Parametrices for pseudodifferential operators with multiple characteristics

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1. Introduction and statement of the main results

The purpose of this paper is to construct parametrices and give solvability (hypoellipticity) and non-solvability results for some classes of pseudodifferential operators for which the characteristic set is a closed manifold in the cotangent space. Some of our results have been announced in [23, 24].

Moyer [18] has reformulated the condition of Nirenberg-Trèves [19] for local solvability in terms of the argument variation of the principal symbol along certain closed curves in the cotangent space. We shall use the notion of argument variation here to study invariant classes of pseudodifferential operators for which the characteristic manifold has codimension 2. For operators with double characteristics we shall also give a complementary result which is valid also when the codimension of the characteristic manifold is larger than 2. We want to point out that Višik and Grušin [6—10, 26, 27], Radkevič [21] and Melin [16] have studied overlapping classes of operators. We have been influenced by these works and we shall use techniques developed by Melin, Višik and Grušin. We also note the interesting example by Trèves-Gilioli [5] and that Boutet de Monvel and Trèves [1] recently have obtained independently a result which is contained in our Theorems 1.6 and 1.7.*

We now start to formulate the precise results. Let Ω be a paracompact C^∞ manifold of dimension n and let $T^*(\Omega) \setminus 0$ be the cotangent space minus the zero section. We adopt the notations $L^m = L^m_{1,0}$ and $S^m = S^m_{1,0}$ for the spaces of pseudodifferential operators and symbols introduced by Hörmander [11, 13].

Definition 1.1. Let $\Sigma \subset T^*(\Omega) \setminus 0$ be a closed conic submanifold and let $m \in \mathbf{R}$, $M \in \mathbf{Z}^+ \cup \{0\}$. Then we define $L^{m,M}(\Omega, \Sigma)$ to be the set of pseudo-

*) Added in proof: More recently they have improved their results. In particular, Boutet de Monvel has shown that the parametrix (in Theorem 1.6 here) is a pseudodifferential operator of type 1/2, 1/2.

differential operators $P \in L^m(\Omega)$ which in every local coordinate system $U \subset \Omega$ has a symbol of the form

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j/2}(x, \xi), \quad p_{m-j/2} \in S^{m-j/2}(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})) \tag{1.1}$$

where the $p_{m-j/2}$ are positively homogeneous of degree $m - j/2$ and satisfy:

(1.2) For every $K \subset\subset U$ there exists a constant $C > 0$, such that

(1.2a) $|p_m(x, \xi)|/|\xi|^m \geq C^{-1}(d(x, \xi))^M$,

(1.2b) $|p_{m-j/2}(x, \xi)|/|\xi|^{m-j/2} \leq C(d(x, \xi))^{M-j}, \quad 0 \leq j \leq M,$
for all $(x, \xi) \in K \times \mathbf{R}^n$, $|\xi| > 1$.

Here $d(x, \xi) = \inf_{(y, \eta) \in \Sigma} (|x - y| + |\eta - \xi/|\xi||)$ is the distance from $(x, \xi/|\xi|)$ to Σ.

We also introduce the set $L_c^{m, M}(\Omega, \Sigma) \subset L^{m, M}(\Omega, \Sigma)$ for which the $p_{m-j/2}$ in (1.1) can be taken to be zero when j is odd. Note that p_m is invariantly defined on $T^*(\Omega) \setminus 0$ and note the composition formula

$$L^{m', M'}(\Omega, \Sigma) \circ L^{m'', M''}(\Omega, \Sigma) \subset L^{m'+m'', M'+M''}(\Omega, \Sigma), \tag{1.3}$$

where in the left hand side we only take compositions $A \circ B$, where one of the factors is properly supported. The space $L^{m, M}$ was essentially introduced in [23].

Now let $\sigma = \sum_1^n d\xi_j \wedge dx_j$ be the symplectic form on $T^*(\Omega)$ and assume that $\Sigma \subset T^*(\Omega) \setminus 0$ is a closed conic non-involutive submanifold of codimension 2. That Σ is non-involutive means that the restriction of σ to Σ is non-degenerate. Let $\varrho \in \Sigma$. Since σ is a non-degenerate alternating form on $T_\varrho(\Sigma)$, the tangent space of Σ at ϱ, we have

$$T_\varrho(T^*(\Omega)) = T_\varrho(\Sigma) \oplus N_\varrho(\Sigma) \tag{1.4}$$

where $N_\varrho(\Sigma)$ denotes the orthogonal space of $T_\varrho(\Sigma)$ with respect to σ. It is clear that σ is non-degenerate also on $N_\varrho(\Sigma)$ and thus induces an orientation there. More precisely, we can choose $e_1, e_2 \in N_\varrho(\Sigma) \setminus \{0\}$ such that $\sigma(e_1, e_2) < 0$ and we identify $N_\varrho(\Sigma)$ with the complex plane linearly, by putting $e_1 = 1$, $e_2 = i$. If γ is a closed curve in $N_\varrho(\Sigma) \setminus \{0\}$, the argument variation $\operatorname{var} \arg \gamma$ is then defined and independent of the choice of e_1, e_2 with $\sigma(e_1, e_2) < 0$. If $\varphi\colon N_\varrho(\Sigma) \setminus \{0\} \to \mathbf{C} \setminus \{0\}$ is a continuous map, we define the index of φ, $\operatorname{ind} \varphi$, by the equation

$$\operatorname{ind} \varphi = (\operatorname{var} \arg \varphi \circ \gamma)/(\operatorname{var} \arg \gamma)$$

where γ is some closed curve in $N_\varrho(\Sigma) \setminus \{0\}$ with $\operatorname{var} \arg \gamma \neq 0$.

Now let $P \in L^{m, M}(\Omega, \Sigma)$ and let p_m be the positively homogeneous principal symbol in Definition 1.1. If $t \in T_\varrho(T^*(\Omega))$, $\varrho \in \Sigma$, let v be some vector field on $T^*(\Omega)$ equal to t at ϱ and put

$$a(t) = a_\varrho(t) = (M!)^{-1}(v^M p_m)_\varrho.$$

By (1.2) $a(t)$ is independent of the choice of v and $a(t) \neq 0$ when

$$t \in T_\varrho(T^*(\Omega)) \setminus T_\varrho(\Sigma).$$

Put $\operatorname{ind}(p_m) = \operatorname{ind}(p_m, \varrho) = \operatorname{ind}\varphi$, where φ is the map

$$N_\varrho(\Sigma) \setminus \{0\} \ni t \mapsto a(t) \in \mathbf{C} \setminus \{0\}.$$

Then $\operatorname{ind} p_m$ is one of the numbers $-M, -M+2, \ldots, +M$. In fact, if we introduce some linear coordinates x, y in $N_\varrho(\Sigma)$, we get

$$a(t) = \sum_0^M a_j x^j y^{M-j}, \quad t = (x, y)$$

and since $a(t) \neq 0$, when $t \in N_\varrho(\Sigma) \setminus \{0\}$, it follows from the fundamental theorem of algebra that

$$a(t) = a_M \prod_{j=1}^M (x - \gamma_j y), \quad t = (x, y) \in N_\varrho(\Sigma), \tag{1.5}$$

where $a_M \neq 0$ and $\operatorname{Im}\gamma_j \neq 0$ for all j. Now our statement follows, since the index of the map $(x, y) \mapsto (x - \gamma_j y)$ is ± 1.

Our first result can now be formulated exactly as Theorem 1.1 in Duistermaat-Sjöstrand [4]. If A, B are linear operators we write $A \equiv B$ if $A - B$ is an integral operator with C^∞ kernel. Adjoints will be taken with respect to the L^2 inner product on $C_0^\infty(\Omega)$ defined by some strictly positive smooth density on Ω. For any set V we write $\operatorname{diag}(V) = \{(\varrho, \varrho) \in V \times V\}$. Let I always denote the identity operator in the appropriate space. We assume the reader is familiar with the notions WF and WF' introduced by Hörmander [11].

THEOREM 1.2. *Let Ω be a paracompact C^∞ manifold of dimension n and let $\Sigma \subset T^*(\Omega) \setminus 0$ be a closed conic non-involutive submanifold of codimension 2. Let $P \in L^{m,M}(\Omega, \Sigma)$, $M > 0$ and suppose that $\Sigma = \Sigma^+ \cup \Sigma^-$,* where

$$\Sigma^\pm = \{\varrho \in \Sigma; \operatorname{ind}(p_m, \varrho) = \pm M\}.$$

Then there exist properly supported operators $F, F^+, F^-\colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ with the following properties:

(1.6) *F is continuous $H_s^{\mathrm{loc}}(\Omega) \to H_{s+m-M/2}^{\mathrm{loc}}(\Omega)$ and $F^\pm$ are continuous $H_s^{\mathrm{loc}}(\Omega) \to H_s^{\mathrm{loc}}(\Omega)$ for all $s \in \mathbf{R}$.*

(1.7) $F^+ + FP \equiv I$

(1.8) $F^- + PF \equiv I$

(1.9) $(F^+)^* \equiv F^+, \quad (F^-)^* \equiv F^-$

(1.10) $WF'(F) = \operatorname{diag}(T^*(\Omega) \setminus 0)$

(1.11) $WF'(F^\pm) = \operatorname{diag}(\Sigma^\pm)$

Finally if $\tilde{F}, \tilde{F}^\pm$ *are also continuous linear* $\mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ *and continuous:* $C^\infty(\Omega) \to C^\infty(\Omega)$, *satisfying* (1.7) — (1.9) *with* $F, F^\pm$ *replaced by* $\tilde{F}, \tilde{F}^\pm$, *and*

$$WF'(\tilde{F}^+) \cap WF'(\tilde{F}^-) = \emptyset, \quad WF'(\tilde{F}) \cap (\Sigma^+ \times \Sigma^-) = \emptyset$$

then $F \equiv \tilde{F}$, $F^\pm \equiv \tilde{F}^\pm$.

As we shall see below, this result is a generalization of Theorem 1.1 in [4] and as in that paper we have the interpretation that F^+ and F^- are in an approximate sense the orthogonal projections onto the nullspace of P and along the image of P respectively. In fact, from (1.6)—(1.11) it follows easily that

$$PF^+ \equiv F^-P \equiv 0, \tag{1.12}$$

$$(F^+)^2 \equiv F^+, \quad (F^-)^2 \equiv F^-, \tag{1.13}$$

$$F^+F \equiv FF^- \equiv 0. \tag{1.14}$$

As in [4], section 5 one can prove, using Theorem 1.2 and functional analysis, that when Ω is compact and $\Sigma^+ = \emptyset$ (or $\Sigma^- = \emptyset$) then one can find $F, F^\pm$ such that (1.6)—(1.9) are valid with equality instead of "$\equiv$" and such that F^+ (F^-) has finite rank. As proved at the end of section 2 of [4], if P is as in Theorem 1.2 and $\Gamma \subset \Sigma^+$ is a closed cone, then there exists $u \in H_0^{\mathrm{loc}}(\Omega)$ such that $Pu \in C^\infty(\Omega)$ and $WF(u) = \Gamma$, provided that P is properly supported or that Γ has a compact projection in the base. For the proof of the uniqueness part of Theorem 1.2 we refer to [4].

To see that our Theorem 1.2 is a generalization of Theorem 1.1 in [4], it suffices to prove the following proposition, which is a special case of the results of Moyer [18].

PROPOSITION 1.3. *Let* $P \in L^m(\Omega)$ *be a classical pseudodifferential operator with principal symbol* p_m *positively homogeneous of degree* m. *Suppose that the Poisson bracket* $\{\operatorname{Re} p_m, \operatorname{Im} p_m\}$ *is* $\neq 0$ *when* p_m *vanishes. If*

$$\Sigma = \{(x, \xi) \in T^*(\Omega) \setminus 0; \ p_m(x, \xi) = 0\},$$

then Σ *is a closed conic non-involutive submanifold of codimension* 2. *Moreover* $P \in L_c^{m,1}(\Omega, \Sigma)$ *and*

$$\operatorname{ind}(p_m, \varrho) = -\operatorname{sign}(\{\operatorname{Re} p_m, \operatorname{Im} p_m\}) \quad \textit{for all} \quad \varrho \in \Sigma.$$

Proof. Recall the general identity

$$\{u, v\} = \sigma(H_u, H_v) = \langle H_u, dv\rangle, \quad u, v \in C^\infty(T^*(\Omega)),$$

where H_u and H_v are the Hamilton fields of u and v. Let e_1, e_2 be the Hamiltonian vectors of $\operatorname{Re} p_m$ and $\operatorname{Im} p_m$ at a point $\varrho \in \Sigma$. Then $\sigma(e_1, e_2) \neq 0$ so $d(\operatorname{Re} p_m)$ and $d(\operatorname{Im} p_m)$ are linearly independent. Thus Σ is a closed manifold of codimension 2 and (e_1, e_2) is a basis in $N_\varrho(\Sigma)$. σ is clearly non-degenerate on $T_\varrho(\Sigma)$ since it is non-degenerate on $N_\varrho(\Sigma)$ and we have thus proved the statements about Σ. With $t = xe_1 + ye_2$, we get

$$\begin{aligned}(1.15) \qquad a(t) &= \langle t, dp_m\rangle = \langle xe_1 + ye_2, d\operatorname{Re} p_m + i\, d\operatorname{Im} p_m\rangle = \\ &= ix\{\operatorname{Re} p_m, \operatorname{Im} p_m\} + y\{\operatorname{Im} p_m, \operatorname{Re} p_m\} = i\{\operatorname{Re} p_m, \operatorname{Im} p_m\}(x + iy).\end{aligned}$$

From this it is clear that $\operatorname{ind}(p_m, \varrho) = -\operatorname{sign} \sigma(e_1, e_2) = -\operatorname{sign}\{\operatorname{Re} p_m, \operatorname{Im} p_m\}$ and the proof is complete.

A natural question to ask is what happens when $|\operatorname{ind} p_m| < M$ on Σ. In this case the situation becomes more complicated and all the terms $p_{m-j/2}$ in (1.1) with $j \leq M$ play an essential role. However we still have a non-hypoellipticity result when M is arbitrary and $\operatorname{codim}(\Sigma) = 2$, and we have a rather precise result when $M = 2$ and $\operatorname{codim}(\Sigma)$ is arbitary.

THEOREM 1.4. *Let $\Sigma \subset T^*(\Omega) \setminus 0$ be a closed conic non-involutive manifold of codimension 2, and let $P \in L^{m,M}(\Omega, \Sigma)$. If $\varrho \in \Sigma$ and $\operatorname{ind}(p_m, \varrho) > 0$, then there exists $u \in H_0^{\mathrm{comp}}(\Omega)$ so that $WF(u) = \{\lambda\varrho;\, \lambda > 0\}$ and $Pu \in C^\infty(\Omega)$.*

A consequence of the theorem is that P is not hypoelliptic. When $M = 2$ this result is of course contained in Theorem 1.2. In this case Theorem 1.4 also intersects with the result of Cardoso and Trèves [2].

