Parametric Resonance in Immersed Elastic Boundaries

John Stockie

Department of Mathematics
Simon Fraser University
Burnaby, British Columbia, Canada

stockie@math.sfu.ca
http://www.math.sfu.ca/~stockie/

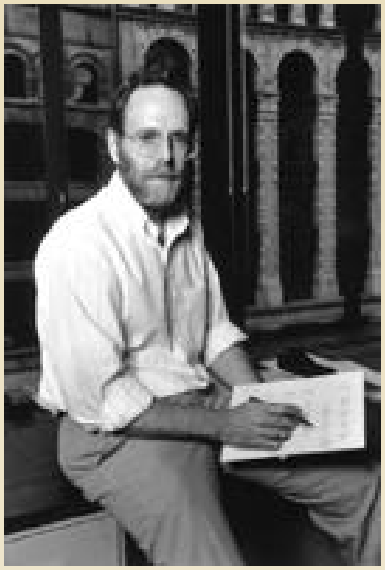
November 2, 2004

Acknowledgments

Ricardo Cortez
Tulane University

Douglas Varela
CalTech

Charles Peskin
Courant Institute



Natural Sciences and Engineering
Research Council of Canada

NSERC CRSNG

TEXpower document class: **http://ls1-www.cs.uni-dortmund.de/~lehmke/texpower/**

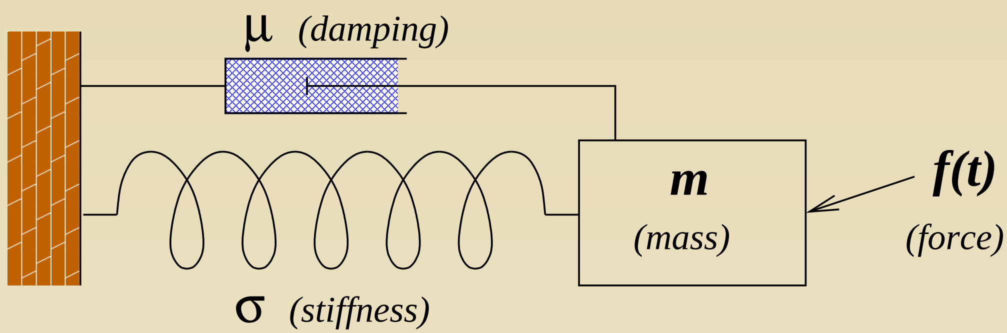
Outline

1. Review of
   - "normal" resonance $\Longrightarrow$ external forcing
   - parametric resonance $\Longrightarrow$ internal forcing
2. Floquet stability analysis for parametrically-excited springs
3. Immersed boundary model
4. Parametric resonance in immersed boundaries
5. Numerical simulations
6. Summary and future work

1. Review of Resonance

For a damped, mass-spring system:

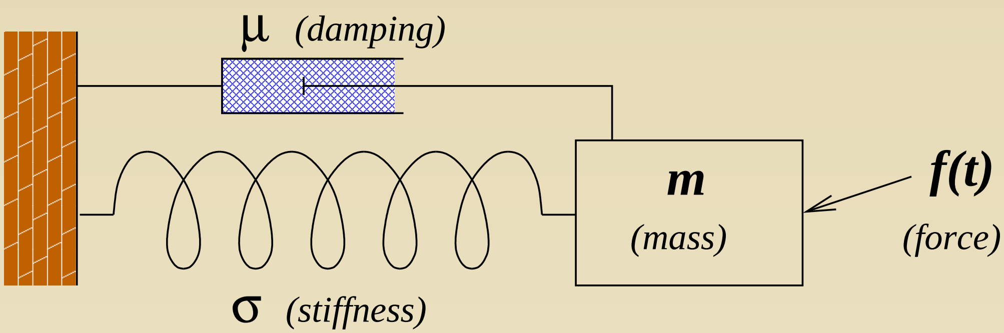
$$m\ddot{x} + \mu\dot{x} + \sigma x = f(t)$$



1. Review of Resonance

For a damped, mass-spring system:

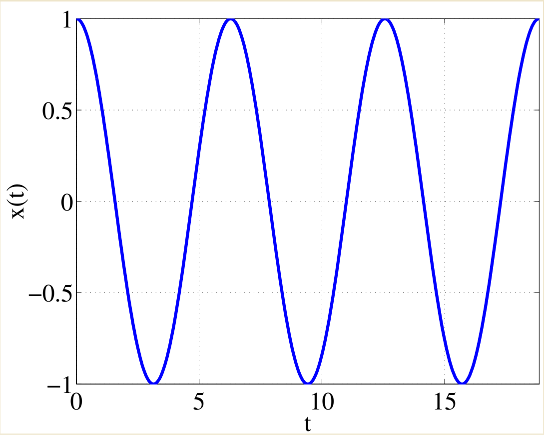
$$m\ddot{x} + \mu\dot{x} + \sigma x = f(t)$$



I. Unforced or Natural Oscillations, $f(t) \equiv 0$:

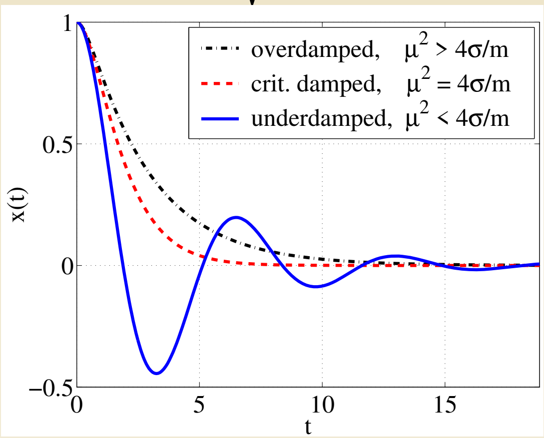
(a) Undamped system, $\mu = 0$:

sustained oscillations with natural frequency $\omega_o = \sqrt{\frac{\sigma}{m}}$



(b) Damped system, $\mu > 0$:

decaying oscillations with natural frequency $\omega_o = \sqrt{\frac{\sigma}{m} - \frac{\mu^2}{4}}$



II. Externally-forced Oscillations:

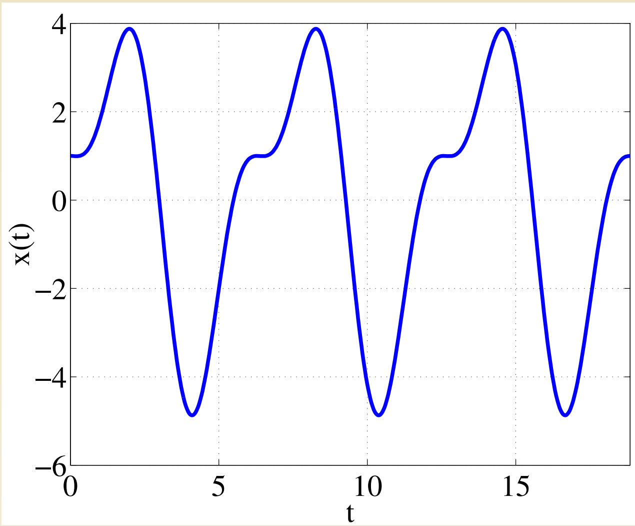
The system is subjected to an external periodic force
(modeled as a **separate term** in the DE)

$$m\ddot{x} + \mu\dot{x} + \sigma x = F_o \cos \omega t$$

Resonance occurs when forcing is close to the natural frequency
$\Longrightarrow$ amplitude of resulting oscillations grows when undamped ($\mu = 0$)

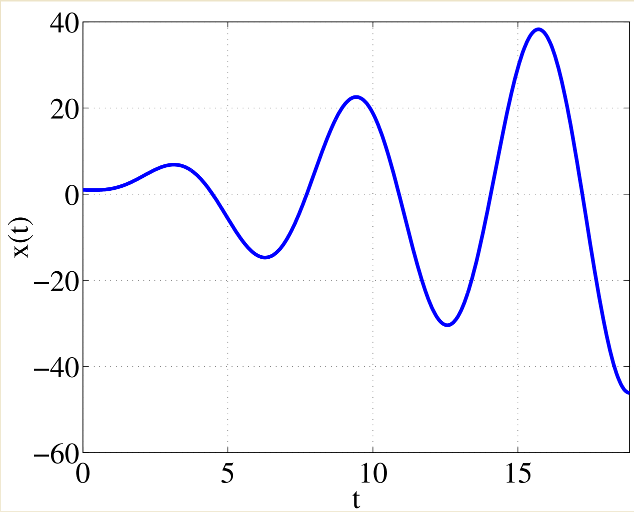
(a) Undamped, non-resonant ($\mu = 0$, $\omega \neq \omega_o$):

persistent, mixed-frequency oscillations



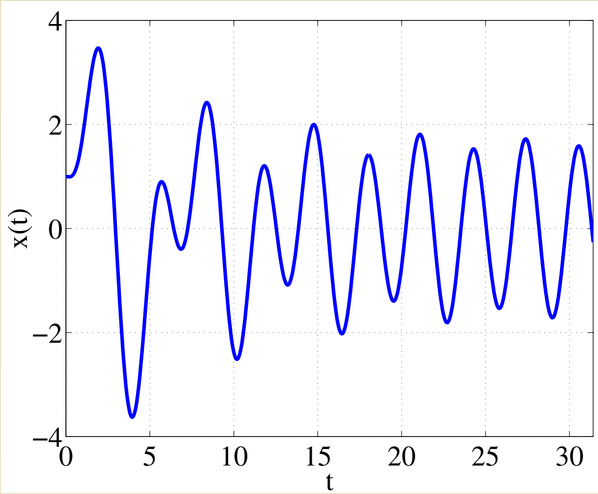
(b) Undamped, resonant ($\mu = 0$, $\omega = \omega_o$):

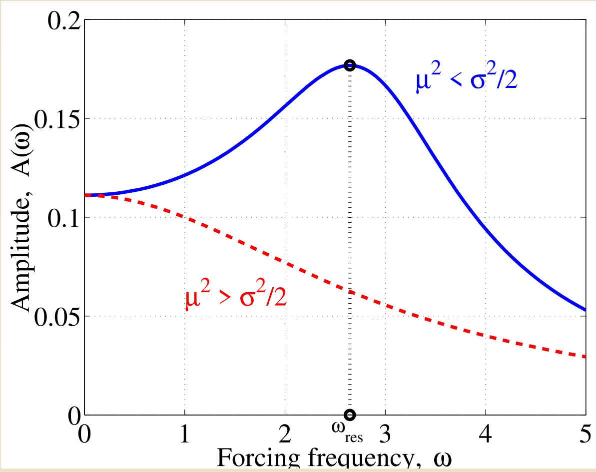
unstable, unbounded oscillations



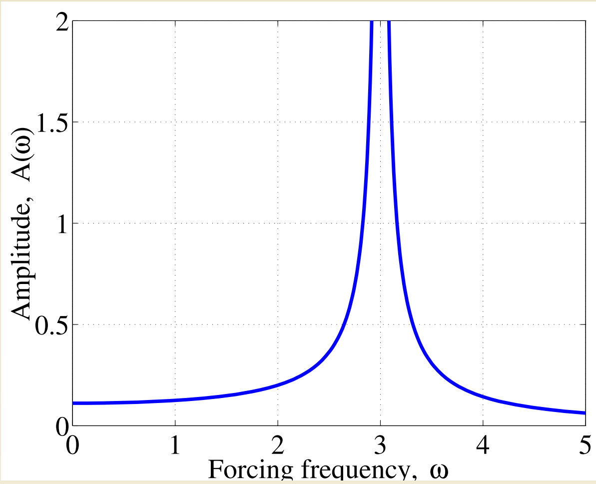
(c) With damping ($\mu > 0$) — “real” systems:

- oscillations are **bounded**, transient dies out, and forcing persists
- **resonance** appears as a peak in max. amplitude, only if $\mu^2 < \frac{1}{2}\sigma^2$





Compare to the frequency–response curve for the **undamped system**:



III. Internal forcing:

Oscillations can also be excited via periodic variation in a **system parameter:**

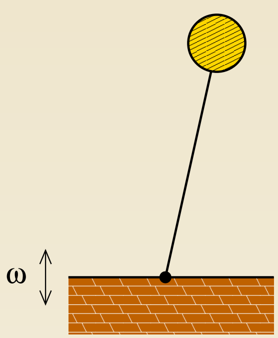
$$m\ddot{x} + \mu\dot{x} + \sigma(t)x = 0$$

$\sigma(t)$ periodic:	$\sigma(t) = \sigma(t+T)$	$\Longrightarrow$	**Hill's equation (1886)**
special case:	$\sigma(t) = \sigma_o(1 + 2\varepsilon\cos\omega t)$	$\Longrightarrow$	**Mathieu equation (1868)**

Compare:

- Solution can become **unstable** whether or not $\mu = 0$ **!**
  $\Longrightarrow$ called **parametric resonance**
- The system responds at frequency $\frac{1}{2}\omega$ [Why?]
- Internal forcing can also **stabilize** systems that are otherwise unstable

e.g., inverted pendulum with gravitational modulation



A Heuristic Look at Parametric Resonance

Take $m = 1$, $\mu = 0$, and treat the periodic term as a forcing term:

$$\boldsymbol{\ddot{x} + \omega_o^2 x = -2\varepsilon\omega_o^2(\cos\omega t)x}$$

where $\boldsymbol{\omega_o^2 = \sigma_o}$.

First solve the homogeneous problem:

$$\ddot{x} + \omega_o^2 x = 0 \qquad \Longrightarrow \qquad x(t) = A\cos(\omega_o t - \varphi)$$

Substitute into the right hand side:

$$\begin{aligned}\ddot{x} + \omega_o^2 x &= -2A\varepsilon\omega_o^2 \cos\omega t\ \cos(\omega_o t - \varphi) \\ &= -A\varepsilon\omega_o^2\Big\{\cos\big[(\omega + \omega_o)t - \varphi\big] + \cos\big[\boldsymbol{(\omega - \omega_o)}t - \varphi\big]\Big\}\end{aligned}$$

Resonance ensues if $\quad \omega - \omega_o = \omega_o \quad$ or $\qquad \omega_o = \frac{1}{2}\omega$

$$\begin{matrix}\text{response}\\ \text{frequency}\end{matrix} = \frac{1}{2}\begin{pmatrix}\text{forcing}\\ \text{frequency}\end{pmatrix}$$

Examples of Parametric Resonance

- Springs or pendula with moving supports

Examples of Parametric Resonance

- Springs or pendula with moving supports
- RLC circuits with periodically-varying inductance or capacitance

Examples of Parametric Resonance

- Springs or pendula with moving supports
- RLC circuits with periodically-varying inductance or capacitance
- A child pumping a swing [Curry, 1976]

Examples of Parametric Resonance

- Springs or pendula with moving supports
- RLC circuits with periodically-varying inductance or capacitance
- A child pumping a swing [Curry, 1976]
- Solar surface heating generating atmospheric convection cells or patterned ground formation [McKay, 1998]

Examples of Parametric Resonance

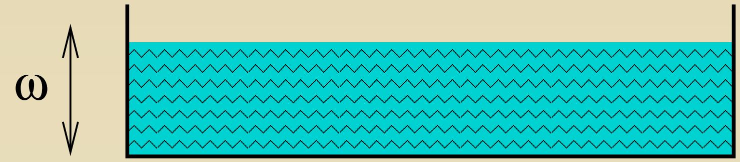
- Springs or pendula with moving supports
- RLC circuits with periodically-varying inductance or capacitance
- A child pumping a swing [Curry, 1976]
- Solar surface heating generating atmospheric convection cells or patterned ground formation [McKay, 1998]
- Surface waves in fluids and granular materials $\Longrightarrow$ Faraday instability
  [Faraday, 1831; Benjamin & Ursell, 1954]

Examples of Parametric Resonance

- Springs or pendula with moving supports
- RLC circuits with periodically-varying inductance or capacitance
- A child pumping a swing [Curry, 1976]
- Solar surface heating generating atmospheric convection cells or patterned ground formation [McKay, 1998]
- Surface waves in fluids and granular materials $\Longrightarrow$ Faraday instability [Faraday, 1831; Benjamin & Ursell, 1954]
- Two-fluid interfaces [Kumar & Tuckerman, 1994]
- Bubble oscillations and sonoluminescence [Brenner, Lohse & Dupont, 1995]
- Many problems in aerodynamics involving **wing stability** and **flutter** (but wing oscillations are small and have no influence on the flow)

There are **many** examples in fluids, but almost no analysis for **fluid-structure interaction problems**.

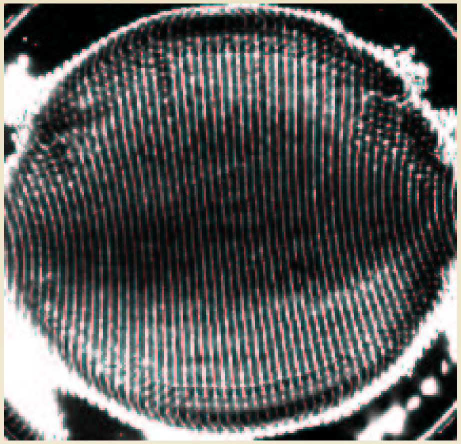
Faraday Waves



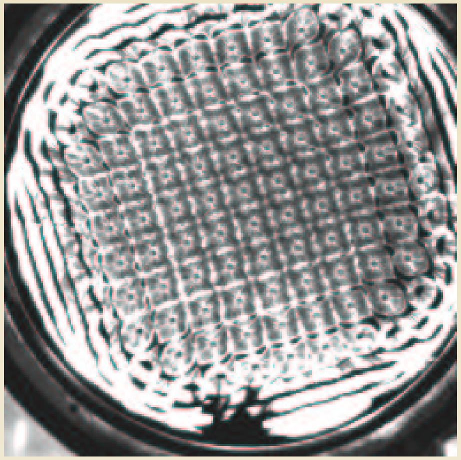
$$F(t) = mg(1 + 2\varepsilon \cos \omega t)$$

Faraday (1831) observed that:

- regular, symmetric patterns arise on the surface
- waves oscillate with **twice the period** of the parametric forcing



Stripes (1-fold symmetry)



Squares (2-fold symmetry)

Hexagons (3-fold symmetry)

Source: Dept. of Physics, University of Toronto
http://mobydick.physics.utoronto.ca/faraday.html

Some Results from Floquet Theory

For homogeneous, periodic, linear systems:

$$\dot{x} = A(t) \cdot x, \qquad \text{where } A(t) \in \mathbb{R}^{n \times n},\ x(t) \in \mathbb{R}^n \qquad (*)$$

$$\text{and} \qquad A(t+T) = A(t), \qquad \text{for some } T > 0.$$

Floquet Theorem (Bloch Theorem in solid state physics):

Every fundamental matrix solution $X(t)$ of (*) has the form:

$$X(t) = e^{Bt} \cdot P(t)$$

$$\text{where} \quad P(t+T) = P(t) \quad \text{and} \quad B = \text{constant}.$$

Corollary: Any solution of (*) is a linear combination of functions of the form:

$$e^{\gamma t} \cdot p(t)$$

"Floquet" or characteristic exponent, $\gamma \in \mathbb{C}$

a polynomial with T-periodic coefficients

2. Floquet Analysis of the Mathieu equation $\ddot{x} + \mu\dot{x} + \sigma_o(1 + 2\varepsilon\cos\omega t)x = 0$

- Cannot have a solution $x(t) = A\cos\omega t + B\sin\omega t$
- Instead, the ω–periodic term excites **all Fourier modes** with frequencies $n\omega$, $n = 1, 2, \ldots$

$\Longrightarrow$ Look for solutions of the form (details in [Nayfeh & Mook, 1979])

$$\boxed{x(t) = e^{\gamma t} \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \quad \text{where} \quad \gamma = \alpha + i\beta}$$

α governs stability:			**β is the frequency (consider only $0 \leq \beta \leq \frac{1}{2}\omega$):**		
$\alpha > 0$	$\longrightarrow$	unstable	$\beta = 0$	$\longrightarrow$	**harmonic**
$\alpha < 0$	$\longrightarrow$	stable	$\beta = \frac{1}{2}\omega$	$\longrightarrow$	**subharmonic**
$\alpha = 0$	$\longrightarrow$	**stability boundary**	$0 < \beta < \frac{1}{2}\omega$	$\longrightarrow$	decaying solutions (ignore these)

Substitute

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{(\gamma + in\omega)t} \qquad \text{into} \qquad \ddot{x} + \mu\dot{x} + \sigma_o(1 + 2\varepsilon\cos\omega t)x = 0$$

Interested only in the stability boundary where $\alpha = 0$ (i.e., $\gamma = 0 + i\beta$):

$$\sum_{n=-\infty}^{\infty} c_n e^{i(\beta+n\omega)t} \left\{ -(\beta + n\omega)^2 + i\mu(\beta + n\omega) + \sigma_o \left[1 + \varepsilon \underbrace{\left(e^{i\omega t} + e^{-i\omega t} \right)}_{\textbf{shifts n by } \pm 1} \right] \right\} = 0$$

Substitute

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{(\gamma + in\omega)t} \qquad \text{into} \qquad \ddot{x} + \mu\dot{x} + \sigma_o(1 + 2\varepsilon\cos\omega t)x = 0$$

