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Parametric Transformations between the Heun and Gauss Hypergeometric Functions

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Abstract. The hypergeometric and Heun functions are classical special functions. Transformation formulas between them are commonly induced by pull-back transformations of their differential equations, with respect to some coverings $\boldsymbol{P}^1 \to \boldsymbol{P}^1$. This gives expressions of Heun functions in terms of better understood hypergeometric functions. This article presents the list of hypergeometric-to-Heun pull-back transformations with a free continuous parameter, and illustrates most of them by a Heun-to-hypergeometric reduction formula. In total, 61 parametric transformations exist, of maximal degree 12.

Key Words and Phrases. Heun equation, Hypergeometric Equation, Heun function, Gauss hypergeometric function.

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1. Introduction

The Gauss hypergeometric function ${}_2\mathrm{F}_1(a, b; c|z)$ and the local Heun function $\mathrm{Hn}(t, q|a, b; c, d|x)$ are classical special functions, holomorphic in a neighborhood of $z = 0$, respectively $x = 0$. They are solutions of canonical second-order Fuchsian differential equations with 3 or 4 singular points (respectively) on the Riemann sphere $\boldsymbol{P}^1$. The Fuchsian equations are the *Gauss hypergeometric equation* and the *Heun equation* [8]. We present these equations and functions soon, in formulas (1.5)–(1.9).

The special functions ${}_2\mathrm{F}_1$ and Hn satisfy many identities such as

$$(1.1)\qquad {}_2\mathrm{F}_1\left(\begin{matrix} 2a, 2b \\ a+b+1/2 \end{matrix}\,\middle|\, x\right) = {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ a+b+1/2 \end{matrix}\,\middle|\, 4x(1-x)\right),$$

$$(1.2)\qquad \mathrm{Hn}\left(\begin{matrix} 1/2 \\ 2ab \end{matrix}\,\middle|\,\begin{matrix} 2a, 2b \\ c, c \end{matrix}\,\middle|\, x\right) = {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, 4x(1-x)\right).$$

The former is Gauss' quadratic transformation of ${}_2\mathrm{F}_1$, and the latter is a well-known generalization with an extra free parameter. It can be viewed as a *Heun-to-hypergeometric*, or *Heun-to-Gauss, reduction*. These transformations are of particular interest when the differential equations have no Liouvillian (i.e., "elementary") solutions.

Transformations between hypergeometric functions were first systematically investigated by Goursat [10]. A complete classification, with a few sets of unpredicted transformations, was recently performed by the first author [26]. Several Heun-to-Gauss reduction formulas we found by Maier [17]. All these transformations are based on a rational map $z = \varphi(x)$, such as $\varphi(x) = 4x(1-x)$.

Generally, the considered transformations are induced by *pull-back transformations* of Fuchsian equations of the form

$$(1.3) \qquad z \mapsto \varphi(x), \qquad y(z) \mapsto Y(x) = \theta(x)\, y(\varphi(x)).$$

Here $\varphi(x)$ is a rational function representing the pullback covering along which a hypergeometric (or more general Fuchsian) equation is *lifted* or *pulled back*. The *gauge prefactor* $\theta(x)$ is a radical function (i.e., a product of powers of rational functions). It is usually chosen to yield a pulled-back equation with fewer singularities and standard values of local exponents. The *degree* of a pull-back transformation is the degree of $\varphi(x)$. The considered *hypergeometric-to-Heun* pull-back transformations will be called *Gauss-to-Heun* transformations (or pull-backs) for brevity. We encounter *Heun-to-Heun* and *Gauss-to-Gauss* (or just *hypergeometric*) transformations as well.

This article focuses on the Heun-to-hypergeometric reductions

$$(1.4) \qquad \mathrm{Hn}\left(\begin{array}{c|c|c} t & a,b & \\ q & c,d & x \end{array}\right) = \theta(x)\,{}_2\mathrm{F}_1\left(\begin{array}{c|c} A,B & \\ C & \varphi(x) \end{array}\right)$$

with at least one free parameter and induced by pull-back transformations (1.3) of respective differential equations. Quadratic transformations such as (1.2) were first found by Kuiken [15]. The several transformations of Maier [17] are all parametric Heun-to-Gauss reductions without the prefactor $\theta(x)$.

In a parallel article [25], the authors classify the pull-back coverings appearing in parametric Heun-to-Gauss reductions (excluding the cases with Liouvillian solutions). The coverings that occur in these pull-back transformations turn out to be *Belyi functions*, in the sense that they have at most three critical values on $\boldsymbol{P}^1$. The coverings characteristically branch only above the singular points $\varphi \in \{0, 1, \infty\}$ of the hypergeometric equation. The four singular points of Heun's equation lie above the same set $\{0, 1, \infty\} \subset \boldsymbol{P}^1$.

The present article spells out the list of parametric Heun equations reducible to hypergeometric ones via pull-back transformations (1.3), and gives an extensive list of transformation formulas between the Heun and hypergeometric functions. Up to Möbius fractional-linear transformations, there are 61 different cases of Heun-to-Gauss reductions. Among these reductions, 28 are compositions of lower degree transformations among Heun and hypergeometric functions. The maximal degree of a parametric Heun-to-Gauss reduction is 12. In the 61 Heun-to-hypergeometric reductions, 48 different Belyi coverings are involved. Herfurtner's list [12] of elliptic surfaces with 4 singular fibers contains 38 of these coverings as Klein's $\mathcal{J}$-invariants of the elliptic surfaces.

Just as [25], this article skips differential equations with Liouvillian solutions. Those parametric hypergeometric and Heun equations have a cyclic or dihedral monodromy.

Here is the content and the structure of the article. Section 2 recalls the list of 61 Heun-to-Gauss reductions (up to Möbius transformations) obtained in [25]. In § 2.2 this list is rewritten in an order convenient for answering the following basic question:

> Is a given Heun function with a free parameter (and without Liouvillian solutions) reducible to Gauss hypergeometric functions by a pull-back transformation?

The newly ordered Heun equations have labels P1 to P61. In § 2.3 we inspect the t-values of the reducible Heun functions and make an arithmetic observation. Section 3 explains how to obtain identities between Heun and hypergeometric functions induced by the listed pull-back transformations. Section 4 is a comprehensive survey on Heun-to-Gauss reductions, including brief overviews of Gauss-to-Gauss and Heun-to-Heun transformations. All these transformations are exemplified in a Maple package [9]. Appendix A gives the criteria for the ordering P1–P61. Appendix B reminds the symmetries of hypergeometric and Heun equations. Appendix C reviews the composite transformations among the Heun-to-Gauss reductions. Appendix D gives additional invariants (of the fractional-linear transformations) to identify the reducible Heun equations completely.

Before presenting the transformation lists and formulas, we briefly recall that the hypergeometric and Heun equations are

$$\frac{d^2y(z)}{dz^2}+\left(\frac{C}{z}+\frac{A+B-C+1}{z-1}\right)\frac{dy(z)}{dz}+\frac{AB}{z(z-1)}\,y(z)=0 \tag{1.5}$$

and, respectively,

$$\frac{d^2y(x)}{dx^2}+\left(\frac{c}{x}+\frac{d}{x-1}+\frac{a+b-c-d+1}{x-t}\right)\frac{dy(x)}{dx} + \frac{abx-q}{x(x-1)(x-t)}\,y(x)=0. \tag{1.6}$$

They are canonical second-order Fuchsian differential equations on the Riemann sphere $\boldsymbol{P}^1$, having 3 and 4 regular singular points respectively [8]. Any second order Fuchsian equation with 3 or 4 singularities can be transformed to them by Möbius transformations. The singular points of these equations are $z=0$, $z=1$, $z=\infty$ and $x=0$, $x=1$, $x=t$, $x=\infty$. The information about the singularities and local exponents is encoded in the Riemann P-schemes for these equations:

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & A & z \\ 1-C & C-A-B & B & \end{matrix}\right\}, \tag{1.7}$$

$$P\left\{\begin{matrix} 0 & 1 & t & \infty & \\ 0 & 0 & 0 & a & x \\ 1-c & 1-d & c+d-a-b & b & \end{matrix}\right\},$$

so that the local exponents at $z=0$ for the hypergeometric equation are 0, $1-C$, etc. Recall that Fuchsian equations with 3 singularities are defined uniquely by their singularities and the local exponents. This is not generally true for Fuchsian equations with more singularities. In particular, q is an *accessory parameter* of Heun's equation.

The local solution at $z=0$ with the local exponent 0 and the value 1 of the hypergeometric equation is the well-known Gauss hypergeometric series:

$${}_2\mathrm{F}_1\left(\begin{matrix} A, B \\ C \end{matrix}\,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{(C)_n n!} z^n. \tag{1.8}$$

The local solution at $x=0$ with the local exponent 0 and the value 1 of Heun's equation is denoted by

$$\mathrm{Hn}\left(\begin{matrix} t \\ q \end{matrix}\,\middle|\,\begin{matrix} a, b \\ c; d \end{matrix}\,\middle|\, x\right). \tag{1.9}$$

The power series $\sum_{n=0}^{\infty} h_n x^n$ for this (*local*) *Heun function* is rather complicated. Its coefficients h_n satisfy a second order linear recurrence relation [19], with the coefficients quadratic in n. Provided that c is not a non-positive integer (to avoid division by zero), $\mathrm{Hn}(x)$ is defined and holomorphic at least on $|x| < \min(1, |t|)$. The Heun function degenerates to the ${}_2\mathrm{F}_1$ function if $d = a+b-c+1$ and $q = abt$. Notice the corresponding degeneration of Heun's equation to (1.5). If $q \neq abt$, the point $x=t$ becomes logarithmic rather than ordinary. The Heun function is identical to the constant 1 if $ab=0$ and $q=0$.

Note that the parameters a, b are symmetric and give the local exponents at $x = \infty$, whereas the parameters c and d are not interchangeable and determine the non-zero local exponents at $x = 0$ and $x = 1$, respectively.

The Heun equation contains a large number of interesting special cases. In particular, the Lamé equation [8] is the most studied case (it is basically $a + b = c = d = 1/2$) and has considerable importance in mathematical physics [8, 22]. The Heun equation appears in problems such as diffusion, wave propagation, magneto-hydrodynamics, heat and mass transfer, particle physics and cosmology of the very early universe. Heun functions are much less understood than the hypergeometric functions. Particularly, no general elementary integral representation of Heun functions is known. It is thus generally desirable to have expressions of Heun's and especially Lamé functions in terms of more elementary functions.

2. Two classifications of Heun-to-hypergeometric reductions

Here we recall the classification of Heun-to-Gauss reductions in [25] up to Möbius transformations, and rewrite the list in the order convenient for finding out whether an encountered Heun function or equation is reducible to a hypergeometric one. An arithmetic observation on the t-parameters of the reducible Heun functions is made in §2.3.

First we introduce some notation. From (1.7) it is clear that the local exponent differences of the hypergeometric equation at the singular points are $1 - C$, $C - A - B$, $A - B$, while the exponent differences of Heun's equation are $1 - c$, $1 - d$, $c + d - a - b$, $a - b$. As in [25], let $E(\alpha, \beta, \gamma)$ denote a hypergeometric equation with the exponent differences α, β, γ, and let $HE(\alpha, \beta, \gamma, \delta)$ denote a Heun equation with the exponent differences α, β, γ, δ.

A pull-back transformation of degree D from a hypergeometric equation $E(\alpha_1, \beta_1, \gamma_1)$ to Heun's equation $HE(\alpha_2, \beta_2, \gamma_2, \delta_2)$ is denoted by $E(\alpha_1, \beta_1, \gamma_1) \overset{D}{\leftarrow} HE(\alpha_2, \beta_2, \gamma_2, \delta_2)$. Sometimes we indicate the pullback covering more specifically by putting a subscript to the degree D. Similarly, a pull-back between hypergeometric equations is denoted by $E(\alpha_1, \beta_1, \gamma_1) \overset{D}{\leftarrow} E(\alpha_2, \beta_2, \gamma_2)$. Like in the similar notation $(\alpha_1, \beta_1, \gamma_1) \overset{D}{\leftarrow} (\alpha_2, \beta_2, \gamma_2)$ in [26], the arrow follows the direction of the covering $\varphi : \boldsymbol{P}_x^1 \to \boldsymbol{P}_z^1$. Our notation points to existence of differential equations with the given exponent differences and related by an indicated pull-back transformation, rather than to existence of pull-backs between any differential equations with the given exponent differences. The order of exponent differences on both sides of the arrow is irrelevant for us, as we do not assign them to particular singularities by this notation.

From now on, let ω denote the cubic root of unity $\exp(2\pi i/3)$.

2.1. The starting classification

The parallel paper [25] classifies the pull-back transformations (1.3) between the hypergeometric and Heun equations with a free continuous parameter up to Möbius transformations. We recall the results in Tables 1 and 2 to be self-contained.

The classification in [25] starts with the hypergeometric equations with a free parameter that could be pulled-back to Heun equations. To get a pulled-back equation with just 4 singular points, some of the local exponent differences must be restricted to the value $1/k$, with k a positive integer. Since we want a free parameter, at least one exponent difference is left unrestricted. The quadratic transformation illustrated in (1.2) has no restrictions on the parameters of the hypergeometric equation, while other pull-back transformations have the following sequences of restricted exponent differences:

$$\left(\frac{1}{2}\right), \left(\frac{1}{3}\right), \left(\frac{1}{2},\frac{1}{3}\right), \left(\frac{1}{2},\frac{1}{4}\right), \left(\frac{1}{2},\frac{1}{5}\right), \left(\frac{1}{2},\frac{1}{6}\right), \left(\frac{1}{3},\frac{1}{3}\right), \left(\frac{1}{3},\frac{1}{4}\right), \left(\frac{1}{4},\frac{1}{4}\right).$$

The classification in [25] skips the restrictions (1) and $(1/2, 1/2)$ as the hypergeometric equation has then Liouvillian solutions. There are infinitely many pull-backs to Heun equations then, it can be noticed soon.

Tables 1 and 2 are renditions of [25, Tables 1, 2, 3]. The first two columns give the exponent differences (up to the sign) of the hypergeometric and Heun equation under a pull-back transformation. Let E be the hypergeometric equation. The third column gives the degree D of the transformation. The fourth column gives the branching pattern of the pull-back covering. The branching pattern is given by 3 partitions of D separated by the equality sign. The partitions specify the branching orders of the covering in the 3 fibers above the singular points of E. The notation $[k]_n$ means the sum of n repeated k's in a partition. It represents n points with the branching order k above a singularity of E with the exponent difference restricted to $1/k$; those n points would be non-singular with an appropriate gauge prefactor $\theta(x)$ in (1.3). The number of bracketed branching orders is equal to the number of restricted exponent differences. The number of non-bracketed branching orders is equal to 4; they represent the 4 singular points of the pulled-back Heun equation. The total number of points in the three fibers is equal to $D+2$, as required for the Belyi coverings $\boldsymbol{P}^1 \to \boldsymbol{P}^1$ by the Hurwitz formula; see [25, Lemma 3.2].

The fifth column identifies the pull-back coverings. The H_k notation refers to the list of 48 Belyi coverings in [25, Table 4]. The coverings there are not normalized by Möbius transformations to have 3 of Heun's 4 singular points at $x = 0, 1, \infty$. Most of the coverings can be found in the explicit formulas of §4 here, as arguments of the hypergeometric functions. The fifth column also tells

Table 1: Gauss-to-Heun transformations with two continuous parameters, or from $E(1/2, 1/3, \alpha)$.

Local exponent differences					
hypergeom.	Heun's	D	Branching pattern above singular points	The covering, its composition	Characterization of §2.2
$\alpha,\ \beta,\ \gamma$	$\alpha,\ \alpha,\ 2\beta,\ 2\gamma$	2	$1+1=2=2$	H_{32}, indecomposable	P1, $j=1728$
$1/2,\ \alpha,\ \beta$	$1/2,\ \alpha,\ 2\alpha,\ 3\beta$	3	$[2]_1+1=2+1=3$	H_{34}, indecomposable	P15, $t=-3$
	$\alpha,\ \alpha,\ 2\alpha,\ 4\beta$	4	$[2]_2=2+1+1=4$	H_{35}, $2\cdot 2$	P3, $j=1728$
	$\alpha,\ 3\alpha,\ \beta,\ 3\beta$		$[2]_2=3+1=3+1$	H_{47}, indecomposable	P20, $t=-8$
	$2\alpha,\ 2\alpha,\ 2\beta,\ 2\beta$		$[2]_2=2+2=2+2$	H_{31}, 2×2	P2, $j=1728$
$1/3,\ \alpha,\ \beta$	$\alpha,\ 2\alpha,\ \beta,\ 2\beta$	3	$[3]_1=2+1=2+1$	H_{34}, indecomposable	P19, $t=-8$
	$\alpha,\ \alpha,\ \alpha,\ 3\beta$		$[3]_1=3=1+1+1$	H_{33}, indecomposable	P51, $j=0$
$1/2,\ 1/3,\ \alpha$	$1/2,\ 1/2,\ 1/3,\ 4\alpha$	4	$[2]_1+1+1=[3]_1+1=4$	H_{36}, indecomposable	P47, $t\in \boldsymbol{Q}(\sqrt{-2})$
	$1/2,\ 2/3,\ \alpha,\ 4\alpha$	5	$[2]_2+1=[3]_1+2=4+1$	H_{29}, indecomposable	P31, $t=32/5$
	$1/2,\ 2/3,\ 2\alpha,\ 3\alpha$		$[2]_2+1=[3]_1+2=3+2$	H_{30}, indecomposable	P25, $t=-4$
	$1/2,\ 1/3,\ 1/3,\ 5\alpha$		$[2]_2+1=[3]_1+1+1=5$	H_{37}, indecomposable	P59, $t\in \boldsymbol{Q}(\sqrt{-15})$
	$1/2,\ 1/2,\ \alpha,\ 5\alpha$	6	$[2]_2+1+1=[3]_2=5+1$	H_{26}, indecomposable	P45, $t\in \boldsymbol{Q}(i)$
	$1/2,\ 1/2,\ 2\alpha,\ 4\alpha$		$[2]_2+1+1=[3]_2=4+2$	H_{27}, $2\cdot 3$	P4, $j=1728$
	$1/2,\ 1/2,\ 3\alpha,\ 3\alpha$		$[2]_2+1+1=[3]_2=3+3$	H_{28}, $2_H\cdot 3_C$	P38, $t\in \boldsymbol{Q}(\sqrt{3})$
	$1/3,\ 2/3,\ \alpha,\ 5\alpha$		$[2]_3=[3]_1+2+1=5+1$	H_{24}, indecomposable	P26, $t=25/9$
	$1/3,\ 2/3,\ 2\alpha,\ 4\alpha$		$[2]_3=[3]_1+2+1=4+2$	H_{25}, $3\cdot 2$	P21, $t=-8$
	$1/3,\ 1/3,\ 1/3,\ 6\alpha$		$[2]_3=[3]_1+1+1+1=6$	H_{38}, $3_C\cdot 2$	P52, $j=0$
	$1/2,\ 1/3,\ \alpha,\ 6\alpha$	7	$[2]_3+1=[3]_2+1=6+1$	H_{21}, indecomposable	P61, $j\in \boldsymbol{Q}(\omega)$
	$1/2,\ 1/3,\ 2\alpha,\ 5\alpha$		$[2]_3+1=[3]_2+1=5+2$	H_{22}, indecomposable	P36, $t=189/64$
	$1/2,\ 1/3,\ 3\alpha,\ 4\alpha$		$[2]_3+1=[3]_2+1=4+3$	H_{23}, indecomposable	P30, $t=-27$
	$2/3,\ \alpha,\ \alpha,\ 6\alpha$	8	$[2]_4=[3]_2+2=6+1+1$	H_{15}, $2\cdot 4$	P7, $j=1728$
	$2/3,\ \alpha,\ 2\alpha,\ 5\alpha$		$[2]_4=[3]_2+2=5+2+1$	H_{16}, indecomposable	P29, $t=27/2$
	$2/3,\ 2\alpha,\ 3\alpha,\ 3\alpha$		$[2]_4=[3]_2+2=3+3+2$	H_{17}, $2\cdot 4$	P8, $j=1728$
	$1/3,\ 1/3,\ \alpha,\ 7\alpha$		$[2]_4=[3]_2+1+1=7+1$	H_{18}, indecomposable	P56, $t\in \boldsymbol{Q}(\omega)$
	$1/3,\ 1/3,\ 2\alpha,\ 6\alpha$		$[2]_4=[3]_2+1+1=6+2$	H_{19}, $4_B\cdot 2$, $2\cdot 4$	P6, $j=1728$
	$1/3,\ 1/3,\ 4\alpha,\ 4\alpha$		$[2]_4=[3]_2+1+1=4+4$	H_{20}, $4\cdot 2$, $2_H\cdot 4_A$	P41, $t\in \boldsymbol{Q}(\sqrt{3})$
	$1/2,\ \alpha,\ \alpha,\ 7\alpha$	9	$[2]_4+1=[3]_3=7+1+1$	H_{11}, indecomposable	P57, $t\in \boldsymbol{Q}(\sqrt{-7})$
	$1/2,\ \alpha,\ 2\alpha,\ 6\alpha$		$[2]_4+1=[3]_3=6+2+1$	H_{12}, $3\cdot 3$	P17, $t=-3$
	$1/2,\ \alpha,\ 3\alpha,\ 5\alpha$		$[2]_4+1=[3]_3=5+3+1$	H_{13}, indecomposable	P33, $t=128/3$
	$1/2,\ 2\alpha,\ 3\alpha,\ 4\alpha$		$[2]_4+1=[3]_3=4+3+2$	H_{14}, $3\cdot 3$	P18, $t=-3$
	$1/3,\ \alpha,\ \alpha,\ 8\alpha$	10	$[2]_5=[3]_3+1=8+1+1$	H_7, indecomposable	P49, $t\in \boldsymbol{Q}(\sqrt{-2})$
	$1/3,\ \alpha,\ 2\alpha,\ 7\alpha$		$[2]_5=[3]_3+1=7+2+1$	H_8, indecomposable	P35, $t=81/32$
	$1/3,\ \alpha,\ 4\alpha,\ 5\alpha$		$[2]_5=[3]_3+1=5+4+1$	H_9, indecomposable	P28, $t=-80$
	$1/3,\ 2\alpha,\ 3\alpha,\ 5\alpha$		$[2]_5=[3]_3+1=5+3+2$	H_{10}, indecomposable	P32, $t=32/5$
	$\alpha,\ \alpha,\ \alpha,\ 9\alpha$	12	$[2]_6=[3]_4=9+1+1+1$	H_1, $3_C\cdot 4$	P55, $j=0$
	$\alpha,\ \alpha,\ 2\alpha,\ 8\alpha$		$[2]_6=[3]_4=8+2+1+1$	H_2, $2\cdot 2\cdot 3$	P14, $j=1728$
	$\alpha,\ 2\alpha,\ 3\alpha,\ 6\alpha$		$[2]_6=[3]_4=6+3+2+1$	H_3, $3\cdot 4$, $4\cdot 3$	P24, $t=-8$
	$\alpha,\ \alpha,\ 5\alpha,\ 5\alpha$		$[2]_6=[3]_4=5+5+1+1$	H_4, $2_H\cdot 6$	P43, $t\in \boldsymbol{Q}(\sqrt{5})$
	$2\alpha,\ 2\alpha,\ 4\alpha,\ 4\alpha$		$[2]_6=[3]_4=4+4+2+2$	H_5, $2\cdot 3_C\cdot 2$, $2\times 2\cdot 3$	P12, $j=1728$
	$3\alpha,\ 3\alpha,\ 3\alpha,\ 3\alpha$		$[2]_6=[3]_4=3+3+3+3$	H_6, $3_C\cdot 4$, $2_H\cdot 2_H\cdot 3_C$	P53, $j=0$

Table 2: Other hypergeometric-to-Heun transformations.