We next formulate our additional result for $M = 2$. In the rest of this section we assume that Σ is a closed conic non-involutive submanifold of $T^*(\Omega) \setminus 0$ of any codimension. When $P \in L_c^{m,2}(\Omega, \Sigma)$ then $a_\varrho(t)$ becomes a quadratic form, so we prefer to write $a_\varrho(t, t)$ instead. The following lemma is a consequence of Lemma 3.1 in section 3.

LEMMA 1.5. *Let $P \in L_c^{m,2}(\Omega, \Sigma)$ and suppose that $\operatorname{ind}(p_m) = 0$ on Σ if* $\operatorname{codim}(\Sigma) = 2$. *Then there exists a C^∞ function*

$$\Sigma \ni \varrho \mapsto z(\varrho) \in \mathbf{C}$$

such that $|z(\varrho)| = 1$ and such that $\operatorname{Re}(z(\varrho)a_\varrho(t, t))$ is positive definite on $N_\varrho(\Sigma)$ for all $\varrho \in \Sigma$.

For every real linear space L we denote by $\tilde{L}$ its complexification. We consider σ and a_ϱ as bilinear forms on $\tilde{T}_\varrho(T^*(\Omega)) \times \tilde{T}_\varrho(T^*(\Omega))$. Since σ is non-degenerate, we can define for every $\varrho \in \Sigma$ a linear map $A_\varrho: \tilde{T}_\varrho(T^*(\Omega)) \to \tilde{T}_\varrho(T^*(\Omega))$ by the equation

$$\sigma(u, A_\varrho v) = a_\varrho(u, v), \quad u, v \in \tilde{T}_\varrho(T^*(\Omega)). \tag{1.16}$$

This definition is inspired by Melin [16]. It is easy to verify that A_ϱ maps $\tilde{N}_\varrho(\Sigma)$ into itself and that A_ϱ is zero on $\tilde{T}_\varrho(\Sigma)$. We also note that A_ϱ is antisymmetric with respect to σ and that $\operatorname{Im} z(\varrho) \cdot \lambda \neq 0$ for every eigenvalue λ of $A_\varrho: \tilde{N}_\varrho(\Sigma) \to \tilde{N}_\varrho(\Sigma)$. (See Lemma 3.8.) For every $\lambda \in \mathbf{C}$ we denote by $\nu(\lambda, \varrho)$ the multiplicity of λ as a zero of $\det(A_\varrho - \lambda I)$. It is easy to see that the set $K_\varrho = \{\lambda \in \mathbf{C}; \operatorname{Im}(\nu(\lambda, \varrho) z(\varrho)\lambda) > 0\}$ does not depend on the choice of the function $z(\varrho)$ in Lemma 1.5.

When $P \in L_c^{m,2}(\Omega, \Sigma)$ we can define the subprincipal symbol $S_P \in C^\infty(\Sigma)$ in local coordinates by the equation

$$S_P(x, \xi) = p_{m-1}(x, \xi) - (2i)^{-1} \sum_1^n \frac{\partial^2 p_m(x, \xi)}{\partial x_j \partial \xi_j}.$$

In section 6 we shall give a very easy proof of the fact that S_P is invariantly defined on Σ. Note that S_P is invariantly defined on $T^*(\Omega) \setminus 0$ if we consider P as operating on the densities of order $1/2$ instead of the functions. This has been proved by Duistermaat and Hörmander [3].

THEOREM 1.6. *Let* $P \in L_c^{m,2}(\Omega, \Sigma)$ *be such that* $\operatorname{ind}(p_m) = 0$ *on* Σ *when* $\operatorname{codim}(\Sigma) = 2$ *and assume that*

$$S_P(\varrho) + (i)^{-1} \sum_{\lambda \in K_\varrho} (\nu(\lambda, \varrho) + 2k_\lambda)\lambda \neq 0 \tag{1.17}$$

for all $\varrho \in \Sigma$ *and integers* $k_\lambda \geq 0$. *Then there exists an operator* $F: \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ *such that* F *is continuous* $H_s^{\mathrm{loc}}(\Omega) \to H_{s+m-1}^{\mathrm{loc}}(\Omega)$ *for all* s *and* $PF \equiv FP \equiv I$. *Moreover* $WF'(F) = \operatorname{diag}(T^*(\Omega) \setminus 0)$.

When the assumptions of the theorem are satisfied we have local solvability and hypoellipticity for P. Moreover, when P is properly supported we have an a priori estimate of the form

$$\|u\|_{m-1} \leq C_K(\|Pu\|_0 + \|u\|_{m-2}), \quad u \in C_0^\infty(K), K \subset\subset \Omega. \tag{1.18}$$

The condition (1.17) is certainly not necessary for hypoellipticity or local solvability. However we have the following converse of Theorem 1.6.

THEOREM 1.7. *Suppose that* $P \in L^{m,2}(\Omega, \Sigma)$ *and that* $\operatorname{ind}(p_m) = 0$ *on* Σ, *when* $\operatorname{codim}(\Sigma) = 2$. *If the condition* (1.17) *is not valid at the point* $\varrho \in \Sigma$, *then for every* $\varepsilon > 0$ *and* $0 < \delta < 1/2$ *there is a* $u \in H_{m-1-\delta}^{\mathrm{loc}}(\Omega) \setminus H_{m-1-\delta+\varepsilon}^{\mathrm{loc}}(\Omega)$ *such that*

$WF(u) = \{\lambda\varrho;\ \lambda > 0\}$ and $Pu \in H_0^{\mathrm{loc}}(\Omega)$. Moreover $Qu \in H^{\mathrm{loc}}_{m-1-\delta+1/2}(\Omega)$ for all $Q \in L^{0,1}(\Omega, \Sigma)$.

It follows in particular that when (1.17) is not valid at a point $\varrho \in \Sigma$, there can be no a priori estimate of the form

$$\|u\|_{m-\delta} \leq C_K(\|Pu\|_0 + \|u\|_{m-1-\delta}),\quad u \in C_0^\infty(K),\ K \subset\subset \Omega, \tag{1.19}$$

when $\delta < 3/2$ and K contains the projection of ϱ. Note the similarity of this result with the result of Hörmander [12], stating that if P is a non-elliptic classical pseudodifferential operator of order m, then (1.19) can not be valid with $\delta < 1/2$. The study of (1.19) with $\delta = 3/2$ is certainly delicate. It is however possible to find implicit necessary and sufficient conditions for this estimate as well as for hypoellipticity and other properties. See Remark 5.11.

Remark 1.8. Duistermaat has pointed out to the author that when p_m is real valued then A_ϱ can be given by the formula:

$$2A_\varrho = \left(\frac{d}{dt}\,(d\varphi_t)_\varrho\right)_{t=0}.$$

Here $\varphi_t\colon T^*(\Omega) \setminus 0 \to T^*(\Omega) \setminus 0$, $t \in \mathbf{R}$, is the group of germs of diffeomorphisms at ϱ, generated by the Hamilton field of p_m and $(d\varphi_t)_\varrho$ is the differential of φ_t at ϱ. Note that $\varphi_t(\varrho) = \varrho$, since the Hamilton field is zero at $\varrho \in \Sigma$.

Our main inspiration for Theorem 1.6 has been the work [16] by Melin. He studies half estimates for operators with non-negative principal part. His necessary and sufficient condition is expressed as an inequality which contains the subprincipal symbol and the quantity $\sum_{\mathrm{Im}\,\lambda>0} \nu(\lambda, \varrho)\lambda$. In section 3 we shall use the idea of Melin to apply linear canonical transformations in the study of certain second order differential operators with polynomial coefficients. We have also been influenced by a result of Radkevič [20] which concerns an operator with real principal part and purely imaginary subprincipal part.

Višik-Grušin [26, 27] and Grušin [6—9] have studied operators on $\mathbf{R}^n$ for which the characteristic set is in a special position. In section 5 we shall apply a refinement of their techniques which is based on the use of vector valued pseudodifferential operators. Such operators have been used before by Trèves [25] and more recently by Grušin [10].

We also note that in the case when $\operatorname{codim}(\Sigma) = 2$, Boutet de Monvel and Trèves [1] have recently (independently) given a necessary and sufficient condition for the a priori estimate (1.18). Their condition is of course equivalent to our condition (1.17). See Remark 6.6.

Remark 1.9. Theorem 1.2 has been announced in [23] in a less precise form. Theorems 1.6, 1.7 have been announced in [24]. We think it is possible with small modifications of the proof to generalize Theorem 1.6 to the case when the restriction of σ to Σ has constant rank. It is also possible to extend the theorem to the case when $P \in L^{m,2}(\Omega, \Sigma)$ so that $p_{m-1/2}$ in (1.1) may be non-zero. See Remark 3.14. We also want to point out that the homogeneity assumption in Definition 1.1 for the terms $p_{m-j/2}$ with $j \geq M+1$ is introduced only for formal reasons and is not necessary in our results.

The plan of the paper is the following: In section 2 we make some preliminaries about ordinary differential operators. In section 3 we calculate the spectrum of certain elliptic second order operators associated with quadratic forms. In section 4, we give some preliminaries about vector valued pseudodifferential operators. In section 5 we combine the results of sections 2 and 4 to give microlocal versions of Theorems 1.2 and 1.4 when Σ is in a special position. Similarly the results of section 3 and 4 give microlocal versions of Theorems 1.6 and 1.7. Finally in section 6 we apply Hörmander's theory of Fourier integral operators to complete the proofs. At the end of that section we also give an example for Theorem 1.6.

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2. Preliminaries about differential operators with polynomial coefficients

All the results in this section can be found in Grušin [6]. We consider an operator on $\mathbf{R}^n$

$$P(x, D) = \sum_{|\alpha+\beta| \leq M} a_{\alpha\beta} x^\alpha D^\beta, \tag{2.1}$$

where $a_{\alpha\beta} \in \mathbf{C}$. Assume that

$$\sum_{|\alpha+\beta|=M} a_{\alpha\beta} x^\alpha \xi^\beta \neq 0 \quad \text{when} \quad 0 \neq (x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n. \tag{2.2}$$

In particular P is elliptic in the usual sense. Let $B^M(\mathbf{R}^n) \subset \mathcal{S}'(\mathbf{R}^n)$ be the space with the norm:

$$\|u\|^2_{B^M} = \sum_{|\alpha+\beta| \leq M} \|x^\alpha D^\beta u\|^2_{L^2(\mathbf{R}^n)}, \tag{2.3}$$

so that $x^\alpha D^\beta \colon B^M(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is a continuous operator when $|\alpha + \beta| \leq M$. Grušin has proved that it is even a compact operator when $|\alpha + \beta| < M$.

THEOREM 2.1. (Grušin). *$P \colon B^M(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ has finite index. If $u \in L^2(\mathbf{R}^n)$ and $Pu \in \mathcal{S}(\mathbf{R}^n)$ then $u \in \mathcal{S}(\mathbf{R}^n)$.*

Grušin proves this theorem in a slightly more general form by constructing a parametrix, which is a pseudodifferential operator for which the symbol behaves well for large x.

In the case when $n = 1$ we also have

LEMMA 2.2. (Grušin). *Let* $m^{\pm}$ *be the number of zeros of the equation*

$$\sum_{\alpha+\beta=M} a_{\alpha\beta}\zeta^{\beta} = 0 \tag{2.4}$$

for which $\operatorname{Im} \zeta \gtrless 0$, *so that* $M = m^+ + m^-$. *Then the index of* $P\colon B^M(\mathbf{R}) \to L^2(\mathbf{R})$ *is* $m^+ - m^-$.

The proof of this lemma is simple. We note that P is modulo a compact operator of the form

$$P(x, D) = a_{0M} \prod_{j=1}^{M} (D - \gamma_j x), \quad \operatorname{Im} \gamma_j \neq 0.$$

Moreover it is easy to verify that the index of

$$D - \gamma_j x\colon B^k(\mathbf{R}) \to B^{k-1}(\mathbf{R})$$

is ± 1 when $\operatorname{Im} \gamma_j \gtrless 0$, for all $k \geq 1$.

The lemma is of course a special case of the Atiyah-Singer index formula.

3. Second order operators with polynomial coefficients

In this section we shall calculate the spectrum of operators of the form

$$\sum_{|\alpha+\beta|=2} a_{\alpha\beta} x^{\alpha} D^{\beta}, \quad a_{\alpha\beta} \in \mathbf{C}.$$

First we prove a well known lemma, from which Lemma 1.5 follows.

LEMMA 3.1. *Let* $a(x, x)$ *be a complex valued quadratic form on* $\mathbf{R}^n$ *such that* $a(x, x) \neq 0$ *when* $x \neq 0$. *In the case* $n = 2$ *we assume that* $\operatorname{var} \arg a(x, x) = 0$ *along every closed curve in* $\mathbf{R}^2 \setminus \{0\}$. *Then there exists* $z \in \mathbf{C} \setminus \{0\}$ *such that* $\operatorname{Re} (za(x, x))$ *is positive definite.*

Proof. The case $n = 1$ is trivial. In the case when $n = 2$, we can find coordinates x_1, x_2 so that

$$a(x, x) = (x_1 - ix_2)(\alpha x_1 + \beta x_2)$$

for some $\alpha, \beta \in \mathbf{C}$. Putting $\zeta = x_1 + ix_2$ we can write

$$a(x, x) = z^{-1}\bar{\zeta}(\zeta + \gamma\bar{\zeta})$$

for some $z^{-1} \in \mathbf{C} \setminus \{0\}$ and $\gamma \in \mathbf{C}$. The conditions in the lemma imply that $|\gamma| < 1$, so it is clear that

$$\operatorname{Re}(za(x, x)) = \operatorname{Re}\left(|\zeta|^2\left(1 + \gamma \frac{\bar{\zeta}}{\zeta}\right)\right) > 0$$

for $x \neq 0$ and the lemma is proved in the case $n = 2$.

We now consider the case $n \geq 3$. We have to prove that $\operatorname{Im} a = \{a(x, x); x \in \mathbf{R}^n\}$ is a closed convex proper cone in $\mathbf{C}$. Since $\operatorname{Im} a$ is evidently closed, it suffices to prove that for all $x, y \in \mathbf{R}^n \setminus \{0\}$ we have $a(x, x) \neq -a(y, y)$. Now $\mathbf{R}^n \setminus \{0\}$ is simply connected, so the restriction of a to a 2-dimensional space, containing the vectors x and y, must satisfy the conditions in the lemma for the case $n = 2$. This case has already been settled so the proof is complete.