Interested only in the stability boundary where $\alpha = 0$ (i.e., $\gamma = 0 + i\beta$):

$$\sum_{n=-\infty}^{\infty} c_n e^{i(\beta + n\omega)t} \left\{ -(\beta + n\omega)^2 + i\mu(\beta + n\omega) + \sigma_o \left[1 + \varepsilon \underbrace{\left(e^{i\omega t} + e^{-i\omega t} \right)}_{\text{shifts } n \text{ by } \pm 1} \right] \right\} = 0$$

$$\Longrightarrow \quad [\sigma_o - (\beta + n\omega)^2 + i\mu(\beta + n\omega)]c_n + \varepsilon\sigma_o(c_{n-1} + c_{n+1}) = 0 \qquad (**)$$

Aim: To find real values of ε:

- split (**) into real and imaginary parts,
- let $c_n = c_n^r + ic_n^i$, and . . .

define $\vec{c} = [\ldots, c^r_{n-1}, c^i_{n-1}, c^r_n, c^i_n, \ldots]^T$ so (**) can be written in matrix form:

$$\begin{bmatrix} & \ddots & & & & \\ 0 & 0 & A_n & -B_n & 0 & 0 \\ 0 & 0 & B_n & A_n & 0 & 0 \\ & & & & \ddots & \end{bmatrix} \vec{c} + \varepsilon\sigma_o \begin{bmatrix} & \ddots & & & & \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ & & & & \ddots & \end{bmatrix} \vec{c} = 0$$

define $\vec{c} = [\ldots, c^r_{n-1}, c^i_{n-1}, c^r_n, c^i_n, \ldots]^T$ so (**) can be written in matrix form:

$$\underbrace{\begin{bmatrix} & \ddots & & & & \\ 0 & 0 & A_n & -B_n & 0 & 0 \\ 0 & 0 & B_n & A_n & 0 & 0 \\ & & & & \ddots & \end{bmatrix}}_{D} \vec{c} + \varepsilon\sigma_o \underbrace{\begin{bmatrix} & \ddots & & & & \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ & & & & \ddots & \end{bmatrix}}_{E} \vec{c} = 0$$

where $\quad A_n = \sigma_o - (\beta + n\omega)^2 \quad$ and $\quad B_n = \mu(\beta + n\omega)$.

More compactly:

$$\boxed{(-\boldsymbol{\sigma_o D^{-1} E})\vec{\boldsymbol{c}} = \left(\frac{\mathbf{1}}{\boldsymbol{\varepsilon}}\right) \vec{\boldsymbol{c}},}$$

An **infinite-dimensional** eigenvalue problem with **eigenvalues $\frac{1}{\varepsilon}$, and eigenvectors $\vec{c}$.**

Details:

- Cut off at a **finite N**, i.e., $\displaystyle\sum_{-\infty}^{\infty} \longrightarrow \sum_{-N}^{N}$ (since $c_n \to 0$ as $|n| \to \infty$)

- Apply **"reality conditions"**:

  - $c_{-n} = c_n^*$, if $\beta = 0$ (harmonic)
  - $c_{-n} = c_{n-1}^*$, if $\beta = \frac{1}{2}\omega$ (subharmonic)

  to eliminate all c_n with $n < 0$ $\Longrightarrow$ a $(2N+2) \times (2N+2)$ linear system

- Parameter values:

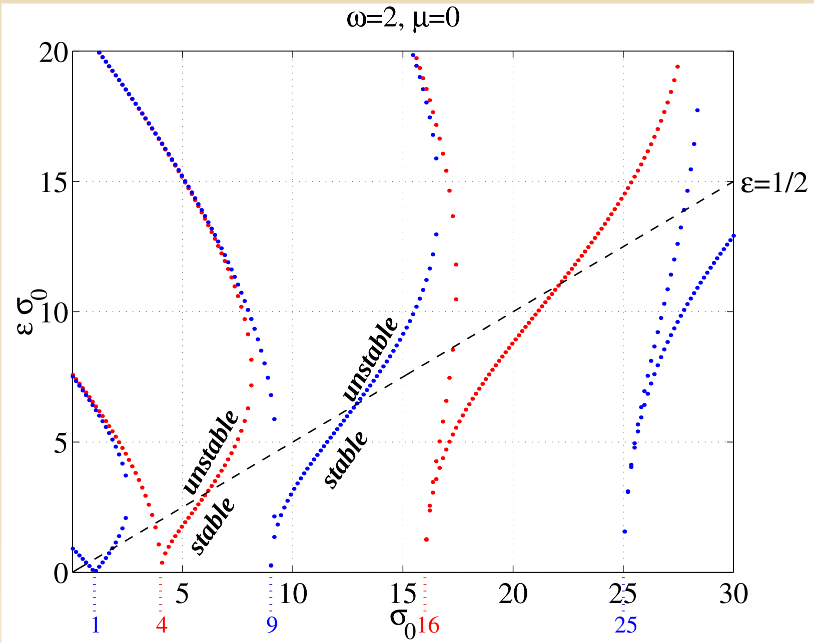
  μ, ω: given
  $\alpha = 0$: for stability boundaries
  $\beta = 0,\ \frac{1}{2}\omega$: for harmonic/subharmonic cases
  $N = 15$: chosen so that $|c_n| \ll 1$ for all $|n| > N$

  ε: comes from solving the eigenvalue problem
  σ_o: is a free parameter

Basic Idea: for each σ_o, we obtain $2N+2$ values of ε
. . . pick the positive, real ones.

Stability of the Mathieu equation
$\ddot{\mathbf{x}} + \mu\dot{\mathbf{x}} + \sigma_o(1 + 2\varepsilon\cos\omega t)\mathbf{x} = 0$

Ince-Strutt Diagram ($\omega = 2,\ \mu = 0$):

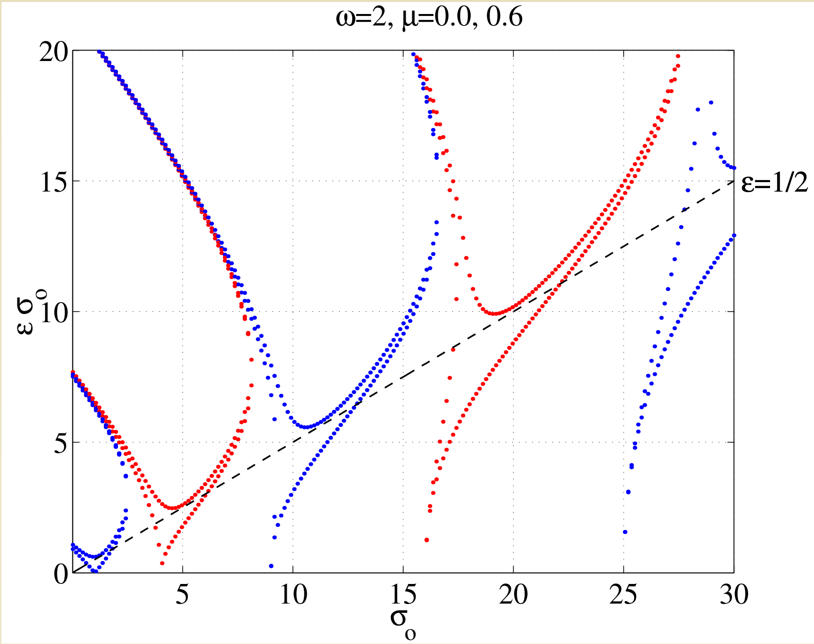


... $=$ harmonic $(\beta = 0)$

... $=$ subharmonic $(\beta = \frac{1}{2}\omega)$

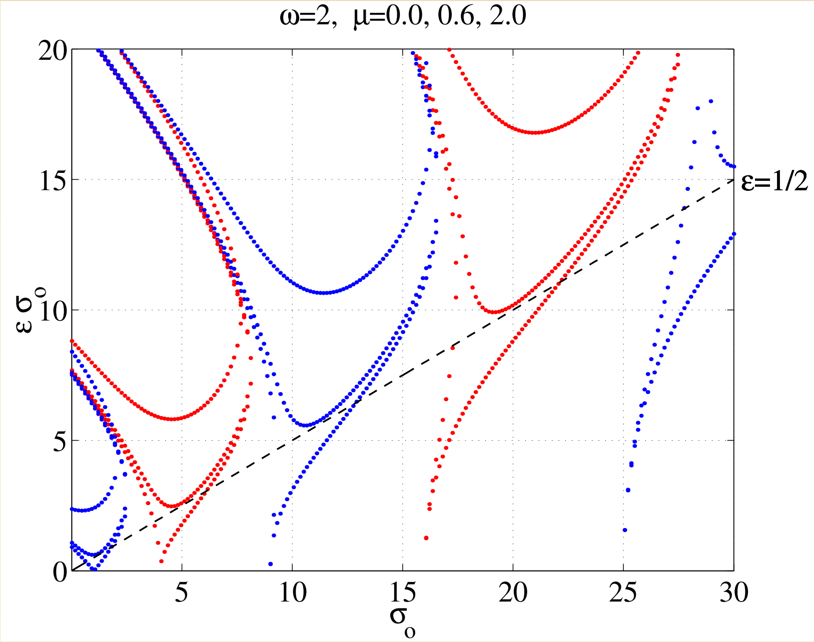
- Eigenvalues divide the plane into stable and unstable regions $\Longrightarrow$ **tongues** of instability
- $\varepsilon \le \frac{1}{2}$ are "physical" eigenvalues
- $\varepsilon = 0$ gives natural modes, $\boldsymbol{\sigma_o = n^2}$

Stability of the Mathieu equation $\ddot{x} + \mu\dot{x} + \sigma_o(1 + 2\varepsilon\cos\omega t)x = 0$



Increase damping further to $\mu = 0.6$
$\Longrightarrow$ no more **physical** instabilities

Stability of the Mathieu equation
$\ddot{\mathbf{x}} + \mu\dot{\mathbf{x}} + \sigma_o(1 + 2\varepsilon \cos \omega t)\mathbf{x} = 0$



Increase damping to $\mu = 2.0$

[movie]

Note:

- even small perturbations ($\varepsilon \approx 0$) can lead to instability if μ is small enough
- damping has a **stabilizing** influence
  $\Longrightarrow$ the $\omega = 2$ problem is stable if $\mu \gtrapprox 0.6$

3. Immersed Boundaries

Move on to fluid-structure interaction problems ...

"Life is . . . fiber-reinforced fluid."

– C. S. Peskin (1999)

3. Immersed Boundaries

Move on to fluid-structure interaction problems ...