Local exponent differences					
hypergeom.	Heun's	D	Branching pattern above singular points	The covering, its composition	Characterization of §2.2
$1/2,\ 1/4,\ \alpha$	$1/2,\ 1/2,\ \alpha,\ 3\alpha$	4	$[2]_1+1+1=[4]_1=3+1$	H_{36}, indecomposable	P48, $t\in \boldsymbol{Q}(\sqrt{-2})$
	$1/2,\ 1/2,\ 2\alpha,\ 2\alpha$		$[2]_1+1+1=[4]_1=2+2$	H_{35}, $2_H\cdot 2$	P37, $t\in \boldsymbol{Q}(\sqrt{2})$
	$1/2,\ 1/4,\ \alpha,\ 4\alpha$	5	$[2]_2+1=[4]_1+1=4+1$	H_{44}, indecomposable	P60, $j\in \boldsymbol{Q}(i)$
	$1/2,\ 1/4,\ 2\alpha,\ 3\alpha$		$[2]_2+1=[4]_1+1=3+2$	H_{29}, indecomposable	P27, $t=-80$
	$1/2,\ \alpha,\ 2\alpha,\ 3\alpha$	6	$[2]_3=[4]_1+2=3+2+1$	H_{25}, $3\cdot 2$	P16, $t=-3$
	$1/4,\ 1/4,\ \alpha,\ 5\alpha$		$[2]_3=[4]_1+1+1=5+1$	H_{42}, indecomposable	P44, $t\in \boldsymbol{Q}(i)$
	$1/4,\ 1/4,\ 3\alpha,\ 3\alpha$		$[2]_3=[4]_1+1+1=3+3$	H_{43}, $2_H\cdot 3$	P39, $t\in \boldsymbol{Q}(\sqrt{3})$
	$\alpha,\ \alpha,\ 2\alpha,\ 4\alpha$	8	$[2]_4=[4]_2=4+2+1+1$	H_{40}, $2\cdot 2\cdot 2$	P13, $j=1728$
	$\alpha,\ \alpha,\ 3\alpha,\ 3\alpha$		$[2]_4=[4]_2=3+3+1+1$	H_{20}, $4\cdot 2$, $2_H\cdot 4_A$	P23, $t=-8$
	$2\alpha,\ 2\alpha,\ 2\alpha,\ 2\alpha$		$[2]_4=[4]_2=2+2+2+2$	H_{41}, $2\times 2\times 2$	P10, $j=1728$
$1/2,\ 1/5,\ \alpha$	$1/2,\ \alpha,\ \alpha,\ 3\alpha$	5	$[2]_2+1=[5]_1=3+1+1$	H_{37}, indecomposable	P58, $t\in \boldsymbol{Q}(\sqrt{-15})$
	$1/2,\ \alpha,\ 2\alpha,\ 2\alpha$		$[2]_2+1=[5]_1=2+2+1$	H_{45}, indecomposable	P42, $t\in \boldsymbol{Q}(\sqrt{5})$
	$1/5,\ \alpha,\ \alpha,\ 4\alpha$	6	$[2]_3=[5]_1+1=4+1+1$	H_{42}, indecomposable	P46, $t\in \boldsymbol{Q}(i)$
	$1/5,\ \alpha,\ 2\alpha,\ 3\alpha$		$[2]_3=[5]_1+1=3+2+1$	H_{24}, indecomposable	P34, $t=128/3$
$1/2,\ 1/6,\ \alpha$	$\alpha,\ \alpha,\ \alpha,\ 3\alpha$	6	$[2]_3=[6]_1=3+1+1+1$	H_{38}, $3_C\cdot 2$	P54, $j=0$
	$\alpha,\ \alpha,\ 2\alpha,\ 2\alpha$		$[2]_3=[6]_1=2+2+1+1$	H_{39}, $3\cdot 2$, $2_H\cdot 3$	P22, $t=-8$
$1/3,\ 1/3,\ \alpha$	$1/3,\ 1/3,\ \alpha,\ 3\alpha$	4	$[3]_1+1=[3]_1+1=3+1$	H_{46}, indecomposable	P5, $j=1728$
	$1/3,\ 1/3,\ 2\alpha,\ 2\alpha$		$[3]_1+1=[3]_1+1=2+2$	H_{47}, indecomposable	P40, $t\in \boldsymbol{Q}(\sqrt{3})$
	$\alpha,\ \alpha,\ 2\alpha,\ 2\alpha$	6	$[3]_2=[3]_2=2+2+1+1$	H_{28}, $2\cdot 3_C$	P11, $j=1728$
$1/3,\ 1/4,\ \alpha$	$1/3,\ \alpha,\ \alpha,\ 2\alpha$	4	$[3]_1+1=[4]_1=2+1+1$	H_{36}, indecomposable	P50, $t\in \boldsymbol{Q}(\sqrt{-2})$
$1/4,\ 1/4,\ \alpha$	$\alpha,\ \alpha,\ \alpha,\ \alpha$	4	$[4]_1=[4]_1=1+1+1+1$	H_{48}, $2_H\cdot 2$	P9, $j=1728$

which coverings are compositions of lower degree coverings, and indicates the compositions by the component degrees. Specific meaning of the composition notation is explained at the beginning of §C.

The coverings H_1 to H_{38} appear in Herfurtner's list [12] of elliptic surfaces over $\boldsymbol{P}^1$ with 4 singular fibers; the $\mathscr{J}(X,Y)$-expressions in [12, Table 3] are projectivized Belyi coverings and give the j-invariants of the elliptic surfaces up to the multiple 1728. The numbering H_1 to H_{38} agrees with [20, Table 1], where Movasati and Reiter observe that 38 of Herfurtner's 50 cases of elliptic surfaces give rise to pull-backs from $E(1/2,1/3,\alpha)$ to Heun equations. In Table 1 here, the coverings H_1 to H_{30} and H_{36}, H_{37}, H_{38} appear in the pull-backs specifically from $E(1/2,1/3,\alpha)$, while the coverings H_{31} to H_{35} appear in pull-back transformations with 2 or 3 parameters. The coverings H_{39} to H_{48} appear in pull-backs to Heun functions from the hypergeometric equations different from $E(1/2,1/3,\alpha)$. The coverings H_{20}, H_{24}, H_{25}, H_{28}, H_{29}, H_{34}, H_{35}, H_{37}, H_{38}, H_{42}, H_{47} appear twice in Tables 1 and 2, while H_{36} three times, as their branching patterns can be parsed for the Heun-to-Gauss reductions in multiple ways.

The last column of Tables 1 and 2 exhibits the P-numbers of the Heun-to-Gauss reductions assigned by a new perspective, described in the next section. Relatedly, the last column adds minimal information about the t-values of the pulled-back Heun equations. A t-value gives the location of the fourth singularity of Heun's equation when the other singularities are normalized to $x = 0, 1, \infty$. Generally, a t-value is a *cross-ratio* of the 4 singular points of the pulled-back Fuchsian equation with 4 singularities. Permutation of the 4 singular points generally produces an orbit of six t-values:

$$t,\ 1-t,\ \frac{1}{t},\ \frac{1}{1-t},\ \frac{t}{t-1},\ 1-\frac{1}{t}. \tag{2.1}$$

As it is well-known, the set of six values can be represented by one number, the j-invariant:

$$j(t) = \frac{256(t^2-t+1)^3}{t^2(t-1)^2}. \tag{2.2}$$

The $\mathscr{J}$-invariant used in [12] is the Belyi function $\mathscr{J}(t) = j(t)/1728$. Its version appears in hypergeometric transformation (4.4) below. The last column of Tables 1, 2 additionally indicates:

- the most frequent j-values 1728 and 0, if $t \in \{-1, 2, 1/2, -\omega, 1+\omega\}$;
- or a representative t-value, if it is in $\boldsymbol{Q}$ and $j(t) \neq 1728$;
- or the number field for the t-values, if $t \notin \boldsymbol{Q}$ and $j(t) \in \boldsymbol{Q} \backslash \{0\}$;
- or the number field for the j-value, if $j(t) \notin \boldsymbol{Q}$.

The last case appears only twice, with the coverings H_{21} and H_{44}. These two coverings are not defined over $\boldsymbol{Q}$ either, but over $\boldsymbol{Q}(\omega)$ and $\boldsymbol{Q}(i)$, respectively. Technically speaking, the notations H_{21} and H_{44} represent pairs of coverings related by the complex conjugation. Therefore the strict count of involved Belyi coverings (or of their *dessin d'enfant*) is 50 rather than 48. For technical purposes, the corresponding Heun-to-Gauss reductions P61 and P60 can be counted as pairs of different transformations as well.

2.2. Classification by Heun equations

The main application of the list of possible Heun-to-Gauss reductions is, of course, finding out whether an encountered Heun function or equation is reducible to a hypergeometric one. Tables 1, 2 are not convenient for matching parameters of an encountered Heun equation, as even the tuples of 4 exponent differences are listed disorderly.

Of all Heun's parameters t, q, a, b, c, d, the most characteristic one is t, which is the location of the fourth singularity. Therefore t or its j-invariant are the most sensible main criteria for sorting Heun-to-Gauss reductions. A full

Table 3: Parametric Heun-to-Gauss reductions equations with a t value in $\{-1,-3,-8\}$ up to fractional-linear transformations.

Id	j-invariant	Exponent differences	Covering Id	Covering D	${}_2F_1$ equation	Other transformations
P1	$2^6 3^3 = 1728$	$\alpha,\ \alpha,\ \beta,\ \gamma$	H_{32}	2	$\alpha,\ \beta/2,\ \gamma/2$	—
P2		$\alpha,\ \alpha,\ \beta,\ \beta$	H_{31}	4	$1/2,\ \alpha/2,\ \beta/2$	P1, 2_H
P3		$\alpha,\ \alpha,\ 2\alpha,\ \beta$	H_{35}	4	$1/2,\ \alpha,\ \beta/4$	P1
P4		$1/2,\ 1/2,\ \alpha,\ 2\alpha$	H_{27}	6	$1/2,\ 1/3,\ \alpha/2$	P1; 2^H
P5		$1/3,\ 1/3,\ \alpha,\ 3\alpha$	H_{46}	4	$1/3,\ 1/3,\ \alpha$	—; P1, P6
P6			H_{19}	8	$1/2,\ 1/3,\ \alpha/2$	P1, P5
P7		$2/3,\ \alpha,\ \alpha,\ 6\alpha$	H_{15}	8	$1/2,\ 1/3,\ \alpha$	P1
P8		$2/3,\ 2\alpha,\ 3\alpha,\ 3\alpha$	H_{17}	8	$1/2,\ 1/3,\ \alpha$	P1
P9		$\alpha,\ \alpha,\ \alpha,\ \alpha$	H_{48}	4	$1/4,\ 1/4,\ \alpha$	2_H; P1, P2, P10, 4_H
P10			H_{41}	8	$1/2,\ 1/4,\ \alpha/2$	P1, P2, P9, 2_H, 4_H
P11		$\alpha,\ \alpha,\ 2\alpha,\ 2\alpha$	H_{28}	6	$1/3,\ 1/3,\ \alpha$	P1; P2, P3, P12, 2_H
P12			H_5	12	$1/2,\ 1/3,\ \alpha/2$	P1, P2, P3, P11, 2_H
P13		$\alpha,\ \alpha,\ 2\alpha,\ 4\alpha$	H_{40}	8	$1/2,\ 1/4,\ \alpha$	P1, P3
P14		$\alpha,\ \alpha,\ 2\alpha,\ 8\alpha$	H_2	12	$1/2,\ 1/3,\ \alpha$	P1, P3
P15	$2^4 13^3/3^2$	$1/2,\ \alpha,\ 2\alpha,\ \beta$	H_{34}	3	$1/2,\ \alpha,\ \beta/3$	—
P16		$1/2,\ \alpha,\ 2\alpha,\ 3\alpha$	H_{25}	6	$1/2,\ 1/4,\ \alpha$	P15
P17		$1/2,\ \alpha,\ 2\alpha,\ 6\alpha$	H_{12}	9	$1/2,\ 1/3,\ \alpha$	P15
P18		$1/2,\ 2\alpha,\ 3\alpha,\ 4\alpha$	H_{14}	9	$1/2,\ 1/3,\ \alpha$	P15
P19	$2^2 7 3^3/3^4$	$\alpha,\ 2\alpha,\ \beta,\ 2\beta$	H_{34}	3	$1/3,\ \alpha,\ \beta$	—
P20		$\alpha,\ 3\alpha,\ \beta,\ 3\beta$	H_{47}	4	$1/2,\ \alpha,\ \beta$	—
P21		$1/3,\ 2/3,\ \alpha,\ 2\alpha$	H_{25}	6	$1/2,\ 1/3,\ \alpha/2$	P19
P22		$\alpha,\ \alpha,\ 2\alpha,\ 2\alpha$	H_{39}	6	$1/2,\ 1/6,\ \alpha$	P19, 2_H
P23		$\alpha,\ \alpha,\ 3\alpha,\ 3\alpha$	H_{20}	8	$1/2,\ 1/4,\ \alpha$	P20, 2_H
P24		$\alpha,\ 2\alpha,\ 3\alpha,\ 6\alpha$	H_3	12	$1/2,\ 1/3,\ \alpha$	P19, P20

ordering is formulated in §A, following the criteria in [13, §A] for sorting a more complicated set of non-parametric Heun-to-Gauss reductions. The resulting list of parametric Heun-to-Gauss reductions is displayed in Tables 3 and 4. It starts with a list of 14 transformations to Heun equations with $t \in \{-1, 2, 1/2\}$. A detailed look shows that the Heun equations for these transformations are the same as for the well-known quadratic transformation $E(\alpha,\beta,\gamma) \overset{2}{\leftarrow} HE(\alpha,\alpha,2\beta,2\gamma)$, up to the parameter identification and the fractional-linear symmetries of §B. We mark the quadratic transformation by P1.

The last column of Tables 3 and 4 displays other reduction transformations possible for the currently indicated Heun equation. A semicolon there separates the transformations that are composition factors of the currently numbered one, from the other possible transformations (after a semicolon, if present). Accordingly, P1 is listed in the last column for the P2–P14 entries, but it is not a composition factor for P5 and P9. The notation 2_H, 2^H, 4_H refers to

Table 4: Other parametric Heun-to-hypergeometric reductions.

Id	j-invariant	(number field for t)	Exponent differences	Covering Id	Covering D	${}_2F_1$ equation	Other transformations
P25	$2^4 3^3 7^3/5^2$		$1/2,\ 2/3,\ 2\alpha,\ 3\alpha$	H_{30}	5	$1/2,\ 1/3,\ \alpha$	—
P26	$13^3 37^3/3^4 5^4$		$1/3,\ 2/3,\ \alpha,\ 5\alpha$	H_{24}	6	$1/2,\ 1/3,\ \alpha$	—
P27	$6481^3/3^8 5^2$		$1/2,\ 1/4,\ 2\alpha,\ 3\alpha$	H_{29}	5	$1/2,\ 1/4,\ \alpha$	—
P28			$1/3,\ \alpha,\ 4\alpha,\ 5\alpha$	H_9	10	$1/2,\ 1/3,\ \alpha$	—
P29	$2^6 7^3 97^3/3^6 5^4$		$2/3,\ \alpha,\ 2\alpha,\ 5\alpha$	H_{16}	8	$1/2,\ 1/3,\ \alpha$	—
P30	$2^4 757^3/3^6 7^2$		$1/2,\ 1/3,\ 3\alpha,\ 4\alpha$	H_{23}	7	$1/2,\ 1/3,\ \alpha$	—
P31	$7^3 127^3/2^2 3^6 5^2$		$1/2,\ 2/3,\ \alpha,\ 4\alpha$	H_{29}	5	$1/2,\ 1/3,\ \alpha$	—
P32			$1/3,\ 2\alpha,\ 3\alpha,\ 5\alpha$	H_{10}	10	$1/2,\ 1/3,\ \alpha$	—
P33	$7^3 2287^3/2^6 3^2 5^6$		$1/2,\ \alpha,\ 3\alpha,\ 5\alpha$	H_{13}	9	$1/2,\ 1/3,\ \alpha$	—
P34			$1/5,\ \alpha,\ 2\alpha,\ 3\alpha$	H_{24}	6	$1/2,\ 1/5,\ \alpha$	—
P35	$4993^3/2^2 3^8 7^4$		$1/3,\ \alpha,\ 2\alpha,\ 7\alpha$	H_8	10	$1/2,\ 1/3,\ \alpha$	—
P36	$19^3 1459^3/2^4 3^6 5^6 7^2$		$1/2,\ 1/3,\ 2\alpha,\ 5\alpha$	H_{22}	7	$1/2,\ 1/3,\ \alpha$	—
P37	$2^3 3^3 11^3$	$(\sqrt{2})$	$1/2,\ 1/2,\ \alpha,\ \alpha$	H_{35}	4	$1/2,\ 1/4,\ \alpha/2$	$2_H;\ 2^H$
P38	$2^4 3^3 5^3$	$(\sqrt{3})$	$1/2,\ 1/2,\ \alpha,\ \alpha$	H_{28}	6	$1/2,\ 1/3,\ \alpha/3$	$2_H;\ 2^H$
P39	$2^2 193^3/3$	$(\sqrt{3})$	$1/4,\ 1/4,\ \alpha,\ \alpha$	H_{43}	6	$1/2,\ 1/4,\ \alpha/3$	2_H
P40	$2^7 5^3/3^3$	$(\sqrt{3})$	$1/3,\ 1/3,\ \alpha,\ \alpha$	H_{47}	4	$1/3,\ 1/3,\ \alpha/2$	—; P41, 2_H
P41				H_{20}	8	$1/2,\ 1/3,\ \alpha/4$	P47, 2_H; P40
P42	$2^4 17^3$	$(\sqrt{5})$	$1/2,\ \alpha,\ 2\alpha,\ 2\alpha$	H_{45}	5	$1/2,\ 1/5,\ \alpha$	—
P43	$2^{14} 31^3/5^3$	$(\sqrt{5})$	$\alpha,\ \alpha,\ 5\alpha,\ 5\alpha$	H_4	12	$1/2,\ 1/3,\ \alpha$	2_H
P44	$2^2 3^3 13^3/5^4$	$(\sqrt{-1})$	$1/4,\ 1/4,\ \alpha,\ 5\alpha$	H_{42}	6	$1/2,\ 1/4,\ \alpha$	—
P45	$-2^4 109^3/5^6$	$(\sqrt{-1})$	$1/2,\ 1/2,\ \alpha,\ 5\alpha$	H_{26}	6	$1/2,\ 1/3,\ \alpha$	—; 2^H
P46			$1/5,\ \alpha,\ \alpha,\ 4\alpha$	H_{42}	6	$1/2,\ 1/5,\ \alpha$	—
P47	$-2^5 19^3/3^6$	$(\sqrt{-2})$	$1/2,\ 1/2,\ 1/3,\ \alpha$	H_{36}	4	$1/2,\ 1/3,\ \alpha/4$	—; 2^H
P48	$2\cdot 47^3/3^8$	$(\sqrt{-2})$	$1/2,\ 1/2,\ \alpha,\ 3\alpha$	H_{36}	4	$1/2,\ 1/4,\ \alpha$	—; 2^H
P49			$1/3,\ \alpha,\ \alpha,\ 8\alpha$	H_7	10	$1/2,\ 1/3,\ \alpha$	—
P50	$-2^6 239^3/3^{10}$	$(\sqrt{-2})$	$1/3,\ \alpha,\ \alpha,\ 2\alpha$	H_{36}	4	$1/3,\ 1/4,\ \alpha$	—
P51	0	$(\sqrt{-3})$	$\alpha,\ \alpha,\ \alpha,\ \beta$	H_{33}	3	$1/3,\ \alpha,\ \beta/3$	—
P52			$1/3,\ 1/3,\ 1/3,\ \alpha$	H_{38}	6	$1/2,\ 1/3,\ \alpha/6$	P51
P53			$\alpha,\ \alpha,\ \alpha,\ \alpha$	H_6	12	$1/2,\ 1/3,\ \alpha/3$	P51
P54			$\alpha,\ \alpha,\ \alpha,\ 3\alpha$	H_{38}	6	$1/2,\ 1/6,\ \alpha$	P51
P55			$\alpha,\ \alpha,\ \alpha,\ 9\alpha$	H_1	12	$1/2,\ 1/3,\ \alpha$	P51
P56	$-2^{11} 11^3/3^3 7^4$	$(\sqrt{-3})$	$1/3,\ 1/3,\ \alpha,\ 7\alpha$	H_{18}	8	$1/2,\ 1/3,\ \alpha$	—
P57	$5^3 43^3/2^6 7^3$	$(\sqrt{-7})$	$1/2,\ \alpha,\ \alpha,\ 7\alpha$	H_{11}	9	$1/2,\ 1/3,\ \alpha$	—
P58	$-269^3/2^{10} 3^5$	$(\sqrt{-15})$	$1/2,\ \alpha,\ \alpha,\ 3\alpha$	H_{37}	5	$1/2,\ 1/5,\ \alpha$	—
P59	$71^3/2^4 3^3 5$	$(\sqrt{-15})$	$1/2,\ 1/3,\ 1/3,\ \alpha$	H_{37}	5	$1/2,\ 1/3,\ \alpha/5$	—
P60	$\frac{(1+i)^{12}(3-2i)^3}{(2-i)^2}$		$1/2,\ 1/4,\ \alpha,\ 4\alpha$	H_{44}	5	$1/2,\ 1/4,\ \alpha$	—
P61	$-\omega 2^4 \frac{(1-2\omega)^3(7+6\omega)^3}{(1+2\omega)^6(3+2\omega)^2}$		$1/2,\ 1/3,\ \alpha,\ 6\alpha$	H_{21}	7	$1/2,\ 1/3,\ \alpha$	—

Heun-to-Heun transformations. It is explained in § C, and the Heun transformations are considered in § 4.3.