Now let $a((x, \xi), (y, \eta))$ be a symmetric bilinear form on $(\mathbf{C}^n \oplus \mathbf{C}^n) \times (\mathbf{C}^n \oplus \mathbf{C}^n)$. We assume that

$$a(x, \xi, x, \xi) \neq 0 \text{ for all } 0 \neq (x, \xi) \in \mathbf{R}^n \oplus \mathbf{R}^n. \tag{3.1}$$

When $n = 1$ we also make the assumption:

$$\text{var arg } a(x, \xi, x, \xi) = 0 \text{ along every closed curve in } \mathbf{R}^n \oplus \mathbf{R}^n \setminus \{0\}. \tag{3.2}$$

By Lemma 3.1 we know that there exists a number $z = z_a$ such that $|z_a| = 1$ and

$$\operatorname{Re} za(x, \xi, x, \xi) > 0 \text{ for all } 0 \neq (x, \xi) \in \mathbf{R}^n \oplus \mathbf{R}^n. \tag{3.3}$$

Following Melin, we consider the linear map $A: \mathbf{C}^n \oplus \mathbf{C}^n \to \mathbf{C}^n \oplus \mathbf{C}^n$ defined by the equation

$$\sigma((x, \xi), A(y, \eta)) = a((x, \xi), (y, \eta)), \quad (x, \xi), (y, \eta) \in \mathbf{C}^n \oplus \mathbf{C}^n, \tag{3.4}$$

where

$$\sigma((x, \xi), (y, \eta)) = \langle y, \xi \rangle - \langle x, \eta \rangle$$

is the symplectic bilinear form. Note that A is antisymmetric with respect to σ. For every $\lambda \in \mathbf{C}$ we let $V_\lambda \in \mathbf{C}^n \oplus \mathbf{C}^n$ be the largest subspace where $A - \lambda I$ is nilpotent and we put $\nu(\lambda) = \dim(V_\lambda)$. In other words, V_λ is the space of generalized eigenvectors with eigenvalue λ.

We recall that a complex subspace $\Lambda \subset \mathbf{C}^n \oplus \mathbf{C}^n$ of dimension n is called a Lagrangian plane if σ vanishes identically on Λ. The following definition is due to Hörmander [14, p. 153].

Definition 3.2. We say that a Lagrangian plane $\Lambda \subset \mathbf{C}^n \oplus \mathbf{C}^n$ is positive if

$$-i\sigma((x, \xi), \overline{(x, \xi)}) > 0 \text{ for all } 0 \neq (x, \xi) \in \Lambda.$$

Note that if Λ is a positive Lagrangian plane then

$$\Lambda \cap (\mathbf{R}^n \oplus \mathbf{R}^n) = 0.$$

PROPOSITION 3.3. *We have* $\nu(\lambda) = \nu(-\lambda)$ *for all* λ *and*

$$\mathbf{C}^n \oplus \mathbf{C}^n = \bigoplus_{\operatorname{Im} z_a \lambda > 0} (V_\lambda \oplus V_{-\lambda}).$$

Moreover

$$V_a^+ = \bigoplus_{\operatorname{Im} z_a \lambda > 0} V_\lambda$$

is a positive Lagrangian plane.

The proof will be given below.

Remark 3.4. It follows in particular that $\{\lambda;\, \nu(\lambda) \neq 0,\, \operatorname{Im} z_a\lambda > 0\}$ is independent of the choice of z_a satisfying (3.4).

The fact that V_a^+ is positive implies that the projection π_x: $V_a^+ \ni (x, \xi) \mapsto x \in \mathbf{C}^n$ is surjective. Therefore

$$V_a^+ = \{(x, B^+x);\; x \in \mathbf{C}^n\}, \tag{3.5}$$

where B^+ is symmetric and $\operatorname{Im} B^+ > 0$. We put

$$b^+(x, x) = \langle x, B^+x\rangle, \quad x \in \mathbf{R}^n. \tag{3.6}$$

To the quadratic form a we associate the differential operator $a(x, D, x, D)$, which is obtained by replacing ξ in $a(x, \xi, x, \xi)$ by $D = (D_1, \ldots, D_n)$. We do this symmetrically so that the coefficient of x_jD_j is equal to the coefficient of D_jx_j. In particular when a is real, the operator $a(x, D, x, D)$ is formally self-adjoint.

Theorem 2.1 (by Grušin) implies that

(3.7) $a(x, D, x, D)$: $B^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ has finite index, and

(3.8) every generalized eigenfunction of $a(x, D, x, D)$ which belongs to $L^2(\mathbf{R}^n)$ belongs to $\mathcal{S}(\mathbf{R}^n)$.

(We say that u is a generalized eigenfunction if there exists $\lambda \in \mathbf{C}$ and $p \in \mathbf{Z}^+$ such that $(a(x, D, x, D) - \lambda I)^p u = 0$.)

Our main result in this section is now:

THEOREM 3.5.

(i) *The index of* $a(x, D, x, D)$: $B^2 \to L^2$ *is zero.*

(ii) *The spectrum of* $a(x, D, x, D)$: $L^2 \to L^2$ *is*

$$S_a = \Big\{-i \sum_{\operatorname{Im} z_a \nu(\lambda)\lambda > 0} (\nu(\lambda) + 2k_\lambda)\lambda\,;\; k_\lambda \in \mathbf{Z}^+ \cup \{0\}\Big\}.$$

(iii) *The linear hull of the generalized eigenfunctions in* L^2 *is* $P(\mathbf{R}^n)e^{i\,b^+(x,x)/2}$, *where* $P(\mathbf{R}^n)$ *denotes the space of complex polynomials in* n *variables, considered as functions on* $\mathbf{R}^n$.

Remark 3.6. Let K be a compact set and suppose that $a_t(x, D, x, D)$ depends continuously on $t \in K$ and satisfies the conditions of the theorem. If $K \ni t \mapsto \lambda_t \in \mathbf{C}$ is a continuous function and $\lambda_t \notin S_{a_t}$ for all t, then $a_t(x, D, x, D) - \lambda_t\colon B^2 \to L^2$ has a uniformly bounded inverse $E_t, t \in K$.

Melin [16] has considered the case when $a(x, \xi, x, \xi)$ is positive semidefinite. His result is that for $\mu \in \mathbf{R}$ the half estimate

$$(u, u)_{L^2} \leq C((a(x, D, x, D) - \mu)u, u)_{L^2}, \quad u \in C_0^\infty(\mathbf{R}^n),$$

is valid for some constant C if and only if

$$\mu \leq i^{-1} \sum_{\operatorname{Im} \lambda > 0} \nu(\lambda)\lambda.$$

In this case all the eigenvalues of A are purely imaginary. In the proof Melin uses real canonical transformations to reduce to the operator $|x|^2 + |D|^2$. In our proof we will have to use complex canonical transformations and we also have to consider the fundamental role of V_a^+.

In the proof of Theorem 3.5 and Proposition 3.3 it is no restriction to assume that $\operatorname{Re} a > 0$ so that $z_a = 1$. Clearly the index of $a(x, D, x, D)\colon B^2 \to L^2$ is 0 in the self-adjoint case, that is, when a is real. Now the homotopy $a_t(x, D, x, D) = (1 - t)a(x, D, x, D) + t(\operatorname{Re} a)(x, D, x, D)$ shows that (i) in Theorem 3.5 is true in the general case.

Next we shall analyze the structure of A and $a(x, \xi, x, \xi)$.

LEMMA 3.7. *If* $\lambda \neq -\mu$ *then* V_λ *and* V_μ *are orthogonal with respect to* σ.

Proof. If u is a generalized eigenvector of A with eigenvalue λ, we let m_u be the smallest number m such that $(A - \lambda)^m u = 0$. Suppose that $u \in V_\lambda, v \in V_\mu$, $\lambda + \mu \neq 0$. If $m_u = m_v = 1$, we have

$$\lambda\sigma(u, v) = \sigma(Au, v) = -\sigma(u, Av) = -\mu\sigma(u, v),$$

so $\sigma(u, v) = 0$. By induction over $m_u + m_v$ one shows easily that $\sigma(u, v) = 0$ for arbitrary $u \in V_\lambda, v \in V_\mu$.

LEMMA 3.8. $V_\lambda \neq 0 \Rightarrow \operatorname{Im} \lambda \neq 0$.

Proof. Let $u \in V_\lambda$ be an eigenvector. Then

$$\lambda\sigma(u, \bar{u}) = \sigma(Au, \bar{u}) = -a(u, \bar{u}).$$

Since $\operatorname{Re} a > 0$ and $-i\sigma(u, \bar{u})$ is real we obtain $\operatorname{Im} \lambda \neq 0$. Note also that $-i\sigma(u, \bar{u})$ has the same sign as $\operatorname{Im} \lambda$.

Since σ is non-degenerate, Lemmas 3.7 and 3.8 imply that V_λ and $V_{-\lambda}$ have the same dimension and that σ is non-degenerate as a bilinear form on $V_\lambda \times V_{-\lambda}$. Moreover we have

$$\mathbf{C}^n \oplus \mathbf{C}^n = \bigoplus_{\operatorname{Im} \lambda > 0} (V_\lambda \oplus V_{-\lambda}). \tag{3.9}$$

To complete the proof of Proposition 3.3 we must prove that V_a^+ is a positive Lagrangian plane. Recall the definition of V_a^+:

$$V_a^+ = \bigoplus_{\operatorname{Im} \lambda > 0} V_\lambda. \tag{3.10}$$

V_a^+ is clearly a Lagrangian plane in view of Lemma 3.7. To prove that it is positive we shall apply a deformation argument. First note that $V_a^+ \cap (\mathbf{R}^n \oplus \mathbf{R}^n) = 0$, since $a(u, u) = \sigma(u, Au) = 0$ on V_a^+ in view of Lemma 3.7.

Suppose that $[0, 1] \ni t \mapsto W_t$ is a continuous family of Lagrangian planes such that $W_t \cap (\mathbf{R}^n \oplus \mathbf{R}^n) = 0$. Then all the W_t are positive or none is. In fact, suppose that W_{t_0} is positive and that W_{t_1} is not. Then there is a t_2 between t_0 and t_1 such that $-i\sigma(u, \bar{u})$ is positive semidefinite but not positive definite on W_{t_2}. Take $0 \neq v \in W_{t_2}$ such that $\sigma(v, \bar{v}) = 0$. The Schwarz inequality

$$|-i\sigma(u, \bar{v})|^2 \leq (-i\sigma(u, \bar{u}))(-i\sigma(v, \bar{v})), \quad u, v \in W_{t_2},$$

implies that $\sigma(u, \bar{v}) = 0$ for all $u \in W_{t_2}$. Since W_{t_2} is Lagrangian we conclude that $\bar{v} \in W_{t_2}$. Thus either $v + \bar{v}$ or iv is a non-zero real vector in W_{t_2}, which contradicts the assumption that $W_{t_2} \cap (\mathbf{R}^n \oplus \mathbf{R}^n) = 0$.

Now let $a_0(x, \xi, x, \xi) = \sum_{j=1}^n x_j^2 + \xi_j^2$ and let A_0 be the corresponding map, defined by (3.4). Then

$$A_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

has the eigenvalues $+i$ and $-i$ and $V_{a_0}^+$ consists entirely of eigenvectors. Thus $V_{a_0}^+$ is positive by an observation in the proof of Lemma 3.8. Put $a_t = (1-t)a_0 + ta$, $0 \leq t \leq 1$ and let A_t and $V_{a_t}^+$ be the corresponding operators and Lagrangian planes defined by (3.4), (3.10). Then $V_{a_t}^+ \cap (\mathbf{R}^n \oplus \mathbf{R}^n) = 0$ and $[0, 1] \ni t \mapsto V_{a_t}^+$ is a continuous family. In fact, $P_t = (2\pi i)^{-1} \int_\gamma (A_t - z)^{-1} dz$ is a projection onto $V_{a_t}^+$ changing continuously with t if γ is the boundary of a disc in the upper half plane containing all eigenvalues there. Since $V_{a_0}^+$ is positive we conclude that $V_{a_1}^+$ is also positive and the proof of Proposition 3.3 is complete.

For every λ with $\operatorname{Im} \lambda > 0$ we choose a basis $u_{1,\lambda}, \ldots, u_{\nu(\lambda),\lambda}$ in V_λ such that A takes the Jordan form:

$$(3.11)\qquad Au_{j,\lambda} = \lambda u_{j,\lambda} + \gamma(j,\lambda)u_{j+1,\lambda},\quad 1 \le j \le \nu(\lambda) - 1,$$
$$Au_{\nu(\lambda),\lambda} = \lambda u_{\nu(\lambda),\lambda}.$$

Here $\gamma(j, \lambda)$ is a function with values 0, 1 and we define $\gamma(0, \lambda) = \gamma(\nu(\lambda), \lambda) = 0$. Let $v_{1,\lambda}, \ldots, v_{\nu(\lambda),\lambda}$ be the dual basis in $V_{-\lambda}$ with respect to σ, so that

$$(3.12)\qquad \sigma(u_{j,\lambda}, v_{k,\mu}) = -\,\delta_k^j \cdot \delta_\mu^\lambda,\quad \sigma(u_{j,\lambda}, u_{k,\mu}) = 0,\quad \sigma(v_{j,\lambda}, v_{k,\mu}) = 0.$$

Then

$$(3.13)\qquad \sigma(u_{j,\lambda}, Av_{k,\mu}) = -\,\sigma(Au_{j,\lambda}, v_{k,\mu}) = -\,\sigma(\lambda u_{j,\lambda} + \gamma(j,\lambda)u_{j+1,\lambda}, v_{k,\mu})$$
$$= (\lambda\delta_k^j + \gamma(j,\lambda)\delta_k^{j+1})\delta_\mu^\lambda.$$

The equations (3.12) express that $(u_{j,\lambda}, v_{j,\lambda})$ is a symplectic basis in $\mathbf{C}^n \oplus \mathbf{C}^n$. We let $X_{j,\lambda}$, $\Xi_{j,\lambda}$ be the corresponding coordinates. (3.13) and (3.4) imply:

$$(3.14)\quad a(u,u) = 2\sum_{\operatorname{Im}\lambda>0}\Big(\sum_1^{\nu(\lambda)} \lambda X_{j,\lambda}\Xi_{j,\lambda} + \sum_1^{\nu(\lambda)-1}\gamma(j,\lambda)X_{j,\lambda}\Xi_{j+1,\lambda}\Big),\quad u = \sum X_{j,\lambda}u_{j,\lambda} + \Xi_{j,\lambda}v_{j,\lambda}.$$

We relabel the standard real symplectic coordinates x_j, ξ_j and write $x_{j,\lambda}$, $\xi_{j,\lambda}$ instead, where $1 \le j \le \nu(\lambda)$, $\operatorname{Im}\lambda > 0$. Let a' be the quadratic form on $\mathbf{C}^n \oplus \mathbf{C}^n$ defined in these coordinates by

$$(3.15)\qquad a'(u,u) = 2\sum_{\operatorname{Im}\lambda>0}\Big(\sum_1^{\nu(\lambda)} \lambda x_{j,\lambda}\xi_{j,\lambda} + \sum_1^{\nu(\lambda)-1}\gamma(j,\lambda)x_{j,\lambda}\xi_{j+1,\lambda}\Big).$$

Then $a' \circ \mathcal{H} = a$, where $\mathcal{H}$ is the canonical transformation

$$(3.16)\qquad \mathbf{C}^n \oplus \mathbf{C}^n \ni (x, \xi) \mapsto (X, \Xi) \in \mathbf{C}^n \oplus \mathbf{C}^n.$$

Defining $V_a^- = \oplus_{\operatorname{Im}\lambda>0} V_{-\lambda}$, we see that

$$(3.17)\qquad \mathcal{H}(V_a^+) = \{(x, 0) \in \mathbf{C}^n \oplus \mathbf{C}^n\},$$
$$\mathcal{H}(V_a^-) = \{(0, \xi) \in \mathbf{C}^n \oplus \mathbf{C}^n\}.$$

We want to reduce the study of $a(x, D, x, D)$ to the study of $a'(x, D, x, D)$. In order to do that, we shall first write $\mathcal{H}$ as a product of elementary canonical transformations. To each one of these we shall associate a "Fourier integral operator". (Cf. Melin [16].)