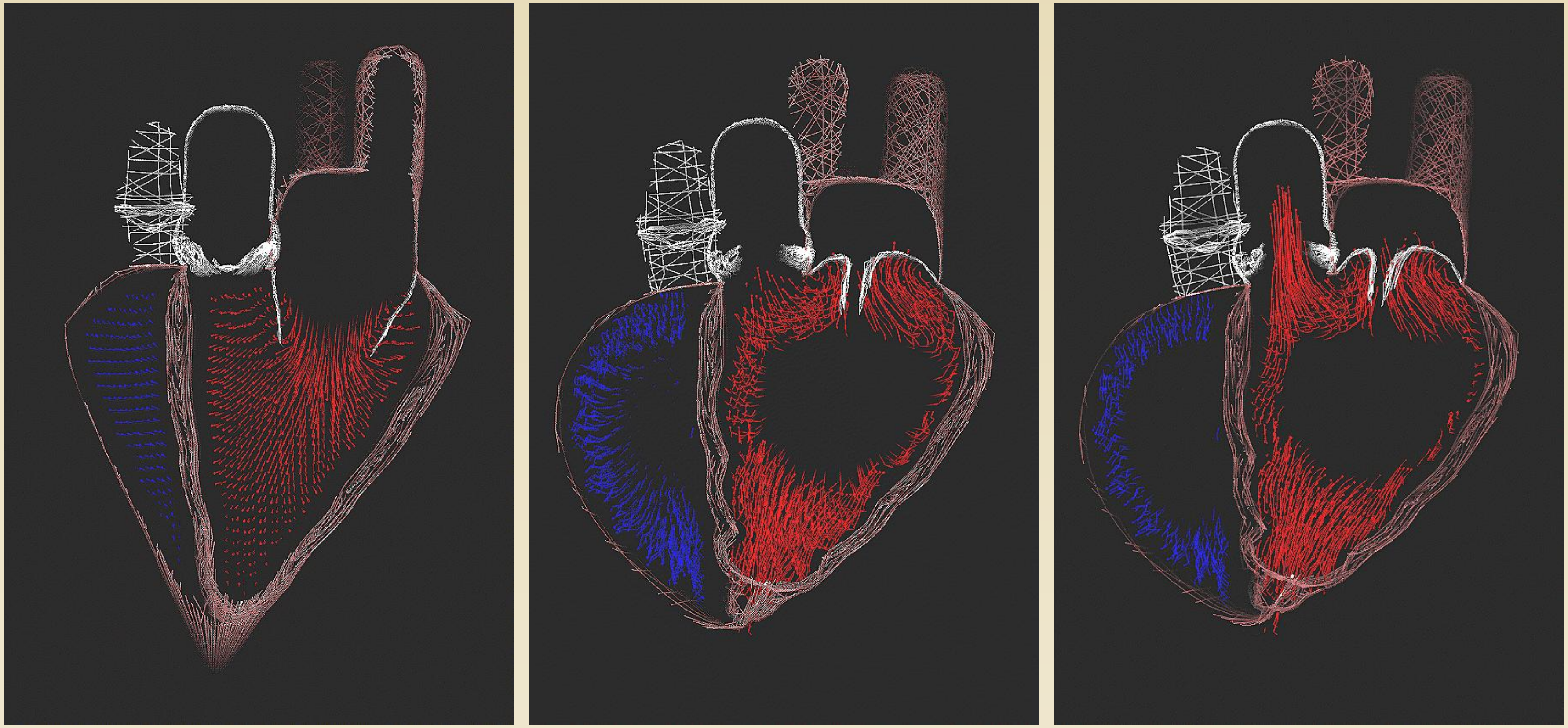
> ***"Life is . . . fiber-reinforced fluid."***
>
> – C. S. Peskin (1999)

Biological fibers, and surfaces constructed of fibers, immersed in fluid:

- heart and blood vessels
- worms and leeches
- flagellae and cilia
- plant cells (esp. wood)
- microtubules and actin filaments
- suspensions of proteins, DNA, polymers, etc.

Example: The Heart

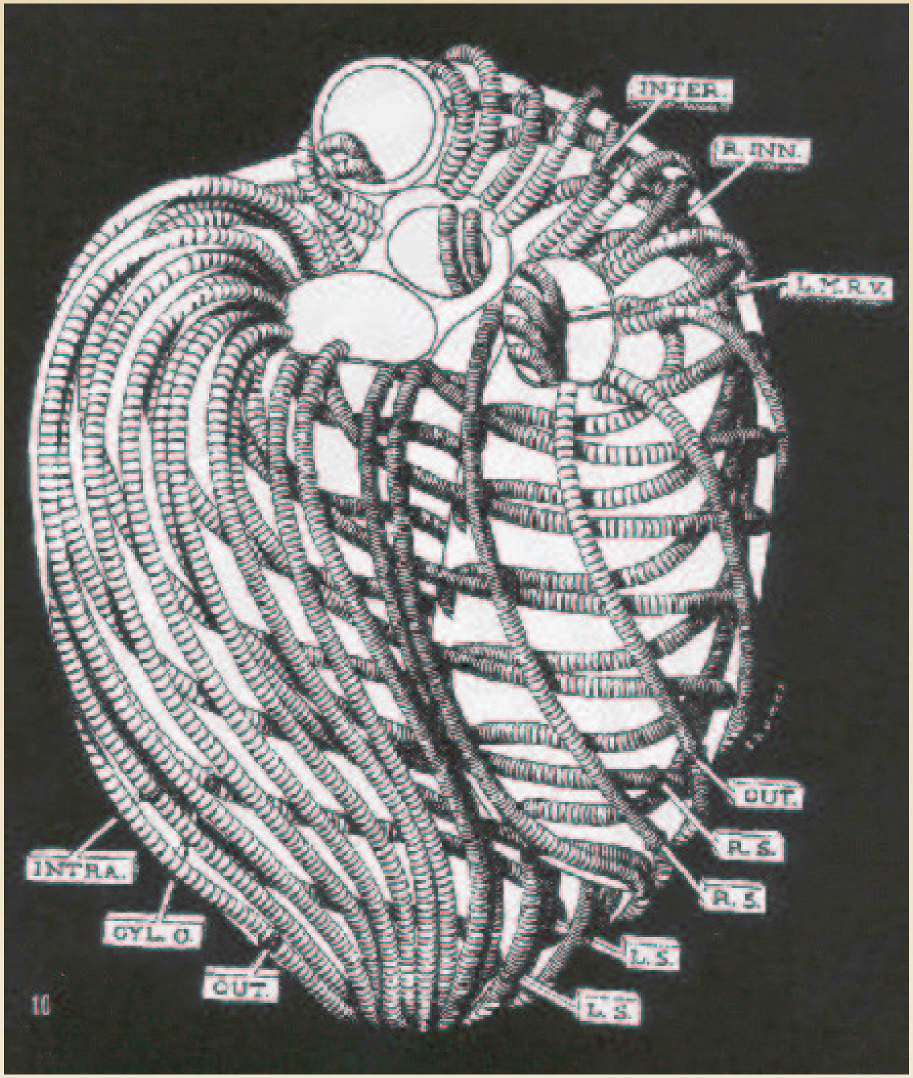
Peskin and McQueen (2000): Simulations of the beating heart



Source: http://www.psc.edu/science/Peskin/Peskin.html

[movie0] [movie1] [movie2] [movie3]

Fiber Architecture of the Heart Muscle Wall



[Dissected pig heart: Carolyn Thomas, 1957]

The Immersed Boundary Model

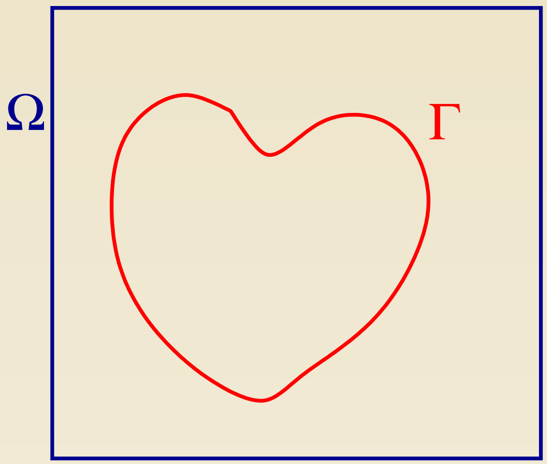
Simplifying assumptions:

- 2D fluid with a single, impermeable, elastic membrane (or fiber)
- fiber has zero volume and mass, and is neutrally buoyant
- fluid lies both inside **and** outside the fiber
- domain is periodic
- in 3D, surfaces are built from an **interwoven mesh** of fibers

The Immersed Boundary Model

Simplifying assumptions:

- 2D fluid with a single, impermeable, elastic membrane (or fiber)
- fiber has zero volume and mass, and is neutrally buoyant
- fluid lies both inside **and** outside the fiber
- domain is periodic
- in 3D, surfaces are built from an **interwoven mesh** of fibers



$\Omega =$ fluid domain
$p(\vec{x}, t) =$ fluid pressure
$\vec{u}(\vec{x}, t) =$ fluid velocity

$\Gamma =$ fiber
$s =$ arclength parameter
$\vec{X}(s, t) =$ fiber position
$\vec{U}(s, t) =$ fiber velocity
$\vec{F}(s, t) =$ fiber force density $= \sigma \partial_s^2 \vec{X}$ ("springs")

Dynamic interaction between fluid and fiber:

- the fiber exerts a **singular force** on adjacent fluid particles:

$$\rho\,(\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u}) = \mu\Delta\vec{u} - \nabla p \; + \int_\Gamma \vec{F}(s,t)\;\delta(\vec{x}-\vec{X}(s,t))\;ds$$

$$\nabla\cdot\vec{u} = 0$$

(Navier-Stokes equations with a singular force)

Dynamic interaction between fluid and fiber:

- the fiber exerts a **singular force** on adjacent fluid particles:

$$\rho\,(\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u}) = \mu\Delta\vec{u} - \nabla p \ + \int_\Gamma \vec{F}(s,t)\ \delta(\vec{x} - \vec{X}(s,t))\ ds$$
$$\nabla\cdot\vec{u} = 0$$

(Navier-Stokes equations with a singular force)

- fiber moves at the fluid velocity – **the no slip condition**:

$$\partial_t \vec{X} = \vec{u}(\vec{X}(s,t),t) = \int_\Omega \vec{u}(\vec{x},t)\ \delta(\vec{x} - \vec{X}(s,t))\ d\vec{x}$$

$\Longrightarrow$ interactions are mediated by **delta functions!**

An alternate formulation eliminates delta functions in favour of **jump conditions** $\Longrightarrow$ immersed interface method [Leveque & Li, 1994]

Previous Work

1. A linear stability analysis for **unforced** fibers, initially deformed and then released [JS & Wetton, 1995]:
   - unconditionally stable $(\alpha > 0)$
   - dependence of solution on parameters $(Re,\ \sigma_o)$ is non-trivial

Previous Work

1. A linear stability analysis for **unforced** fibers, initially deformed and then released [JS & Wetton, 1995]:
   - unconditionally stable $(\alpha > 0)$
   - dependence of solution on parameters $(Re,\ \sigma_o)$ is non-trivial

2. A nonlinear stability analysis for a passive circular membrane in an **inviscid fluid** [Cortez & Varela, 1997]

Previous Work

1. A linear stability analysis for **unforced** fibers, initially deformed and then released [JS & Wetton, 1995]:
   - unconditionally stable $(\alpha > 0)$
   - dependence of solution on parameters $(Re,\ \sigma_o)$ is non-trivial

2. A nonlinear stability analysis for a passive circular membrane in an **inviscid fluid** [Cortez & Varela, 1997]

Question:
What happens when a fiber is pulsed periodically
(like a heart muscle fiber)?