The other cases of different transformations with (generically) the same Heun equation are within the sequences P15–P18, P19–P24, P40–P41 and P51–P55. In particular, the reducible Heun equations with $j(t) = 0$ can be obtained by the cubic transformation P51. Likewise, all transformations to Heun equations with $t \in \{-3, 4, -1/3, 4/3, 1/4, 3/4\}$ are specializations of the two-parametric P15. However, there are two unrelated transformations P19, P20 giving Heun equations with $t \in \{-8, 9, -1/8, 9/8, 1/9, 8/9\}$.

The number of different j-invariants in the reducible Heun equations is 32, counting pairs of conjugate j-values of P60 and P61 as two different numbers. The number of different Heun equations up to Möbius transformations is 38. The following theorem is the main result of this paper.

Theorem 2.1. *Suppose that Heun's equation* (1.6) *is* (*a specialization of a*) *parametric pull-back transformation of a hypergeometric equation, and has no Liouvillian solutions. Then the j-invariant* (2.2) *and the* 4 *local exponent differences of Heun's equation gives one of the following situations*:

(i) $j(t) = 1728$, *and at least* 2 *exponent differences are equal up to multiplication by* -1;

(ii) $j(t) = 0$, *and at least* 3 *exponent differences are equal up to multiplication by* -1;

(iii) $j(t) = 35152/9$, *and the Heun equation is* $HE(\pm 1/2, \alpha, \pm 2\alpha, \beta)$ *for some* $\alpha, \beta \in \boldsymbol{C}$;

(iv) $j(t) = 1556068/81$, *and the Heun equation is* $HE(\alpha, \pm 2\alpha, \beta, \pm 2\beta)$ *or* $HE(\alpha, \pm 3\alpha, \beta, \pm 3\beta)$ *for some* $\alpha, \beta \in \boldsymbol{C}$;

(v) *the j-invariant is listed in the second column of Table* 4 *among the entries* P25–P40, P42–P50, P56–P59, *and the exponent differences satisfy the respective pattern in the third column up to multiplication by* -1.

(vi) *up to the conjugation* $i \mapsto -i$, $\omega \mapsto -\omega - 1$, *the j-invariant is listed in the* P60 *or* P61 *entry of Table* 4, *and the exponent differences satisfy the respective pattern in the third column up to multiplication by* -1.

Proof. A detailed inspection of Tables 3 and 4, and additional analysis of Heun equations with the same j-invariant and matching pattern of exponent differences proves the statement. □

Theorem 2.1 gives necessary conditions for a given Heun equation to be reducible to a hypergeometric one by the considered pull-back transformations. The value of the accessory parameter q gives another necessary restriction. It can be checked either by relating to the formulas of § 4, or considering additional invariants of the fractional-linear transformations. The additional invariants are derived §D. Theorem 2.1 is complimented with sufficient conditions in Theorem D.2.

2.3. Arithmetic observation of the t-values

The second column of Table 5 gives all possible values of the t-parameter of the reducible Heun equations considered here. A look at the rational

Table 5: The t-values of reducible Heun equations.

Id	t-values	$a+b=c$
P1/P14	$-1,\ 2,\ \frac{1}{2}$	$1+1=2$
P15/P18	$-3,\ 4,\ -\frac{1}{3},\ \frac{4}{3},\ \frac{1}{4},\ \frac{3}{4}$	$1+3=2^2$
P19/P24	$-8,\ 9,\ -\frac{1}{8},\ \frac{9}{8},\ \frac{1}{9},\ \frac{8}{9}$	$1+2^3=3^2$
P25	$-4,\ 5,\ -\frac{1}{4},\ \frac{5}{4},\ \frac{1}{5},\ \frac{4}{5}$	$1+2^2=5$
P26	$-\frac{16}{9},\ \frac{25}{9},\ -\frac{9}{16},\ \frac{25}{16},\ \frac{9}{25},\ \frac{16}{25}$	$3^2+4^2=5^2$
P27/P28	$-80,\ 81,\ -\frac{1}{80},\ \frac{81}{80},\ \frac{1}{81},\ \frac{80}{81}$	$1+2^4\cdot 5=3^4$
P29	$-\frac{25}{2},\ \frac{27}{2},\ -\frac{2}{25},\ \frac{27}{25},\ \frac{2}{27},\ \frac{25}{27}$	$2+5^2=3^3$
P30	$-27,\ 28,\ -\frac{1}{27},\ \frac{28}{27},\ \frac{1}{28},\ \frac{27}{28}$	$1+3^3=2^2\cdot 7$
P31/P32	$-\frac{27}{5},\ \frac{32}{5},\ -\frac{5}{27},\ \frac{32}{27},\ \frac{5}{32},\ \frac{27}{32}$	$5+3^3=2^5$
P33/P34	$-\frac{125}{3},\ \frac{128}{3},\ -\frac{3}{125},\ \frac{128}{125},\ \frac{3}{128},\ \frac{125}{128}$	$3+5^3=2^7$
P35	$-\frac{49}{32},\ \frac{81}{32},\ -\frac{32}{49},\ \frac{81}{49},\ \frac{32}{81},\ \frac{49}{81}$	$2^5+7^2=3^4$
P36	$-\frac{125}{64},\ \frac{189}{64},\ -\frac{64}{125},\ \frac{189}{125},\ \frac{64}{189},\ \frac{125}{189}$	$2^6+5^3=3^3\cdot 7$
P37	$-16\pm 12\sqrt{2},\ 17\pm 12\sqrt{2},\ \frac{1}{2}\pm\frac{3\sqrt{2}}{8}$	$(1-\sqrt{2})^2+(\sqrt{2})^5=(1+\sqrt{2})^2$
P38	$-7\pm 4\sqrt{3},\ 8\pm 4\sqrt{3},\ \frac{1}{2}\pm\frac{\sqrt{3}}{4}$	$1+(2+\sqrt{3})^2=(2-\sqrt{3})(1+\sqrt{3})^4$
P39	$-96\pm 56\sqrt{3},\ 97\pm 56\sqrt{3},\ \frac{1}{2}\pm\frac{7\sqrt{3}}{24}$	$1+\sqrt{3}(2-\sqrt{3})(1+\sqrt{3})^6=(2+\sqrt{3})^4$
P40/P41	$-26\pm 15\sqrt{3},\ 27\pm 15\sqrt{3},\ \frac{1}{2}\pm\frac{5\sqrt{3}}{18}$	$(2+\sqrt{3})^2+(2-\sqrt{3})=(\sqrt{3})^3(1+\sqrt{3})$
P42	$-8\pm 4\sqrt{5},\ 9\pm 4\sqrt{5},\ \frac{1}{2}\pm\frac{\sqrt{5}}{4}$	$\left(\frac{1+\sqrt{5}}{2}\right)^3+\left(\frac{1-\sqrt{5}}{2}\right)^3=2^2$
P43	$\frac{-123\pm 55\sqrt{5}}{2},\ \frac{125\pm 55\sqrt{5}}{2},\ \frac{1}{2}\pm\frac{11\sqrt{5}}{50}$	$\left(\frac{1+\sqrt{5}}{2}\right)^5+\left(\frac{\sqrt{5}-1}{2}\right)^5=(\sqrt{5})^3$
P44	$\frac{-7\pm 24i}{25},\ \frac{32\pm 24i}{25},\ \frac{1}{2}\pm\frac{3i}{8}$	$(2+i)^2+(1+i)^6=(2-i)^2$
P45/P46	$\frac{8\pm 44i}{125},\ \frac{117\pm 44i}{125},\ \frac{1}{2}\pm\frac{11i}{4}$	$(2+i)^3+(1+i)^4=-(2-i)^3$
P47	$\frac{4\pm 10\sqrt{-2}}{27},\ \frac{23\pm 10\sqrt{-2}}{27},\ \frac{1}{2}\pm\frac{5\sqrt{-2}}{4}$	$(1+\sqrt{-2})^3+(\sqrt{-2})^3=(1-\sqrt{-2})^3$
P48/P49	$\frac{17\pm 56\sqrt{-2}}{81},\ \frac{64\pm 56\sqrt{-2}}{81},\ \frac{1}{2}\pm\frac{7\sqrt{-2}}{16}$	$(1-\sqrt{-2})^4+(\sqrt{-2})^7=(1+\sqrt{-2})^4$
P50	$\frac{2\pm 22\sqrt{-2}}{243},\ \frac{241\pm 22\sqrt{-2}}{243},\ \frac{1}{2}\pm\frac{11\sqrt{-2}}{2}$	$(1+\sqrt{-2})^5+(1-\sqrt{-2})^5=-(\sqrt{-2})^2$
P51/P55	$\frac{1}{2}\pm\frac{\sqrt{-3}}{2}$	$(-\omega)+(1+\omega)=1$
P56	$\frac{27\pm 39\sqrt{-3}}{98},\ \frac{71\pm 39\sqrt{-3}}{98},\ \frac{1}{2}\pm\frac{13\sqrt{-3}}{18}$	$(3+2\omega)^2+(1+\omega)(1+2\omega)^3=\omega(1-2\omega)^2$
P57	$\frac{-87\pm 91\sqrt{-7}}{256},\ \frac{343\pm 91\sqrt{-7}}{256},\ \frac{1}{2}\pm\frac{13\sqrt{-7}}{98}$	$\left(\frac{1+\sqrt{-7}}{2}\right)^7+(\sqrt{-7})^3=\left(\frac{1-\sqrt{-7}}{2}\right)^7$
P58	$\frac{243\pm 171\sqrt{-15}}{1024},\ \frac{781\pm 171\sqrt{-15}}{1024},\ \frac{1}{2}\pm\frac{19\sqrt{-15}}{54}$	$\left(\frac{1+\sqrt{-15}}{2}\right)^4\left(\frac{3-\sqrt{-15}}{2}\right)+3^3=\left(\frac{1-\sqrt{-15}}{2}\right)^4\left(\frac{-3-\sqrt{-15}}{2}\right)$
P59	$\frac{-7\pm 33\sqrt{-15}}{128},\ \frac{135\pm 33\sqrt{-15}}{128},\ \frac{1}{2}\pm\frac{11\sqrt{-15}}{90}$	$\left(\frac{1+\sqrt{-15}}{2}\right)^3+3\sqrt{-15}=\left(\frac{1-\sqrt{-15}}{2}\right)^3$
P60	$-2i,\ 1+2i,\ 1-\frac{i}{2},\ \frac{i}{2},\ \frac{1-2i}{5},\ \frac{4+2i}{5}$	$1+(1+i)^2=(1+2i)$
P61	$\frac{1+3\omega}{4},\ \frac{3-3\omega}{4},\ \frac{-8-12\omega}{7},\ \frac{15+12\omega}{7},\ \frac{1-4\omega}{9},\ \frac{8+4\omega}{9}$	$(1+3\omega)+(1-\omega)^2(2+\omega)=2^2$

t-values in the P19–P36 cases reveals several nicely factorizable integers like 81, 32, 125, 128 in the numerators or denominators of the t-values. Algebraic t-values have nice factorization expressions as well. For example, in the cases P43, P50, P57 we have

$$\frac{-123+55\sqrt{5}}{2} = -\left(\frac{1-\sqrt{5}}{2}\right)^{10}, \qquad \frac{241+22\sqrt{-2}}{243} = -\frac{(1+\sqrt{-2})^{10}}{3^5},$$
$$\frac{-87+91\sqrt{-7}}{2} = \left(\frac{1-\sqrt{-7}}{2}\right)^{14}.$$

Often all six t-values in an orbit under fractional-linear transformations factorize remarkably.

A compact expression for the classical orbit (2.1) of six t-values is an identity

$$a+b=c, \tag{2.3}$$

where the vector (a,b,c) of numbers is a multiple of $(t,1-t,1)$. The orbit of six t-values is recovered as $\{a/c, b/c, c/a, c/b, -a/b, -b/a\}$. When t is a rational number, a convenient (a,b,c) triple is obtained by clearing the denominators of $(t,1-t,1)$, so that a,b,c in (2.3) are pairwise co-prime integers. For example, the *abc* identity for P26 is $9+16=25$, reminding the most famous Pythagorean triangle. If the t-values are in an algebraic number field that is a *principal ideal domain*, the numbers a, b, c can be chosen to be "co-prime" as well, but there is freedom to multiply (2.3) by units. For a general number field K, identities (2.3) up to constant multiples should be considered as points $(a:b:c)$ on the projective line $\boldsymbol{P}^1$ over K.

The third column of Table 5 spells out arithmetic *abc*-identities defining the t-values of the encountered Heun equations, including those over algebraic number fields. Many of the identities are indeed attractive, as only factors of small norm are involved. This is not accidental. Our coverings are Belyi functions, and it is known [3] that Belyi functions degenerate only modulo primes (or prime ideals) of small size. The cross-ratio t degenerates to 0, 1 or ∞ only modulo primes of bad reduction for the involved Belyi covering, because only then Heun's 4 singular points coalesce. This is the reason why the numbers in the listed $a+b=c$ identities are highly factorizable, or only a few primes are involved.

Equations like (2.3) with a, b, c prescribed to involve only a small set S of primes are known in number theory as *S-unit equations*. They typically have only finitely many solutions up to scalar multiplication [23]. Diophantine equations for "highly factorizable" integers enjoy wide popular interest. In particular, solving Fermat's equation $x^n + y^n = z^n$ in integers was a famous open problem for centuries. After Wiles' resolution of Fermat's problem in 1995, a prominent generalizing arithmetic conjecture is the *abc-conjecture* of Masser and Osterlé [11]. It states that for any real $\varepsilon > 0$ there should be finitely many identities (2.3) with co-prime integers a, b, c such that the *quality ratio*

$$Q(a,b,c) := \frac{\log \max(|a|,|b|,|c|)}{\log \operatorname{rad}(abc)} \tag{2.4}$$

is greater than $1 + \varepsilon$. Here the *radical* $\operatorname{rad}(n)$ is the product of prime numbers dividing n. For example, the quality ratio of $3 + 125 = 128$, which gives the t-values for P33/P34, is equal to $\log(128)/\log(30) \approx 1.426565$. Currently there are over 200 examples known [6] with the quality ratio greater than 1.4.

For comparison, the *abc*-theorem [11, Proposition 2] for polynomials states that for any identity (2.3) with co-prime polynomials $a, b, c \in \boldsymbol{C}[x]$ of maximal degree D, the number of different roots of the product abc is at least $D + 1$. This is a familiar consequence of the Hurwitz formula, as in [25, Lemma 3.2]. The bound $D + 1$ is attained when the rational function a/c is a Belyi function and its value at $x = \infty$ is 0, 1 or ∞ on $\boldsymbol{P}^1$.

There is a generalization of the *abc*-conjecture over number fields [4], where the definition of the quality ratio in (2.4) is adjusted as follows. The numerator is replaced by the logarithm of the *height* of $(a : b : c) \in \boldsymbol{P}^1(K)$, and $\operatorname{rad}(abc)$ is replaced by the product of (the absolute value of) the field discriminant and the norms of the *prime ideals* (or *places, non-archimedean norms*) which reduce $(a : b : c)$ to a trivial point like $(1 : 0 : 1)$. For example, the quality ratio for the P48/P49 identity is computed as $\log \max(3^4, 2^7, 3^4)/\log(8 \cdot 3 \cdot 2 \cdot 3) \approx 1.074487$, while the quality ratio for the P43 identity is equal to $\log \max(1^5, 1^5, 5^3)/\log(5 \cdot 5) = 1.5$. Among the encountered number fields, only $\boldsymbol{Q}(\sqrt{-15})$ is not a principal ideal domain. This field defines two transformations with the same covering H_{37}. The quality ratio for the P58 identity is equal to $\log \max(2^9, 2^9, 3^5)/\log(15 \cdot 2 \cdot 2 \cdot 3) \approx 1.201305$, while for the P59 identity it is equal to $\log \max(4^3, 9 \cdot 15, 4^3)/\log(15 \cdot 2 \cdot 3 \cdot 5 \cdot 2) \approx 0.721110$.

In total, Table 5 contains 12 identities $a + b = c$ of the quality ratio greater than 1. Two identities (for P33/P34 and P43) have the quality ratio greater than 1.4. The encountered t-values are of relatively small size, and the *abc* identities are not groundbreaking. However, the identity for P45/P46 recently brought a \$50 prize to Fred W. Helenius [21]. Considering Belyi functions and cross ratios of 4 points in the three branching fibers may be a fruitful strategy for finding interesting *abc* triples, especially over algebraic number fields. The non-parametric "hyperbolic" hypergeometric-to-Heun transformations [13] give more known high quality examples, such as

$$1 + 2^5 5^2 3 = 7^4, \qquad \left(\frac{1+\sqrt{-7}}{2}\right)^{13} + \sqrt{-7} = \left(\frac{1-\sqrt{-7}}{2}\right)^{13},$$

$$\left(\frac{\sqrt{5}-1}{2}\right)^{12} + 2^4 3^2 \sqrt{5} = \left(\frac{1+\sqrt{5}}{2}\right)^{12}$$

with the respective quality ratios 1.455673, 1.707222, 1.697794, and a new identity in $\boldsymbol{Q}(\sqrt{-14})$ with the quality ratio 1.581910.

3. Deriving Heun-to-hypergeometric identities

With the list of suitable Belyi coverings at hand, pull-back transformations (1.3) between hypergeometric and Heun equations are obtained by normalizing the Belyi functions with Möbius transformations (so that the singularities of the pulled-back equation would indeed be at $x = 0, 1, \infty, t$), and by choosing suitable gauge prefactors $\theta(x)$. The parameters a, b, c, d and A, B, C of the related differential equations (1.5), (1.6) are determined by the exponent differences assigned to the singular points. The accessory parameter q can be determined by Lemma 3.2 here below, or by considering the first power series terms of a two-term Heun-to-Gauss identity. A pull-back transformation can be composed with the fractional-linear symmetries of the hypergeometric and Heun equations, described in § B.

The role of the prefactor $\theta(x)$ is to get rid of *irrelevant singularities* and shift a local exponent at each $x = 0$, $x = 1$, $x = t$ to the value 0, as prescribed by a Riemann scheme in (1.7). The direct pull-back transformation with $\theta(x) = 1$ would typically give a Fuchsian equation with several non-logarithmic singular points where the exponent difference is equal to 1; we call them *irrelevant singularities*. They can be turned into non-singular points by shifting their exponents to the values 0 and 1. The possible irrelevant singular points are above $z = \infty$, and above the finite singular points ($z = 0$, $z = 1$) where the restricted exponents of the hypergeometric equation are 0, $-1/k$ rather than 0, $1/k$. Besides, the relevant singular points above $z = \infty$ would typically have only non-zero local exponents, and a local exponent for all of them except $x = \infty$ has to be shifted to the value 0. The prefactor will have the form $\theta(x) = \prod_i (x - \sigma_i)^{-\xi_i}$, where σ_i are all the x-points where the local exponents need to be shifted, and ξ_i is the local exponent at σ_i to be shifted to 0. The local exponents at $x = \infty$ then shift by the sum of all ξ_i's. It is convenient to use the P-notation of the Riemann scheme, as demonstrated in appendix formula (B.6).

The prefactor is not needed when there is only one point $x = \infty$ above $z = \infty$. If the local exponents at the restricted points $z \in \{0, 1\}$ are then chosen to be 0, $1/k$ (rather than 0, $-1/k$), there are no irrelevant singularities under the direct pullback. The rational function $\varphi(x)$ defining the covering is then a polynomial. Maier [17] classified all parametric transformations between hypergeometric and Heun equations without a prefactor. A list of seven transformations was obtained. Using our identification, these are five Maier's indecomposable

$$\text{(3.1)} \qquad \text{P1:} \quad E(\alpha,\beta,\gamma) \xleftarrow{2} HE(\alpha,\alpha,2\beta,2\gamma),$$

$$\text{P15:} \quad E(1/2,\alpha,\beta) \xleftarrow{3} HE(1/2,\alpha,2\alpha,3\beta),$$

$$\text{P47:} \quad E(1/2,1/3,\alpha) \xleftarrow{4} HE(1/2,1/2,1/3,4\alpha),$$

$$\text{P51:} \quad E(1/3,\alpha,\beta) \xleftarrow{3} HE(\alpha,\alpha,\alpha,3\beta),$$

$$\text{P59:} \quad E(1/2,1/3,\alpha) \xleftarrow{5} HE(1/2,1/3,1/3,5\alpha),$$

and two composite transformations:

$$\text{(3.2)} \qquad \text{P3:} \quad E(1/2,\alpha,\beta) \xleftarrow{2} E(\alpha,\alpha,2\beta) \xleftarrow{2} HE(\alpha,\alpha,2\alpha,4\beta),$$

$$\text{P52:} \quad E(1/2,1/3,\alpha) \xleftarrow{2} E(1/3,1/3,2\alpha) \xleftarrow{3} HE(1/3,1/3,1/3,6\alpha).$$

The Belyi coverings are H_{32} to H_{38} in a mixed up order. A proper normalization by the fractional-linear symmetries of § B is required to avoid the prefactor. In addition, several more transformations without a prefactor are given in [17] for the degenerate Heun equation with $ab = q = 0$. The function $\varphi(x)$ does not have to be a polynomial then, as the points above $z = \infty$ immediately have a local exponent 0.

Two-term identities between the Heun and hypergeometric functions are derived by identifying standard local solutions at the corresponding points of the related Heun and hypergeometric equations. By fractional-linear transformations, any singular x-point can be chosen as $x = 0$ and its projection as $z = 0$. Then we are identifying the standard Heun and hypergeometric series at $x = 0$. This determines a two-term identity up to fractional-linear transformations (B.1) and (B.3)–(B.4). The prefactor $\theta(x)$ has to be normalized to the value $\theta(x) = 1$ at $x = 0$. If the exponent difference at $z = 0$ is an unrestricted parameter, changing its sign gives essentially the same two-term identity. More generally, the following choices of $x = 0$ give the same two-term identities up to the fractional-linear transformations and change of parameters:

(i) the x-points with the same branching index and above the same point of $\boldsymbol{P}^1_z$;

(ii) the x-points with the same branching index, if they are in different fibers with the same branching pattern, and either the local exponents at the corresponding z-points are the same, or the exponent differences at both z-points are free parameters.