Let B^+ be the symmetric matrix defined by (3.5) and let $\mathcal{H}_1$ be the canonical transformation:

$$(3.18)\qquad \mathcal{H}_1\colon (\mathbf{C}^n \oplus \mathbf{C}^n) \ni (x, \xi) \mapsto (x, \xi - B^+x) \in \mathbf{C}^n \oplus \mathbf{C}^n.$$

Then $\mathcal{H}_1(V_a^+) = \{(x, 0) \in \mathbf{C}^n \oplus \mathbf{C}^n\}$ and since $\mathcal{H}_1(V_a^-)$ is transversal to this space, we have $\mathcal{H}_1(V_a^-) = \{(B^-\xi, \xi) \in \mathbf{C}^n \oplus \mathbf{C}^n\}$ where B^- is some symmetric matrix. Let $\mathcal{H}_2$ be the canonical transformation:

$$(3.19)\qquad \mathcal{H}_2\colon \mathbf{C}^n \oplus \mathbf{C}^n \ni (x, \xi) \mapsto (x - B^-\xi, \xi) \in \mathbf{C}^n \oplus \mathbf{C}^n.$$

Then

$$\mathcal{H}_2\mathcal{H}_1(V_a^+) = \{(x, 0) \in \mathbf{C}^n \oplus \mathbf{C}^n\}, \tag{3.20}$$
$$\mathcal{H}_2\mathcal{H}_1(V_a^-) = \{(0, \xi) \in \mathbf{C}^n \oplus \mathbf{C}^n\}.$$

Put $\mathcal{H}_3 = \mathcal{H}(\mathcal{H}_2\mathcal{H}_1)^{-1}$ so that

$$\mathcal{H} = \mathcal{H}_3\mathcal{H}_2\mathcal{H}_1. \tag{3.21}$$

Then (3.17), (3.20) imply that $\mathcal{H}_3$ is of the form

$$\mathcal{H}_3\colon \mathbf{C}^n \oplus \mathbf{C}^n \ni (x, \xi) \mapsto (C^{-1}x, {}^tC\xi) \in \mathbf{C}^n \oplus \mathbf{C}^n,$$

for some invertible complex matrix C. In fact, $\mathcal{H}_3$ leaves the spaces $x = 0$ and $\xi = 0$ invariant.

Recall that $P(\mathbf{R}^n)$ is the space of complex polynomials regarded as functions on $\mathbf{R}^n$. To $\mathcal{H}_1$ we associate the operator $K_1\colon P(\mathbf{R}^n)e^{ib^+(x,x)/2} \to P(\mathbf{R}^n)$, defined by

$$K_1 u = e^{-ib^+(x,x)/2}u. \tag{3.22}$$

To $\mathcal{H}_2$ we associate the operator $K_2\colon P(\mathbf{R}^n) \to P(\mathbf{R}^n)$ given by

$$K_2 u = \mathcal{F}^{-1}(e^{ib^-(\xi,\xi)/2}\mathcal{F}u), \quad b^-(\xi, \xi) = \langle \xi, B^-\xi\rangle. \tag{3.23}$$

Here $\mathcal{F}$ is the Fourier transformation. To $\mathcal{H}_3$ we associate $K_3\colon P(\mathbf{R}^n) \to P(\mathbf{R}^n)$ given by

$$K_3 u(x) = u(Cx). \tag{3.24}$$

Finally we put

$$K = K_3K_2K_1\colon P(\mathbf{R}^n)e^{ib^+(x,x)/2} \to P(\mathbf{R}^n). \tag{3.25}$$

Note that K is a bijection.

We write

$$a'(x, D, x, D) = a_1(x, D, x, D) + a_2(x, D, x, D), \tag{3.26}$$

where

$$a_1(x, D, x, D) = \sum_{\operatorname{Im}\lambda>0}\sum_{1}^{\nu(\lambda)} \lambda(x_{j,\lambda}D_{j,\lambda} + D_{j,\lambda}x_{j,\lambda}), \tag{3.27}$$
$$a_2(x, D, x, D) = 2\sum_{\operatorname{Im}\lambda>0}\sum_{1}^{\nu(\lambda)-1} \gamma(j, \lambda)x_{j,\lambda}D_{j+1,\lambda}.$$

PROPOSITION 3.9. *$a'(x, D, x, D) \circ K = K \circ a(x, D, x, D)$ on $P(\mathbf{R}^n)e^{ib^+(x,x)/2}$.*

Proof. Let (b_{jk}) be the matrix of B^+. Then

$$D_{x_j}K_1u = D_{x_j}ue^{-ib^+(x,x)/2} = e^{-ib^+(x,x)/2}(D_{x_j}u - \sum_{k=1}^n b_{jk}x_ku).$$

If we let x_j also denote multiplication with x_j, our equation takes the simpler form:

$$D \circ K_1 = K_1 \circ (D - B^+x), \quad \text{where} \quad x = (x_1, \ldots, x_n), \quad D = (D_1, \ldots, D_n).$$

Moreover we have trivially

$$x \circ K_1 = K_1 \circ x.$$

If b is an arbitrary quadratic form on $\mathbf{C}^n \oplus \mathbf{C}^n$ we therefore have

$$\begin{aligned} b(x, D, x, D) \circ K_1 &= K_1 \circ b(x, D - B^+x, x, D - B^+x) = \\ &= K_1 \circ (b \circ \mathcal{H}_1)(x, D, x, D). \end{aligned} \tag{3.28}$$

Using the identities:

$$x \circ \mathcal{F} = \mathcal{F} \circ D, \quad \mathcal{F}^{-1} \circ x = D \circ \mathcal{F}^{-1}, \quad \mathcal{F} \circ x = -D \circ \mathcal{F}, \quad x \circ \mathcal{F}^{-1} = -\mathcal{F}^{-1} \circ D,$$

we get

$$b(x, D, x, D) \circ \mathcal{F}^{-1} = \mathcal{F}^{-1} \circ b(-D, x, -D, x),$$

$$b(-D, x, -D, x) \circ e^{ib^-(x,x)/2} = e^{ib^-(x,x)/2} \circ b(-D - B^-x, x, -D - B^-x, x),$$

$$b(-D - B^-x, x, -D - B^-x, x) \circ \mathcal{F} = \mathcal{F} \circ b(x - B^-D, D, x - B^-D, D).$$

Composing these three equations, we get

$$b(x, D, x, D) \circ K_2 = K_2 \circ (b \circ \mathcal{H}_2)(x, D, x, D). \tag{3.29}$$

Using that $D \circ K_3 = K_3 \circ ({}^tCD)$, $x \circ K_3 = K_3 \circ (C^{-1}x)$ we get

$$b(x, D, x, D) \circ K_3 = K_3 \circ (b \circ \mathcal{H}_3)(x, D, x, D). \tag{3.30}$$

Composition of (3.28)—(3.30) shows that

$$a'(x, D, x, D) \circ K = K \circ (a' \circ \mathcal{H})(x, D, x, D) = K \circ a(x, D, x, D)$$

and the proposition is proved.

If L is a complex linear space, $F: L \to L$ a linear operator and $\lambda \in \mathbf{C}$, we denote by $E(F, L, \lambda)$ the space of generalized eigenvectors with eigenvalue λ.

PROPOSITION 3.10. *Every monomial is a generalized eigenvector of*

$$a'(x, D, x, D): P(\mathbf{R}^n) \to P(\mathbf{R}^n).$$

The spectrum of a' *is* S_a (defined in Theorem 3.5) *and* $\dim (E(a', P(\mathbf{R}^n), \lambda)) < \infty$ *for all* $\lambda \in \mathbf{C}$.

Proof. We recall the definition of a_1 and a_2 in (3.27) and we write

$$a_1(x, D, x, D) = \sum_{\operatorname{Im} \lambda > 0} a^\lambda(x, D, x, D),$$

where

$$a^{\lambda}(x, D, x, D) = \sum_{1}^{\nu(\lambda)} \lambda(x_{j,\lambda}D_{j,\lambda} + D_{j,\lambda}x_{j,\lambda}).$$

If $p_\lambda(x)$ is a homogeneous polynomial of degree k_λ in the variables

$$x_{1,\lambda}, \ldots, x_{\nu(\lambda),\lambda}, \ \operatorname{Im} \lambda > 0,$$

we have

$$a^{\lambda}(x, D, x, D)p_\lambda(x) = -i(\nu(\lambda)\lambda + 2k_\lambda \lambda)p_\lambda(x).$$

If $p(x) = \prod_{\operatorname{Im}\lambda > 0} p_\lambda(x)$ is a product of such polynomials then

$$a_1(x, D, x, D)p(x) = -i\left(\sum_{\operatorname{Im} \nu(\lambda)\lambda > 0} \nu(\lambda)\lambda + 2k_\lambda \lambda\right)p(x).$$

Thus the monomials constitute a basis of eigenvectors in $P(\mathbf{R}^n)$ of a_1 and we have an explicit description of all the $E(a_1, P(\mathbf{R}^n), \lambda)$. Now it is easy to check that $a_2(x, D, x, D)$ is nilpotent on every $E(a_1, P(\mathbf{R}^n), \lambda)$ and since $a' = a_1 + a_2$ the proposition follows. Note that the real part of the eigenvalue tends to $+\infty$ with $\sum \nu(\lambda)k_\lambda$.

Combination of Proposition 3.9 and 3.10 gives:

PROPOSITION 3.11. *The operator* $a(x, D, x, D)$: $P(\mathbf{R}^n)e^{ib^+(x,x)/2} \to P(\mathbf{R}^n)e^{ib^+(x,x)/2}$ *has a basis of generalized eigenvectors. The spectrum is* S_a *defined in Theorem* 3.5 *and for every eigenvalue the corresponding space of generalized eigenfunctions is finite dimensional.*

(ii) and (iii) of Theorem 3.5 will follow from this proposition and the following two simple lemmas.

LEMMA 3.12. *If* $b(x, x)$ *is a quadratic form with* $\operatorname{Im} b > 0$, *then* $P(\mathbf{R}^n)e^{ib}$ *is dense in* $L^2(\mathbf{R}^n)$.

Proof. Suppose that $f \in L^2(\mathbf{R}^n)$ is orthogonal to $P(\mathbf{R}^n)e^{ib}$, that is

$$\int p(x)e^{ib(x,x)}\overline{f(x)}dx = 0$$

for all $p \in P(\mathbf{R}^n)$. Let F be the Fourier transform of $\overline{f(x)}e^{ib(x,x)}$. Then F is an entire function and by the above equation all the derivatives of F at the origin are zero. Thus $F = 0$ and therefore $f = 0$.

LEMMA 3.13. *Let* $u, v \in L^2(\mathbf{R}^n)$ *be generalized eigenfunctions of* a *and* a^* *respectively with the eigenvalues* λ *and* μ. *Then if* $\lambda \neq \bar{\mu}$, *we have* $(u, v) = 0$.

Proof. By Theorem 2.1, we know that $u, v \in \mathcal{S}(\mathbf{R}^n)$ and therefore the proof of Lemma 3.7 can be applied.

Now let L_λ be the closed linear hull in $L^2(\mathbf{R}^n)$ of $\bigcup_{\mu \neq \lambda} E(a, P(\mathbf{R}^n)e^{ib^+/2}, \mu)$. Then by Lemma 3.12

$$L_\lambda + E(a, P(\mathbf{R}^n)e^{ib^+/2}, \lambda) = L^2(\mathbf{R}^n)$$

since $E(a, P(\mathbf{R}^n)e^{ib^+/2}, \lambda)$ is finite dimensional. Thus

$$\operatorname{codim} L_\lambda \leq \dim E(a, P(\mathbf{R}^n)e^{ib^+/2}, \lambda).$$

By Lemma 3.13 we have

$$E(a^*, L^2(\mathbf{R}^n), \bar{\lambda}) \perp L_\lambda$$

so that

$$\dim E(a^*, L^2(\mathbf{R}^n), \bar{\lambda}) \leq \dim E(a, P(\mathbf{R}^n)e^{ib^+/2}, \lambda) \leq \dim E(a, L^2(\mathbf{R}^n), \lambda).$$

By symmetry we also have

$$\dim E(a, L^2(\mathbf{R}^n), \lambda) \leq \dim E(a^*, L^2(\mathbf{R}^n), \bar{\lambda}),$$

so it follows that

$$E(a, P(\mathbf{R}^n)e^{ib^+/2}, \lambda) = E(a, L^2(\mathbf{R}^n), \lambda).$$

This equality and Proposition 3.11 imply (ii) and (iii) of Theorem 3.6, which is therefore completely proved.

Remark 3.14. With the same methods we can determine the spectrum of $a(x, D, x, D) + f(x, D)$, where f is an arbitrary complex linear form. In fact, $a(x, \xi, x, \xi)$ is non-degenerate in view of Lemma 3.8 so there exists $(x_0, \xi_0) \in \mathbf{C}^n \oplus \mathbf{C}^n$ and $\lambda_0 \in \mathbf{C}$, such that

$$a(x, \xi, x, \xi) + f(x, \xi) = a(x - x_0, \xi - \xi_0, x - x_0, \xi - \xi_0) + \lambda_0.$$

Moreover to the affine canonical transformation $\mathcal{H}_4\colon (x, \xi) \mapsto (x + x_0, \xi + \xi_0)$ we can associate the operator K_4 defined by

$$(K_4 u)(x) = u(x - x_0)e^{i\langle x, \xi_0\rangle}.$$

We omit the details. If one likes to extend Theorem 1.6 to the case when $P \in L^{m,2}$ instead of $L_c^{m,2}$, then one has to study an operator of the form $a(x, D, x, D) + f(x, D)$ instead of $a(x, D, x, D)$. The reader will see this in section 5.