Previous Work

1. A linear stability analysis for **unforced** fibers, initially deformed and then released [JS & Wetton, 1995]:
   - unconditionally stable ($\alpha > 0$)
   - dependence of solution on parameters (Re, σ_o) is non-trivial

2. A nonlinear stability analysis for a passive circular membrane in an **inviscid fluid** [Cortez & Varela, 1997]

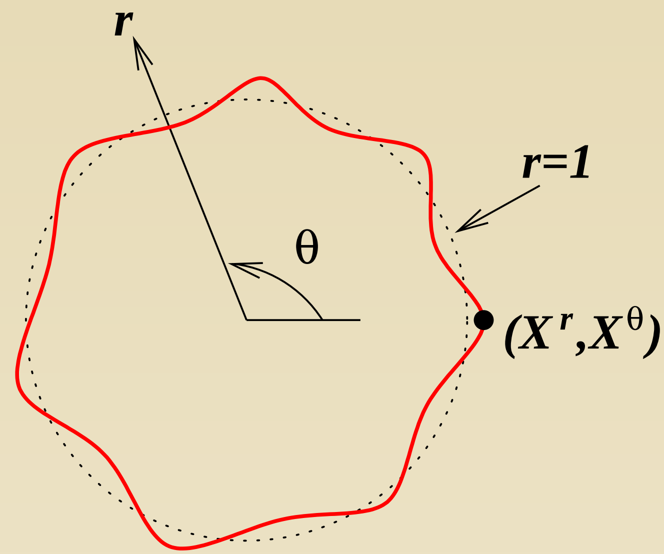
Question:
What happens when a fiber is pulsed periodically
(like a heart muscle fiber)?

3. A related problem [Wang, 2003]:
   - Floquet-type analysis for buckling instabilities in a headbox, assuming **small deformations**
   - flow oscillations drive the structure BUT the structure has **no effect** on the fluid

4. Stability Analysis for Immersed Boundaries

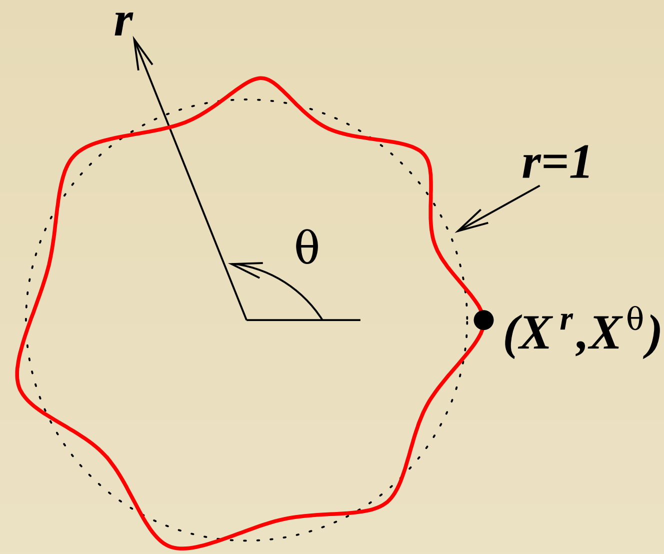
Investigate the stability of a perturbed **circular** fiber with periodically-varying stiffness

$$\sigma(t) = \sigma_o(1 + 2\varepsilon \cos \omega t)$$



4. Stability Analysis for Immersed Boundaries

Investigate the stability of a perturbed **circular** fiber with periodically-varying stiffness



$$\sigma(t) = \sigma_o(1 + 2\varepsilon\cos\omega t)$$

Simplifications:

- Assume **small** perturbations about $r = 1$
- Linearize the Navier–Stokes equations $\Longrightarrow$ **Stokes' equations**
- Convert to stream function & vorticity: $u,\ v \quad\longrightarrow\quad \psi,\ \xi$
- Integrate the Navier-Stokes equations across the fiber
  $\Longrightarrow$ eliminates δ–functions in favour of **jump conditions**

The Stream Function–Vorticity Formulation

Inner ($r < 1$) and outer ($r > 1$) solutions both obey:

$$\Delta\psi = -\xi$$

$$\partial_t \xi = \frac{\mu}{\rho}\Delta\xi$$

Immersed boundary equation:

$$\partial_t \vec{X} = \left(\partial_\theta \psi, -\partial_r \psi\right)\big|_{r=1}$$

Jump conditions, with $[\![\cdot]\!] \doteq (\cdot)|_{r=1^+} - (\cdot)|_{r=1^-}$:

$$[\![\psi]\!] = 0, \qquad [\![\xi]\!] = \frac{p^2\sigma_o}{i\omega\mu}\left(\psi(1) - \partial_r\psi(1)\right)$$

$$[\![\partial_r\psi]\!] = 0, \qquad [\![\partial_r\xi]\!] = \frac{p^2(p^2-1)\sigma_o}{i\omega\mu}\psi(1)$$

Boundary conditions:

ψ and ξ bounded as $r \to \infty$

periodic matching at $\theta = 0$ and 2π

Floquet–type solutions:

For ξ (and ψ):

$$\xi(r,\theta,t) = \underbrace{e^{ip\theta}}_{\substack{\text{periodic}\\ \text{in } \theta}} \sum_{n=-\infty}^{\infty} \xi_n(r) \underbrace{e^{(\gamma+in\omega)t}}_{\text{same as before}}$$

For X^r (and X^θ):

$$X^r(\theta,t) = e^{ip\theta} \sum_{n=-\infty}^{\infty} X_n^r \, e^{(\gamma+in\omega)t}$$

Substitute and obtain a **Bessel equation** for ξ_n:

$$z^2\xi_n''(z) + z\xi_n'(z) + (z^2 - p^2)\xi_n(z) = 0$$

$$\text{where} \quad z \doteq -(\Omega_n\, r)^2 \quad \text{and} \quad \Omega_n \doteq \sqrt{\frac{\rho(\gamma + in\omega)}{\mu}}$$

Solution: can be written in terms of Bessel functions, J_p and H_p:

$$\xi_n(r) = \begin{cases} a_n H_p(i\Omega_n r), & \text{if } r > 1 \text{ (outer)} \\ b_n J_p(i\Omega_n r), & \text{if } r < 1 \text{ (inner)} \end{cases}$$

and $$\psi_n(r) = \begin{cases} \text{combinations of } J_p,\ H_p,\ r^{-p} \text{ and } r^p \\ \ldots \end{cases}$$

Applying jump conditions yields the two equations:

$$\begin{aligned} 0 = i &\left\{ \frac{\mu^2}{\sigma_o}\,\Omega_n^3 \left[\frac{H_p(i\Omega_n)}{H_{p-1}(i\Omega_n)} - \frac{J_p(i\Omega_n)}{J_{p+1}(i\Omega_n)} \right] + ip \right\} X_n^r \\ &+ \left\{ \frac{\mu^2}{\sigma_o}\,\Omega_n^3 \left[\frac{H_p(i\Omega_n)}{H_{p-1}(i\Omega_n)} + \frac{J_p(i\Omega_n)}{J_{p+1}(i\Omega_n)} \right] - ip^2 \right\} X_n^\theta \\ &+ i\varepsilon p \left(X_{n-1}^r - X_{n+1}^r \right) - \varepsilon p^2 \left(X_{n-1}^\theta - X_{n+1}^\theta \right) \\ 0 = i &\left\{ \frac{\mu^2}{\sigma_o}\,\Omega_n^4 \left[2 - \frac{H_{p+1}(i\Omega_n)}{H_{p-1}(i\Omega_n)} - \frac{J_{p-1}(i\Omega_n)}{J_{p+1}(i\Omega_n)} \right] + 2p(p^2-1) \right\} X_n^r \\ &+ \frac{\mu^2}{\sigma_o}\,\Omega_n^4 \left[\frac{H_{p+1}(i\Omega_n)}{H_{p-1}(i\Omega_n)} - \frac{J_{p-1}(i\Omega_n)}{J_{p+1}(i\Omega_n)} \right] X_n^\theta + 2\varepsilon p(p^2-1) \left(X_{n-1}^r - X_{n+1}^r \right) \end{aligned}$$

View as a linear system in the unknowns $\left[Re(X_n^r), Im(X_n^r), Re(X_n^\theta), Im(X_n^\theta) \right]$ **!**

After some messy algebra, obtain another eigenvalue problem

$$(-D^{-1}E)\,\vec{c} = \left(\frac{1}{\varepsilon}\right)\,\vec{c}$$

where D and E are infinite, real matrices consisting of 4×4 blocks.

Basic Idea (same as for the Mathieu equation):

- Cut off at finite N
- Look for stability boundaries, $\gamma = 0$ and $\frac{1}{2}i\beta$
- Parameter space is now the p–ε plane (σ_o is scaled out)
- Results are reported in terms of the nondimensional parameters

$$\boldsymbol{\kappa = \frac{\sigma_o}{\rho\omega_o^2 R^2}} \qquad \textbf{and} \qquad \boldsymbol{\nu = \frac{\mu}{\rho\omega_o R^2}}$$

Question:

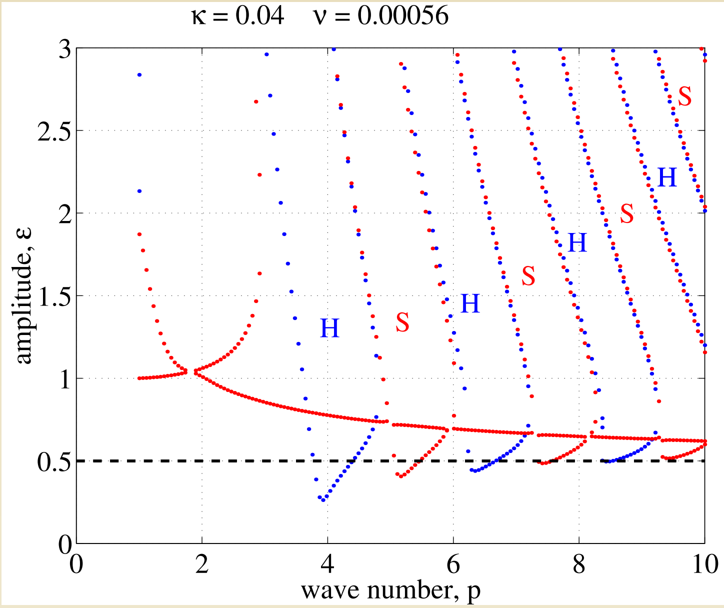
Are there regions of instability for physically-reasonable parameter values?