A pull-back transformation (1.3) between the hypergeometric and (or) Heun equations might fail to produce two-term identities between the hypergeometric and (or) Heun solutions only if all singular points of the transformed equation lie above non-singular points of the starting equation. The singularities of the transformed equation are then apparent, and the pull-back covering is typically not a Belyi function so that [25, Proposition 3.3] likely applies. An example of such a transformation in given in [28, Remark 5.9]; it is a composition of $E(1/2,1/2,1/2) \overset{4}{\leftarrow} E(1,1,1)$ and $E(1,1,1) \overset{3}{\leftarrow} E(3,2,2)$, with general ramification fibers in the second transformation. On the other hand, transformation identities between the hypergeometric and Heun functions might formally exist without a pull-back between their equations. For example, the linear function

$$ {}_2\mathrm{F}_1\left(\begin{matrix} -1, b \\ c \end{matrix} \middle| z \right) $$

can be formally transformed to any (hypergeometric or Heun) polynomial. Part 2 of [26, Lemma 2.1] indicates that this situation can occur only if we start with a hypergeometric function actually satisfying a first order Fuchsian equation.

If the exponent difference at $z = 0$ is restricted to $1/k$ with $k \in \boldsymbol{Z}$, the choices 0, $1/k$ and 0, $-1/k$ of local exponents give different identities. Changing the sign of the exponent difference at $z = 0$ gives an identity between the other two local solutions (with non-zero local exponents) at $x = 0$ and $z = 0$. The following lemma gives the alternative identity at the same $x = 0$. It is a reformulation of [26, Lemma 2.3] that captures the same situation for hypergeometric identities.

Lemma 3.1. *Suppose that we have the identity* (1.4) *coming from a pull-back transformation between the corresponding hypergeometric and Heun equations. Then* $\varphi(x)^{1-C} \sim Kx^{1-c}$ *as* $x \to 0$ *for some constant* K, *and the following identity holds*:

$$ \mathrm{Hn}\left(\begin{matrix} t \\ q_1 \end{matrix} \middle| \begin{matrix} 1+a-c, 1+b-c \\ 2-c; d \end{matrix} \middle| x \right) = \Theta(x)\,{}_2\mathrm{F}_1\left(\begin{matrix} 1+A-C, 1+B-C \\ 2-C \end{matrix} \middle| \varphi(x) \right), $$

where $q_1 = q - (c-1)(a+b-c-d+dt+1)$ *and* $\Theta(x) = \theta(x)\varphi(x)^{1-C}/Kx^{1-c}$.

Proof. The lemma is proved by a straightforward identification of the other canonical local solutions of both equations at $x = 0$ and $z = 0$. □

The accessory parameter q of the pulled-back Heun equation can be determined at the latest stage, by considering power series expansions at $x = 0$ in a supposed two-term identity and comparing the first couple of terms in the power series. The value of q is given by the following lemma. Note particularly, that if the covering $\varphi(x)$ branches at $x = 0$ and the prefactor $\theta(x)$ is absent, then $q = 0$ (because $\lambda = \mu = 0$); check formulas (4.9) and (4.24) below.

Lemma 3.2. *Suppose that we have the identity* (1.4) *coming from a pull-back transformation between the corresponding hypergeometric and Heun equations with*

$$\varphi(x) = \lambda x + O(x^2), \qquad \theta(x) = 1 + \mu x + O(x^2)$$

as $x \to 0$. *Then* $q = ct(\mu + AB\lambda/C)$.

Proof. Expanding both sides of (1.4) in the power series at $x = 0$ gives

$$1 + \frac{q}{ct}x + O(x^2) = 1 + \mu x + \frac{AB}{C}\lambda x + O(x^2). \qquad \square$$

4. Identities between Heun and hypergeometric functions

In this section we briefly survey pull-back transformations between the Heun or hypergeometric functions, and then we present parametric Gauss-to-Heun transformations. Only parametric pull-backs from hypergeometric equations with Liouvillian solutions are not considered here. All 61 considered Heun-to-Gauss reductions are exemplified in Maple package [9].

We remark that we do not consider identities like

$$\text{Hn}\left(\begin{array}{c} 4 \\ 1/2 \end{array} \middle| \begin{array}{c} 1/2, 1/2 \\ 1; 1/2 \end{array} \middle| -\frac{4s(s-1)(s+2)(s+1)}{(2s+1)^2}\right) = \sqrt{1+2s}\,{}_2\text{F}_1\left(\begin{array}{c} 1/2, 1/2 \\ 1 \end{array} \middle| \frac{s^3(s+2)}{2s+1}\right), \tag{4.1}$$

with rational functions in both arguments, or algebraic radicals in an argument, even if they contain a free parameter. Formula (4.1) is a reparametrized version (without argument radicals) of Joyce's identity [14], cited in [24, formula (24)] as well. Nor we consider relations of Heun's equations with an apparent singularity to ${}_3\text{F}_2$ and other generalized hypergeometric functions, nor relations to sums of contiguous ${}_2F_1$ functions, illustrated in [16] and [19, §5].

4.1. Transformations between hypergeometric functions

Pull-back transformations between the hypergeometric equations give algebraic transformations between hypergeometric functions of the form

$$(4.2)\qquad {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x\right) = \theta(x)\,{}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix} \middle| \varphi(x)\right).$$

The classical transformations were obtained by Gauss, Goursat, Riemann, Kummer. Here is an example of a cubic transformation with one free parameter a:

$$(4.3)\qquad {}_2F_1\left(\begin{matrix} 3a, 1/3 - a \\ 2a + 5/6 \end{matrix} \middle| x\right) = (1 - 4x)^{-3a}\,{}_2F_1\left(\begin{matrix} a, a + 1/3 \\ 2a + 5/6 \end{matrix} \middle| \frac{27x}{(4x - 1)^3}\right).$$

This is the transformation $E(1/2, 1/3, \alpha) \overset{3}{\leftarrow} E(1/2, \alpha, 2\alpha)$, with $\alpha = 1/6 - 2a$. The classical transformations have degree at most 6, namely $E(1/2, 1/3, \alpha) \overset{6}{\leftarrow} E(\alpha, \alpha, 4\alpha)$ and $E(1/2, 1/3, \alpha) \overset{6}{\leftarrow} E(2\alpha, 2\alpha, 2\alpha)$. The latter formula is given by

$$(4.4)\qquad {}_2F_1\left(\begin{matrix} 6a, 2a + 1/3 \\ 4a + 2/3 \end{matrix} \middle| x\right) = (1 - x + x^2)^{-3a}\,{}_2F_1\left(\begin{matrix} a, a + 1/3 \\ 2a + 5/6 \end{matrix} \middle| \frac{27}{4}\,\frac{x^2(x - 1)^2}{(x^2 - x + 1)^3}\right).$$

Pull-back transformations between the hypergeometric equations, and subsequently, algebraic transformations of the Gauss hypergeometric functions, are systematically classified in [26]. (Transformation (4.4) is presented in [26, formula (28)] with a misprint in the lower parameter $2a + 5/6$. Here is a list of other inaccuracies in [26]: (ii) the case $a \neq 0$ in (29) should be multiplied by -1; (iii) proof of Theorem 6.1 should refer to [28, Theorem 5.1]; (iv) uniqueness claims on pg. 162 and Remark 7.1 are dubious, especially with $\ell/k \in \boldsymbol{Z}$; see [28, (5.47)] and [25, §5.4]. Furthermore, the question of Remark 7.1 about existence of Gauss-to-Gauss pull-backs that do not yield two-term hypergeometric formulas is answered in [28, Remark 5.7] positively with the example $E(1/2, 1/2, 1/2) \overset{12}{\leftarrow} E(3, 2, 2)$, as mentioned here in §3 right affer the listing (i)–(ii).)

The well-known quadratic transformations of Gauss hypergeometric functions have two free parameters:

$$(4.5)\qquad {}_2\mathrm{F}_1\left(\begin{matrix}2a,2b\\ a+b+1/2\end{matrix}\,\middle|\,x\right)={}_2\mathrm{F}_1\left(\begin{matrix}a,b\\ a+b+1/2\end{matrix}\,\middle|\,4x(1-x)\right),$$

$$(4.6)\qquad {}_2\mathrm{F}_1\left(\begin{matrix}2a,a-b+1/2\\ a+b+1/2\end{matrix}\,\middle|\,x\right)=(1-x)^{-2a}{}_2\mathrm{F}_1\left(\begin{matrix}a,b\\ a+b+1/2\end{matrix}\,\middle|\,-\frac{4x}{(x-1)^2}\right),$$

$$(4.7)\qquad {}_2\mathrm{F}_1\left(\begin{matrix}2a,b\\ 2b\end{matrix}\,\middle|\,x\right)=\left(1-\frac{x}{2}\right)^{-2a}{}_2\mathrm{F}_1\left(\begin{matrix}a,a+1/2\\ b+1/2\end{matrix}\,\middle|\,\frac{x^2}{(2-x)^2}\right).$$

The first two formulas are related by fractional-linear transformations (B.1), whereas (4.7) is not equivalent up to two-term fractional-linear transformations (on either $\boldsymbol{P}^1_x$ or $\boldsymbol{P}^1_z$), as noted by Askey [2] and Maier [19, Remark 4.1.2]. The dividing difference is the choice of the point $x=0$: it is a non-branching point in (4.5)–(4.6) but a branching point in the last formula.

But formula (4.7) can be derived from (4.5) by the following argument. The functions

$$x^{-2a}{}_2\mathrm{F}_1\left(\begin{matrix}2a,a-b+1/2\\ 2a-2b+1\end{matrix}\,\middle|\,\frac{1}{x}\right),\qquad (1-2x)^{-2a}{}_2\mathrm{F}_1\left(\begin{matrix}a,a+1/2\\ 1+a-b\end{matrix}\,\middle|\,\frac{1}{(1-2x)^2}\right)$$

are among the 24 Kummer solutions of the differential equations for the left-hand side and the right-hand side of (4.5) respectively. Therefore the two functions satisfy the same Fuchsian equation of order 2. We multiply both functions by x^{2a}, make the substitutions $x\mapsto 1/x$ and $b\mapsto a-b+1/2$ and obtain the two functions in (4.7) up to a constant multiple on the right-hand side. Those two functions satisfy the same Fuchsian equation of order 2, have the same value and the same local exponent at a regular singular point (with a non-integer exponent difference in general), so they must be equal in a neighborhood of $x=0$, and (4.7) follows.

An example of a non-classical Gauss-to-Gauss transformation is $E(1/2,1/3,1/7)\overset{10}{\leftarrow}E(1/3,1/7,2/7)$ given by

$$(4.8)\qquad {}_2\mathrm{F}_1\left(\begin{matrix}5/42,19/42\\ 5/7\end{matrix}\,\middle|\,x\right)$$

$$=\left(1-\frac{19}{9}x-\frac{343}{243}x^2+\frac{16807}{6561}x^3\right)^{-1/28}$$

$$\times\,{}_2\mathrm{F}_1\left(\begin{matrix}1/84,29/84\\ 6/7\end{matrix}\,\middle|\,\frac{x^2(1-x)(49x-81)^7}{4(16807x^3-9261x^2-13851x+6561)^3}\right).$$

The degree 10 rational function is one of our Belyi coverings H_8 up to the Möbius transformations. This is not surprising, as specialization of the exponent difference to $\alpha = 1/7$ turns the Heun equation for P32 to $E(1/3, 1/7, 2/7)$. In the same way, all 61 Heun-to-Gauss parametric transformations can be specialized to Gauss-to-Gauss transformations classified in [26].

The identities like (4.8) can be verified by checking the power series at $x = 0$. But the common region of convergence usually appears to be small. For example, (4.8) does not hold at $x = 1$ or $x = 81/49$ for the standard analytic branches of ${}_2F_1$ functions, as can be checked numerically.

4.2. Quadratic hypergeometric-to-Heun transformations (P1)

Quadratic Gauss-to-Heun transformations apply to Gauss hypergeometric functions without any restriction of their parameters. The underlying reason is that a quadratic covering branches only above 2 points, and if the branching is above the singularities of the hypergeometric equation, there are exactly 4 points above those singularities. Here are explicit formulas:

$$\mathrm{Hn}\left(\begin{matrix} -1 \\ 0 \end{matrix} \,\middle|\, \begin{matrix} 2a, 2b \\ 2c-1; a+b-c+1 \end{matrix} \,\middle|\, x \right) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, x^2 \right), \tag{4.9}$$

$$\mathrm{Hn}\left(\begin{matrix} 2 \\ 4ab \end{matrix} \,\middle|\, \begin{matrix} 2a, 2b \\ c; 2a+2b-2c+1 \end{matrix} \,\middle|\, x \right) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, x(2-x) \right), \tag{4.10}$$

$$\mathrm{Hn}\left(\begin{matrix} 1/2 \\ 2ab \end{matrix} \,\middle|\, \begin{matrix} 2a, 2b \\ c; c \end{matrix} \,\middle|\, x \right) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, 4x(1-x) \right). \tag{4.11}$$

They were first found by Kuiken in [15]. Other possible polynomials $\varphi(x)$ for quadratic transformations between the hypergeometric and Heun equations are $1 - x^2$, $(1-x)^2$, $(2x-1)^2$. Fractional-linear transformations of P-symbols for the 192 Heun functions and the related Kummer's 24 hypergeometric functions give a set of another 30 rational functions of degree 2 that transform the general hypergeometric equation to Heun's equations (with a prefactor, in general). The 30 rational functions are given in [15] in the context of the degenerate case $ab = q = 0$.

Like for hypergeometric quadratic transformations (4.5)–(4.7), we have two different choices for $x = 0$: a branching point and a non-branching point. Accordingly, identities (4.10) and (4.11) are related by fractional-linear transformations (B.1), (B.4)–(B.3), whereas identity (4.9) cannot be related to them by the fractional-linear transformations. To derive (4.9) from (4.10), one can observe that the functions

$$\mathrm{Hn}\left(\begin{matrix} -1 \\ 0 \end{matrix} \,\middle|\, \begin{matrix} 2a, 2b \\ 2a+2b-2c+1; c \end{matrix} \,\middle|\, 1-x \right), \qquad {}_2F_1\left(\begin{matrix} a, b \\ a+b-c+1 \end{matrix} \,\middle|\, (1-x)^2 \right)$$

satisfy the same Heun equation as both sides of (4.10), and have the same local exponent and value at $x = 1$. Therefore they must be generally equal; formula (4.9) is then obtained after the substitution $x \mapsto 1 - x$, $c \mapsto a + b - c + 1$.

4.3. Heun-to-Heun transformations

Existence of quadratic and quartic transformations was pointed out by Erdélyi in [8, Vol. 3]. Examples of these transformations are given by Maier in [19, §4]. Here are two alternative formulas of the quadratic transformations:

$$\text{(4.12)} \qquad \text{Hn}\left(\begin{array}{c} s^2 \\ (1+s)^2 q - 2abs \end{array} \middle| \begin{array}{c} 2a, 2a-b+1 \\ b; 2a-b+1 \end{array} \middle| x \right) = \left(1 + \frac{x}{s}\right)^{-2a} \text{Hn}\left(\begin{array}{c} 4s/(1+s)^2 \\ q \end{array} \middle| \begin{array}{c} a, a+1/2 \\ b; 1/2 \end{array} \middle| \frac{4xs}{(x+s)^2}\right),$$

$$\text{(4.13)} \qquad \text{Hn}\left(\begin{array}{c} s^2/(2s-1) \\ \{2abs + 4qs(s-1)\}/(2s-1) \end{array} \middle| \begin{array}{c} 2a, b \\ b; b \end{array} \middle| x \right) = \left(1 - \frac{x}{s}\right)^{-2a} \text{Hn}\left(\begin{array}{c} \{4s(1-s)\}^{-1} \\ q \end{array} \middle| \begin{array}{c} a, a+1/2 \\ b; 2a-b+1 \end{array} \middle| \frac{x(x-1)}{(x-s)^2}\right).$$

The two formulas are related by a series of fractional-linear transformations (and reparametrizations). The transformation of local exponents is given by

$$\text{(4.14)} \qquad HE(1/2, 1/2, \alpha, \beta) \overset{2_H}{\leftarrow} HE(\alpha, \alpha, \beta, \beta).$$

All choices for $x = 0$ give two-term identities related by fractional-linear transformations (B.3)–(B.4).

A quartic transformation is obtained by composing two versions of the quadratic transformation:

$$\text{(4.15)} \qquad HE(1/2, 1/2, 1/2, \alpha) \overset{2_H}{\leftarrow} HE(1/2, 1/2, \alpha, \alpha) \overset{2_H}{\leftarrow} HE(\alpha, \alpha, \alpha, \alpha).$$

Up to a Möbius transformation, the starting equation is a general Lamé equation $HE(1/2, 1/2, 1/2, \alpha)$. Remarkably, transformation of the parameters t and q simplifies greatly. Let us compose two versions of (4.13) with $s \mapsto 1/2s$, $b \mapsto 2a + 1/2$ and $a \mapsto 2a$, $b \mapsto 2a + 1/2$. After setting $t = s^2/(2s-1)$ we recognize the same transformation as in [19, Theorem 4.2]:

$$(4.16)\qquad \mathrm{Hn}\left(\begin{matrix} t \\ 4q \end{matrix}\,\middle|\, \begin{matrix} 4a, 2a+1/2 \\ 2a+1/2; 2a+1/2 \end{matrix}\,\middle|\, x\right)$$
$$= \left(1-\frac{x^2}{t}\right)^{-2a} \mathrm{Hn}\left(\begin{matrix} t \\ q \end{matrix}\,\middle|\, \begin{matrix} a, a+1/2 \\ 2a+1/2; 1/2 \end{matrix}\,\middle|\, \frac{4tx(x-1)(x-t)}{(x^2-t)^2}\right),$$

The composite degree 4 covering happens to be the Belyi covering H_{31}.

Other Heun-to-Heun transformations are possible only for the very special case of Lamé equation $HE(1/2,1/2,1/2,1/2)$. This can be seen by considering necessary branching patterns. Other pull-back coverings cannot be Belyi functions (as we wish only 4 singular points), hence they ramify above all 4 singular z-points. The 4 fibers would contain at least $2D+2$ different points, and we want at least $2D-2$ of them to be non-singular after a pull-back. But each fiber has at most $\lfloor D/2 \rfloor$ non-singular points, quickly leading to $HE(1/2,1/2,1/2,1/2)$. As recalled in [24, § 3], Carlitz [5] solved this equation by giving an explicit basis of solutions. The two independent solutions of Carlitz are

$$(4.17)\qquad y_{\pm}(x) = \exp\left(\pm\sqrt{q}\int_0^x \frac{du}{\sqrt{u(u-1)(u-t)}}\right).$$

This is an integral of a holomorphic differential on the general *Legendre elliptic curve*

$$(4.18)\qquad w^2 = u(u-1)(u-t).$$

Any isogeny between Legendre elliptic curves transforms the holomorphic differentials to each other up to a scalar multiple, since the space of holomorphic differentials on elliptic curves is one-dimensional. Vice versa, the particular branching pattern of the coverings $\varphi(x)$ ensures the transformations $u \mapsto \varphi(x)$ of holomorphic differentials. It follows that any degree transformations of $HE(1/2,1/2,1/2,1/2)$ exist, and they correspond to the isogenies of Legendre elliptic curves. In particular, here are the cubic and an alternative quartic transformations:

$$(4.19)\qquad \mathrm{Hn}\left(\begin{matrix} s^3(s-2)/(1-2s) \\ q(1-2s)^2 \end{matrix}\,\middle|\, \begin{matrix} 0, 1/2 \\ 1/2; 1/2 \end{matrix}\,\middle|\, x\right)$$
$$= \mathrm{Hn}\left(\begin{matrix} s(s-2)^3/(1-2s)^3 \\ q \end{matrix}\,\middle|\, \begin{matrix} 0, 1/2 \\ 1/2; 1/2 \end{matrix}\,\middle|\, \frac{x(x+s(s-2))^2}{((1-2s)x+s^2)^2}\right),$$

$$(4.20)\quad \mathrm{Hn}\left(\begin{matrix} s^4 \\ -q(s-1)^4 \end{matrix}\,\middle|\,\begin{matrix} 0, 1/2 \\ 1/2; 1/2 \end{matrix}\,\middle|\, x\right)$$
$$= \mathrm{Hn}\left(\begin{matrix} (s+1)^4/(s-1)^4 \\ q \end{matrix}\,\middle|\,\begin{matrix} 0, 1/2 \\ 1/2; 1/2 \end{matrix}\,\middle|\, \frac{x(s+1)^4(x+s^2)^2}{(x-1)(x-s^4)(x-s^2)^2}\right).$$

They correspond to generic isogenies of degree 3 and 4 between Legendre elliptic curves. The t-values are related by algebraic equations of the modular curves corresponding to the congruence subgroups $\Gamma_0(3) \cap \Gamma(2)$ and $\Gamma_0(4) \cap \Gamma(2)$ of $PSL(2, \boldsymbol{Z})$, respectively, while the pull-back coverings are the isogeny transformations of the u-coordinate of (4.18), with u replaced by x. Equivalent statements hold for isogeny transformations of any degree D. Parametric quadratic transformations (4.12)–(4.13) applied to $HE(1/2, 1/2, 1/2, 1/2)$ correspond to the generic isogeny of degree 2, while quartic transformation (4.16) then represents the multiplication by 2 map on (4.18). Both quartic transformations (4.16) and (4.20) are compositions of two quadratic Heun-to-Heun transformations.

4.4. Heun-to-hypergeometric reductions with two parameters

In the following subsections we present the possible Gauss-to-Heun transformation formulas up to the fractional-linear transformations (B.1), (B.3)–(B.4). As explained with the items (i)–(ii) in § 3, the number of different two-term identities is determined by the number of singularities with different exponent differences in the same fiber and the number of non-symmetric branching fibers.