4. Preliminaries about vector valued pseudodifferential operators

Vector valued pseudodifferential operators have been used before by Trèves [25] and more recently by Grušin [10]. See also Sjöstrand [22]. Assume that V_1

and V_2 are complex Hilbert spaces and let $\mathcal{L}(V_1, V_2)$ be the Banach space of bounded linear operators $V_1 \to V_2$. We define $S^m_{\varrho,\delta}(\mathbf{R}^n \times \mathbf{R}^k; V_1, V_2)$ as the space of C^∞ functions p on $\mathbf{R}^n \times \mathbf{R}^k$ with values in $\mathcal{L}(V_1, V_2)$ such that for all $K \subset\subset \mathbf{R}^n$ and multiindices α, β there is a constant C, depending on K, α, β such that

$$\|D^\alpha_x D^\beta_\xi p(x, \xi)\|_{\mathcal{L}(V_1, V_2)} \leq C(1 + |\xi|)^{m+\delta|\alpha|-\varrho|\beta|}, \quad \text{for all} \quad (x, \xi) \in K \times \mathbf{R}^k. \tag{4.1}$$

With such symbols we define $L^m_{\varrho,\delta}(\mathbf{R}^n; V_1, V_2)$ to be the space of pseudo-differential operators $C^\infty_0(\mathbf{R}^n; V_1) \to C^\infty(\mathbf{R}^n; V_2)$, given by one of the usual integral formulas. It is easy to check that all the calculus for scalar operators extends to the vector valued case. In particular we have the usual composition formula and the results about H_s-continuity. When $\varrho = 1$ and $\delta = 0$ we sometimes write L^m_1 or L^m for $L^m_{1,0}$.

We shall have to consider the case when V_1 or V_2 is equal to the space $B^M_\xi(\mathbf{R}^k)$, given by the norm:

$$\|u\|^2_{B^M_\xi} = \sum_{|\alpha+\beta| \leq M} (1 + |\xi|)^{M+|\alpha|-|\beta|} \|x^\alpha D^\beta u\|^2_{L^2(R^k)}. \tag{4.2}$$

In this case the norm in (4.1) depends on ξ, but all the calculus remains valid because we have the inequality

$$\|u\|_{B^M} \leq \|u\|_{B^M_\xi} \leq (1 + |\xi|)^M \|u\|_{B^M}, \tag{4.3}$$

so if for instance $P \in L^m_{\varrho,\delta}(\mathbf{R}^n; V_1, B^M_\xi)$ then $P \in L^m_{\varrho,\delta}(\mathbf{R}^n; V_1, B^M)$. Note that there is a Hilbert space structure on B^M given by the scalar product

$$(u, v)_{B^M} = \sum_{|\alpha+\beta| \leq M} (x^\alpha D^\beta u, x^\alpha D^\beta v)_{L^2}. \tag{4.4}$$

Now let $1 \leq k < n$ and for $x \in \mathbf{R}^n$ write $x = (x', x'')$ with $x' \in \mathbf{R}^{n-k}$, $x'' \in \mathbf{R}^k$. We define $H_{(m,s)}(\mathbf{R}^n)$ as usual with the norm

$$\|u\|^2_{(m,s)} = \int |\hat{u}(\xi)|^2 (1 + |\xi|^2)^m (1 + |\xi'|^2)^s d\xi.$$

Then the injection

$$C^\infty_0(\mathbf{R}^n) \ni u \mapsto (x' \mapsto u(x', \cdot)) \in C^\infty_0(\mathbf{R}^{n-k}; L^2(\mathbf{R}^k)) \tag{4.5}$$

extends to an isomorphism:

$$H_{(0,s)}(\mathbf{R}^n) \to H_s(\mathbf{R}^{n-k}; L^2(\mathbf{R}^k)). \tag{4.6}$$

LEMMA 4.1. *Let* $P(x, D) = \sum_{|\alpha+\beta| \leq M} x''^\alpha a_{\alpha\beta}(x, D) D^\beta_{x''}$ *where*

$$a_{\alpha\beta}(x, D) \in L^{\sigma+(M+|\alpha|-|\beta|)/2}(\mathbf{R}^n)$$

and $a_{\alpha\beta}(x, \xi) = 0$ *in the domain* $|x''| + |\xi''/|\xi'|| \geq C$ *for some constant* $C > 0$. *Then we can regard* P *as an element of* $L^\sigma(\mathbf{R}^{n-k}; B^M_{\xi'}(\mathbf{R}^k), L^2(\mathbf{R}^k))$.

Proof. The $D_{x'}^{\alpha'} D_{\xi'}^{\beta'} P(x, \xi', D_{x''})$ are of the same type as P, so it suffices to prove that

$$(1 + |\xi'|)^{-\sigma} P(x, \xi', D_{x''}) \colon B_{\xi'}^{M}(\mathbf{R}^k) \to L^2(\mathbf{R}^k)$$

is uniformly bounded when $x' \in K \subset\subset \mathbf{R}^{n-k}$, $\xi' \in \mathbf{R}^{n-k}$. After commuting x''^{α} and $a_{\alpha\beta}$, we can write

$$P(x, \xi', D_{x''}) = \sum_{|\alpha+\beta| \leq M} b_{\alpha\beta}(x, \xi', D_{x''}) x''^{\alpha} D_{x''}^{\beta},$$

where the $b_{\alpha\beta}$ have the same properties as the $a_{\alpha\beta}$. It therefore suffices to prove that

$$(1 + |\xi'|)^{-\sigma-(M+|\alpha|-|\beta|)/2} b_{\alpha\beta}(x, \xi', D_{x''}) \colon L^2(\mathbf{R}^k) \to L^2(\mathbf{R}^k)$$

is uniformly bounded for $(x', \xi') \in K \times \mathbf{R}^{n-k}$. This is however a consequence of Theorem 3.5 in Hörmander [13], since

$$b_{\alpha\beta}(x', x'', \xi', \xi'')(1 + |\xi'|)^{-\sigma-(M+|\alpha|-|\beta|)/2}$$

belongs to a bounded subset of $S^0(\mathbf{R}^k \times \mathbf{R}^k)$ when $(x', \xi') \in K \times \mathbf{R}^{n-k}$ and is $= 0$ when $|x''| \geq C$.

Note that the conclusion in the lemma remains trivially valid if we replace the $a_{\alpha\beta}(x, D)$ by some new $a_{\alpha\beta}(x', D') \in L^{\sigma+(M+|\alpha|-|\beta|)/2}(\mathbf{R}^{n-k})$.

LEMMA 4.2. *Let* $P = \sum_{|\alpha+\beta| \leq M} x''^{\alpha} a_{\alpha\beta}(x, D) D_{x''}^{\beta}$ *with* $a_{\alpha\beta} \in L^{(M+|\alpha|-|\beta|)/2}(\mathbf{R}^n)$ *and suppose that* $a_{\alpha\beta}(x', 0, \xi', 0) = 0$. *Let* $(0, (\xi_0', 0)) \in T^*(\mathbf{R}^n) \setminus 0$ *and let* $V \subset T^*(\mathbf{R}^{n-k}) \setminus 0$ *be a conic neighbourhood of* $(0, \xi_0')$ *with compact projection on the base. Then for every* $\varepsilon > 0$ *there exists*

$$P_\varepsilon(x, D) \in L^M(\mathbf{R}^n) \cap L^0(\mathbf{R}^{n-k}; B_{\xi'}^M(\mathbf{R}^k), L^2(\mathbf{R}^k))$$

such that

(i) $WF(P_\varepsilon(x, D)) \cap \{(x, \xi) \in T^*(\mathbf{R}^n) \setminus 0; \xi' = 0\} = \emptyset$,

(ii) $P_\varepsilon(x, \xi) \sim P(x, \xi)$ *in a conic neighbourhood of* $x'' = \xi'' = 0$,

(iii) $\|P_\varepsilon(x, \xi', D_{x''})\|_{\mathcal{L}(B_{\xi'}^M, L^2)} < \varepsilon$ *when* $(x', \xi') \in V$ *and* ξ' *is large enough.*

Proof. It is no restriction to assume that $a_{\alpha\beta}(x, \xi) = 0$ when $|x''| + |\xi''/|\xi'|| \geq 1$. Commuting the x'' with the $a_{\alpha\beta}(x, D)$ and applying Lemma 4.1 we get

$$P = \sum_{|\alpha+\beta| \leq M} a_{\alpha\beta}(x, D) x''^{\alpha} D_{x''}^{\beta} + Q,$$

where $Q \in L^{M-1}(\mathbf{R}^n) \cap L^{-1/2}(\mathbf{R}^{n-k}; B_{\xi'}^M(\mathbf{R}^k), L^2(\mathbf{R}^k))$ and

$$WF(Q) \cap \{(x, \xi) \cap T^*(\mathbf{R}^n) \setminus 0; \xi' = 0\} = \emptyset.$$

Using Taylor's formula, we can write

$$a_{\alpha\beta}(x, D) = \sum_{|\gamma|=1} x''^{\gamma} b_{\alpha\beta\gamma}(x, D) + c_{\alpha\beta\gamma}(x, D) D^{\gamma}_{x''},$$

where $b_{\alpha\beta\gamma} \in L^{(M+|\alpha|-|\beta|)/2}(\mathbf{R}^n)$, $c_{\alpha\beta\gamma} \in L^{(M+|\alpha|-|\beta|)/2-1}(\mathbf{R}^n)$ and $b_{\alpha\beta\gamma}(x, \xi) = c_{\alpha\beta\gamma}(x, \xi) = 0$ when $x'' + |\xi''/|\xi'|| \geq 1$. (Here we use that $a_{\alpha\beta}(x', 0, \xi', 0) = 0$.)

Now we see as in the proof of the preceding lemma that if $\chi \in C_0^\infty(\mathbf{R}^k)$ is $= 1$ near the origin, then

$$\|x''^{\gamma} \chi(x''/\delta) b_{\alpha\beta\gamma}(x, \xi', D_{x''}) x''^{\alpha} D^{\beta}_{x''}\|_{\mathcal{L}(B^M_{\xi'}, L^2)}$$

and

$$\|c_{\alpha\beta\gamma}(x, \xi', D_{x''}) D^{\gamma}_{x''} \chi(D_{x''}/\delta|\xi'|) x''^{\alpha} D^{\beta}_{x''}\|_{\mathcal{L}(B^M_{\xi'}, L^2)}$$

are uniformly as small as we like in V for large ξ', when $\delta > 0$ is small enough. Thus the operator

$$(4.7) \quad P_\varepsilon = Q + \sum_{\substack{|\alpha+\beta| \leqq M \\ |\gamma|=1}} (x''^{\gamma} \chi(x''/\delta) b_{\alpha\beta\gamma}(x, D) + c_{\alpha\beta\gamma}(x, D) D^{\gamma}_{x''} \chi(D_{x''}/\delta|D_{x'}|)) x''^{\alpha} D^{\beta}_{x''}$$

has the required properties if δ is small enough.

We shall now estimate the wave front sets of some classes of vector valued pseudodifferential operators. The following lemma is an easy consequence of Theorem 2.6 in Hörmander [13] which can be extended to vector valued pseudodifferential operators.

LEMMA 4.3. *Let $P \in L^m(\mathbf{R}^n; V_1, V_2)$, $K_1 \subset\subset \mathbf{R}^n \setminus \{0\}$, $K \subset\subset \mathbf{R}^n$ and let K_2 be a bounded subset of $C_0^\infty(K; V_1)$. For $u \in C_0^\infty(\mathbf{R}^n; V_1)$ we put*

$$v_{\lambda,\xi,u}(x) = \lambda^{-m} e^{-i\lambda<x,\xi>} P(x, D_x)(u(x) e^{i\lambda<x,\xi>}),$$

so that $v_{\lambda,\xi,u} \in C^\infty(\mathbf{R}^n; V_2)$. Then

$$\{v_{\lambda,\xi,u};\ \lambda \geq 1, \xi \in K_1, u \in K_2\}$$

is a bounded subset of $C^\infty(\mathbf{R}^n; V_2)$ and for every $N > 0$ the set

$$\{\lambda^N v_{\lambda,\xi,u};\ \lambda \geq 1, \xi \in K_1, u \in K_2\}$$

defines by restriction a bounded subset of $C^\infty(\complement \bar{K}; V_2)$.

Of course we could have given a complete asymptotic formula for $v_{\lambda,\xi,u}$ with estimates for the remainder terms. However Lemma 4.3 is all that we need in the following.

PROPOSITION 4.4. *Let $P \in L^m(\mathbf{R}^{n-k}; V_1, V_2)$, where V_1 and V_2 are equal to $B^M_{\xi'}(\mathbf{R}^k)$ or $L^2(\mathbf{R}^k)$. Then, considering P as an operator $C_0^\infty(\mathbf{R}^n) \to \mathcal{D}'(\mathbf{R}^n)$, we have $(x', \xi') = (y', \eta')$ if $(x, \xi, y, \eta) \in WF'(P)$ and $|\xi'| + |\eta'| \neq 0$.*

Proof. For simplicity we assume that $V_1 = V_2 = L^2(\mathbf{R}^k)$. Suppose that $(x_0, \xi_0, y_0, \eta_0) \in (\mathbf{R}^n \times \mathbf{R}^n) \times (\mathbf{R}^n \times \mathbf{R}^n)$ and that $(x_0', \xi_0') \neq (y_0', \eta_0')$, $|\xi_0'| + |\eta_0'| \neq 0$. We shall show that $(x_0, \xi_0, y_0, \eta_0) \notin WF'(P)$. By passing to the adjoint if necessary, we can assume that $\eta_0' \neq 0$.

Let $\varphi, \psi \in C_0^\infty(\mathbf{R}^n)$ be such that $\varphi(x_0) = \psi(y_0) = 1$ and

(4.8) The projections of $\operatorname{supp} \varphi$ and $\operatorname{supp} \psi$ along $\mathbf{R}^k$ do not intersect if $x_0' \neq y_0'$.

