d_*f)=0$. For the parametric case, consider replacing axiom (iii) by

(iii′) $d_*(d_*f)=0$ for parameter-independent functions f.

Consider an exterior derivative operator, d_*, on parametric differential forms satisfying (i), (ii), (iii′), and (iv) for parametric forms, vector fields, and functions. We have the following familiar coordinate expressions:

(1) since the coordinate functions do not depend on the parameter, we have, by (ii) and (iii′)

$$\begin{aligned}d_*(dx^{i_1}\cdots dx^{i_p})&=(d_*^2x^{i_1})\wedge dx^{i_2}\,dx^{i_3}\cdots dx^{i_p}-dx^{i_1}\wedge(d_*^2x^{i_2})\wedge dx^{i_3}\cdots dx^{i_p}\cdots\\&\quad+(-1)^{p-1}\,dx^{i_1}\cdots dx^{i_{p-1}}\wedge(d_*^2x^{i_p})\\&=0,\end{aligned}$$

(2)

$$d_*(f dx^{i_1}\cdots dx^{i_p})=d_*f\wedge dx^{i_i}\cdots dx^{i_p}+f d_*(dx^{i_1}\cdots dx^{i_p})=d_*f\wedge dx^{i_1}\cdots dx^{i_p},$$

and

(3) using (iv), d_* on any parametric p-form has the coordinate expression

$$d_*(\theta)=d_*(\theta_{i_1\cdots i_p})\wedge dx^{i_1}\cdots dx^{i_p},$$

which can also be written

$$d_*(\theta)=d\theta+\omega\wedge\dot{\theta}.$$

It thus follows just as in the standard case that these axioms uniquely define the *parametric exterior derivative operator* d_*.

What about $d_*(d_*f)$ on arbitrary parametric functions? According to this set of axioms we have

$$d_*^2f=d_*(f_{*i}\,dx^i)=f_{*ij}dx^j\wedge dx^i=-f_{*ji}\,dx^j\wedge dx^i.$$

Therefore, $2d_*^2f=(f_{*ij}-f_{*ji})dx^j\wedge dx^i$, which turns out to involve only parameter derivatives of f. This is the intrinsic version of the *deficiency*, which now measures the failure of d_*^2 to be identically zero.

Definition 8: The deficiency, $\mathscr{D}$, is the derivative operator defined by

$$\mathscr{D}(X,Y)f=2d_*^2f(X,Y)$$

for $X,Y\in\chi_*(\Sigma)$ and $f\in\mathscr{F}_*(\Sigma)$.

In terms of a coordinate basis we have

$$\mathscr{D}(X,Y)f=2d_*^2f(X^i\partial_i,Y^j\partial_j)=X^iY^j(f_{*ji}-f_{*ij})=X^iY^j(M_{j*i}-M_{i*j})\dot{f}=:X^iY^j\mathscr{D}_{ji}\dot{f},$$

which is precisely the second term in Eq. (3).

IV. A BRACKET OPERATOR

We can now easily define the bracket of two parametric vector fields intrinsically. We want our intrinsic definition to correspond to the projected bracket, i.e., the first term of Eq. (3). But the deficiency gives us a way to describe the second term there. Thus, for two parametric vector fields X and Y, define

$$[X,\ Y]_* f = X(Y(f\ )) - Y(X(f\ )) - \mathscr{D}(X,Y)f.$$

Working this out in a coordinate basis, we have

$$[X,\ Y]_* f = (X^i Y^j{}_{*i} - Y^i X^j{}_{*i}) f_{*j} + X^i Y^j (f_{*ji} - f_{*ij}) - X^i Y^j (f_{*ji} - f_{*ij}) = (X^i Y^j{}_{*i} - Y^i X^j{}_{*i}) f_{*j},$$

which is of course the first term in Eq. (3) as desired. If $\{x^i\}$ are coordinates on Σ, then $[\partial_i,\ \partial_j]_* = 0$ as one would like.

The *parametric bracket operator* $[,]_*$ just defined fails to satisfy the Jacobi identity, but rather satisfies a generalized (and somewhat messy) form of this identity involving the deficiency. However, many of the usual properties do hold without modification. For instance, the standard expressions for the exterior derivatives of differential forms in terms of Lie bracket are still valid in the parametric case.

Theorem 9: *If θ is a (parametric) one-form, then*

$$2d_*\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,\ Y]_*)$$

for all (parametric) vector fields X and Y.