4.4.1. The transformation P15: $E(1/2, \alpha, \beta) \stackrel{3}{\leftarrow} HE(1/2, \alpha, 2\alpha, 3\beta)$

Up to fractional-linear transformations, we have these identities:

$$(4.21)\quad \mathrm{Hn}\left(\begin{matrix} 1/4 \\ 9ab/4 \end{matrix}\,\middle|\,\begin{matrix} 3a, 3b \\ 1/2; a+b+1/2 \end{matrix}\,\middle|\, x\right) = {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ 1/2 \end{matrix}\,\middle|\, x(4x-3)^2\right),$$

$$(4.22)\quad \mathrm{Hn}\left(\begin{matrix} 1/4 \\ (9ab+3a+3b-1)/4 \end{matrix}\,\middle|\,\begin{matrix} 3a, 3b \\ 3/2; a+b+1/6 \end{matrix}\,\middle|\, x\right)$$
$$= \left(1 - \frac{4x}{3}\right) {}_2\mathrm{F}_1\left(\begin{matrix} a+1/3, b+1/3 \\ 3/2 \end{matrix}\,\middle|\, x(4x-3)^2\right),$$

$$(4.23)\quad \mathrm{Hn}\left(\begin{matrix} 3/4 \\ 27ab/4 \end{matrix}\,\middle|\,\begin{matrix} 3a, 3b \\ a+b+1/2; 1/2 \end{matrix}\,\middle|\, x\right) = {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ a+b+1/2 \end{matrix}\,\middle|\, x(4x-3)^2\right),$$

$$(4.24)\qquad \mathrm{Hn}\left(\begin{matrix}-3\\0\end{matrix}\,\middle|\,\begin{matrix}3a,3b\\2a+2b;1/2\end{matrix}\,\middle|\,x\right)={}_2\mathrm{F}_1\left(\begin{matrix}a,b\\a+b+1/2\end{matrix}\,\middle|\,\frac{x^2(x+3)}{4}\right),$$

$$(4.25)\qquad \mathrm{Hn}\left(\begin{matrix}4/3\\6a^2+6ab-a\end{matrix}\,\middle|\,\begin{matrix}3a,2a+b\\3a+3b-1/2;1/2\end{matrix}\,\middle|\,x\right)$$

$$=\left(1-\frac{3x}{4}\right)^{-2a}{}_2\mathrm{F}_1\left(\begin{matrix}a,b\\a+b+1/2\end{matrix}\,\middle|\,\frac{x^3}{(4-3x)^2}\right).$$

The five formulas represent the five non-equivalent choices for the exponent difference at $x=0$. The choices for the local exponent at $x=0$ are $1/2$, $-1/2$, α, 2α, 3β respectively. The first two identities are related by Lemma 3.1. The arguments of the first four transformations are polynomials. Note that the cubic argument in (4.23) is the same as in (4.21)–(4.22) even if the fiber for $x=0$ is different. However, the branching pattern in both fibers and the branching order for $x=0$ is the same, so the same configuration of the singular points $x=0,1,\infty$ is possible (even if the local exponents at the respective points are different). The argument in (4.24) is related to $x(4x-3)^2$ by the affine transformation $x\mapsto(x+3)/4$, giving us other point as $x=0$ on essentially the same covering.

4.4.2. The transformation P19: $E(1/3,\alpha,\beta)\overset{3}{\leftarrow}HE(\alpha,2\alpha,\beta,2\beta)$

Up to fractional-linear transformations, we have these identities:

$$(4.26)\qquad \mathrm{Hn}\left(\begin{matrix}9\\18a^2-9ab+6a\end{matrix}\,\middle|\,\begin{matrix}3a,2a+b\\a+b+1/3;2a-2b+1\end{matrix}\,\middle|\,x\right)$$

$$=(1-x)^{-2a}{}_2\mathrm{F}_1\left(\begin{matrix}a,b\\a+b+1/3\end{matrix}\,\middle|\,-\frac{x(x-9)^2}{27(x-1)^2}\right),$$

$$(4.27)\qquad \mathrm{Hn}\left(\begin{matrix}8/9\\4a^2+4ab-2a/3\end{matrix}\,\middle|\,\begin{matrix}3a,2a+b\\2a+2b-1/3;a+b+1/3\end{matrix}\,\middle|\,x\right)$$

$$=\left(1-\frac{9x}{8}\right)^{-2a}{}_2\mathrm{F}_1\left(\begin{matrix}a,b\\a+b+1/3\end{matrix}\,\middle|\,\frac{27x^2(x-1)}{(9x-8)^2}\right).$$

The non-restricted exponents α, β and their fibers are symmetric, hence the only non-symmetric choices for the point $x=0$ are represented by the exponent differences α, 2α. There are thus only two transformation formulas up to two-term fractional-linear transformations.

4.4.3. The transformation P20: $E(1/2,\alpha,\beta) \overset{4}{\leftarrow} HE(\alpha,3\alpha,\beta,3\beta)$

Up to fractional-linear transformations, we have these identities:

$$(4.28)\qquad \mathrm{Hn}\left(\begin{matrix} 9/8 \\ 9a^2+9ab-3a/2 \end{matrix}\,\middle|\,\begin{matrix} 4a, 3a+b \\ 3a+3b-1/2; a+b+1/2 \end{matrix}\,\middle|\, x\right)$$

$$= \left(1-\frac{8x}{9}\right)^{-3a} {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ a+b+1/2 \end{matrix}\,\middle|\, \frac{64x^3(x-1)}{(8x-9)^3}\right),$$

$$(4.29)\qquad \mathrm{Hn}\left(\begin{matrix} -1/8 \\ 3a^2-5ab+3a/2 \end{matrix}\,\middle|\,\begin{matrix} 4a, 3a+b \\ a+b+1/2; 3a+3b-1/2 \end{matrix}\,\middle|\, x\right)$$

$$= (1+8x)^{-3a} {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ a+b+1/2 \end{matrix}\,\middle|\, \frac{64x(1-x)^3}{(8x+1)^3}\right).$$

As with P19, the are only two non-symmetric choices for the point $x=0$, represented by the exponent differences α, 3α.

4.4.4. The transformation P51: $E(1/3,\alpha,\beta) \overset{3}{\leftarrow} HE(\alpha,\alpha,\alpha,3\beta)$

Up to fractional-linear transformations, we have these identities:

$$(4.30)\qquad \mathrm{Hn}\left(\begin{matrix} -\omega \\ 3(1-\omega)ab \end{matrix}\,\middle|\,\begin{matrix} 3a, 3b \\ a+b+1/3; a+b+1/3 \end{matrix}\,\middle|\, x\right)$$

$$= {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ a+b+1/3 \end{matrix}\,\middle|\, 3(2\omega+1)x(x-1)(x+\omega)\right),$$

$$(4.31)\qquad \mathrm{Hn}\left(\begin{matrix} \omega+1 \\ 3(\omega+2)ab \end{matrix}\,\middle|\,\begin{matrix} 3a, a+b+1/3 \\ 3b; 2a-b+2/3 \end{matrix}\,\middle|\, x\right)$$

$$= \left(1+\frac{\omega-1}{3}x\right)^{-3a} {}_2\mathrm{F}_1\left(\begin{matrix} a, a+1/3 \\ b+2/3 \end{matrix}\,\middle|\, \frac{x^3}{(x-\omega-2)^3}\right),$$

where ω is the root of $\omega^2+\omega+1=0$. The choices for the local exponent at $x=0$ are α and 3β. To relate the argument in (4.30) to [17, formula (3.6a)], note that

$$3(2\omega+1)x(x-1)(x+\omega) = 1-(1-(\omega+2)x)^3.$$

4.4.5. Two composite transformations (P2 and P3)

As indicated in Table 1, there are two composite Gauss-to-Heun transformations with two parameters. They have degree 4, and transform $E(1/2, \alpha, \beta)$ to $HE(2\alpha, 2\alpha, 2\beta, 2\beta)$ or $HE(\alpha, \alpha, 2\alpha, 4\beta)$.

The transformation P2 can be expressed as a composition of two quadratic transformations in three ways:

$$\text{(4.32)} \qquad \text{P2:} \quad E(1/2, \alpha, \beta) \xleftarrow{2} E(\alpha, \alpha, 2\beta) \xleftarrow{2} HE(2\alpha, 2\alpha, 2\beta, 2\beta),$$

$$E(1/2, \alpha, \beta) \xleftarrow{2} E(2\alpha, \beta, \beta) \xleftarrow{2} HE(2\alpha, 2\alpha, 2\beta, 2\beta),$$

$$E(1/2, \alpha, \beta) \xleftarrow{2} HE(1/2, 1/2, 2\alpha, 2\beta) \xleftarrow{2_H} HE(2\alpha, 2\alpha, 2\beta, 2\beta).$$

In the third expression, the transformation P1 is composed with Heun-to-Heun transformation (4.14). Up to two-term fractional-linear transformations, we have one identity:

$$\text{(4.33)} \qquad \text{Hn}\left(\begin{matrix} -1 \\ 0 \end{matrix} \,\middle|\, \begin{matrix} 4a, 2a-2b+1 \\ 2a+2b; 2a-2b+1 \end{matrix} \,\middle|\, x\right)$$

$$= (1-x^2)^{-2a} {}_2\text{F}_1\left(\begin{matrix} a, b \\ a+b+1/2 \end{matrix} \,\middle|\, -\frac{4x^2}{(x^2-1)^2}\right),$$

as the choice of the exponent differences 2α or 2β for $x = 0$ gives equivalent formulas. The identity is a composition of (4.6) and (4.9). The first two expressions in (4.32) imply a relation between $E(\alpha, \alpha, 2\beta)$ and $E(2\alpha, \beta, \beta)$, and hypergeometric identities such as

$$\text{(4.34)} \qquad {}_2\text{F}_1\left(\begin{matrix} a, b \\ 2a \end{matrix} \,\middle|\, x(2-x)\right) = (1-x)^{-b} {}_2\text{F}_1\left(\begin{matrix} 2a-b, b \\ a+1/2 \end{matrix} \,\middle|\, \frac{x^2}{4(x-1)}\right).$$

This is a bi-quadratic transformation with two free parameters. A few transformations of this kind are presented in [1, p. 128–130].

The transformation P3 can be composed in one way:

$$\text{(4.35)} \qquad \text{P3:} \quad E(1/2, \alpha, \beta) \xleftarrow{2} E(\alpha, \alpha, 2\beta) \xleftarrow{2} HE(\alpha, \alpha, 2\alpha, 4\beta).$$

There are indeed three non-equivalent choices for the exponent difference at $x = 0$, namely α, 2α, 3β. Here are the respective formulas, up to fractional-linear transformations:

$$\text{(4.36)} \qquad \text{Hn}\left(\begin{matrix} 1/2 \\ 8ab \end{matrix} \,\middle|\, \begin{matrix} 4a, 4b \\ a+b+1/2; a+b+1/2 \end{matrix} \,\middle|\, x\right)$$

$$= {}_2\text{F}_1\left(\begin{matrix} a, b \\ a+b+1/2 \end{matrix} \,\middle|\, 16x(1-x)(1-2x)^2\right),$$

$$\text{(4.37)}\quad \mathrm{Hn}\left(\begin{matrix}-1\\0\end{matrix}\,\middle|\,\begin{matrix}4a,4b\\2a+2b;a+b+1/2\end{matrix}\,\middle|\,x\right)={}_2\mathrm{F}_1\left(\begin{matrix}a,b\\a+b+1/2\end{matrix}\,\middle|\,4x^2(1-x^2)\right),$$

$$\text{(4.38)}\quad \mathrm{Hn}\left(\begin{matrix}-1\\0\end{matrix}\,\middle|\,\begin{matrix}4a,2b\\4b-1;2a-b+1\end{matrix}\,\middle|\,x\right)$$
$$=\left(1-\frac{x^2}{2}\right)^{-2a}{}_2\mathrm{F}_1\left(\begin{matrix}a,a+1/2\\b+1/2\end{matrix}\,\middle|\,\frac{x^4}{(x^2-2)^2}\right).$$

The three identities are compositions of, respectively, (4.5) and (4.11), (4.5) and (4.9), or (4.7) and (4.9).

4.5. One-parameter transformations

The one-parameter transformations are P4–P14, P16–P18, P21–P50 and P52–P61. All these transformations are exemplified in Maple package [9]. Here we give examples only of *indecomposable* one-parameter transformations. Composite transformations are less interesting, especially compositions with Gauss-to-Gauss transformations that do not affect the t and q parameters. Appendix C spells out all compositions among the coverings P1–P61, and exemplifies the compositions P9, P37–P39, P43 that are obtained only by composing with Heun-to-Heun transformations.

Here are the indecomposable pull-back transformations, together with the covering and an illustrating formula for each. We took a few pragmatic choices of style in presenting formulas. We normalize the points $x=0$, $x=\infty$, but not necessarily $x=1$. Therefore the argument of some Heun functions is a constant multiple of x, while algebraic numbers (or longer expressions) are avoided in the coverings and on the right-hand sides of those formulas. Some algebraic numbers are written in denominators rather than numerators, when that makes a formula more compact.

P5: $E(1/3,1/3,\alpha)\overset{4}{\leftarrow}HE(1/3,1/3,\alpha,3\alpha)$, with H_{46}:

$$\mathrm{Hn}\left(\begin{matrix}-1\\4a(6a-1)/3\end{matrix}\,\middle|\,\begin{matrix}4a,4a+1/3\\2/3;2a+2/3\end{matrix}\,\middle|\,x\right)$$
$$=(1-2x)^{-3a}{}_2\mathrm{F}_1\left(\begin{matrix}a,a+1/3\\2/3\end{matrix}\,\middle|\,-\frac{x(x-2)^3}{(2x-1)^3}\right);$$

P25: $E(1/2,1/3,\alpha)\overset{5}{\leftarrow}HE(1/2,2/3,2\alpha,3\alpha)$, with H_{30}:

$$\mathrm{Hn}\left(\begin{matrix}1/5\\2a/3\end{matrix}\,\middle|\,\begin{matrix}5a,1/2-a\\1/3;1/2\end{matrix}\,\middle|\,x\right)=(1-5x)^{-2a}{}_2\mathrm{F}_1\left(\begin{matrix}a,1/6-a\\2/3\end{matrix}\,\middle|\,\frac{x^2(9x-5)^3}{4(5x-1)^2}\right);$$

P26: $E(1/2,1/3,\alpha) \overset{6}{\leftarrow} HE(1/3,2/3,\alpha,5\alpha)$, with
$H_{24}(x) = 27x^2(x-1)(3x+125)^3/\{4(9x-25)^5\}$:

$$\mathrm{Hn}\left(\begin{matrix}25/9\\5a/3\end{matrix}\,\middle|\,\begin{matrix}6a,4a+1/6\\1/3;2/3\end{matrix}\,\middle|\,x\right) = \left(1-\frac{9x}{25}\right)^{-5a}{}_2\mathrm{F}_1\left(\begin{matrix}a,1/6-a\\2/3\end{matrix}\,\middle|\,H_{24}(x)\right);$$

P27: $E(1/2,1/4,\alpha) \overset{5}{\leftarrow} HE(1/2,1/4,2\alpha,3\alpha)$, with
$H_{29}(x) = x^2(x+80)^3/(5x-32)^4$:

$$\mathrm{Hn}\left(\begin{matrix}-80\\-25a(8a+1)\end{matrix}\,\middle|\,\begin{matrix}5a,5a+1/4\\4a+1/2;1/2\end{matrix}\,\middle|\,x\right) = \left(1-\frac{5x}{32}\right)^{-4a}{}_2\mathrm{F}_1\left(\begin{matrix}a,a+1/4\\2a+3/4\end{matrix}\,\middle|\,H_{29}(x)\right);$$

P28: $E(1/2,1/3,\alpha) \overset{10}{\leftarrow} HE(1/3,\alpha,4\alpha,5\alpha)$, with
$H_9(x) = x(9x^3-90x^2+105x+40)^3/\{64(x-9)(9x-1)^4\}$:

$$\mathrm{Hn}\left(\begin{matrix}1/81\\50a(20a+1)/81\end{matrix}\,\middle|\,\begin{matrix}10a,5/6\\2/3;2a+5/6\end{matrix}\,\middle|\,\frac{x}{9}\right)$$
$$= \left(1-\frac{x}{9}\right)^{-a}(1-9x)^{-4a}{}_2\mathrm{F}_1\left(\begin{matrix}a,1/6-a\\2/3\end{matrix}\,\middle|\,H_9(x)\right);$$

P29: $E(1/2,1/3,\alpha) \overset{8}{\leftarrow} HE(2/3,\alpha,2\alpha,5\alpha)$, with
$H_{16}(x) = 4x^2(x^2-8x+10)^3/\{27(2x-1)^2(4x-27)\}$:

$$\mathrm{Hn}\left(\begin{matrix}2/27\\56a/81\end{matrix}\,\middle|\,\begin{matrix}8a,5/6-2a\\1/3;2a+5/6\end{matrix}\,\middle|\,\frac{4x}{27}\right)$$
$$= \left(1-\frac{4x}{27}\right)^{-a}(1-2x)^{-2a}{}_2\mathrm{F}_1\left(\begin{matrix}a,1/6-a\\2/3\end{matrix}\,\middle|\,H_{16}(x)\right);$$

P30: $E(1/2,1/3,\alpha) \overset{7}{\leftarrow} HE(1/2,1/3,3\alpha,4\alpha)$, with
$H_{23}(x) = -4x(27x^2-28x+7)^3/(7x-4)^3$:

$$\mathrm{Hn}\left(\begin{matrix}27/28\\a(97-294a)/24\end{matrix}\,\middle|\,\begin{matrix}7a,2/3-a\\2/3;1/2\end{matrix}\,\middle|\,\frac{27x}{16}\right)$$
$$= \left(1-\frac{7x}{4}\right)^{-3a}{}_2\mathrm{F}_1\left(\begin{matrix}a,1/6-a\\2/3\end{matrix}\,\middle|\,H_{23}(x)\right);$$

P31: $E(1/2,1/3,\alpha) \overset{5}{\leftarrow} HE(1/2,2/3,\alpha,4\alpha)$, with H_{29}:

$$\mathrm{Hn}\left(\begin{matrix}32/5\\4a/3\end{matrix}\,\middle|\,\begin{matrix}5a,3a+1/6\\1/3;1/2\end{matrix}\,\middle|\,x\right) = \left(1-\frac{5x}{32}\right)^{-4a}{}_2\mathrm{F}_1\left(\begin{matrix}a,1/6-a\\2/3\end{matrix}\,\middle|\,\frac{x^2(x+80)^3}{(5x-32)^4}\right);$$

P32: $E(1/2, 1/3, \alpha) \overset{10}{\leftarrow} HE(1/3, 2\alpha, 3\alpha, 5\alpha)$, with
$H_{10}(x) = 4x(9x^3 - 60x^2 + 130x - 90)^3/\{9(3x-8)^2(4x-9)^3\}$:

$$\mathrm{Hn}\left(\begin{matrix} 27/32 \\ 25a(11-30a)/48 \end{matrix} \middle| \begin{matrix} 10a, 5/6 \\ 2/3; 4a+2/3 \end{matrix} \middle| \frac{3x}{8}\right) = \left(1-\frac{3x}{8}\right)^{-2a}\left(1-\frac{4x}{9}\right)^{-3a} {}_2\mathrm{F}_1\left(\begin{matrix} a, 1/6-a \\ 2/3 \end{matrix} \middle| H_{10}(x)\right);$$

P33: $E(1/2, 1/3, \alpha) \overset{9}{\leftarrow} HE(1/2, \alpha, 3\alpha, 5\alpha)$, with
$H_{13}(x) = 27x(4x-3)^5/\{4(x^3 - 12x^2 - 54x - 2)^3\}$:

$$\mathrm{Hn}\left(\begin{matrix} 3/128 \\ 81a(1+51a)/128 \end{matrix} \middle| \begin{matrix} 9a, 3a+1/2 \\ 2a+5/6; 1/2 \end{matrix} \middle| \frac{x}{32}\right) = \left(1+27x+6x^2-\frac{x^3}{2}\right)^{-3a} {}_2\mathrm{F}_1\left(\begin{matrix} a, a+1/3 \\ 2a+5/6 \end{matrix} \middle| H_{13}(x)\right);$$

P34: $E(1/2, 1/5, \alpha) \overset{6}{\leftarrow} HE(1/5, \alpha, 2\alpha, 3\alpha)$, with
$H_{24}(x) = 27x^2(x-1)(3x+125)^3/\{4(9x-25)^5\}$:

$$\mathrm{Hn}\left(\begin{matrix} -125/3 \\ -30a(1+10a) \end{matrix} \middle| \begin{matrix} 6a, 6a+1/5 \\ 4a+2/5; 2a+7/10 \end{matrix} \middle| x\right) = \left(1-\frac{9x}{25}\right)^{-5a} {}_2\mathrm{F}_1\left(\begin{matrix} a, a+1/5 \\ 2a+7/10 \end{matrix} \middle| H_{24}(x)\right);$$

P35: $E(1/2, 1/3, \alpha) \overset{10}{\leftarrow} HE(1/3, \alpha, 2\alpha, 7\alpha)$, with
$H_8(x) = 4x(x^3 - 12x^2 + 42x - 42)^3/\{27(4x-27)(3x-8)^2\}$:

$$\mathrm{Hn}\left(\begin{matrix} 32/81 \\ 2a(179-686a)/81 \end{matrix} \middle| \begin{matrix} 10a, 7/6-4a \\ 2/3; 2a+5/6 \end{matrix} \middle| \frac{4x}{27}\right) = \left(1-\frac{4x}{27}\right)^{-a}\left(1-\frac{3x}{8}\right)^{-2a} {}_2\mathrm{F}_1\left(\begin{matrix} a, 1/6-a \\ 2/3 \end{matrix} \middle| H_8(x)\right);$$

P36: $E(1/2, 1/3, \alpha) \overset{7}{\leftarrow} HE(1/2, 1/3, 2\alpha, 5\alpha)$, with
$H_{22}(x) = 4x(4x^2 - 35x + 70)^3/\{27(28x-125)^2\}$:

$$\mathrm{Hn}\left(\begin{matrix} 125/189 \\ 8a(38-147a)/81 \end{matrix} \middle| \begin{matrix} 7a, 5/6-3a \\ 2/3; 1/2 \end{matrix} \middle| \frac{4x}{27}\right) = \left(1-\frac{28x}{125}\right)^{-2a} {}_2\mathrm{F}_1\left(\begin{matrix} a, 1/6-a \\ 2/3 \end{matrix} \middle| H_{22}(x)\right);$$

P40: $E(1/3,1/3,\alpha) \stackrel{4}{\leftarrow} HE(1/3,1/3,2\alpha,2\alpha)$, with H_{47}:

$$\mathrm{Hn}\left(\begin{matrix} 15\sqrt{3}-26 \\ 8a(12a+1)/\{3(5+3\sqrt{3})\} \end{matrix}\,\middle|\,\begin{matrix} 4a, 4a+1/3 \\ 2/3; 4a+1/3 \end{matrix}\,\middle|\,\frac{x}{5+3\sqrt{3}}\right)$$
$$= (1-2x)^{-3a}\,{}_2\mathrm{F}_1\left(\begin{matrix} a, a+1/3 \\ 2/3 \end{matrix}\,\middle|\,\frac{x(x+4)^3}{4(2x-1)^3}\right);$$

P42: $E(1/2,1/5,\alpha) \stackrel{5}{\leftarrow} HE(1/2,\alpha,2\alpha,2\alpha)$, with $H_{45}(x) = x(x^2-10x+5)^2/(x+1)^5$:

$$\mathrm{Hn}\left(\begin{matrix} 9+4\sqrt{5} \\ 5a(10a+1)/(10-4\sqrt{5}) \end{matrix}\,\middle|\,\begin{matrix} 5a, 3a+3/10 \\ 1/2; 4a+2/5 \end{matrix}\,\middle|\,\frac{5x}{5-2\sqrt{5}}\right)$$
$$= (1+x)^{-5a}\,{}_2\mathrm{F}_1\left(\begin{matrix} a, a+1/5 \\ 1/2 \end{matrix}\,\middle|\,H_{45}(x)\right);$$