$\varphi(x)e^{i<x'', \xi''>}$ and $\psi(x)e^{i<x'', \xi''>}$ can be regarded as elements of $C_0^\infty(\mathbf{R}_{x'}^{n-k}; L^2(\mathbf{R}_{x''}^k))$ depending on the parameter ξ''. We denote these elements by $\varphi'(x', \xi'')$, $\psi'(x', \xi'')$. Clearly they belong to a bounded subset of $C_0^\infty(\mathbf{R}^{n-k}; L^2(\mathbf{R}^k))$ when ξ'' varies in $\mathbf{R}^k$.

Put $\Phi(x, y) = \overline{\varphi(x)}\psi(y)$ and let $P(x, y)$ be the distribution kernel of P. If $(\ ,\ )_{L^2(\mathbf{R}^k)}$ is the scalar product in $L^2(\mathbf{R}^k)$ we see that the Fourier transform of ΦP at $(\lambda\xi, -\lambda\eta)$ is

$$(4.9)\quad \widehat{\Phi P}(\lambda\xi, -\lambda\eta) = \int (P(x', D')(\psi'(x', \lambda\eta'')e^{i<x', \lambda\eta'>}), \varphi'(x', \lambda\xi''))_{L^2(\mathbf{R}^k)} e^{-i<x', \lambda\xi'>} dx'.$$

Let $K_1 \subset \mathbf{R}^{n-k}$, $K_2 \subset \mathbf{R}^{n-k} \setminus \{0\}$ be compact neighbourhoods of ξ_0' and η_0' respectively such that $K_1 \cap K_2 = \emptyset$ if $\xi_0' \neq \eta_0'$. If $(\xi, \eta) \in (K_1 \times \mathbf{R}^k) \times (K_2 \times \mathbf{R}^k)$ it follows from the preceding lemma that

$$(4.10)\qquad P(x', D')(\psi'(x', \lambda\eta'')e^{i<x', \lambda\eta'>}) = \lambda^m e^{i<x', \lambda\eta'>} v_{\lambda, \eta}(x'),$$

where $v_{\lambda, \eta}$ belongs to a bounded subset of $C^\infty(\mathbf{R}^{n-k}; L^2(\mathbf{R}^k))$ for $\lambda > 1$ and $\|v_{\lambda, \eta}(x')\|_{L^2(\mathbf{R}^k)}$ is uniformly rapidly decreasing as $\lambda \to +\infty$ when x' belongs to a compact set, not intersecting the projection of $\operatorname{supp} \psi$ along $\mathbf{R}^k$. In the case when $x_0' \neq y_0'$ it follows from (4.8) that

$$(P(x', D')(\psi'(x', \lambda\eta'')e^{i<x', \lambda\eta'>}), \varphi'(x', \lambda\xi''))_{L^2(\mathbf{R}^k)}$$

is uniformly rapidly decreasing when $\lambda \to +\infty$. Thus in this case $\widehat{\Phi P}(\lambda\xi, -\lambda\eta)$ is uniformly rapidly decreasing for $(\xi, \eta) \in (K_1 \times \mathbf{R}^k) \times (K_2 \times \mathbf{R}^k)$.

In the case when $\xi_0' \neq \eta_0'$ we use (4.10) and write

$$(4.11)\qquad \widehat{\Phi P}(\lambda\xi, -\lambda\eta) = \int w_{\lambda, \xi, \eta} \cdot \lambda^m e^{i<x', \lambda(\eta' - \xi')>} dx',$$

where

$$w_{\lambda, \xi, \eta}(x') = (v_{\lambda, \eta}(x'), \varphi'(x', \lambda\xi''))_{L^2(\mathbf{R}^k)}$$

belongs to a bounded subset of $C_0^\infty(\mathbf{R}^{n-k})$ when $\lambda \geq 1$, $(\xi, \eta) \in (K_1 \times \mathbf{R}^k) \times (K_2 \times \mathbf{R}^k)$. In this case $\xi' - \eta' \in K_1 - K_2 \subset \mathbf{R}^{n-1} \setminus \{0\}$ and by using suitable partial integrations, we see that the integral (4.11) is uniformly rapidly decreasing when $\lambda \to +\infty$.

We have thus shown that $(x_0, \xi_0, y_0, \eta_0) \notin WF'(P)$ if $(x_0', \xi_0') \neq (y_0', \eta_0')$,

$|\xi_0'| + |\eta_0'| \neq 0$ and the proposition is proved.

In exactly the same way one can prove

PROPOSITION 4.5. *Let* $P \in L^m(\mathbf{R}^{n-k}; \mathbf{C}, B^M_{\xi'}(\mathbf{R}^k))$. *Then, considering* P *as an operator* $C_0^\infty(\mathbf{R}^{n-k}) \to \mathscr{D}'(\mathbf{R}^n)$, *we have* $(x', \xi') = (y', \eta')$ *if* $(x, \xi, y', \eta') \in WF'(P)$ *and* $|\xi'| + |\eta'| \neq 0$.

Before ending this section, we review the results in the appendix in [22]. With a fixed k; $1 \leq k < n$, we define $T^m(\mathbf{R}^n)$ to be the space of "pseudodifferential operators" $a(x, D')$, where $a(x, \xi') \in S^m(\mathbf{R}^n \times \mathbf{R}^{n-k})$. These are not pseudodifferential operators in the usual sence. However we have

LEMMA 4.6. *Suppose that* $P \in T^{m'}(\mathbf{R}^n)$ *and* $\chi \in L^{m''}(\mathbf{R}^n)$ *are properly supported and that* $WF(\chi) \cap \{(x, \xi) \in T^*(\mathbf{R}^n) \setminus 0;\ \xi' = 0\} = \emptyset$. *Then* $P\chi$ *and* χP *belong to* $L^{m'+m''}(\mathbf{R}^n)$ *and the usual composition formula is valid for the symbols.*

LEMMA 4.7. *Suppose that* $\chi \in L^0(\mathbf{R}^n)$ *and that* $WF(\chi) \cap \{(x, \xi) \in T^*(\mathbf{R}^n) \setminus 0;\ \xi' = 0\} = \emptyset$. *Then* χ *is continuous* $H^{\mathrm{comp}}_{(m', s')}(\mathbf{R}^n) \to H^{\mathrm{loc}}_{(m'', s'')}(\mathbf{R}^n)$ *if* $m' + s' = m'' + s''$.

5. Microlocal results when Σ is in a special position

In this section we shall apply techniques, influenced by Višik-Grušin [26, 27] and Grušin [6—9]. The main difference is that we use vector valued pseudodifferential operators instead of freezing coefficients and taking partial Fourier transform. This gives sharper results, in particular we can estimate the wave front sets of the parametrices. Our method can be considered as a generalization of the method in [22]. Note that Grušin [10] has more recently applied vector valued pseudodifferential operators in a similar context.

We shall first prove microlocal versions of Theorems 1.2 and 1.4 when Σ is given by $x_n = \xi_n = 0$ and $P \in L^M(\mathbf{R}^n)$ belongs microlocally to $L^{M,M}(\mathbf{R}^n, \Sigma)$ in a conic neighbourhood of a point $\varrho_0 = (0, (\xi_0', 0)) \in T^*(\mathbf{R}^n) \setminus 0$, $|\xi_0'| = 1$. Applying Taylor's formula we obtain that

$$P(x, D) \equiv \sum_{\alpha+\beta \leq M} x_n^\alpha a_{\alpha\beta}(x, D) D_n^\beta \tag{5.1}$$

in a conic neighbourhood of ϱ_0, where $a_{\alpha\beta} \in L^{(M+\alpha-\beta)/2}(\mathbf{R}^n)$ have positively homogeneous principal parts $a^0_{\alpha\beta}$. Here we use the terminology introduced in [4]: If $A, B\colon C_0^\infty(\mathbf{R}^n) \to \mathscr{D}'(\mathbf{R}^n)$ are continuous linear operators with

$$WF'(A) \cup WF'(B) \subset \operatorname{diag}(T^*(\mathbf{R}^n) \setminus 0)$$

and $V \subset T^*(\mathbf{R}^n) \setminus 0$ is a conic open set, we say that $A \equiv B$ in V if

$$WF'(A - B) \cap \operatorname{diag}(V) = \emptyset.$$

Let p_M be the principal symbol of P which is positively homogeneous of degree M and put

$$L_0(x, D) = \sum_{\alpha+\beta \leq M} x_n^\alpha a^0_{\alpha\beta}(x', 0, D', 0) D_n^\beta \in L^0(\mathbf{R}^{n-1}; B^M_{\xi'}, L^2). \tag{5.2}$$

Suppose that $\operatorname{ind}(p_M) = N$ at Σ near ϱ_0. Then it follows from Lemma 2.2 that the index of

$$L_0(x', \xi') = L_0(x, \xi', D_n) \colon B^M_{\xi'}(\mathbf{R}) \to L^2(\mathbf{R}),\ \xi' \neq 0,$$

is N in a conic neighbourhood of $(0, \xi_0')$. Let $u_1(x_n), \ldots, u_{m^+}(x_n) \in \mathcal{S}(\mathbf{R})$ and $v_1(x_n), \ldots, v_{m^-}(x_n) \in \mathcal{S}(\mathbf{R})$ be orthogonal bases for the kernel and the cokernel of $L_0(0, x_n, \xi_0', D_n)$. Note that $m^+ \leq M, m^- \leq M, m^+ - m^- = N$. Let $R^+(x', \xi') \colon B^M_{\xi'}(\mathbf{R}) \to \mathbf{C}^{m^+}$ and $R^-(x', \xi') \colon \mathbf{C}^{m^-} \to L^2(\mathbf{R})$ be defined by

$$(R^+(x', \xi')u)_j = (u(x_n), |\xi'|^{M/2+1/4} u_j(|\xi'|^{1/2} x_n))_{L^2(\mathbf{R}_{x_n})},$$

$$R^-(x', \xi')(z_1, \ldots, z_{m^-}) = \sum_{j=1}^{m^-} z_j |\xi'|^{1/4} v_j(|\xi'|^{1/2} x_n).$$

Then

$$R^+(x', \xi') \in S^0(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}; B^M_{\xi'}(\mathbf{R}), \mathbf{C}^{m^+}),$$

$$R^-(x', \xi') \in S^0(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}; \mathbf{C}^{m^-}, L^2(\mathbf{R}))$$

after having been suitably modified for small ξ'. We shall consider

$$\mathcal{L}_0(x', \xi') \in S^0(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}; B^M_{\xi'}(\mathbf{R}) \oplus \mathbf{C}^{m^-}, L^2(\mathbf{R}) \oplus \mathbf{C}^{m^+})$$

given by the matrix

$$\mathcal{L}_0(x', \xi') = \begin{pmatrix} L_0(x', \xi') & R^-(x', \xi') \\ R^+(x', \xi') & 0 \end{pmatrix}.$$

LEMMA 5.1. *There exists* $\mathcal{E}^0_0(x', \xi') \in S^0(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}; L^2 \oplus \mathbf{C}^{m^+}, B^M_{\xi'} \oplus \mathbf{C}^{m^-})$ *so that* $\mathcal{E}^0_0(x', \xi')$ *is the inverse of* $\mathcal{L}_0(x', \xi')$ *in a conic neighbourhood of* $(0, \xi_0)$ *for* $|\xi'| \geq 1$.

Proof. $\mathcal{L}_0(x', \xi')$ has evidently an inverse in a small neighbourhood of $(0, \xi_0')$. To see that this is true also in a conic neighbourhood, we make a change of variables due to Višik-Grušin and use the homogeneity. We put

$$t = x_n |\xi'|^{1/2},\quad \tilde{u}(t) = |\xi'|^{M/2 - 1/4} u(x_n),\quad \tilde{v}(t) = |\xi'|^{-1/4} v(x_n).$$

Then the system $\mathscr{L}_0(x', \xi')(u, z) = (v, y)$:

$$\begin{cases} L_0(x, \xi', D_n)u + \sum_{j=1}^{m^-} z_j v_j(|\xi'|^{1/2}x_n)|\xi'|^{1/4} = v \\ \int u(x_n)|\xi'|^{M/2+1/4}\overline{u_j(|\xi'|^{1/2}x_n)}dx_n = y_j, \quad 1 \leq j \leq m^+ \end{cases}$$

transforms into

$$\begin{cases} L_0(x', t, \xi'/|\xi'|, D_t)\tilde{u} + \sum_1^{m^-} z_j v_j(t) = \tilde{v}(t) \\ \int \tilde{u}(t)\overline{u_j(t)}dt = y_j, \quad 1 \leq j \leq m^+. \end{cases}$$

Moreover $\|\tilde{u}(t)\|_{B^M} = (1 + o(1))\|u\|_{B^M_{\xi'}}$ for large ξ' and $\|\tilde{v}(t)\|_{L^2} = \|v(x_n)\|_{L^2}$, so we see that $\mathscr{L}_0(x', \xi')$ has a uniformly bounded inverse $\mathscr{E}_0^0(x', \xi')$ in a conic neighbourhood of $(0, \xi_0')$ when $|\xi'| \geq 1$. To show that $\mathscr{E}_0^0(x', \xi')$ is actually a symbol, we can estimate $\|D_x^\alpha D_{\xi'}^\beta \mathscr{E}_0^0(x', \xi')\|$ by taking successively higher and higher derivatives of the relation $\mathscr{E}_0^0(x', \xi') \circ \mathscr{L}_0(x', \xi') = I$ and using the relation $\mathscr{L}_0(x', \xi') \circ \mathscr{E}_0^0(x', \xi') = I$. We omit the details since this is the same argument as in Hörmander [13] p. 166. The proof is complete.

If we write $\mathscr{E}_0^0$ in matrix form:

$$\mathscr{E}_0^0(x', \xi') = \begin{pmatrix} E_0(x', \xi') & E_0^+(x', \xi') \\ E_0^-(x', \xi') & E_0^{-+}(x', \xi') \end{pmatrix},$$

it follows from the change of variables in the proof above that

$$E_0^{-+}(x', \xi') \in S^0(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}; \mathbf{C}^{m^+}, \mathbf{C}^{m^-})$$

is positively homogeneous of degree 0 in ξ' for $|\xi'| \geq 1$ and (x', ξ') in a conic neighbourhood of $(0, \xi_0')$. Moreover

$$(E_0^+(x', \xi')y)(x_n) = |\xi'|^{-M/2+1/4}(E_0^+(x', \xi'/|\xi'|)y)(x_n|\xi'|^{1/2}) \tag{5.3}$$

for all $y \in \mathbf{C}^{m^+}$, $x_n \in \mathbf{R}$, and (x', ξ') in the same domain.