Given a parametric vector field X, we can define an $\mathscr{R}$-linear mapping $\mathscr{L}_{*_X} : \chi_*(\Sigma) \to \chi_*(\Sigma)$ by $\mathscr{L}_{*_X} Y = [X,\ Y]_*$. Since

$$\begin{aligned}\mathscr{L}_{*_X}(fY)_g &= [X,\ fY]_* g \\ &= X(f\ )(Y(g)) - fY(X(f\ )) - \mathscr{D}(X,fY)g \\ &= X(f\ )Y(g) + fXY(g) - fY(X(g)) - f2d_*^2 g(X,Y) \\ &= (X(f\ )Y + f\mathscr{L}_{*_X} Y)g\end{aligned}$$

for all $f,g \in \mathscr{F}_*(\Sigma)$ and $X,Y \in \chi_*(\Sigma)$, $\mathscr{L}_{*_X}$ may be extended uniquely to a parametric tensor derivation on Σ, the *parametric Lie derivative*. (See Theorem 15 in Chap. 2 of Ref. 5.)

The standard expression relating Lie differentiation, exterior differentiation, and the interior product generalizes directly to the parametric setting. Specifically, letting $i_X\alpha$ denote the obvious extension to parametric fields of the usual interior product of a differential form by a vector field X, we have the following result.

Theorem 10: *When acting on differential forms, parametric Lie differentiation satisfies the operator equation*

$$\mathscr{L}_{*_X} = i_X d_* + d_* i_X$$

for any parametric vector field X.

Proof: It is straightforward to show that the right-hand side of this equation defines a derivation. It thus suffices to check the action of both sides on functions and certain one-forms. (It in fact suffices to check the action of both sides for parameter independent one-forms of the form df, since all parametric forms can be written as the product of a parametric function and a parameter-independent differential form. It is nevertheless instructive to keep track of the deficiency in the more general calculation presented here.) We have

$$\mathscr{L}_{*_X} f = X(f) = d_* f(X) = i_X d_* f = (i_X d_* f + d_* i_X) f,$$

where the last equality uses the fact that $i_X f \equiv 0$. Furthermore

$$\begin{aligned}(\mathscr{L}_{*_X} d_* f)(Y) &= \mathscr{L}_{*_X}(d_* f(Y)) - d_* f(\mathscr{L}_{*_X}(Y)) \\ &= \mathscr{L}_{*_X}(Y(f)) - d_* f([X, Y]_*) \\ &= X(Y(f)) - [X, Y]_*(f) \\ &= Y(X(f)) + \mathscr{D}(X,Y)(f) \\ &= Y(\mathscr{L}_{*_X} f) + 2 d_*^2 f(X,Y) \\ &= d_*(\mathscr{L}_{*_X} f)(Y) + (i_X d_*^2 f)(Y).\end{aligned}$$

Thus

$$\mathscr{L}_{*_X} d_* f = d_*(\mathscr{L}_{*_X} f) + i_X d_*^2 f = d_* i_X d_* f + i_X d_*^2 f$$

and the theorem is proven. ■

V. PARAMETRIC CONNECTIONS

We will now introduce the notion of a connection on a parametric manifold. Although the following definition looks identical to the definition of a standard affine connection on a manifold, this is an illusion created by the choice of notation. Specifically, we have been using $X(f)$ to denote the action of a parametric vector field on a parametric function. The underlying operator for such an action is *not* partial differentiation, but parametric differentiation via the operator ∂_{*i}. In this sense, one can view a parametric connection as a *generalized* connection on a manifold. (In Ref. 6, Otsuki describes *generalized connections* which do not always reduce to partial differentiation on functions.) That is, we generalize the notion of a vector field acting on a function.

Definition 11: An (affine) parametric connection, ∇_* *on* Σ *is a mapping* $\nabla_*: \chi_*(\Sigma) \times \chi_*(\Sigma) \rightarrow \chi_*(\Sigma)$*, denoted by* $\nabla_*(X,Y) = \nabla_{*X} Y$*, which satisfies the following properties:*

(i) *linearity over* $\mathscr{F}_*(\Sigma)$: $\nabla_{*(fX+gY)} Z = f \nabla_{*X} Y + g \nabla_{*Y} Z$,

(ii) *linearity*: $\nabla_{*X}(Y+Z) = \nabla_{*X} Y + \nabla_{*X} Z$,

(iii) *derivation*: $\nabla_{*X}(fY) = X(f) Y + f \nabla_{*X} Y$ *for all* $X,Y,Z \in \chi_*(\Sigma)$, $f,g \in \mathscr{F}_*(\Sigma)$, *and* $X(f)$ *refers to the parametric action of* X *on* f.

As before, given $X \in \chi_*(\Sigma)$ one can consider the $\mathscr{R}$-linear mapping $\nabla_{*X}: \chi_*(\Sigma) \rightarrow \chi_*(\Sigma)$. Condition (iii) above and Ref. 5 guarantee that ∇_{*X} may be extended uniquely to a parametric tensor derivation on Σ. Thus, we may treat ∇_{*X} as a covariant derivative operator on any parametric tensor.