P44: $E(1/2,1/4,\alpha) \stackrel{6}{\leftarrow} HE(1/4,1/4,\alpha,5\alpha)$, with $H_{42}(x) = 256x^5/\{(x+5)^4(5x^2+6x+5)\}$:

$$\mathrm{Hn}\left(\begin{matrix} (-7+24i)/25 \\ 5a(40a-1)/(6-8i) \end{matrix}\,\middle|\,\begin{matrix} 6a, 4a+1/4 \\ 10a-1/4; 3/4 \end{matrix}\,\middle|\,\frac{5x}{4i-3}\right)$$
$$= \left(1+\frac{x}{5}\right)^{-4a}\left(1+\frac{6x}{5}+x^2\right)^{-a}{}_2\mathrm{F}_1\left(\begin{matrix} a, a+1/4 \\ 2a+3/4 \end{matrix}\,\middle|\,H_{42}(x)\right);$$

P45: $E(1/2,1/3,\alpha) \stackrel{6}{\leftarrow} HE(1/2,1/2,\alpha,5\alpha)$, with $H_{26}(x) = 1728x/(x^2+10x+5)^3$:

$$\mathrm{Hn}\left(\begin{matrix} (117+44i)/125 \\ a(49-228a)/(11-2i) \end{matrix}\,\middle|\,\begin{matrix} 6a, 5/6-4a \\ 2a+5/6; 1/2 \end{matrix}\,\middle|\,\frac{x}{2i-11}\right)$$
$$= \left(1+2x+\frac{x^2}{5}\right)^{-3a}{}_2\mathrm{F}_1\left(\begin{matrix} a, a+1/3 \\ 2a+5/6 \end{matrix}\,\middle|\,H_{26}(x)\right);$$

P46: $E(1/2,1/5,\alpha) \stackrel{6}{\leftarrow} HE(1/5,\alpha,\alpha,4\alpha)$, with $H_{42}(x) = x^4(25x^2-22x+5)/\{4(2x-1)^5\}$:

$$\mathrm{Hn}\left(\begin{matrix} (117+44i)/125 \\ 10a(40a-1)/(11-2i) \end{matrix}\,\middle|\,\begin{matrix} 6a, 6a+1/5 \\ 8a-1/5; 2a+7/10 \end{matrix}\,\middle|\,\frac{25x}{11-2i}\right)$$
$$= (1-2x)^{-5a}\,{}_2\mathrm{F}_1\left(\begin{matrix} a, a+1/5 \\ 2a+7/10 \end{matrix}\,\middle|\,H_{42}(x)\right);$$

P47: $E(1/2, 1/3, \alpha) \xleftarrow{4} HE(1/2, 1/2, 1/3, 4\alpha)$, with H_{36}:

$$\mathrm{Hn}\left(\begin{matrix}(23+10\sqrt{-2})/27\\ 32a(1-6a)/\{3(5-\sqrt{-2})\}\end{matrix}\,\middle|\,\begin{matrix}4a, 2/3-4a\\ 2/3; 1/2\end{matrix}\,\middle|\,\frac{x}{5-\sqrt{-2}}\right)$$
$$= {}_2\mathrm{F}_1\left(\begin{matrix}a, 1/6-a\\ 2/3\end{matrix}\,\middle|\,-\frac{x(x-4)^3}{27}\right);$$

P48: $E(1/2, 1/4, \alpha) \xleftarrow{4} HE(1/2, 1/2, \alpha, 3\alpha)$, with H_{36}:

$$\mathrm{Hn}\left(\begin{matrix}(17+56\sqrt{-2})/81\\ a(17-40a)/(7-4\sqrt{-2})\end{matrix}\,\middle|\,\begin{matrix}4a, 3/4-2a\\ 2a+3/4; 1/2\end{matrix}\,\middle|\,\frac{x}{-7+4\sqrt{-2}}\right)$$
$$= \left(1+\frac{x}{3}\right)^{-4a} {}_2\mathrm{F}_1\left(\begin{matrix}a, a+1/4\\ 2a+3/4\end{matrix}\,\middle|\,\frac{256x}{(x+3)^4}\right);$$

P49: $E(1/2, 1/3, \alpha) \xleftarrow{10} HE(1/3, \alpha, \alpha, 8\alpha)$, with
$H_7(x) = -4x(x^3-6x^2+15x-12)^3/\{27(3x^2-14x+27)\}$:

$$\mathrm{Hn}\left(\begin{matrix}(17+56\sqrt{-2})/81\\ 4a(13-64a)/(7-4\sqrt{-2})\end{matrix}\,\middle|\,\begin{matrix}10a, 4/3-6a\\ 2/3; 2a+5/6\end{matrix}\,\middle|\,\frac{3x}{7-4\sqrt{-2}}\right)$$
$$= \left(1-\frac{14x}{27}+\frac{x^2}{9}\right)^{-a} {}_2\mathrm{F}_1\left(\begin{matrix}a, 1/6-a\\ 2/3\end{matrix}\,\middle|\,H_7(x)\right);$$

P50: $E(1/3, 1/4, \alpha) \xleftarrow{4} HE(1/3, \alpha, \alpha, 2\alpha)$, with H_{36}:

$$\mathrm{Hn}\left(\begin{matrix}(241+22\sqrt{-2})/243\\ 8a(7-8a)/(22-\sqrt{-2})\end{matrix}\,\middle|\,\begin{matrix}4a, 5/6\\ 2/3; 2a+7/12\end{matrix}\,\middle|\,\frac{18x}{22-\sqrt{-2}}\right)$$
$$= (1-x)^{-4a} {}_2\mathrm{F}_1\left(\begin{matrix}a, a+1/4\\ 2/3\end{matrix}\,\middle|\,\frac{x(3x-4)^3}{27(x-1)^4}\right);$$

P56: $E(1/2, 1/3, \alpha) \xleftarrow{8} HE(1/3, 1/3, \alpha, 7\alpha)$, with
$H_{18}(x) = 1728x/(x^4-14x^3+63x^2-70x-7)^2$:

$$\mathrm{Hn}\left(\begin{matrix}(55+39\omega)/49\\ 2a(71-348a)/\{3(5-3\omega)\}\end{matrix}\,\middle|\,\begin{matrix}8a, 7/6-6a\\ 2a+5/6; 2/3\end{matrix}\,\middle|\,\frac{x}{5-3\omega}\right)$$
$$= \left(1+10x-9x^2+2x^3-\frac{x^4}{7}\right)^{-2a} {}_2\mathrm{F}_1\left(\begin{matrix}a, a+1/2\\ 2a+5/6\end{matrix}\,\middle|\,H_{18}(x)\right);$$

P57: $E(1/2,1/3,\alpha) \overset{9}{\leftarrow} HE(1/2,\alpha,\alpha,7\alpha)$, with
$H_{11}(x) = x(2x^4 - 12x^3 + 42x^2 - 70x + 63)^2/\{27(4x^2 - 13x + 32)\}$:

$$\mathrm{Hn}\left(\begin{matrix} (-87+91\sqrt{-7})/256 \\ 2a(31-147a)/(13-7\sqrt{-7}) \end{matrix}\,\middle|\, \begin{matrix} 9a, 7/6-5a \\ 1/2; 2a+5/6 \end{matrix} \,\middle|\, \frac{8x}{13-7\sqrt{-7}}\right)$$
$$= \left(1 - \frac{13x}{32} + \frac{x^2}{8}\right)^{-a} {}_2\mathrm{F}_1\left(\begin{matrix} a, 1/6-a \\ 1/2 \end{matrix}\,\middle|\, H_{11}(x)\right);$$

P58: $E(1/2,1/5,\alpha) \overset{5}{\leftarrow} HE(1/2,\alpha,\alpha,3\alpha)$, with
$H_{37}(x) = x(x^2 - 10x + 30)^2/(x-4)^5$:

$$\mathrm{Hn}\left(\begin{matrix} (781+171\sqrt{-15})/1024 \\ 10a(23-45a)/(95-9\sqrt{-15}) \end{matrix}\,\middle|\, \begin{matrix} 5a, 9/10-a \\ 1/2; 2a+7/10 \end{matrix} \,\middle|\, \frac{20x}{95-9\sqrt{-15}}\right)$$
$$= \left(1 - \frac{x}{4}\right)^{-5a} {}_2\mathrm{F}_1\left(\begin{matrix} a, a+1/5 \\ 1/2 \end{matrix}\,\middle|\, H_{37}(x)\right);$$

P59: $E(1/2,1/3,\alpha) \overset{5}{\leftarrow} HE(1/2,1/3,1/3,5\alpha)$, with H_{37}:

$$\mathrm{Hn}\left(\begin{matrix} (-7+33\sqrt{-15})/128 \\ 25a(1-6a)/(11-3\sqrt{-15}) \end{matrix}\,\middle|\, \begin{matrix} 5a, 5/6-5a \\ 1/2; 2/3 \end{matrix} \,\middle|\, \frac{6x}{11-3\sqrt{-15}}\right)$$
$$= {}_2\mathrm{F}_1\left(\begin{matrix} a, 1/6-a \\ 1/2 \end{matrix}\,\middle|\, \frac{x(3x^2-10x+15)^2}{64}\right);$$

P60: $E(1/2,1/4,\alpha) \overset{5}{\leftarrow} HE(1/2,1/4,\alpha,4\alpha)$, with
$H_{44}(x) = x(x+3-4i)^4/\{(1+2i)^5(x-1)^4\}$:

$$\mathrm{Hn}\left(\begin{matrix} 1+2i \\ a(5/4+(7+24i)a) \end{matrix}\,\middle|\, \begin{matrix} 5a, 3a+1/4 \\ 3/4; 8a \end{matrix} \,\middle|\, x\right)$$
$$= (1-x)^{-4a} {}_2\mathrm{F}_1\left(\begin{matrix} a, 1/4-a \\ 3/4 \end{matrix}\,\middle|\, H_{44}(x)\right);$$

P61: $E(1/2,1/3,\alpha) \overset{7}{\leftarrow} HE(1/2,1/3,\alpha,6\alpha)$, with
$H_{21}(x) = 4x(x^2 - (5+4\omega)x + 1 + 5\omega)^3/\{3(\omega+2)((2\omega-1)x+9)\}$:

$$\mathrm{Hn}\left(\begin{matrix} (3-12\omega)/7 \\ 2a(7-2(\omega-18)a)/(3+\omega) \end{matrix}\,\middle|\, \begin{matrix} 7a, 1-5a \\ 2/3; 1/2 \end{matrix} \,\middle|\, \frac{x}{1+2\omega}\right)$$
$$= \left(1 - \frac{(1-2\omega)x}{9}\right)^{-a} {}_2\mathrm{F}_1\left(\begin{matrix} a, 1/6-a \\ 2/3 \end{matrix}\,\middle|\, H_{21}(x)\right).$$

A. Appendix: Sorting criteria for Heun-to-Gauss reductions

Here we formulate the ordering of parametric Heun-to-Gauss reductions in Tables 3, 4. It largely follows the criteria in [13] for sorting (Belyi functions for) non-parametric "hyperbolic" Heun-to-Gauss reductions. The criteria here are simplified, because the largest extensions of $\boldsymbol{Q}$ for the t and j-values are quadratic. On the other hand, we have to adjust the criteria in (b1)–(b2) to accommodate free parameters. The settled sorting criteria are the following:

(a) the first criterium is the j-invariant;
(b) the second criterium is the local exponent differences of the Heun equation;
(c) the last criterium is the degree of the covering.

In a similar hierarchical manner, the j-invariants are sorted by the following criteria:

(a1) the number field where the j-invariant is defined;
(a2) the number field where the t-values are defined;
(a3) the leading coefficient of the minimal polynomial in $\boldsymbol{Z}[x]$ for the j-invariant.

Note that for $j \in \boldsymbol{Q}$ the number in (a3) is the denominator of j. The number fields (either for the j-invariant or the t-values) are ordered by the following criteria:

(f1) the field degree, hence $\boldsymbol{Q}$ precedes quadratic extensions;
(f2) quadratic extensions $\boldsymbol{Q}(\sqrt{a})$ are ordered as follows:
(f1a) real quadratic fields (with $a > 0$) precede imaginary quadratic fields (with $a < 0$);
(f1b) the fields with the same sign of a are ordered by the increasing $|a|$.

The positive integers in (a3) are ordered as follows:

(i1) the product of the primes dividing the integer;
(i2) by the increasing value.

Except for the (i1)-part and for using the absolute value in (f1b), all other numeric specifics are ordered in the increasing order. The sets of local exponent differences are ordered as follows:

(b1) in each tuple the 4 exponent differences are ordered by *putting the free parameters at the end*, and the numeric values of the (positive, rational) restricted exponent differences are ordered firstly their denominators, then secondly by the numerators.

(b2) the tuples are *first compared by the number of restricted exponent differences* (*hence the tuples with more free parameters have precedence*), then lexicographically, from their first elements, and the elements are matched first by their denominators then by the numerators.

These criteria break all ties in the list of 61 transformations, as mentioned. In particular, no sorting criteria is necessary for the accessory parameters q (or their invariants). The highlighted text in (b1)–(b2) accounts for the presence of free parameters, absent in the criteria for the non-parametric list in [13].

B. Appendix: Fractional-linear transformations

The hypergeometric equation (1.5) with general A, B, C has the following local bases of solutions:

$$\text{at } z=0:\quad {}_2\mathrm{F}_1\left(\left.\begin{matrix}A,B\\C\end{matrix}\,\right|\,z\right),\qquad z^{1-C}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}1+A-C,1+B-C\\2-C\end{matrix}\,\right|\,z\right);$$

$$\text{at } z=1:\quad {}_2\mathrm{F}_1\left(\left.\begin{matrix}A,B\\1+A+B-C\end{matrix}\,\right|\,1-z\right),$$

$$(1-z)^{C-A-B}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}C-A,C-B\\1+C-A-B\end{matrix}\,\right|\,1-z\right);$$

$$\text{at } z=\infty:\quad z^{-A}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}A,1+A-C\\1+A-B\end{matrix}\,\right|\,\frac{1}{z}\right),\qquad z^{-B}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}B,1+B-C\\1-A+B\end{matrix}\,\right|\,\frac{1}{z}\right).$$

The following Pfaff and Euler fractional-linear transformations [1, Th. 2.2.5] can be applied to the 6 local solutions:

$$\begin{aligned}
{}_2\mathrm{F}_1\left(\left.\begin{matrix}A,B\\C\end{matrix}\,\right|\,z\right) &= (1-z)^{-A}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}A,C-B\\C\end{matrix}\,\right|\,\frac{z}{z-1}\right)\\
&= (1-z)^{-B}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}C-A,B\\C\end{matrix}\,\right|\,\frac{z}{z-1}\right)\\
&= (1-z)^{C-A-B}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}C-A,C-B\\C\end{matrix}\,\right|\,z\right).
\end{aligned}\tag{B.1}$$

This gives $6 \times 4 = 24$ different hypergeometric solutions for a general hypergeometric equation in total; they are referred to as the 24 *Kummer's solutions*. The automorphism group of the hypergeometric equation is the Coxeter group $\mathscr{A}_3$ of order 24 [7]. It contains the permutation group $\mathscr{S}_3$. If the parameters a, b are not considered as symmetric, the automorphism group extends to a semidirect product of $\mathscr{S}_3$ and $(\boldsymbol{Z}/2\boldsymbol{Z})^3$, of order $3! \times 2^3 = 48$. The action of this group is represented by a permutation of the 3 singular points and interchange of the local exponents at those points. Accordingly, the group permutes the local exponents $(1-C, C-A-B, B-A)$ and multiplies (some of them) by -1. The permutation of the singular points is realized by the Möbius transformations mapping z to z, $1-z$, $1/z$, $1/(1-z)$, $z/(z-1)$ and $1-1/z$. For integer values of the parameters A, B or the local exponents, the structure of Kummer's 24 solutions degenerates [27].

The automorphism group of the Heun equation (1.6) is the Coxeter group $\mathscr{D}_4$, of order 192; see [18]. It contains the permutation group $\mathscr{S}_4$, and extends to a semidirect product of $\mathscr{S}_4$ and $(\boldsymbol{Z}/2\boldsymbol{Z})^4$ when the parameters a, b are distinguished. Here are bases of local solutions at $x=0$, $x=1$, $x=t$ and $x=\infty$ for a general Heun equation:

$$\text{(B.2)}\quad \text{at } x=0:\quad \mathrm{Hn}\left(\begin{matrix} t \\ q \end{matrix} \,\middle|\, \begin{matrix} a,b \\ c;d \end{matrix} \,\middle|\, x\right), \qquad x^{1-c}\,\mathrm{Hn}\left(\begin{matrix} t \\ q_1 \end{matrix} \,\middle|\, \begin{matrix} a-c+1, b-c+1 \\ 2-c;d \end{matrix} \,\middle|\, x\right);$$

$$\text{at } x=1:\quad \mathrm{Hn}\left(\begin{matrix} 1-t \\ ab-q \end{matrix} \,\middle|\, \begin{matrix} a,b \\ d;c \end{matrix} \,\middle|\, 1-x\right),$$

$$(1-x)^{1-d}\,\mathrm{Hn}\left(\begin{matrix} 1-t \\ q_2 \end{matrix} \,\middle|\, \begin{matrix} a-d+1, b-d+1 \\ 2-d;c \end{matrix} \,\middle|\, 1-x\right);$$

$$\text{at } x=t:\quad \mathrm{Hn}\left(\begin{matrix} 1-1/t \\ ab-q/t \end{matrix} \,\middle|\, \begin{matrix} a,b \\ a+b-c-d+1;c \end{matrix} \,\middle|\, 1-\frac{x}{t}\right),$$

$$\left(1-\frac{x}{t}\right)^{c+d-a-b}\mathrm{Hn}\left(\begin{matrix} 1-1/t \\ q_3 \end{matrix} \,\middle|\, \begin{matrix} c+d-a, c+d-b \\ c+d-a-b+1;c \end{matrix} \,\middle|\, 1-\frac{x}{t}\right);$$

$$\text{at } x=\infty:\quad x^{-a}\,\mathrm{Hn}\left(\begin{matrix} 1/t \\ q_4 \end{matrix} \,\middle|\, \begin{matrix} a, a-c+1 \\ a-b+1;d \end{matrix} \,\middle|\, \frac{1}{x}\right),$$

$$x^{-b}\,\mathrm{Hn}\left(\begin{matrix} 1/t \\ q_5 \end{matrix} \,\middle|\, \begin{matrix} b, b-c+1 \\ b-a+1;d \end{matrix} \,\middle|\, \frac{1}{x}\right);$$

where

$$q_1 = q - (c-1)(a+b-c-d+dt+1),$$

$$q_2 = ab - q - (d-1)(a+b-ct-d+1),$$

$$q_3 = ab - q/t + (c/t - c - d)(a+b-c-d),$$

$$q_4 = q/t + a(a - b/t - c - d + d/t + 1),$$

$$q_5 = q/t + b(b - a/t - c - d + d/t + 1).$$

Each of these functions can be expressed as the Heun series with all six t-parameter values in (2.1):

$$\text{(B.3)}\quad \operatorname{Hn}\left(\begin{matrix} t \\ q \end{matrix}\,\middle|\,\begin{matrix} a,b \\ c;d \end{matrix}\,\middle|\, x\right)$$

$$= \operatorname{Hn}\left(\begin{matrix} 1/t \\ q/t \end{matrix}\,\middle|\,\begin{matrix} a,b \\ c;a+b-c-d+1 \end{matrix}\,\middle|\, \frac{x}{t}\right)$$

$$= (1-x)^{-a}\operatorname{Hn}\left(\begin{matrix} t/(t-1) \\ (act-q)/(t-1) \end{matrix}\,\middle|\,\begin{matrix} a,a-d+1 \\ c;a-b+1 \end{matrix}\,\middle|\, \frac{x}{x-1}\right)$$

$$= \left(1-\frac{x}{t}\right)^{-a}\operatorname{Hn}\left(\begin{matrix} 1/(1-t) \\ (q-ac)/(t-1) \end{matrix}\,\middle|\,\begin{matrix} a,c+d-b \\ c;a-b+1 \end{matrix}\,\middle|\, \frac{x}{x-t}\right)$$

$$= \left(1-\frac{x}{t}\right)^{-a}\operatorname{Hn}\left(\begin{matrix} 1-t \\ ac-q \end{matrix}\,\middle|\,\begin{matrix} a,c+d-b \\ c;d \end{matrix}\,\middle|\, \frac{(1-t)x}{x-t}\right)$$

$$= (1-x)^{-a}\operatorname{Hn}\left(\begin{matrix} 1-1/t \\ ac-q/t \end{matrix}\,\middle|\,\begin{matrix} a,a-d+1 \\ c;a+b-c-d+1 \end{matrix}\,\middle|\, \frac{(t-1)x}{t(x-1)}\right).$$

Besides, there are 4 transformations which do not change the argument x nor the parameter t:

$$\text{(B.4)}\quad \operatorname{Hn}\left(\begin{matrix} t \\ q \end{matrix}\,\middle|\,\begin{matrix} a,b \\ c;d \end{matrix}\,\middle|\, x\right)$$

$$= (1-x)^{1-d}\operatorname{Hn}\left(\begin{matrix} t \\ q-c(d-1)t \end{matrix}\,\middle|\,\begin{matrix} a-d+1,b-d+1 \\ c;2-d \end{matrix}\,\middle|\, x\right)$$

$$= \left(1-\frac{x}{t}\right)^{c+d-a-b}\operatorname{Hn}\left(\begin{matrix} t \\ q_6 \end{matrix}\,\middle|\,\begin{matrix} c+d-a,c+d-b \\ c;d \end{matrix}\,\middle|\, x\right)$$

$$= (1-x)^{1-d}\left(1-\frac{x}{t}\right)^{c+d-a-b}\operatorname{Hn}\left(\begin{matrix} t \\ q_7 \end{matrix}\,\middle|\,\begin{matrix} c-a+1,c-b+1 \\ c;2-d \end{matrix}\,\middle|\, x\right),$$

where $q_6 = q - c(a+b-c-d)$, $q_7 = q - c(a+b-c-d+dt-t)$. In total, there are $6 \times 4 = 24$ two-term fractional-linear transformations of the Heun functions, and $8 \times 24 = 192$ different Heun series solutions of a general Heun equation, as described by Maier [18]. If the parameters a, b are distinguished, the full set of 2×192 fractional-linear transformations is represented by the permutation and the -1 action on the exponent differences $(1-c, 1-d, c+d-a-b, b-a)$.