From Theorem 2.1 and the identity

$$L_0(x', \xi')E_0^+(x', \xi') + R^-(x', \xi')E_0^{-+}(x', \xi') = 0$$

(which is valid in a conic neighbourhood of $(0, \xi_0')$ when $|\xi'| \geq 1$ by Lemma 5.1) it follows that the range of $E_0^+(x', \xi')$ is contained in $\mathscr{S}(\mathbf{R})$ when (x', ξ') is in a conic neighbourhood of $(0, \xi_0')$ and $|\xi'| \geq 1$. We also note that

$$E_0^{-+}(0, \xi_0') = 0. \tag{5.4}$$

In fact, for all $y = (y_1, \ldots, y_{m^+}) \in \mathbf{C}^{m^+}$ the system

$$\mathcal{L}_0(0, \xi_0')(u, z) = (0, y) \in L^2(\mathbf{R}) \oplus \mathbf{C}^{m^+}$$

has the unique solution $(u, z) = (\sum_1^{m^+} y_j u_j, 0)$ (where the functions u_j are introduced above). Thus $z = E_0^{-+}(0, \xi_0')y = 0$ for all $y \in \mathbf{C}^{m^+}$.

Applying Lemma 4.2 to $P - L_0$ and adding L_0 to the operator obtained, we can for every $\varepsilon > 0$ find an operator

$$L(x, D) = L_\varepsilon(x, D) \in L^0(\mathbf{R}^{n-1}; B^M_{\xi'}(\mathbf{R}), L^2(\mathbf{R}))$$

such that

(5.5) $\|L_0(x, \xi', D_n) - L_\varepsilon(x, \xi', D_n)\|_{\mathcal{L}(B^M_{\xi'}, L^2)} \leq \varepsilon$ in a conic neighbourhood of $(0, \xi_0')$ for ξ' sufficiently large.

(5.6) $L_\varepsilon - L_0 \in L^M(\mathbf{R}^n)$ and $WF(L_\varepsilon - L_0) \cap \{(x, \xi) \in T^*(\mathbf{R}^n) \setminus 0;\ \xi' = 0\} = \emptyset$.

(5.7) $L_\varepsilon(x, \xi) \sim P(x, \xi)$ in a conic neighbourhood of ϱ_0.

In order to study P we shall first consider the operator

$$\mathcal{L}(x', D') = \mathcal{L}_\varepsilon(x', D') = \begin{pmatrix} L_\varepsilon(x', D') & R^-(x', D') \\ R^+(x', D') & 0 \end{pmatrix} \in L^0(\mathbf{R}^{n-1}; B^M_{\xi'} \oplus \mathbf{C}^{m^-}, L^2 \oplus \mathbf{C}^{m^+})$$

Proposition 5.2. *If $\varepsilon > 0$ is small enough, there exists*

$$\mathcal{E}(x', D') = \mathcal{E}_\varepsilon(x', D') \in L^0(\mathbf{R}^{n-1}; L^2 \oplus \mathbf{C}^{m^+}, B^M_{\xi'} \oplus \mathbf{C}^{m^-}),$$

properly supported such that in a conic neighbourhood of $(0, \xi_0')$:

$$\mathcal{L}_\varepsilon(x', D') \circ \mathcal{E}_\varepsilon(x', D') \equiv I \bmod L^{-\infty}(\mathbf{R}^{n-1}; L^2 \oplus \mathbf{C}^{m^+}, L^2 \oplus \mathbf{C}^{m^+})$$

$$\mathcal{E}_\varepsilon(x', D') \circ \mathcal{L}_\varepsilon(x', D') \equiv I \bmod L^{-\infty}(\mathbf{R}^{n-1}; B^M_{\xi'} \oplus \mathbf{C}^{m^-}, B^M_{\xi'} \oplus \mathbf{C}^{m^-}).$$

Here we have applied the following terminology: When $A, B \in L^m(\mathbf{R}^n; V_1, V_2)$ and $\varrho \in T^*(\mathbf{R}^n) \setminus 0$, we say that $A \equiv B$ near ϱ if $\sigma_A - \sigma_B \in S^{-\infty}$ in a conic neighbourhood of ϱ and σ_A and σ_B are the symbols of A and B respectively. This agrees with the terminology for scalar operators.

Proof. If $\varepsilon > 0$ is small enough, it follows from Lemma 5.1 and (5.5) that $\mathcal{L}_\varepsilon(x', \xi')$ has an inverse $\mathcal{E}^0_\varepsilon(x', \xi') \in S^0$ for large ξ' in a conic neighbourhood of $(0, \xi_0')$. The construction of $\mathcal{E}_\varepsilon(x', D')$ is therefore formally the same as the construction of a parametrix of an elliptic operator in the scalar case, so we omit the details.

From now on we drop the subscript ε. For the proof of Theorem 1.7 we shall need the following proposition.

Proposition 5.3. *If we write*

$$\mathscr{E} = \begin{pmatrix} E & E^+ \\ E^- & E^{-+} \end{pmatrix}$$

then $E^{-+} \in L^0(\mathbf{R}^{n-1}; \mathbf{C}^{m^+}, \mathbf{C}^{m^-})$ *has modulo* $S^{-1/2}$ *a symbol* $E_0^{-+}(x', \xi')$ *which is positively homogeneous of degree* 0 *in a conic neighbourhood of* $(0, \xi_0')$ *and moreover* $E_0^{-+}(0, \xi_0') = 0$.

Proof. We shall simply examine the construction of $\mathscr{E}$. Let

$$\mathscr{E}_0^0(x', \xi') \in S^0(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}; L^2 \oplus \mathbf{C}^{m^+}, B_{\xi'}^M \oplus \mathbf{C}^{m^-})$$

be as in Lemma 5.1 so that $\mathscr{E}_0^0(x', \xi')$ is an inverse of $\mathscr{L}_0(x', \xi')$ in a conic neighbourhood of $(0, \xi_0')$ for $|\xi'| \geq 1$. By the proof of Proposition 5.2, $\mathscr{E}$ has a principal symbol $\mathscr{E}^0(x', \xi')$ which is an inverse of $\mathscr{L}(x', \xi')$ $(= \mathscr{L}_\varepsilon(x', \xi'))$ in a conic neighbourhood of $(0, \xi_0')$ for large ξ'. We restrict our computations below to this domain. We can assume that $\varepsilon > 0$ above was chosen so small that the norm in $\mathscr{L}(L^2 \oplus \mathbf{C}^{m^+}, L^2 \oplus \mathbf{C}^{m^+})$ of $\mathscr{K}(x', \xi') = (\mathscr{L}_0(x', \xi') - \mathscr{L}(x', \xi'))\mathscr{E}_0^0(x', \xi')$ is $< 1/2$. Then

$$\mathscr{L}(x', \xi')\mathscr{E}_0^0(x', \xi') = I - \mathscr{K}(x', \xi'),$$

so therefore

$$\mathscr{E}^0 = \mathscr{E}_0^0(I + \mathscr{K} + \mathscr{K}^2 + \ldots)$$

where we have dropped (x', ξ') to avoid heavy notations. (The calculations below deal with symbols and not with the corresponding operators.) Now

$$\mathscr{K} = (\mathscr{L}_0 - \mathscr{L})\mathscr{E}_0^0 = \begin{pmatrix} L_0 - L & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E^0 & E_0^+ \\ E_0^- & E_0^{-+} \end{pmatrix} = \begin{pmatrix} (L_0 - L)E_0 & (L_0 - L)E_0^+ \\ 0 & 0 \end{pmatrix}$$

so we get

$$\mathscr{E}^0 = \mathscr{E}_0^0 + \mathscr{E}_0^0 \begin{pmatrix} \sum_1^\infty ((L_0 - L)E_0)^\nu & \sum_1^\infty ((L_0 - L)E_0)^{\nu-1}(L_0 - L)E_0^+ \\ 0 & 0 \end{pmatrix}.$$

The entry in the lower right hand corner of $\mathscr{E}^0$ is therefore

$$E_0^{-+} + \sum_0^\infty E_0^-((L_0 - L)E_0)^\nu(L_0 - L)E_0^+.$$

Since $\mathscr{E}^0(x', \xi')$ is a principal symbol of $\mathscr{E}(x', D')$ it suffices to prove that

$$(L_0 - L)E_0^+ \in S^{-1/2}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}; \mathbf{C}^{m^+}, L^2)$$

for then

$$\sum_0^\infty E_0^-((L_0 - L)E_0)^\nu(L_0 - L)E_0^+ \in S^{-1/2}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}; \mathbf{C}^{m^+}, \mathbf{C}^{m^-})$$

and the proposition follows from the properties of E_0^{-+} noted after the proof of Lemma 5.1. (Recall that the norm of $(L - L_0)E_0(x', \xi')$ is $< 1/2$.)

Using (5.2)—(5.5) and Taylor expanding $a^0_{\alpha\beta}(x, \xi) - a^0_{\alpha\beta}(x', 0, \xi', 0)$ we can write

$$(L_0 - L)(x', \xi') = A(x', \xi')x_n + B(x', \xi')D_n + C(x', \xi'),$$

where $A \in S^0(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}; B^M_{\xi'}, L^2)$ and $B, C \in S^{-1}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}; B^M_{\xi'}, L^2)$ in view of Lemma 4.1. From (5.3) and the fact that the finite dimensional range of E_0^+ is contained in $\mathcal{S}(\mathbf{R})$ it follows that

$$|\xi'|^{1/2} x_n E_0^+(x', \xi') \in S^0(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}; \mathbf{C}^{m^+}, B^M_{\xi'}),$$
$$|\xi'|^{-1/2} D_n E_0^+(x', \xi') \in S^0(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}; \mathbf{C}^{m^+}, B^M_{\xi'}).$$

Thus $(L_0 - L)E_0^+ = (Ax_n + BD_n + C)E_0^+ \in S^{-1/2}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}; \mathbf{C}^{m^+}, L^2)$ and the proof is complete.

We shall now use Proposition 5.2 to prove a similar result (Proposition 5.4) for the operator

$$\mathcal{P} = \begin{pmatrix} P(x, D) & R^-(x', D') \\ R^+(x', D') & 0 \end{pmatrix} : C_0^\infty(\mathbf{R}^n) \oplus C_0^\infty(\mathbf{R}^{n-1}; \mathbf{C}^{m^-}) \to C^\infty(\mathbf{R}^n) \oplus C^\infty(\mathbf{R}^{n-1}; \mathbf{C}^{m^+}).$$

That proposition will be the main result of this section and the microlocal versions of Theorems 1.2, 1.4, 1.6, 1.7 will follow rather easily. Considering R^+ and R^- as operators $C_0^\infty(\mathbf{R}^n) \to C^\infty(\mathbf{R}^{n-1}; \mathbf{C}^{m^+})$ and $C_0^\infty(\mathbf{R}^{n-1}; \mathbf{C}^{m^-}) \to C^\infty(\mathbf{R}^n)$, it is easy to verify that

$$\begin{aligned} &WF'(R^-) \subset \{((x', 0, \xi', 0), (x', \xi')) \in (T^*(\mathbf{R}^n) \setminus 0) \times (T^*(\mathbf{R}^{n-1}) \setminus 0)\}, \\ &WF'(R^+) \subset \{((x', \xi'), (x', 0, \xi', 0)) \in (T^*(\mathbf{R}^{n-1}) \setminus 0) \times (T^*(\mathbf{R}^n) \setminus 0)\}. \end{aligned} \tag{5.8}$$

Using (5.7) we note that if $\chi \in L^0(\mathbf{R}^n)$ is properly supported and $WF(\chi)$ is contained in a sufficiently small conic neighbourhood of $\varrho_0 = (0, (\xi_0', 0))$, then

$$P\chi \equiv L\chi, \quad \chi L \equiv \chi P. \tag{5.9}$$

Now take $\chi \in L^0(\mathbf{R}^n)$, properly supported so that

$$\chi \equiv I \text{ in a conic neighbourhood of } \varrho_0 \tag{5.10}$$

and $WF(\chi)$ is in a small conic neighbourhood ϱ_0. Using that P is elliptic outside Σ in a small conic neighbourhood of ϱ_0, we see that there exists $\tilde{\chi} \in L^0(\mathbf{R}^n)$ with the same properties as χ and such that:

(5.11) The projection along the $x_n - \xi_n$-plane of $WF(\tilde{\chi}P - P\chi)$ does not contain $\varrho_0' = (0, \xi_0') \in T^*(\mathbf{R}^{n-1}) \setminus 0$.

Moreover we have

$$\tilde{\chi}P \equiv \tilde{\chi}L, \quad P\tilde{\chi} \equiv L\tilde{\chi}. \tag{5.9'}$$

We write the "parametrix" $\mathscr{E}(x', D')$ in Proposition 5.2 as a matrix

$$\mathscr{E}(x', D') = \begin{pmatrix} E(x', D') & E^+(x', D') \\ E^-(x', D') & E^{-+}(x', D') \end{pmatrix}$$

and we put

$$\mathscr{G} = \begin{pmatrix} \chi E \tilde{\chi} & \chi E^+ \\ E^- \tilde{\chi} & E^{-+} \end{pmatrix} = \begin{pmatrix} G & G^+ \\ G^- & G^{-+} \end{pmatrix} : C^\infty(\mathbf{R}^n) \oplus C^\infty(\mathbf{R}^{n-1}; \mathbf{C}^{m^+}) \to C^\infty(\mathbf{R}^n) \oplus C^\infty(\mathbf{R}^{n-1}; \mathbf{C}^{m^-}) .$$

Proposition 5.4. *We have*

(5.12) $WF'(G) \subset \operatorname{diag}(T^*(\mathbf{R}^n) \setminus 0)$,

(5.13) $WF'(G^+) \subset \{((x', 0, \xi', 0), (x', \xi')) \in (T^*(\mathbf{R}^n) \setminus 0) \times (T^*(\mathbf{R}^{n-1}) \setminus 0)\}$,

(5.14) $WF'(G^-) \subset \{((x', \xi'), (x', 0, \xi', 0)) \in (T^*(\mathbf{R}^{n-1}) \setminus 0) \times (T^*(\mathbf{R}^n) \setminus 0)\}$.

For all $\psi \in L^0(\mathbf{R}^n)$, $\psi^+ \in L^0(\mathbf{R}^{n-1}; \mathbf{C}^{m^+}, \mathbf{C}^{m^+})$, $\psi^- \in L^0(\mathbf{R}^{n-1}; \mathbf{C}^{m^-}, \mathbf{C}^{m^-})$ *with their wavefront sets sufficiently close to* ϱ_0 *and* ϱ_0' *respectively, we have*

$$\mathscr{P}\mathscr{G}\begin{pmatrix} \psi & 0 \\ 0 & \psi^+ \end{pmatrix} \equiv \begin{pmatrix} \psi & 0 \\ 0 & \psi^+ \end{pmatrix}, \quad \begin{pmatrix} \psi & 0 \\ 0 & \psi^- \end{pmatrix} \mathscr{G}\mathscr{P} \equiv \begin{pmatrix} \psi & 0 \\ 0 & \psi^- \end{pmatrix}. \tag{5.15}$$

Moreover, for all $s \in \mathbf{R}$, G *is continuous* $H_s^{\mathrm{loc}}(\mathbf{R}^n) \to H_{s+M/2}^{\mathrm{loc}}(\mathbf{R}^n)$, G^+ *is continuous* $H_s^{\mathrm{loc}}(\mathbf{R}^{n-1}) \to H_{s+M/2}^{\mathrm{loc}}(\mathbf{R}^n)$, G^- *is continuous* $H_s^{\mathrm{loc}}(\mathbf{R}^n) \to H_s^{\mathrm{loc}}(\mathbf{R}^{n-1})$ *and* G^{-+} *is continuous* $H_s^{\mathrm{loc}}(\mathbf{R}^{n-1}) \to H_s^{\mathrm{loc}}(\mathbf{R}^{n-1})$.