Stability Diagrams for the Forced Problem

Case I:

$\kappa = 0.04,\ \nu = 0.00056$

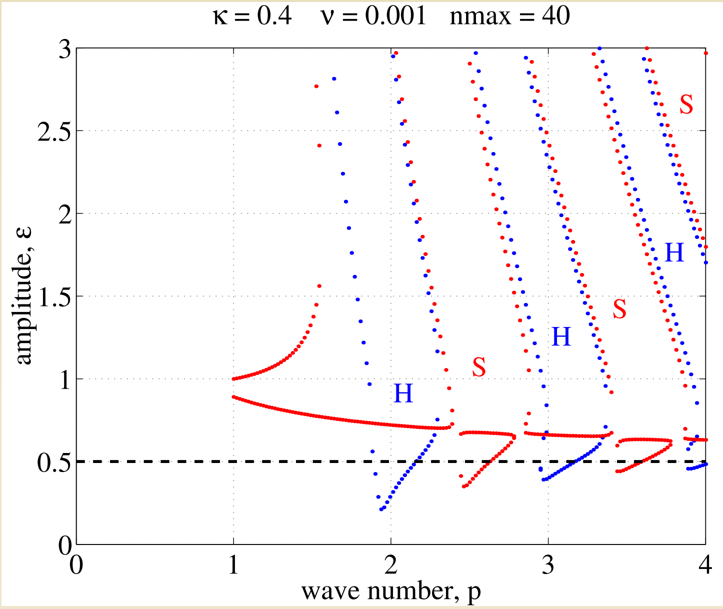
$p = 4$ mode is unstable



Case II:

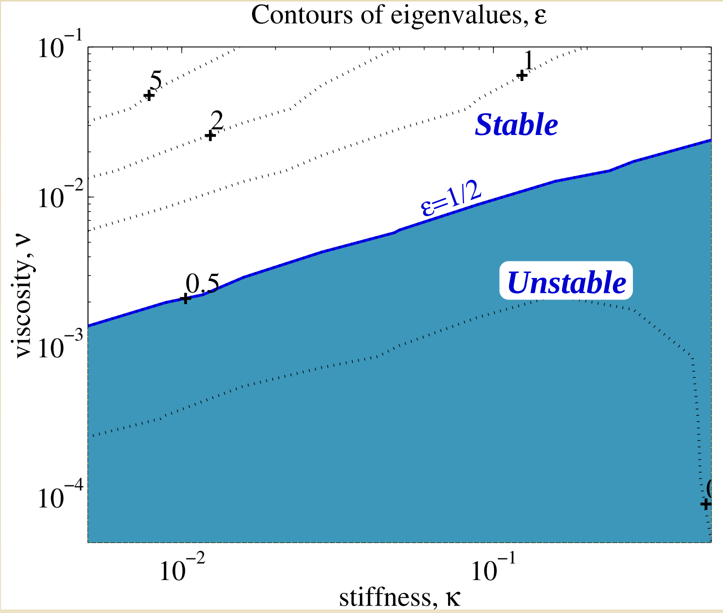
$\kappa = 0.4,\ \nu = 0.001$

$p = 2$ mode is unstable



[κ] [ν]

Compile the minimum ε over a range of κ and ν:



A straight–line fit to the $\varepsilon = \frac{1}{2}$ stability boundary yields

$$\boldsymbol{\nu = 0.0389 \cdot \kappa^{0.626}}$$

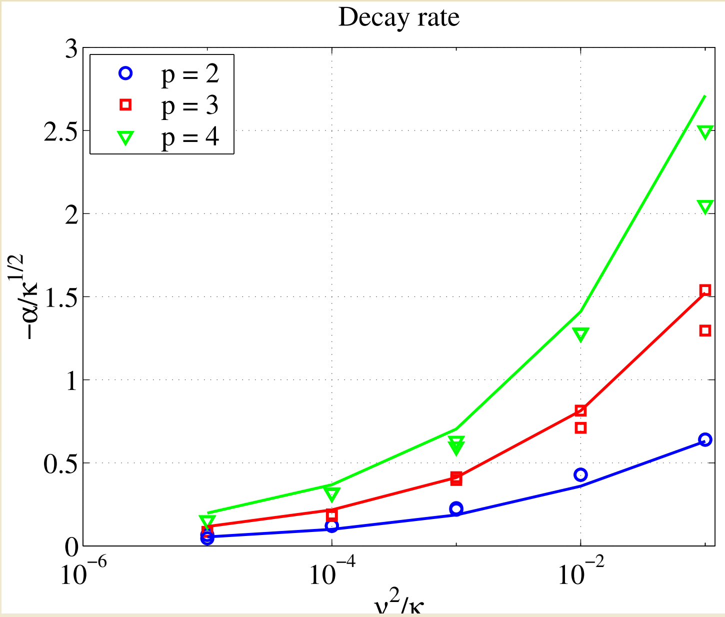
(. . . interesting . . . can this be explained?)

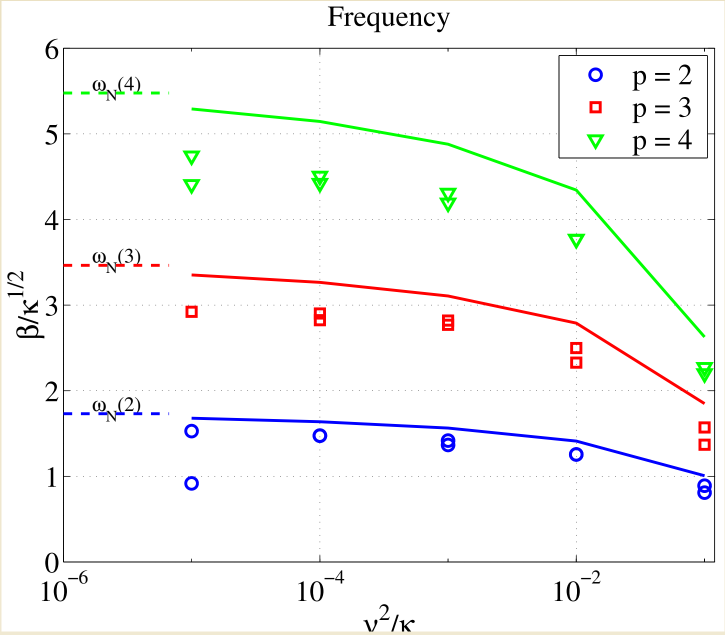
Next, compare to both computations and previous analyses . . .

Natural modes – unforced problem

- consider a fiber with no forcing, $\varepsilon = 0$
- take the $n = 0$ mode
- the eigenvalue equations reduce to a single **dispersion relation**

The decay rate and frequency can be compared to direct numerical simulations





This represents an exact (asymptotic) solution that can be used to validate 2D computations!!!

Natural modes – small ν limit

In the limit as $\nu \longrightarrow 0$, we obtain

Decay rate: $\alpha = 0$

Frequency: $\beta = \omega_N \sqrt{\kappa}$ where $\omega_N(p) = p\sqrt{(p^2 - 1)/2}$.

$\Longrightarrow$ matches the inviscid analysis of [Cortez & Varela, 1997]

Natural modes – small ν limit

In the limit as $\nu \to 0$, we obtain

Decay rate: $\alpha = 0$

Frequency: $\beta = \omega_N \sqrt{\kappa}$ where $\omega_N(p) = p\sqrt{(p^2-1)/2}$.

$\Longrightarrow$ matches the inviscid analysis of [Cortez & Varela, 1997]

Asymptotic expansion of the dispersion relation in ν yields:

Decay rate: $\alpha \sim -\frac{p}{2\sqrt{2}} \omega_N^{1/2} \kappa^{1/4} \sqrt{\nu}$

Frequency: $\beta \sim \omega_N \sqrt{\kappa} - \frac{p}{2\sqrt{2}} \omega_N^{3/4} \kappa^{1/4} \sqrt{\nu}$

$\Longrightarrow$ matches the viscous analysis of [JS & Wetton, 1995]

5. Numerical Simulations

"Usual" implementation of the **immersed boundary method** [Peskin, 1977]

- second order centered differences in space
- split step projection method
- delta function approximation reduces spatial accuracy to first order
- first order and explicit in time

Notice: small discrepancies between analysis and numerics owing to

- artificial dissipation from the numerical scheme
- numerical errors from the first order time–stepping

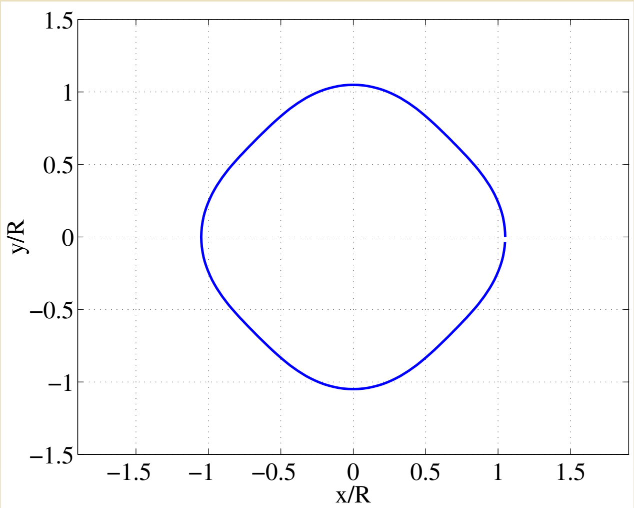
We have also run fully second-order simulations using the **blob projection method** [Cortez & Minion, 2000]

Unforced fiber simulations

Fiber is initially perturbed with a p–mode of magnitude 0.05

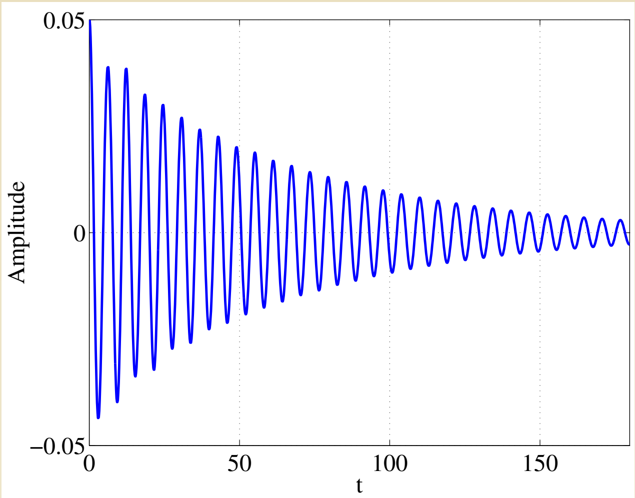
Case I: $\kappa = 0.04$, $\nu = 0.00056$, $p = 4$