We next wish to show that given a parametric metric h on Σ, then there exists a unique parametric connection on Σ which is compatible with h and torsion-free. Hence, we need to define these last two properties.

Let h be a parametric metric on Σ, denoted by $\langle , \rangle$. Metric compatibility is defined in the usual way.

Definition 12: A parametric connection is said to be compatible with the parametric metric h provided

$$X(\langle Y,Z \rangle) = \langle \nabla_{*X} Y, Z \rangle + \langle Y, \nabla_{*X} Z \rangle.$$

Definition 13: The parametric torsion, T_**, of* ∇_* *is defined by*

$$T_*(X,Y)=\nabla_{*X}Y-\nabla_{*Y}X-[X,\ Y]_*.$$

If $T_*(X,Y)=0$ *for all* $X,Y\in\chi_*(\Sigma)$, *then* ∇_* *is said to be torsion-free.*

The following result generalizes to parametric connections the standard existence and uniqueness theorem for the Levi–Civita connection. The proof is identical to the proof of the standard result.[7]

Theorem 14: *There exists a unique torsion-free parametric connection compatible with h.*

Proof: Suppose that such a ∇_* exists. Then we have

$$X(\langle Y,Z\rangle)=\langle\nabla_{*X}Y,Z\rangle+\langle Y,\nabla_{*X}Z\rangle,$$

$$Y(\langle Z,X\rangle)=\langle\nabla_{*Y}Z,X\rangle+\langle Z,\nabla_{*Y}X\rangle,$$

$$-Z(\langle X,Y\rangle)=-\langle\nabla_{*Z}X,Y\rangle-\langle X,\nabla_{*Z}Y\rangle.$$

Adding the above equations yields

$$X(\langle Y,Z\rangle)+Y(\langle Z,X\rangle)-Z(\langle X,Y\rangle)=-\langle[Z,\ X]_*,Y\rangle+\langle[Y,\ Z]_*,X\rangle+\langle[X,\ Y]_*,Z\rangle+2\langle Z,\nabla_{*Y}X\rangle.$$

Therefore, $\nabla_{*Y}X$ is uniquely determined by

$$\langle Z,\nabla_{*Y}X\rangle=\tfrac{1}{2}(X(\langle Y,Z\rangle)+Y(\langle Z,X\rangle)-Z(\langle X,Y\rangle)+\langle[Z,\ X]_*,Y\rangle-\langle[Y,\ Z]_*,X\rangle-\langle[X,\ Y]_*,Z\rangle). \tag{4}$$

One may also use this equation to define ∇_*, thus proving its existence. ■

We can use Eq. (4) to write out the unique parametric connection ∇_* in a coordinate basis. If we let $h_{ij}=\langle\partial_i,\partial_j\rangle$, we can define the connection symbols by $\nabla_{*\partial_i}\partial_j=\gamma^k{}_{ij}\partial_k$. Equation (4) now gives us

$$\gamma^l{}_{ij}h_{lk}=\tfrac{1}{2}(h_{jk*i}+h_{ki*j}-h_{ij*k})$$

or

$$\gamma^k{}_{ij}=\tfrac{1}{2}h^{km}(h_{jm*i}+h_{mi*j}-h_{ij*m}).$$

Therefore, the connection symbols associated with ∇_* agree with the connection symbols associated with the projected covariant derivative D constructed in Ref. 1, which in turn agrees with Perjés.[3]

We now try to construct the curvature tensor associated with ∇_*. The most obvious definition of a curvature operator would be the operator

$$S(X,Y)Z=\nabla_{*X}\nabla_{*Y}Z-\nabla_{*Y}\nabla_{*X}Z-\nabla_{*[X,Y]_*}Z.$$

However, this turns out not to be function linear due to the fact that $[X,\ Y]_*f\neq XY(f)-YX(f)$. This can, however, be easily corrected, since we know why S is not function linear (the presence of deficiency). First, one must extend the action of $\mathscr{D}(X,Y)$ to tensors of rank $(p\text{-}q)$ by differentiating the components of an arbitrary tensor with respect to the parameter t. Since the action of ∂_t on p-forms is covariant, the result is a $(p\text{-}q)$ tensor. Therefore, we define

$$Z(X,Y)W=\nabla_{*X}\nabla_{*Y}W-\nabla_{*Y}\nabla_{*X}W-\nabla_{*[X,Y]_*}W-\mathscr{D}(X,Y)W$$

and it is easily checked that this is function linear as required. Such a definition makes use of the various derivative operators present in a parametric theory. Not only does the parametric manifold Σ have the natural parametric derivative operator ∇_*, but the covariant operation of differentiation with respect to the parameter is also present, since the deficiency operator is built out of this parametric derivative.