The two-term fractional-linear transformations (B.3) and (B.4) fix the exponent difference at $x = 0$ in this representation, characteristically. The transformations in (B.4) represent interchange of the local exponents at $x = 1$ and at $x = t$. By applying (B.3), all Heun functions in (B.2) can be transformed to have the same t-parameter. In particular, we have these four functions as solutions of the same Heun equation:

$$\text{(B.5)} \quad \mathrm{Hn}\left(\begin{array}{c|c|c} t & a, b & \\ q & c; d & x \end{array}\right), \qquad x^{-a}\,\mathrm{Hn}\left(\begin{array}{c|c|c} t & a, a-c+1 & t \\ tq_4 & a+b-c-d+1; a-b+1 & \overline{x} \end{array}\right),$$

$$(x-t)^{-a}\,\mathrm{Hn}\left(\begin{array}{c|c|c} t & a, c+d-b & t(x-1) \\ q-(b-d)t & d; c & \overline{x-t} \end{array}\right),$$

$$(x-1)^{-a}\,\mathrm{Hn}\left(\begin{array}{c|c|c} t & a, a-d+1 & x-t \\ q_8 & a+b-c-d+1; a-b+1 & \overline{x-1} \end{array}\right),$$

with $q_8 = q + a(a-c-d+1)t$. They correspond to the permutations of the singularities in two pairs. For example, the first two functions are related by the interchange of $x = 0$ and $x = \infty$ and the interchange of $x = 1$ and $x = t$. Any 3 of the functions in (B.2) are related by a linear three-term *connection formula* (since the order of Heun's equation is 2), though their coefficients are not known yet in general (unlike for Kummer's solutions of the hypergeometric equation).

Transformations of the hypergeometric and Heun equations can be conveniently presented as transformations of Riemann's P-symbols; for example

$$\text{(B.6)} \qquad P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & A & z \\ 1-C & C-A-B & B & \end{array}\right\}$$

$$= (1-z)^{C-A-B} P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & C-B & z \\ 1-C & A+B-C & C-A & \end{array}\right\}$$

$$= P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & A & 1-z \\ C-A-B & 1-C & B & \end{array}\right\}.$$

C. Appendix: Composite transformations

As Tables 1 and 2 indicate, many parametric Heun-to-Gauss reductions are compositions of lower degree transformations between the hypergeometric and Heun equations. Out of the 61 considered transformations, 28 are composite. Here we explain the composition notation in Tables 1, 2, and recount the transformations more thoroughly.

The numbers in the decomposition notation show the degree of component transformations. The factor 2_H denotes the quadratic Heun-to-Heun transformation (4.14) discussed in §4.3. A few other indexed numbers denote particular coverings of low degree: 3_C denotes the cyclic covering H_{33} with the branching pattern $3 = 3 = 1 + 1 + 1$, while 4_A stands for the covering H_{36} with the pattern $4 = 3 + 1 = 2 + 1 + 1$, and 4_B stands for the covering H_{46} with the pattern $3 + 1 = 3 + 1 = 3 + 1$. The unindexed numbers 3 and 4 denote the frequent coverings H_{34} $(3 = 2 + 1 = 2 + 1)$ and H_{47} $(3 + 1 = 3 + 1 = 2 + 2)$, respectively. The product notation has to be followed from right to left to trace the composition from the starting hypergeometric equation. In a composition, exactly one factor represents an indecomposable Gauss-to-Heun transformation; it is the first one from the left which is not 2_H. The other factors to the right represent pull-backs between hypergeometric equations.

In Tables 3 and 4, 2_H denotes an applicability of the quadratic Heun-to-Heun transformation following the arrow in (4.14), while 2^H denotes applicability of this quadratic transformation from left to right in (4.14), and 4_H denotes applicability of the composite quartic transformation (4.15) following the arrows.

The product 2×2 in Tables 1 and 2 denotes a composition of quadratic transformations that can be realized in multiple ways, possibly including 2_H. Mainly, it indicates involvement of the degree 4 transformation P2 realized by the covering H_{31}. As presented in (4.32), the transformation P2 can be split into quadratic transformations in three ways. The same covering H_{31} realizes the quartic Heun-to-Heun transformation (4.15).

The other composite transformation with 2 free parameters is P3, realized by the Belyi covering H_{35}. The composition is given in (4.35). The same covering H_{35} realizes this composite transformation:

$$\text{(C.1)} \quad \text{P37: } E(1/2, 1/4, \alpha) \overset{2}{\leftarrow} HE(1/2, 1/2, 1/2, 2\alpha) \overset{2_H}{\leftarrow} HE(1/2, 1/2, 2\alpha, 2\alpha).$$

The specialization $\beta = 1/4$ of P2 gives a composite transformation of the same appearance $E(1/2, 1/4, \alpha) \overset{2}{\leftarrow} HE(1/2, 1/2, 1/2, 2\alpha) \overset{2_H}{\leftarrow} HE(1/2, 1/2, 2\alpha, 2\alpha)$. But the coverings are different, and the pulled-back Heun equations have different sets of t-parameters.

The transformations P10 and P12 have the most complicated composition lattices. They are realized by the Belyi coverings H_{41}, H_5, respectively, and include specialized three-way splittings (4.32) of P2:

(C.2)

$$
\begin{array}{ccccccc}
 & \swarrow & E(1/2,\alpha,\alpha) \nwarrow & \Leftarrow & E(\alpha,\alpha,2\alpha) \nwarrow & & \\
\text{P10:}\ E(1/2,1/4,\alpha) & \longleftarrow & E(1/4,1/4,2\alpha) & \longleftarrow & HE(1/2,1/2,2\alpha,2\alpha) & \longleftarrow & HE(2\alpha,2\alpha,2\alpha,2\alpha) \\
 & \nwarrow & HE(1/2,1/2,1/2,2\alpha) \swarrow & \Leftarrow & HE(1/2,1/2,2\alpha,2\alpha) & \swarrow &
\end{array}
$$

(C.3)

$$
\begin{array}{ccccccc}
 & \swarrow & E(1/3,1/3,2\alpha) & \overset{3_C}{\longleftarrow} & E(2\alpha,2\alpha,2\alpha) \nwarrow & & \\
\text{P12:}\ E(1/2,1/3,\alpha) & \overset{3}{\longleftarrow} & E(1/2,\alpha,2\alpha) \swarrow & \longleftarrow & E(\alpha,\alpha,4\alpha) & \longleftarrow & HE(2\alpha,2\alpha,4\alpha,4\alpha) \\
 & & \nwarrow & HE(1/2,1/2,2\alpha,4\alpha) & \swarrow & &
\end{array}
$$

The degree indications 2, 2_H are not shown with the arrows, only 3 and 3_C. The two Heun equations $HE(1/2,1/2,2\alpha,2\alpha)$ in the P10 lattice are different, indicating that both P2 and P37 appear as composition factors of P10. The quadratic transformations $HE(1/2,1/2,1/2,2\alpha) \overset{2_H}{\longleftarrow} HE(1/2,1/2,2\alpha,2\alpha)$ branch over 2 of the 3 singularities with the exponent difference 1/2, giving three choices. One choice leads to a specialized P2 component $E(1/2,1/4,\alpha) \overset{4}{\longleftarrow} HE(1/2,1/2,2\alpha,2\alpha)$, and two choices lead to (C.1) as indicated by the double arrow in the lower part of (C.2). Similarly, the upper component $E(1/2,\alpha,\alpha) \Leftarrow E(\alpha,\alpha,2\alpha)$ has two choices of branching above 1/2, α points of $E(1/2,\alpha,\alpha)$.

Formally, there are always three choices for the first leg of the quartic Heun-to-Heun transformation (4.15). This quartic transformation is involved as a component of P38 and P53 as well.

Here are all compositions for the parametric Gauss-to-Heun transformations in Tables 1 and 2 (and their coverings), except for the just considered P2, P3, P10, P12, P37.

$$\text{P4}(H_{27}):\quad E(1/2,1/3,\alpha) \overset{3}{\longleftarrow} E(1/2,\alpha,2\alpha) \overset{2}{\longleftarrow} HE(1/2,1/2,2\alpha,4\alpha);$$

$$\text{P6}(H_{19}):\quad E(1/2,1/3,\alpha) \overset{4}{\longleftarrow} E(1/3,\alpha,3\alpha) \overset{2}{\longleftarrow} HE(1/3,1/3,2\alpha,6\alpha);$$

$$\text{P7}(H_{15}):\quad E(1/2,1/3,\alpha) \overset{4}{\longleftarrow} E(1/3,\alpha,3\alpha) \overset{2}{\longleftarrow} HE(2/3,\alpha,\alpha,6\alpha);$$

$$\text{P8}(H_{17}):\quad E(1/2,1/3,\alpha) \overset{4}{\longleftarrow} E(1/3,\alpha,3\alpha) \overset{2}{\longleftarrow} HE(2/3,2\alpha,3\alpha,3\alpha);$$

$$\text{P9}(H_{48}):\quad E(1/4,1/4,\alpha) \overset{2}{\longleftarrow} HE(1/2,1/2,\alpha,\alpha) \overset{2_H}{\longleftarrow} HE(\alpha,\alpha,\alpha,\alpha);$$

$$\text{P11}(H_{28}):\quad E(1/3,1/3,\alpha) \overset{3_C}{\longleftarrow} E(\alpha,\alpha,\alpha) \overset{2}{\longleftarrow} HE(\alpha,\alpha,2\alpha,2\alpha);$$

$$\mathrm{P13}(H_{40}):\quad E(1/2,1/4,\alpha)\overset{2}{\leftarrow}E(1/2,\alpha,\alpha)\overset{2}{\leftarrow}E(\alpha,\alpha,2\alpha)\overset{2}{\leftarrow}HE(\alpha,\alpha,2\alpha,4\alpha);$$

$$\mathrm{P14}(H_{2}):\quad E(1/2,1/3,\alpha)\overset{3}{\leftarrow}E(1/2,\alpha,2\alpha)\overset{2}{\leftarrow}E(\alpha,\alpha,4\alpha)\overset{2}{\leftarrow}HE(\alpha,\alpha,2\alpha,8\alpha);$$

$$\mathrm{P16}(H_{25}):\quad E(1/2,1/4,\alpha)\overset{2}{\leftarrow}E(1/2,\alpha,\alpha)\overset{3}{\leftarrow}HE(1/2,\alpha,2\alpha,3\alpha);$$

$$\mathrm{P17}(H_{12}):\quad E(1/2,1/3,\alpha)\overset{3}{\leftarrow}E(1/2,\alpha,2\alpha)\overset{3}{\leftarrow}HE(1/2,\alpha,2\alpha,6\alpha);$$

$$\mathrm{P18}(H_{14}):\quad E(1/2,1/3,\alpha)\overset{3}{\leftarrow}E(1/2,\alpha,2\alpha)\overset{3}{\leftarrow}HE(1/2,2\alpha,3\alpha,4\alpha),$$

$$E(1/2,1/3,\alpha)\overset{2}{\leftarrow}E(1/3,1/3,2\alpha)\overset{4_B}{\leftarrow}HE(1/3,1/3,2\alpha,6\alpha);$$

$$\mathrm{P21}(H_{25}):\quad E(1/2,1/3,\alpha)\overset{2}{\leftarrow}E(1/3,1/3,2\alpha)\overset{3}{\leftarrow}HE(1/3,2/3,2\alpha,4\alpha);$$

$$\mathrm{P22}(H_{39}):\quad E(1/2,1/6,\alpha)\overset{2}{\leftarrow}E(1/3,\alpha,\alpha)\overset{3}{\leftarrow}HE(\alpha,\alpha,2\alpha,2\alpha),$$

$$E(1/2,1/6,\alpha)\overset{3}{\leftarrow}HE(1/2,1/2,\alpha,2\alpha)\overset{2_H}{\leftarrow}HE(\alpha,\alpha,2\alpha,2\alpha);$$

$$\mathrm{P23}(H_{20}):\quad E(1/2,1/4,\alpha)\overset{2}{\leftarrow}E(1/2,\alpha,\alpha)\overset{4}{\leftarrow}HE(\alpha,\alpha,3\alpha,3\alpha),$$

$$E(1/2,1/4,\alpha)\overset{4_A}{\leftarrow}HE(1/2,1/2,\alpha,3\alpha)\overset{2_H}{\leftarrow}HE(\alpha,\alpha,3\alpha,3\alpha);$$

$$\mathrm{P24}(H_{3}):\quad E(1/2,1/3,\alpha)\overset{3}{\leftarrow}E(1/2,\alpha,2\alpha)\overset{4}{\leftarrow}HE(\alpha,2\alpha,3\alpha,6\alpha),$$

$$E(1/2,1/3,\alpha)\overset{4}{\leftarrow}E(1/3,\alpha,3\alpha)\overset{3}{\leftarrow}HE(\alpha,2\alpha,3\alpha,6\alpha);$$

$$\mathrm{P38}(H_{28}):\quad E(1/2,1/3,\alpha)\overset{3_C}{\leftarrow}HE(1/2,1/2,1/2,3\alpha)\overset{2_H}{\leftarrow}HE(1/2,1/2,3\alpha,3\alpha);$$

$$\mathrm{P39}(H_{43}):\quad E(1/2,1/4,\alpha)\overset{3}{\leftarrow}HE(1/2,1/2,1/4,3\alpha)\overset{2_H}{\leftarrow}HE(1/4,1/4,3\alpha,3\alpha);$$

$$\mathrm{P41}(H_{20}):\quad E(1/2,1/3,\alpha)\overset{2}{\leftarrow}E(1/3,1/3,2\alpha)\overset{4}{\leftarrow}HE(1/3,1/3,4\alpha,4\alpha),$$

$$E(1/2,1/3,\alpha)\overset{4_A}{\leftarrow}HE(1/2,1/2,1/3,4\alpha)\overset{2_H}{\leftarrow}HE(1/3,1/3,4\alpha,4\alpha);$$

$$\mathrm{P43}(H_{4}):\quad E(1/2,1/3,\alpha)\overset{6}{\leftarrow}HE(1/2,1/2,\alpha,5\alpha)\overset{2_H}{\leftarrow}HE(\alpha,\alpha,5\alpha,5\alpha);$$

$$\mathrm{P52}(H_{38}):\quad E(1/2,1/3,\alpha)\overset{2}{\leftarrow}E(1/3,1/3,2\alpha)\overset{3_C}{\leftarrow}HE(1/3,1/3,1/3,6\alpha);$$

$$\text{P53}(H_6): \quad E(1/2, 1/3, \alpha) \overset{4}{\leftarrow} E(1/3, \alpha, 3\alpha) \overset{3_C}{\leftarrow} HE(3\alpha, 3\alpha, 3\alpha, 3\alpha),$$

$$E(1/2, 1/3, \alpha) \overset{3_C}{\leftarrow} HE(1/2, 1/2, 1/2, 3\alpha)$$

$$\overset{2_H}{\leftarrow} HE(1/2, 1/2, 3\alpha, 3\alpha) \overset{2_H}{\leftarrow} HE(3\alpha, 3\alpha, 3\alpha, 3\alpha);$$

$$\text{P54}(H_{38}): \quad E(1/2, 1/6, \alpha) \overset{2}{\leftarrow} E(1/3, \alpha, \alpha) \overset{3_C}{\leftarrow} HE(\alpha, \alpha, \alpha, 3\alpha);$$

$$\text{P55}(H_1): \quad E(1/2, 1/3, \alpha) \overset{4}{\leftarrow} E(1/3, \alpha, 3\alpha) \overset{3_C}{\leftarrow} HE(\alpha, \alpha, \alpha, 9\alpha).$$

Finally, we present a few exemplifying formulas for the compositions with 2_H that cannot be obtained by composing with Gauss-to-Gauss transformations. The additional Heun-to-Heun transformation changes the t-parameter except for P9.

P9: $E(1/4, 1/4, \alpha) \overset{4}{\leftarrow} HE(\alpha, \alpha, \alpha, \alpha)$, with H_{48}:

$$\text{Hn}\left(\begin{matrix} -1 \\ 0 \end{matrix} \middle| \begin{matrix} 4a, 2a + 1/2 \\ 2a + 1/2; 2a + 1/2 \end{matrix} \middle| x \right) = (1 - ix)^{-4a} {}_2\text{F}_1\left(\begin{matrix} a, a + 1/4 \\ 2a + 1/2 \end{matrix} \middle| \frac{8ix(x^2 - 1)}{(x + i)^4} \right);$$

P37: $E(1/2, 1/4, \alpha) \overset{4}{\leftarrow} HE(1/2, 1/2, 2\alpha, 2\alpha)$, with H_{35}:

$$\text{Hn}\left(\begin{matrix} 17 + 12\sqrt{2} \\ 2a(1 + 8a)/(3 - 2\sqrt{2}) \end{matrix} \middle| \begin{matrix} 4a, 4a + 1/2 \\ 1/2; 4a + 1/2 \end{matrix} \middle| \frac{x}{3 - 2\sqrt{2}} \right)$$

$$= (1 + x)^{-4a} {}_2\text{F}_1\left(\begin{matrix} a, a + 1/4 \\ 1/2 \end{matrix} \middle| \frac{16x(x - 1)^2}{(x + 1)^4} \right);$$

P38: $E(1/2, 1/3, \alpha) \overset{6}{\leftarrow} HE(1/2, 1/2, 3\alpha, 3\alpha)$, with $H_{28}(x) = 36x(x^2 + 3)^2/(x^2 + 6x - 3)^3$:

$$\text{Hn}\left(\begin{matrix} 4\sqrt{3} - 7 \\ 3a(1 + 12a)/(3 + 2\sqrt{3}) \end{matrix} \middle| \begin{matrix} 6a, 6a + 1/2 \\ 1/2; 6a + 1/2 \end{matrix} \middle| \frac{x}{3 + 2\sqrt{3}} \right)$$

$$= \left(1 - 2x - \frac{x^2}{3}\right)^{-3a} {}_2\text{F}_1\left(\begin{matrix} a, a + 1/3 \\ 1/2 \end{matrix} \middle| H_{28}(x) \right);$$

P39: $E(1/2, 1/4, \alpha) \overset{6}{\leftarrow} HE(1/4, 1/4, 3\alpha, 3\alpha)$, with $H_{43}(x) = 108x(x - 1)^4/(x^2 + 14x + 1)^3$:

$$\text{Hn}\left(\begin{matrix} 97 + 56\sqrt{3} \\ 9a(1 + 24a)/(14 - 8\sqrt{3}) \end{matrix} \middle| \begin{matrix} 6a, 6a + 1/4 \\ 3/4; 6a + 1/4 \end{matrix} \middle| \frac{x}{4\sqrt{3} - 7} \right)$$

$$= (1 + 14x + x^2)^{-3a} {}_2\text{F}_1\left(\begin{matrix} a, 1/4 - a \\ 3/4 \end{matrix} \middle| H_{43}(x) \right);$$

P43: $E(1/2, 1/3, \alpha) \overset{12}{\leftarrow} HE(\alpha, \alpha, 5\alpha, 5\alpha)$, with
$H_4(x) = 1728x^5(x^2 - 11x - 1)/(x^4 - 12x^3 + 14x^2 + 12x + 1)^3$:

$$\mathrm{Hn}\left(\begin{matrix} (-123 + 55\sqrt{5})/2 \\ 12a(1+60a)/(11+5\sqrt{5}) \end{matrix} \middle| \begin{matrix} 12a, 2a+5/6 \\ 10a+1/6; 2a+5/6 \end{matrix} \middle| \frac{2x}{11+5\sqrt{5}} \right)$$
$$= (1 + 12x + 14x^2 - 12x^3 + x^4)^{-3a} {}_2\mathrm{F}_1\left(\begin{matrix} a, a+1/3 \\ 2a+5/6 \end{matrix} \middle| H_4(x) \right).$$

Note that the transformation P9 is not defined over $\boldsymbol{Q}$ even if $t \in \boldsymbol{Q}$. Other example of this type is P11, with the same $t = -1$:

$$\text{(C.4)} \qquad \mathrm{Hn}\left(\begin{matrix} -1 \\ 0 \end{matrix} \middle| \begin{matrix} 6a, 2a+2/3 \\ 4a+1/3; 2a+2/3 \end{matrix} \middle| x \right)$$
$$= (1 - (\omega+1)x^2)^{-3a} {}_2\mathrm{F}_1\left(\begin{matrix} a, a+1/3 \\ 2a+2/3 \end{matrix} \middle| \frac{3(1+2\omega)x^2(x^2-1)}{(x^2+\omega)^3} \right).$$

Here the covering is H_{28}, the same as in P38 (just above) but normalized differently. Composition with 2_H occurs in P38 but not in P11: $E(1/3, 1/3, \alpha) \overset{6}{\leftarrow} HE(\alpha, \alpha, 2\alpha, 2\alpha)$.

D. Appendix: Invariants of fractional-linear transformations

Here we derive invariants of the fractional-linear transformations of § B acting on Heun equations and functions. At the end, Theorem D.2 proves sufficient conditions for Heun's equation to be reducible to a hypergeometric equation by the considered parametric transformations.

Additional invariants of the fractional-linear transformations are needed not only to determine the accessory parameter q, but also to ensure that the t-value is in a right correspondence with the assignment of local exponent differences to the 4 singular points. For example, the permutation of the singularities $x = 0$, $x = 1$ changes the t-value to $1 - t$; hence we have $\mathrm{Hn}(t, q|a, b; c; d|x)$ and $\mathrm{Hn}(1-t, ab-q \,|\, a, b; d; c \,|\, 1-x)$ in the same orbit, but generally not $\mathrm{Hn}(t, \tilde{q}|a, b; d; c|\tilde{x})$ for any $\tilde{q}$, $\tilde{x}$.

When talking about equal (or different) local exponent differences in this appendix, we mean equal up to multiplication by -1 (or different even after multiplication of some by -1).

Theorem D.1. *Consider Heun's equation E_0 as in* (1.6), *and let*

$$\text{(D.1)} \qquad e_0 = 1 - c, \qquad e_1 = 1 - d, \qquad e_t = c + d - a - b, \qquad e_\infty = b - a$$

denote the local exponent differences.

(a) *If some two local exponent differences are equal, there is a fractional-linear transformation of* E_0 *with the same parameters a, b, c, d, but with a different t-parameter.*

(b) *If three local exponent differences are equal, there are fractional-linear transformations of* E_0 *with the same parameters a, b, c, d and any t in the orbit* (2.1).