Initial



[movie]

Amplitude of $p = 4$ mode

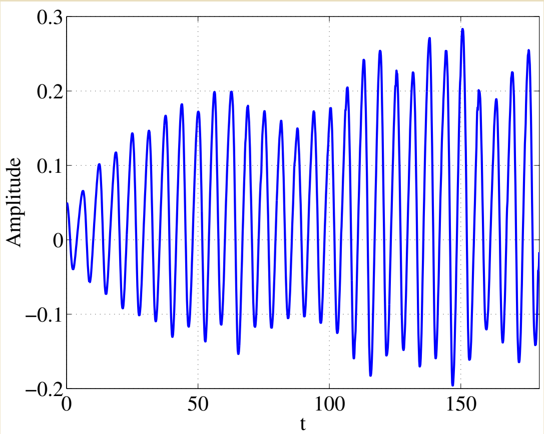


Forced fiber simulations

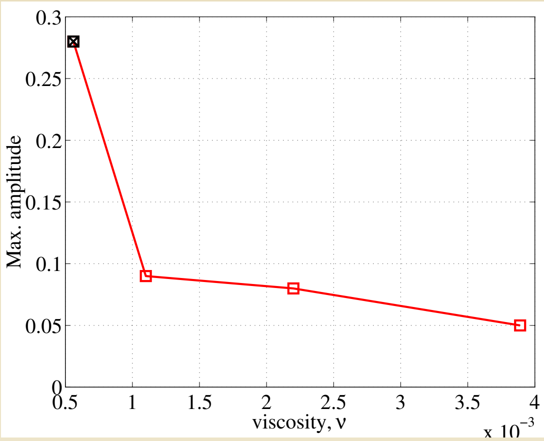
Fiber is initialized and forced with the same resonant p–mode

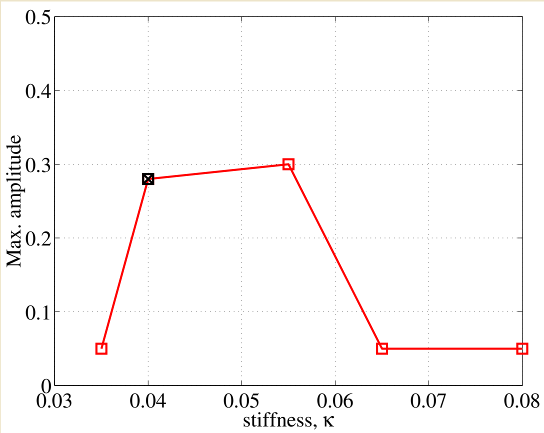
Case I: $\kappa = 0.04$, $\nu = 0.00056$, $p = 4$

Amplitude of $p = 4$ mode



Vary parameters

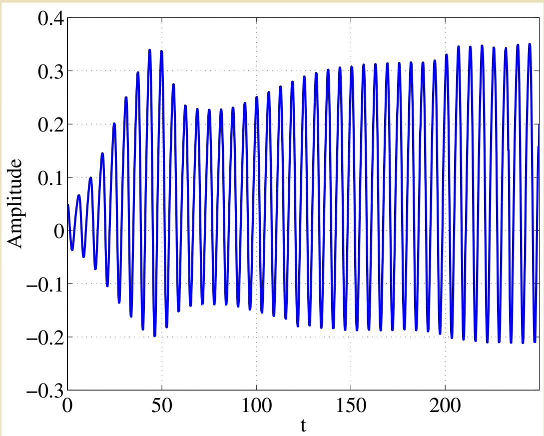




[movie]

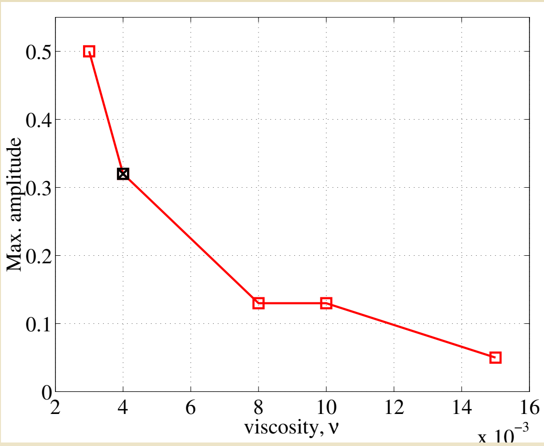
Case II: $\kappa = 0.5$, $\nu = 0.001$, $p = 2$

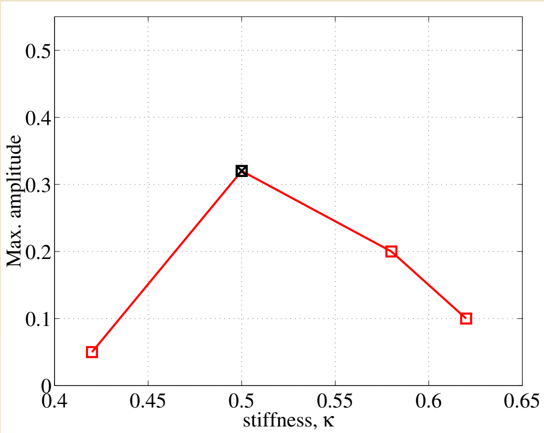
Amplitude of $p = 2$ mode



[movie]

Vary Parameters





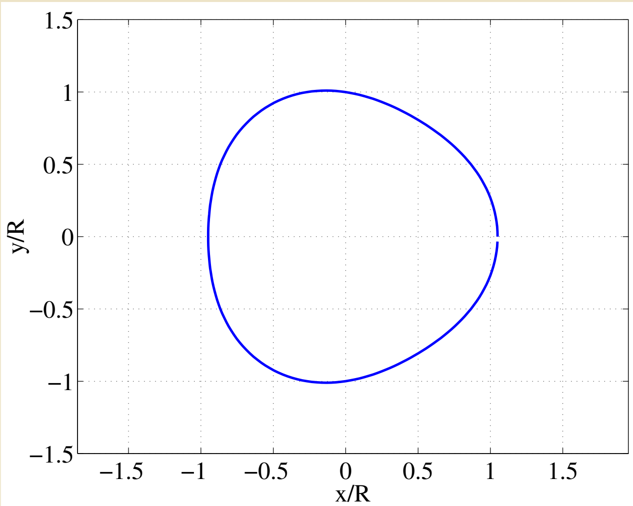
Energy Transfer from Resonant Modes

Because of numerical errors, the fiber motion is not a pure p–mode

- most of the energy remains in the resonant mode
- **Even p:** small perturbations appear in all np–modes
- **Odd p:** asymmetry leads to a small **"drift"** or $p = 1$ mode

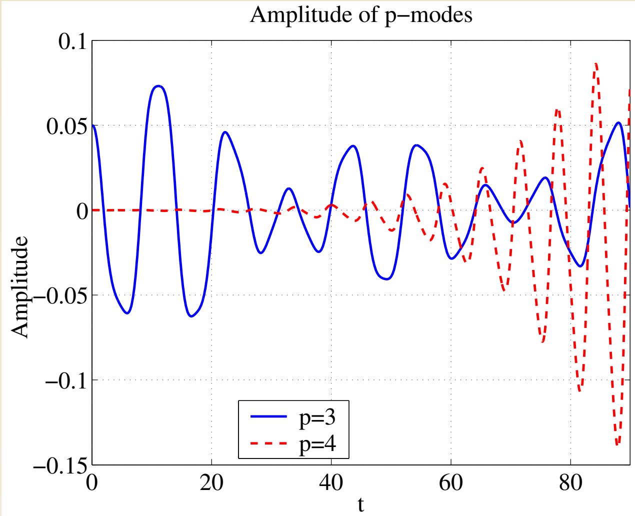
$\Longrightarrow$ **all p-modes are excited!** $[\sin(p \pm q) = \sin p \cos q \pm \cos p \sin q\,]$

Case I, initialized with $p = 3$ mode



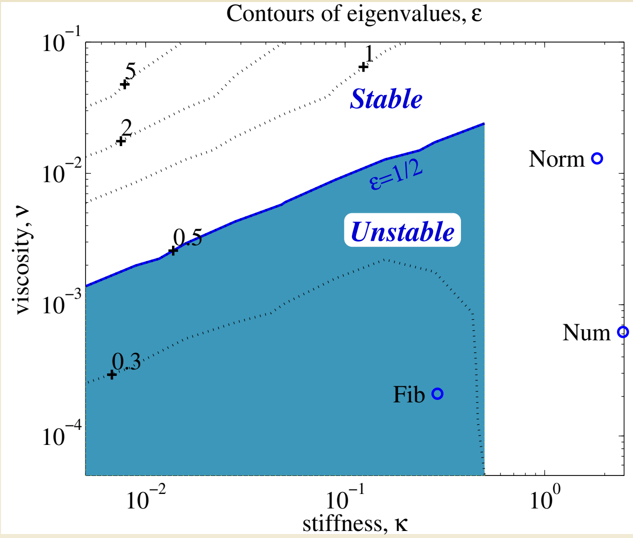
[movie]

Energy transfers to the resonant $p = 4$ mode



Application to Atrial Fibrillation

Parameter	Units	Human heart normal	Human heart fibrillated	3D numerics (Peskin-McQueen)
ρ	g/cm^3	1.0	1.0	1.0
μ	g/cm s	0.04	0.04	1.0
σ_o	g/cm s^2	1000?	1000?	1000?
ω_o	rad/s (bt/min)	6.3 (60)	18.3 (170)	7.3 (70)
R	cm	3.2	3.2	3.2
$\kappa = \frac{\sigma_o}{\rho\omega^2 R^2}$	–	2.47	0.29	1.83
$\nu = \frac{\mu}{\rho\omega R^2}$	–	0.00062	0.00021	0.013



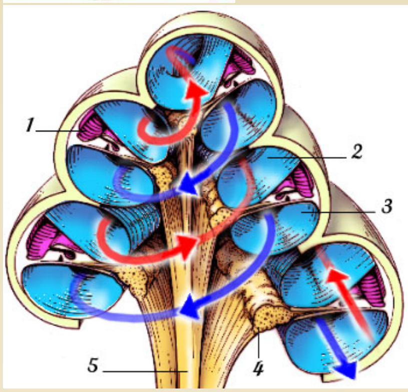
- Most studies of atrial fibrillation point to **electrophysiological** causes.
- Our analysis suggests a possible fluid–mechanical contribution to fibrillation through **feedback** into propagation of electrical signals (?)