Given coordinates x^i, the components of Z may be computed as follows:

$$Z^l{}_{kij}\partial_l = Z(\partial_i,\partial_j)\partial_k = \nabla_{*\partial_i}\nabla_{*\partial_j}\partial_k - \nabla_{*\partial_j}\nabla_{*\partial_i}\partial_k - 0 - 0$$
$$= (\gamma^l{}_{jk*i} - \gamma^l{}_{ik*j} + \gamma^l{}_{mi}\gamma^m{}_{jk} - \gamma^l{}_{mn}\gamma^m{}_{ik})\partial_l .$$

Z is thus precisely the Zel'manov curvature reintroduced by Perjés[3] and discussed in more detail in Ref. 1.

VI. DISCUSSION

We have shown how to recapture the projective flavor of the Gauss–Codazzi formalism without introducing any projection operators. After defining the correct action of parametric vector fields on parametric functions, Eq. (2), and recapturing this action in the guise of an exterior derivative operator, the correct generalizations of Lie bracket, torsion, and affine connection naturally followed. Furthermore, in such an intrinsic setting the Zel'manov curvature tensor (used by Einstein, Bergmann, Zel'manov, and Perjés) is the most natural generalization of the Riemann curvature tensor.

However, as pointed out in Ref. 1, the Zel'manov curvature does *not* seem to be the natural choice in the generalized Gauss–Codazzi setting. Rather, the Gauss–Codazzi formalism leads to the "projected" curvature tensor ${}^\perp R$. Can one reproduce ${}^\perp R$ intrinsically? In terms of a coordinate basis, the difference between ${}^\perp R$ and Z is[1]

$${}^\perp R^l{}_{kij} - Z^l{}_{kij} = (M_{j*i} - M_{i*j})h^{lm}\left(M^2 M_{m*k} - M^2 M_{k*m} + \frac{\partial h_{km}}{\partial t}\right) = \mathscr{D}_{ji}h^{lm}\left(M^2\mathscr{D}_{mk} + \frac{\partial h_{km}}{\partial t}\right),$$

which involves both the deficiency $\mathscr{D}$ and the threading lapse function M. As discussed in Ref. 1, the appearance of M is due to the presence of a parameter t whose relationship to proper "time" is arbitrary. While we have an intrinsic definition for the deficiency, we cannot recover the lapse function without explicitly introducing it. If one is willing to add this additional structure, then one can of course also define ${}^\perp R$ "intrinsically," at least in terms of its components.

Abandoning ${}^\perp R$ for Z results in a curvature operator that can be defined entirely in terms of Σ and the parametric structure ω. However, we know in advance that Z will not possess all of the symmetries of the Riemann curvature tensor. In Ref. 1 it was shown that ${}^\perp R$ is the unique curvature satisfying Gauss's equation and, hence, enjoying all of the inherited symmetries of the Riemann tensor (where the first Bianchi identity for ${}^\perp R$ resembled the identity in the presence of torsion), whereas Z only enjoys some of these symmetries, namely,[3]

(i) $Z(X,Y)W = -Z(Y,X)W$ and

(ii) $Z(X,Y)W + Z(Y,W)X + Z(W,X)Y = 0$.

In the absence of deficiency, a parametric manifold can be viewed as a one-parameter family of hypersurfaces embedded in $\Sigma\times\mathscr{R}$ orthogonal to $\omega(t)-dt$, i.e., such that $\omega(t)-dt$ annihilates all vector fields tangent to the hypersurfaces. The metric on $\Sigma\times\mathscr{R}$ is not fully determined, but requires a specification of the relationship between the parameter t and arclength along the orthogonal curves, i.e., the lapse function M. Nevertheless, the notion of orthogonal curves is well defined.

Another special case is when the physical fields, including both the parametric metric and the parametric structure, do not depend on the parameter t. In this case, the action of vector fields on (physical) functions reduces to ordinary partial differentiation, and the parametric connection reduces to the Levi–Civita connection of the "parametric" metric, which is now a (usual) metric on the manifold of orbits. Parametric manifolds in this setting are thus equivalent to the formalism given by Ref. 8 for space–times with (not necessarily hypersurface-orthogonal) Killing vectors. Some of these notions are related to recent work by Harris and co-workers on orbit spaces[9] and static space–times.[10]

But even when only the parametric structure is independent of the parameter, in the sense that $\omega(t)$ in fact has no t dependence, the structure described here reduces to something more familiar. Parametric exterior differentiation can be viewed as a connection on the fiber bundle $\Sigma\times\mathscr{R}$ over Σ precisely when the *horizontal subspaces* defined by $\omega-dt$ do not depend on t. This means that parametric manifolds can be viewed as a *generalized fiber bundle*. As Perjés has already pointed out,[3] this could lead to a generalization of Yang–Mills (gauge) theory. Work on these issues is continuing.

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