(c) *The following entities are invariants of the action of the fractional-linear transformations on Heun equations (and functions):*

- *The elementary symmetric functions* E_1, E_2, E_3, E_4 *in the squares* e_0^2, e_1^2, e_t^2, e_∞^2, *determined by the polynomial identity*

$$X^4 - E_1X^3 + E_2X^2 - E_3X + E_4 = (X - e_0^2)(X - e_1^2)(X - e_t^2)(X - e_\infty^2). \tag{D.2}$$

- *The j-invariant* $j(t)$ *as in* (2.2).
- *If* $j \neq 0$, *the values*

$$k_1 = \frac{t^2 - t + 1}{t(t-1)}((e_0^2 - e_1^2)(e_t^2 - e_\infty^2)t - (e_0^2 - e_t^2)(e_1^2 - e_\infty^2)), \tag{D.3}$$

$$k_2 = \frac{(e_0^2e_t^2 + e_1^2e_\infty^2)t^2 - (e_0^2e_\infty^2 + e_1^2e_t^2)t + e_0^2e_1^2 + e_t^2e_\infty^2}{t^2 - t + 1} + e_0^2e_\infty^2 + e_1^2e_t^2. \tag{D.4}$$

- *The value* $Q_1 = J_1Q_0$, *where*

$$J_1 = \frac{(t+1)(t-2)(2t-1)(t^2 - t + 1)}{t^2(t-1)^2}, \tag{D.5}$$

$$Q_0 = 12q - 6(e_0 - 1)(e_1t + e_t) + (e_\infty^2 - 2(e_0 - 1)(e_0 - 2))(t+1) - e_1^2(2t - 1) + e_t^2(t - 2).$$

(d) *The invariants j,* k_1 *determine the t-values corresponding to an orderly assignment* $(e_0, e_1, e_t, e_\infty)$ *of the exponent differences to the singular points, except when* $j \in \{0, 1728\}$ *or*

$$j = \frac{1728F_4^3}{(E_2^3 - 9F_6)^2}, \qquad k_1 = -\frac{3}{2}\frac{F_4^2}{E_2^3 - 9F_6}, \tag{D.6}$$

with $F_4 = E_2^2 - 3E_1E_3 + 12E_4$, $F_6 = E_1E_2E_3/2 - 3E_3^2/2 - 3E_1^2E_4/2 + 4E_2E_4$.

(e) *If a pair of local exponent differences is equal, the exceptional case in* (d) *has* $j = 1728$.

(f) *Algebraic relations between j,* k_1, k_2 *are generated by these generic identifications of the j-invariant:*

(D.7)
$$\frac{j}{256} = \frac{-k_1^2 + 9k_1k_2 - 6E_2k_1 - 9F_4}{3k_2^2 - 4E_2k_2 + 4(E_1E_3 - 4E_4)} = k_1\frac{3k_1k_2 - 2E_2k_1 - 3F_4}{F_4k_2 - 2\tilde{F}_6},$$

where $\tilde{F}_6 = E_1E_2E_3/2 - 9E_3^2/2 - 9E_1^2E_4/2 + 16E_2E_4$.

(g) *The invariants* k_1, k_2 *determine the* t*-values corresponding to an orderly assignment* $(e_0, e_1, e_t, e_\infty)$ *of the exponent differences.*

(h) *If all local exponent differences are different, then* t *is unique*:

(D.8)
$$t = \frac{1}{2} + \frac{k_1(k_2 - (e_0^2 + e_t^2)(e_1^2 + e_\infty^2))(k_2 - (e_0^2 + e_\infty^2)(e_1^2 + e_t^2)) - F_4(k_2 - e_0^2e_1^2 - e_t^2e_\infty^2) + \tilde{F}_6}{(e_0^2 - e_1^2)(e_0^2 - e_t^2)(e_0^2 - e_\infty^2)(e_1^2 - e_t^2)(e_1^2 - e_\infty^2)(e_t^2 - e_\infty^2)}.$$

If there is exactly one pair of equal exponent differences, there are two corresponding t*-values. They are determined by* (D.3) *or by* (D.4).

Proof. As mentioned in § B, the fractional-linear transformations permute the exponent differences and multiply them by -1. The transformations that leave t invariant are those that multiply the exponent differences by -1 and interchange them in two pairs; see (B.4) and (B.5). These transformations leave $e_0^2e_1^2 + e_t^2e_\infty^2$, $e_0^2e_t^2 + e_1^2e_\infty^2$, $e_0^2e_\infty^2 + e_1^2e_t^2$ invariant as well.

Part (a) is demonstrated by the first three equalities in (B.4) in the cases $e_1 = e_t$, $e_1 = e_\infty$, $e_t = e_\infty$, respectively. Part (b) is demonstrated by all formulas (B.4) in the case $e_1 = e_t = e_\infty$. Other possible equalities of exponent differences are obtained by applying the transformations that leave t invariant.

The invariants E_1, E_2, E_3, E_4 and $j(t)$ are clear. To obtain other invariants that do not involve q, we consider the shortened orbit sums

$$S_1 = (e_0^2e_1^2 + e_t^2e_\infty^2)\left(\frac{1}{t} + \frac{1}{1-t}\right) + (e_0^2e_t^2 + e_1^2e_\infty^2)\left(t + \frac{t}{t-1}\right) + (e_0^2e_\infty^2 + e_1^2e_t^2)\left(1 - t + \frac{t-1}{t}\right),$$

$$S_2 = (e_0^2e_1^2 + e_t^2e_\infty^2)\left(\frac{1}{t^2} + \frac{1}{(1-t)^2}\right) + (e_0^2e_t^2 + e_1^2e_\infty^2)\left(t^2 + \frac{t^2}{(t-1)^2}\right) + (e_0^2e_\infty^2 + e_1^2e_t^2)\left((1-t)^2 + \frac{(t-1)^2}{t^2}\right).$$

Then $k_1 = S_1 - E_2$, $k_2 = 256(S_2 + E_2)/j$, adjusted for brevity. To obtain Q_1, let S_3 denote the full orbit sum of 192×2 values of the product tq. Then

(D.9)
$$Q_1 = \frac{3}{16}S_3 - \frac{j(E_1 + 8)}{256}.$$

To check the invariance of Q_1 directly, it is useful to note this general action of the fractional-linear transformations on the "semi-invariant" Q_0:

(D.10)
$$\begin{aligned}
&Q_0 \mapsto Q_0, && \text{if } t \text{ remains the same;}\\
&Q_0 \mapsto -Q_0, && \text{if } t \text{ is transformed to } 1-t;\\
&Q_0 \mapsto Q_0/t, && \text{if } t \text{ is transformed to } 1/t;\\
&Q_0 \mapsto -Q_0/t, && \text{if } t \text{ is transformed to } (t-1)/t;\\
&Q_0 \mapsto Q_0/(1-t), && \text{if } t \text{ is transformed to } t/(t-1);\\
&Q_0 \mapsto Q_0/(t-1), && \text{if } t \text{ is transformed to } 1/(1-t).
\end{aligned}$$

Now consider part (d). The algebraic relation between j, k_1 is obtained by eliminating t from (2.2) and (D.3). It is of degree 6 in k_1, naturally. Computation shows that the discriminant with respect to k_1 vanishes only for $j \in \{0, 1728\}$ and for the j-value in (D.6). Non-vanishing discriminant gives a one-to-one correspondence between the t and k_1 values for the same j-invariant. The ambiguous case (D.6) represents a nodal singularity on the plane algebraic curve defined by the relation between j, k_1. It does not distinguish the following t-values:

(D.11)
$$t_1 = -\frac{E_2 - 3e_0^2e_\infty^2 - 3e_1^2e_t^2}{E_2 - 3e_0^2e_t^2 - 3e_1^2e_\infty^2}, \qquad t_2 = -\frac{E_2 - 3e_0^2e_1^2 - 3e_t^2e_\infty^2}{E_2 - 3e_0^2e_\infty^2 - 3e_1^2e_t^2}.$$

If a pair of local exponent differences is equal, these t-values are in $\{-1, 2, 1/2\}$, showing part (e).

Part (f) and formula (D.8) follow from Gröbner basis computations that eliminate t and j. Evidently, t is determined uniquely by the ordered tuple $(e_0, e_1, e_t, e_\infty)$ when the local exponent differences are not equal, and there are at least two fitting t-values when there is an equality by part (a). Formula (D.3) becomes quadratic in t in the latter case. Parts (g) and (h) follow. □

The invariant Q_1 clearly determines q unless $j \in \{0, 1728\}$. On the other hand, the encountered Heun equations with $j \in \{0, 1728\}$ all have $Q_0 = 0$, thus investigation of additional invariants is not needed. Generally, the expression Q_0 with $j \in \{0, 1728\}$ might change even if t remains the same, contrary to the gist of (D.10). For example, if $t = -1$ then $t = 1/t$ but $Q_0 \mapsto -Q_0$. The action on Q_0 is then determined not only by the action on t, but also by the permutation of e_0^2, e_1^2, e_t^2, e_∞^2.

As it turns out, complications with k_1, k_2 for $j \in \{0, 1728\}$ do not have to be considered either, partly because the encountered Heun equations with $j \in \{0, 1728\}$ have some exponent differences equal. The ambiguous case (D.6) is bound to happen for most other encountered Heun equations, because they have a free parameter.

The invariants can be expressed in terms of the parameters a, b, c, d rather then the exponent differences e_0, e_1, e_t, e_∞. For example, the invariant Q_1 can be computed using

$$\text{(D.12)} \qquad Q_0 = 12q - 6ab + (a^2 + b^2 - 2cd + 2c + 2d - 1)(2t - 1) \\ - (c^2 + 2ad + 2bd)(t - 2) - (d^2 + 2ac + 2bc)(t + 1).$$

The following theorem formulates the sufficient conditions for Theorem 2.1.

Theorem D.2. *Heun's equation* (1.6) *is* (*a specialization of*) *a parametric pull-back transformation of a hypergeometric equation if it satisfies one of the conditions* (i)–(v) *of Theorem* 2.1, *and the following respective conditions*:

(i) $Q_0 = 0$, *and* $k_1 = 3(\alpha^2 - \beta^2)(\alpha^2 - \gamma^2)$ *for* $HE(\alpha, \alpha, \beta, \gamma)$;

(ii) $Q_0 = 0$;

(iii) $Q_1 = (5 \cdot 7 \cdot 13/2^5 3^2)(4\alpha^2 + 8\beta^2 - 3)$, $k_1 = -13(\alpha^4 - 5\alpha^2\beta^2/4 + \beta^2/16)$, $k_2 = 36\alpha^4/13 + 5\alpha^2\beta^2 + 5\alpha^2/13 + 9\beta^2/52$;

(iv) $Q_1 = (5 \cdot 7 \cdot 17 \cdot 73/2^5 3^4)(23\alpha^2 + 23\beta^2 - 4)$, $k_1 = (73/2)(\alpha^4 - 4\alpha^2\beta^2 + \beta^4)$, $k_2 = (324/73)(\alpha^4 + 274\alpha^2\beta^2/81 + \beta^4)$ *for* $HE(\alpha, 2\alpha, \beta, 2\beta)$, *and* $Q_1 = (5 \cdot 7 \cdot 17 \cdot 73/2^4 3^4)(23\alpha^2 + 23\beta^2 - 2)$, $k_1 = -73(\alpha^4 - 10\alpha^2\beta^2 + \beta^4)$, $k_2 = (576/73) \cdot (\alpha^4 + 185\alpha^2\beta^2/16 + \beta^4)$ *for* $HE(\alpha, 3\alpha, \beta, 3\beta)$;

(v) *for the cases* P25 *to* P36, *the invariants* Q_1, k_1, k_2 *are as in Table* 6; *for the cases* P37–P40, P42–P50, P56–P59, *the invariants* Q_1, k_1 *are as in Table* 6;

(vi) *the invariants* Q_1, k_1, k_2 *are as in Table* 6, *or conjugated* $i \mapsto -i$, $\omega \mapsto -\omega - 1$ *if the* j*-invariant of Table* 4 *is conjugated.*

Proof. If $j = 0$ as in (ii), the invariants Q_1, k_1, k_2 generally fail. But for the encountered Heun equations with $j = 0$, part (b) of Theorem D.1 applies, and the semi-invariant value $Q_0 = 0$ determines the accessory parameter.

If $j = 1728$ as in (i), the semi-invariant value $Q_0 = 0$ determines the accessory parameter just as well. The invariant k_1 has only two possible values: $3(\alpha^2 - \beta^2)(\alpha^2 - \gamma^2)$ and $-3(\alpha^2 - \beta^2)(\alpha^2 - \gamma^2)/2$. The latter k_1-value gives a confusion between two t-values in $\{-1, 2, 1/2\}$, but the encountered Heun equations have the former k_1-value. This k_1-value gives an equation of the form $(t - \xi)^2 = 0$ and determines the correct $t \in \{-1, 2, 1/2\}$ without the aid from k_2.

In case (iv), the two different Heun equations have to be considered separately. Note that the transformation P24 is a specialization of both P19 and P20, and the invariants specialize consistently to $Q_1 = (5 \cdot 7 \cdot 17 \cdot 73/2^4 3^4) \cdot (115\alpha^2 - 2)$, $k_1 = 1679\alpha^4$, $k_2 = 36432\alpha^4/73$.

Table 6: Invariants for sufficient identification of reducible Heun equations.

Id	Invariant Q_1	Invariant k_1	Invariant k_2
P25	$\frac{3^5\cdot 7}{2^5 5^2}(72\alpha^2-1)$	$\frac{7}{3}\left(81\alpha^4-\frac{173}{16}\alpha^2+\frac{1}{4}\right)$	$\frac{100}{189}\left(81\alpha^4+\frac{121}{16}\alpha^2+\frac{1}{4}\right)$
P26	$\frac{7\cdot 13\cdot 17\cdot 37\cdot 41}{2^7 3^6 5^4}(1881\alpha^2-28)$	$-\frac{481}{400}\left(25\alpha^4-\frac{229}{9}\alpha^2+\frac{4}{81}\right)$	$\frac{81}{481}\left(25\alpha^4+\frac{70549}{729}\alpha^2+\frac{4}{81}\right)$
P27	$\frac{7^2\cdot 23\cdot 41\cdot 79\cdot 6481}{2^{11}3^8 5^2}(1696\alpha^2-83)$	$-\frac{6481}{81}\left(36\alpha^4-\frac{643}{256}\alpha^2+\frac{1}{64}\right)$	$\frac{6400}{6481}\left(36\alpha^4+\frac{52493}{20480}\alpha^2+\frac{1}{64}\right)$
P28	$\frac{7\cdot 23\cdot 41\cdot 79\cdot 6481}{2^7 3^{10} 5^2}(22977\alpha^2-440)$	$\frac{6481}{144}\alpha^2\left(691\alpha^2-\frac{43}{9}\right)$	$\frac{1}{6481}\alpha^2\left(2726825\alpha^2+\frac{166577}{9}\right)$
P29	$\frac{2^2\cdot 7\cdot 13\cdot 23\cdot 29\cdot 97}{3^8 5^4}(1215\alpha^2-16)$	$\frac{679}{9}\alpha^2\left(17\alpha^2-\frac{404}{225}\right)$	$\frac{20}{679}\alpha^2\left(3775\alpha^2+\frac{3274}{9}\right)$
P30	$\frac{5^2\cdot 11\cdot 13\cdot 29\cdot 757}{2^5 3^8 7^2}(2952\alpha^2-71)$	$\frac{757}{27}\left(144\alpha^4-\frac{715}{144}\alpha^2+\frac{1}{36}\right)$	$\frac{784}{757}\left(144\alpha^4+\frac{131365}{28224}\alpha^2+\frac{1}{36}\right)$
P31	$\frac{7\cdot 11\cdot 37\cdot 59\cdot 127}{2^{11}3^8 5^2}(14004\alpha^2-323)$	$-\frac{889}{216}\left(4\alpha^4-\frac{52}{9}\alpha^2+\frac{1}{36}\right)$	$\frac{100}{889}\left(4\alpha^4+\frac{970}{9}\alpha^2+\frac{1}{36}\right)$
P32	$\frac{7\cdot 11\cdot 37\cdot 59\cdot 127}{2^9 3^8 5^2}(4014\alpha^2-95)$	$889\alpha^2\left(\alpha^2-\frac{1}{162}\right)$	$\frac{36}{889}\alpha^2\left(8450\alpha^2+\frac{5641}{162}\right)$
P33	$\frac{7\cdot 11\cdot 23\cdot 61\cdot 131\cdot 2287}{2^{15}3^2 5^6}(12580\alpha^2-237)$	$\frac{16009}{40}\alpha^2\left(23\alpha^2-\frac{63}{100}\right)$	$\frac{250}{16009}\alpha^2\left(15309\alpha^2+\frac{40709}{100}\right)$
P34	$\frac{7\cdot 11\cdot 23\cdot 61\cdot 131\cdot 2287}{2^{13}3^2 5^8}(31450\alpha^2-1953)$	$-\frac{16009}{100}\alpha^2\left(7\alpha^2-\frac{3}{100}\right)$	$\frac{8}{16009}\alpha^2\left(88749\alpha^2+\frac{40621}{100}\right)$
P35	$\frac{5\cdot 13\cdot 17\cdot 113\cdot 4993}{2^9 3^{10} 7^4}(8262\alpha^2-67)$	$\frac{4993}{9}\alpha^2\left(\alpha^2-\frac{17}{882}\right)$	$\frac{2}{4993}\alpha^2\left(672868\alpha^2+\frac{64153}{9}\right)$
P36	$\frac{11\cdot 19\cdot 23\cdot 61\cdot 157\cdot 1459}{2^{13}3^8 5^6 7^2}(50184\alpha^2-635)$	$\frac{27721}{8000}\left(100\alpha^4-\frac{1849}{324}\alpha^2+\frac{1}{36}\right)$	$\frac{35721}{27721}\left(100\alpha^4+\frac{4322681}{1285956}\alpha^2+\frac{1}{36}\right)$
P37	$\frac{3^5\cdot 7\cdot 11}{2^5}(4\alpha^2-1)$	$-33\left(\alpha^2-\frac{1}{4}\right)^2$	not needed
P38	$\frac{3^3\cdot 5\cdot 11}{2^4}(4\alpha^2-1)$	$15\left(\alpha^2-\frac{1}{4}\right)^2$	not needed
P39	$\frac{7\cdot 193\cdot 383}{2^8\cdot 3}(176\alpha^2-71)$	$-193\left(\alpha^2-\frac{1}{16}\right)^2$	not needed
P40	$\frac{5\cdot 53\cdot 109}{3^5}(63\alpha^2-23)$	$53\left(\alpha^2-\frac{1}{9}\right)^2$	not needed
P42	$\frac{3\cdot 5\cdot 17\cdot 31}{2^5}(12\alpha^2-1)$	$-51\alpha^2\left(4\alpha^2-\frac{1}{4}\right)$	not needed
P43	$\frac{2^5\cdot 11\cdot 31\cdot 251}{5^3}(58\alpha^2-1)$	$71424\alpha^4$	not needed
P44	$\frac{3^5\cdot 13\cdot 17}{2^8 5^4}(1-16\alpha^2)$	$39\left(\alpha^2-\frac{1}{16}\right)\left(\alpha^2-\frac{1}{400}\right)$	not needed
P45	$\frac{11\cdot 109\cdot 157}{2^4 5^6}(5-356\alpha^2)$	$-\frac{109}{5}\left(\alpha^2-\frac{1}{4}\right)\left(\alpha^2-\frac{1}{100}\right)$	not needed
P46	$\frac{3^2\cdot 11\cdot 109\cdot 157}{2^2 5^8}(4-75\alpha^2)$	$\frac{327}{25}\alpha^2\left(\alpha^2-\frac{1}{25}\right)$	not needed
P47	$\frac{5\cdot 19\cdot 43}{2^3 3^8}(11-36\alpha^2)$	$\frac{95}{972}\left(\alpha^2-\frac{1}{4}\right)$	not needed
P48	$\frac{7\cdot 47\cdot 337}{2^7 3^8}(44\alpha^2-1)$	$\frac{47}{9}\left(\alpha^2-\frac{1}{4}\right)\left(\alpha^2-\frac{1}{36}\right)$	not needed
P49	$\frac{7\cdot 47\cdot 337}{2^5 3^{10}}(2-513\alpha^2)$	$-\frac{329}{9}\alpha^2\left(\alpha^2-\frac{1}{9}\right)$	not needed
P50	$\frac{2\cdot 5\cdot 11\cdot 239\cdot 251}{3^{12}}(4-27\alpha^2)$	$\frac{239}{81}\alpha^2\left(\alpha^2-\frac{1}{9}\right)$	not needed
P56	$\frac{2\cdot 5\cdot 11\cdot 13\cdot 103}{3^5 7^4}(1-153\alpha^2)$	$-22\left(\alpha^2-\frac{1}{9}\right)\left(\alpha^2-\frac{1}{441}\right)$	not needed
P57	$\frac{3\cdot 5\cdot 11\cdot 13\cdot 37\cdot 43}{2^{15}7^3}(1-244\alpha^2)$	$-\frac{645}{8}\alpha^2\left(\alpha^2-\frac{1}{4}\right)$	not needed
P58	$\frac{5\cdot 19\cdot 269\cdot 499}{2^{19}3^5}(68\alpha^2+3)$	$\frac{269}{64}\alpha^2\left(\alpha^2-\frac{1}{4}\right)$	not needed
P59	$\frac{11\cdot 71\cdot 167}{2^{13}3^5\cdot 5}(17-72\alpha^2)$	$\frac{355}{2304}\left(\alpha^2-\frac{1}{9}\right)$	not needed
P60	$\frac{229-53i}{2\cdot 5^2}\left(\frac{4-i}{16}-(10-7i)\alpha^2\right)$	$\frac{2+3i}{2}\left(16\alpha^4-\frac{28+9i}{8}\alpha^2+\frac{1}{64}\right)$	$\frac{17-6i}{13}\left(16\alpha^4+\frac{9+12i}{5}\alpha^2+\frac{1}{64}\right)$
P61	$\frac{531-6130\omega}{2^3 3^3(2-\omega)^2}\left(\frac{5-4\omega}{36}-(19-8\omega)\alpha^2\right)$	$\frac{11\omega-23}{9}\left(9\alpha^4+\frac{2-21\omega}{4+8\omega}\alpha^2+\frac{1}{144}\right)$	$\frac{12-20\omega}{15+19\omega}\left(9\alpha^4-\frac{157+34\omega}{10+16\omega}\alpha^2+\frac{1}{144}\right)$

In the cases P37–P40, P42–P50, P56–P59, we have two equal exponent differences. The k_1-invariant gives then ambiguity only for $t \in \{-1, 2, 1/2\}$, while the actual t-values are algebraic. Hence the invariant k_2 is not needed.

In all other cases, the full invariants set Q_1, k_1, k_2 is used. □

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