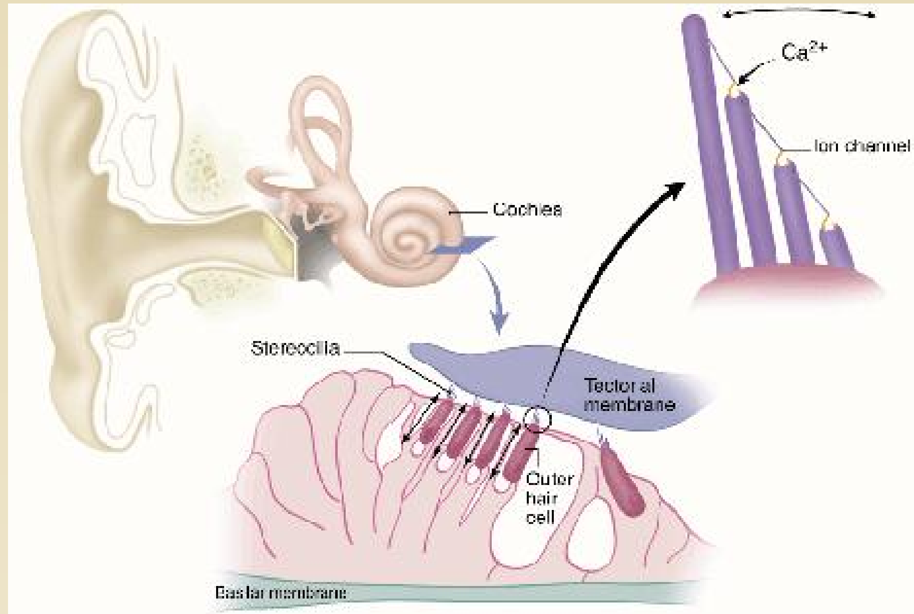
However, we still don't expect instabilities to develop in normal hearts!
Possible explanations:

- nonlinearities tend to stabilize:
  - the fiber, introducing a rest length $R_o \neq 0$
  - the fluid, moving to higher Reynolds number
- 3D effects are missing:
  - added stability of an interwoven fiber mesh
  - thickness of the heart wall
  - a heart beat is actually a spiral wave, not a homogeneous pulse

Application to Cochlear Mechanics

The cochlea is a spiral-wound, fluid-filled tube, which propagates waves along the basilar membrane (BM):



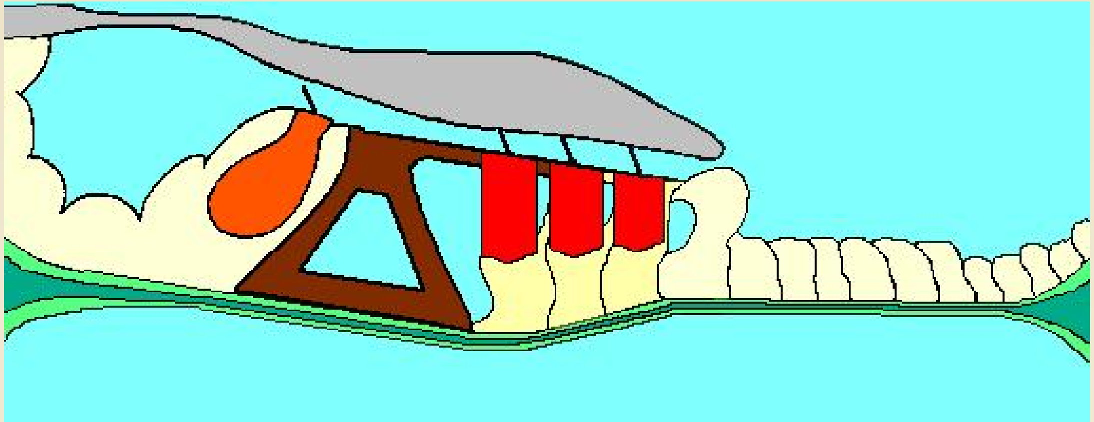


Recent research aims to explain the amazing sensitivity of the BM to sound waves in the presence of large viscous damping:

- (experiments) outer hair cells change length in response to shearing
- modulates the stiffness of the BM
- introduces a mechanical feedback that amplifies BM motions

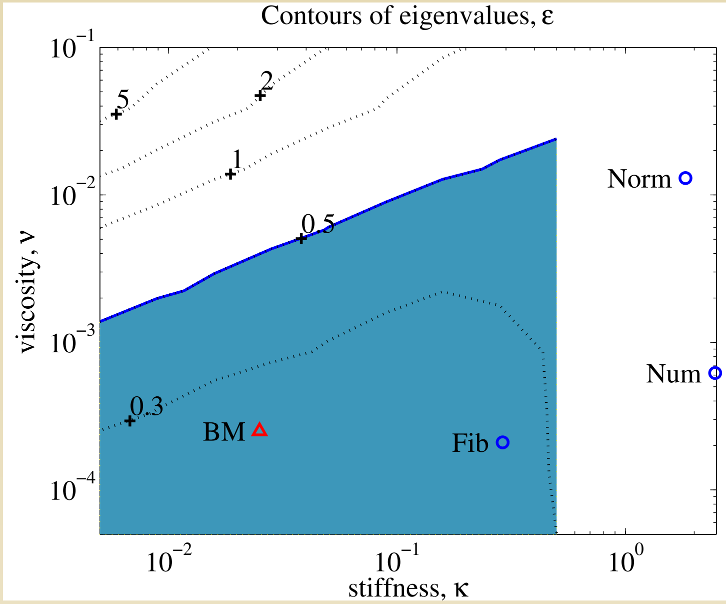
Cochlea and basilar membrane

Outer hair cell



Organ of Corti

Parameter	Units	Cochlea
ρ	$\mathrm{g/cm^3}$	1.0
μ	$\mathrm{g/cm\,s}$	0.01
σ_o	$\mathrm{g/cm\,s^2}$	1000
ω	rad/s	1000+
R	cm	0.2
$\kappa = \frac{\sigma_o}{\rho\omega^2 R^2}$	–	0.025
$\nu = \frac{\mu}{\rho\omega R^2}$	–	0.00025



- Current models of cochlear mechanics focus on mechanical resonance [Martin et al., 2000], [Nobili et al., 1998]
- Our analysis suggests that fluid-mechanical feedback may also play a significant role
- **BUT** we're still missing 3D effects, nonlinearities in BM stiffness, and coupling to mechanical effects

6. Summary

- performed a Floquet analysis of a parametrically-forced immersed boundary
- derived an analytical (leading-order, asymptotic) solution which can be used to validate 2D numerical simulations – **the first such exact solution!**
- matched results with previous analyses of the unforced problem in the small-viscosity limit
- identified parameter ranges in which forced fiber dynamics are unstable

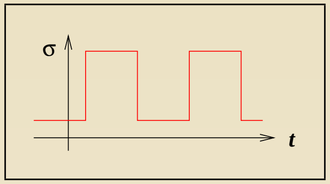
Warning:
(for people simulating active, biological interfaces)
watch out for parametric resonances, which can easily be mistaken for numerical instabilities!

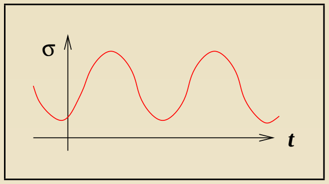
- suggested possible parametric resonance in biological systems such as the human heart and cochlea

Future Work

- a 3D numerical study of forced immersed boundaries
- spatially-varing stiffness, $\boldsymbol{\sigma(\theta, t)}$ $\Longrightarrow$ simulates spiral wave propagation
- extend to fiber spring force with non-zero resting length, R_o:

$$\vec{f}(s,t) = \sigma\, \partial_s^2 \vec{X} \qquad \Longrightarrow \qquad \vec{f}(s,t) = \sigma\, \partial_s \left[\partial_s \vec{X} \left(1 - \mathbf{R_o} / \left| \partial_s \vec{X} \right| \right) \right]$$

- "step–function" forcing 

is better than 

- optimal control of forced immersed boundaries, with application to **pacemaker design**

  $\Longrightarrow$ external forcing $f(t) = A \cos \Omega t$, with Ω a control parameter

References

[1] T. B. Benjamin and F. Ursell. The stability of the plane free surface of a liquid in vertical periodic motion. *Proc. Roy. Soc. Lond. A*, 225(1163):505–515, 1954.

[2] R. P. Beyer, Jr. A computational model of the cochlea using the immersed boundary method. *J. Comput. Phys.*, 98:145–162, 1992.

[3] M. P. Brenner, D. Lohse, and T. F. Dupont. Bubble shape oscillations and the onset of sonoluminescence. *Phys. Rev. Lett.*, 75:954–957, 1995.

[4] R. Cortez and M. Minion. The blob projection method for immersed boundary problems. *J. Comput. Phys.*, 161(2):428–453, 2000.

[5] R. Cortez and D. A. Varela. The dynamics of an elastic membrane using the impulse method. *J. Comput. Phys.*, 138:224–247, 1997.

[6] S. M. Curry. How children swing. *Amer. J. Phys.*, 44(10):924–926, 1976.

[7] M. Faraday. On a peculiar class of acoustical figures; and on certain forms assumed by groups of particles upon vibrating elastic surfaces. *Phil. Trans. Roy. Soc.*, 121:299–340, 1831.

[8] G. Floquet. Sur les équations différentielles linéaires à coefficients périodiques. *Ann. Sci. École Norm. Sup.*, 12(2):47–88, 1883.

[9] K. Kumar and L. S. Tuckerman. Parametric instability of the interface between two fluids. *J. Fluid Mech.*, 279:49–68, 1994.

[10] P. Martin, A. D. Mehta, and A. J. Hudspeth. Negative hair-bundle stiffness betrays a mechanism for mechanical amplification by the hair cell. *Proc. Natl. Acad. Sci. USA*, 97(22):12026–12031, 2000.

[11] G. McKay. Onset of double-diffusive convection in a saturated porous layer with time-periodic surface heating. *Continuum Mech. Thermodyn.*, 10:241–251, 1998.

[12] D. M. McQueen and C. S. Peskin. A three-dimensional computer model of the human heart for studying cardiac fluid dynamics. *Computer Graphics*, 34(1):56–60, 2000.

[13] R. Nobili, F. Mammano, and J. Ashmore. How well do we understand the cochlea? *Trends Neurosci.*, 21:159–167, 1998.

[14] C. S. Peskin. Numerical analysis of blood flow in the heart. *J. Comput. Phys.*, 25:220–252, 1977.

[15] J. M. Stockie and B. T. R. Wetton. Stability analysis for the immersed fiber problem. *SIAM J. Appl. Math.*, 55(6):1577–1591, 1995.

[16] X. Wang. Instability analysis of some fluid-structure interaction problems. *Comput. Fluids*, 32:121–138, 2003.