Proof. We shall first prove the continuity properties. We have

$$E \in L^0(\mathbf{R}^{n-1}; L^2(\mathbf{R}), B_{\xi'}^M(\mathbf{R})) \subset L^{-M/2}(\mathbf{R}^{n-1}; L^2(\mathbf{R}), L^2(\mathbf{R})),$$

$$E^+ \in L^0(\mathbf{R}^{n-1}; \mathbf{C}^{m^+}, B_{\xi'}^M(\mathbf{R})) \subset L^{-M/2}(\mathbf{R}^{n-1}; \mathbf{C}^{m^+}, L^2(\mathbf{R})),$$

$$E^- \in L^0(\mathbf{R}^{n-1}; L^2(\mathbf{R}), \mathbf{C}^{m^-}), \quad E^{-+} \in L^0(\mathbf{R}^{n-1}; \mathbf{C}^{m^+}, \mathbf{C}^{m^-}),$$

so it is clear that E, E^+, E^-, E^{-+} have the analogous continuity properties if we everywhere replace the spaces $H_s(\mathbf{R}^n)$ by $H_{(0,s)}(\mathbf{R}^n)$. (C.f. (4.6).) The fact that $WF(\chi)$ and $WF(\tilde{\chi})$ do not intersect the normals of the planes $x_n = \text{const.}$ therefore gives us the stronger continuity properties for G, G^+, G^-, G^{-+} in view of Lemma 4.7.

We next proceed to the proof of (5.15). We write

$$\mathscr{G} = \begin{pmatrix} \chi & 0 \\ 0 & I \end{pmatrix} \mathscr{E} \begin{pmatrix} \tilde{\chi} & 0 \\ 0 & I \end{pmatrix}.$$

With ψ, ψ^+ as in the proposition we obtain

$$\mathcal{P}\mathcal{G}\begin{pmatrix}\psi & 0\\ 0 & \psi^+\end{pmatrix} = \mathcal{P}\begin{pmatrix}\chi & 0\\ 0 & I\end{pmatrix}\mathcal{E}\begin{pmatrix}\tilde\chi & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}\psi & 0\\ 0 & \psi^+\end{pmatrix}.$$

(5.11), (5.9′) and (5.8) imply that

$$\mathcal{P}\mathcal{G}\begin{pmatrix}\psi & 0\\ 0 & \psi^+\end{pmatrix} \equiv \begin{pmatrix}\tilde\chi & 0\\ 0 & I\end{pmatrix}\mathcal{L}\mathcal{E}\begin{pmatrix}\tilde\chi & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}\psi & 0\\ 0 & \psi^+\end{pmatrix}.$$

Proposition 5.2 and Lemma 4.7 give then:

$$\mathcal{P}\mathcal{G}\begin{pmatrix}\psi & 0\\ 0 & \psi^+\end{pmatrix} \equiv \begin{pmatrix}\tilde\chi & 0\\ 0 & I\end{pmatrix}^2\begin{pmatrix}\psi & 0\\ 0 & \psi^+\end{pmatrix} \equiv \begin{pmatrix}\psi & 0\\ 0 & \psi^+\end{pmatrix}$$

which proves the first part of (5.15). The second part is proved similarly and we omit the details.

We next prove (5.13). From (5.15), we get in particular that

$$(PG^+ + R^-G^{-+})\psi^+ \equiv 0. \tag{5.16}$$

We know from Proposition 4.5 that

$$WF'(G^+) \subset \{((x, \xi), (y', \eta')); x' = y', \xi' = \eta' \neq 0, (x, \xi) \in WF(\chi)\}. \tag{5.17}$$

Now suppose that $((x, \xi), (y', \eta')) \in WF'(G^+\psi^+)$ and that $|x_n| + |\xi_n| \neq 0$. Then, since P is elliptic in $WF(\chi) \setminus \Sigma$, we know that $((x, \xi), (y', \eta')) \in WF'(PG^+\psi^+)$. This is however in contradiction with (5.16), since

$$WF'(R^-G^{-+}\psi^+) \subset \{((x', 0, \xi', 0), (x', \xi')) \in (T^*(\mathbf{R}^n) \setminus 0)\times(T^*(\mathbf{R}^{n-1}) \setminus 0)\}.$$

Thus we conclude that if $((x, \xi), (y', \eta')) \in WF'(G^+\psi^+)$ then $x_n = \xi_n = 0$ and we have therefore proved (5.13) with G^+ replaced by $G^+\psi^+$. Similarly one can prove (5.14) with G^- replaced by ψ^-G^-.

To prove (5.12) we write the following equations, which are consequences of (5.15):

$$(PG + R^-G^-)\psi \equiv \psi, \tag{5.18}$$

$$\psi(GP + G^+R^+) \equiv \psi. \tag{5.19}$$

The same argument as above shows now that (5.12) holds with G replaced by $\psi G\psi$. Thus if we replace the χ and $\tilde\chi$ in the construction of $\mathcal{G}$ by some new χ and $\tilde\chi$ with the same properties but with smaller wavefront sets, we see that (5.12)—(5.14) hold and that the other statements in the proposition remain true. This completes the proof.

We shall now use Proposition 5.4 to prove a microlocal version of Theorem 1.2, when $\Sigma \subset T^*(\mathbf{R}^n) \setminus 0$ is given by $x_n = \xi_n = 0$. Thus we assume $P \in L^M(\mathbf{R}^n)$ is as above and that $\operatorname{ind}(p_M) = M$ on Σ in a neighbourhood of ϱ_0. Then the index of $L_0(x, \xi', D_n)\colon B^M_{\xi'}(\mathbf{R}) \to L^2(\mathbf{R})$ is M so we have $m^+ = M$ and $m^- = 0$ in the construction of R^+ and R^- above and the operators G^-, G^{-+}, R^- disappear.

Lemma 5.5. *$(G^+(x', D'))^*G^+(x', D') \in L^{-M}(\mathbf{R}^{n-1}; \mathbf{C}^M, \mathbf{C}^M)$ is elliptic in a conic neighbourhood of $\varrho_0' = (0, \xi_0')$.*

Proof. It is clear that $(G^+(x', D'))^*G^+(x', D') \in L^{-M}(\mathbf{R}^{n-1}; \mathbf{C}^M, \mathbf{C}^M)$ because $G^+(x', D') \in L^0(\mathbf{R}^{n-1}; \mathbf{C}^M, B^M_{\xi'}(\mathbf{R})) \subset L^{-M/2}(\mathbf{R}^{n-1}; \mathbf{C}^M, L^2(\mathbf{R}))$ and thus

$$G^+(x', D')^* \in L^{-M/2}(\mathbf{R}^{n-1}; L^2(\mathbf{R}), \mathbf{C}^M).$$

(5.15) gives us the identity:

$$R^+(x', D')G^+(x', D')\psi^+(x', D') \equiv \psi^+(x', D').$$

If $G^+(x', \xi')$ is the symbol of $G^+(x', D')$ we then have in a conic neighbourhood of ϱ_0':

$$R^+(x', \xi')G^+(x', \xi') \equiv I \bmod S^{-1}(\mathbf{R}^{n-1}\times \mathbf{R}^{n-1}; \mathbf{C}^M, \mathbf{C}^M). \tag{5.20}$$

From the construction of R^+ we see that $R^+(x', \xi') \in S^{M/2}(\mathbf{R}^{n-1}\times \mathbf{R}^{n-1}; L^2, \mathbf{C}^M)$ so we get from (5.20) the inequality:

$$\|y\|_{\mathbf{C}^M} \leq C|\xi'|^{M/2}\|G^+(x', \xi')y\|_{L^2(\mathbf{R})},\ y \in \mathbf{C}^M, \tag{5.21}$$

with some constant C in a conic neighbourhood of ϱ_0' for large ξ'. (5.21) implies that

$$\|y\|_{\mathbf{C}^M} \leq C^2|\xi'|^M\|G^+(x', \xi')^*G^+(x', \xi')y\|_{\mathbf{C}^M},\ y \in \mathbf{C}^M$$

and the lemma follows since $G^+(x', \xi')^*G^+(x', \xi')$ is a principal symbol of

$$G^+(x', D')^*G^+(x', D').$$

Theorem 5.6. *Let Σ be given by $x_n = \xi_n = 0$ and let $P \in L^M(\mathbf{R}^n)$ belong to $L^{M,M}(\mathbf{R}^n, \Sigma)$ in a conic neighbourhood of ϱ_0. If $\operatorname{ind}(p_M) = +M$ at ϱ_0 then there exist operators $F, F^+\colon \mathcal{D}'(\mathbf{R}^n) \to \mathcal{D}'(\mathbf{R}^n)$ with the following properties*

(5.22) *F is continuous $H^{\mathrm{loc}}_s(\mathbf{R}^n) \to H^{\mathrm{loc}}_{s+M/2}(\mathbf{R}^n)$ for all $s \in \mathbf{R}$,*

(5.23) *F^+ is continuous $H^{\mathrm{loc}}_s(\mathbf{R}^n) \to H^{\mathrm{loc}}_s(\mathbf{R}^n)$ for all $s \in \mathbf{R}$,*

(5.24) $(\varrho_0, \varrho_0) \in WF'(F) \subset \operatorname{diag}(T^*(\mathbf{R}^n) \setminus 0)$,

(5.25) $(\varrho_0, \varrho_0) \in WF'(F^+) \subset \operatorname{diag}(\Sigma)$,

(5.26) *Near ϱ_0 we have $PF \equiv I, F^+ + FP \equiv I, F^+ \equiv (F^+)^*$.*

Proof. Put

$$F^+ = G^+A'(G^+)^*, \tag{5.27}$$

where $A' \in L^M(\mathbf{R}^{n-1}; \mathbf{C}^M, \mathbf{C}^M)$ is a properly supported parametrix of $(G^+)^*G^+$ near ϱ_0'. We can assume that

$$(A')^* = A'. \tag{5.28}$$

Put

$$F = (I - F^+)G. \tag{5.29}$$

Then (5.22), (5.23) and the inclusion parts of (5.24), (5.25) follow from the construction. (5.26) also follows easily if we write (5.15) more explicitly in the form:

$$PG \equiv I,\ R^+G \equiv 0,\ PG^+ \equiv 0,\ R^+G^+ \equiv I,\ G^+R^+ + GP \equiv I, \quad \textit{all near } \varrho_0 \textit{ and } \varrho_0' \textit{ respectively.} \tag{5.30}$$

We omit the details.

That $(\varrho_0, \varrho_0) \in WF'(F)$ follows since $PF \equiv I$ near ϱ_0. To prove that $(\varrho_0, \varrho_0) \in WF'(F^+)$, we take $u \in \mathcal{D}'(\mathbf{R}^{n-1}; \mathbf{C}^M)$ so that $WF(u) = \{\lambda\varrho_0'; \lambda > 0\}$. Then $WF(G^+u) = \{\lambda\varrho_0; \lambda > 0\}$, because $R^+G^+ \equiv I$ near ϱ_0'. On the other hand we have $PG^+u \in C^\infty$ and from (5.26) we therefore get: $F^+(G^+u) \equiv G^+u \bmod C^\infty$. This implies that $(\varrho_0, \varrho_0) \in WF'(F^+)$, and Theorem 5.6 is proved.

We shall next use Proposition 5.4 to prove a microlocal version of Theorem 1.4 when Σ is given by $x_n = \xi_n = 0$. For that purpose we need a lemma, which we suppose is well known, but which we have not found in the litterature.

LEMMA 5.7. *Let $Q \in L^0(\mathbf{R}^n; \mathbf{C}^m, \mathbf{C}^k)$ and let $\varrho \in T^*(\mathbf{R}^n) \setminus 0$. If $m > k$ then there exists $u \in H_0^{\mathrm{comp}}(\mathbf{R}^n; \mathbf{C}^m)$ such that $Qu \in C^\infty$ and $WF(u) = \{\lambda\varrho; \lambda > 0\}$.*

The proof of this lemma is rather long, so we have put it in an appendix.

We now assume that $P \in L^M(\mathbf{R}^n)$ satisfies the conditions of Theorem 1.4 in a conic neighbourhood of $\varrho_0 = (0, (\xi_0', 0))$, and we also assume that Σ is given by $x_n = \xi_n = 0$. The fact that $\operatorname{ind}(p_M) > 0$ at ϱ_0 implies that the numbers m^+ and m^- in the definition of $R^+(x', \xi')$ and $R^-(x', \xi')$ satisfy the inequality

$$m^+ > m^-. \tag{5.31}$$

Thus we can apply Lemma 5.7 to see that there exists $u \in H_0^{\mathrm{comp}}(\mathbf{R}^{n-1}; \mathbf{C}^{m^+})$ such that $WF(u) = \{\lambda\varrho_0'; \lambda > 0\}$ and

$$G^{-+}u \in C^\infty. \tag{5.32}$$

We put $v = G^+u \in H_{M/2}^{\mathrm{comp}}(\mathbf{R}^n)$. It follows from (5.13) that $WF(v) \subset \{\lambda\varrho_0; \lambda > 0\}$. On the other hand, as a consequence of (5.15), we have the identity $R^+v \equiv u \bmod C^\infty$, so $WF(v)$ can not be empty. Thus $WF(v) = \{\lambda\varrho_0;\ \lambda > 0\}$. Moreover $Pv \in C^\infty(\mathbf{R}^n)$ because of (5.32) and the identity

$$(PG^+ + R^-G^{-+})u \equiv 0 \bmod C^\infty,$$

which is also a consequence of (5.15). We have thus proved

THEOREM 5.8. *Suppose that $P \in L^M(\mathbf{R}^n)$ satisfies the conditions of Theorem 1.4 in a conic neighbourhood of ϱ_0 and that Σ is given by $x_n = \xi_n = 0$. Then there exists $v \in H^{\mathrm{comp}}_{M/2}(\mathbf{R}^n)$ such that $WF(v) = \{\lambda\varrho_0;\ \lambda > 0\}$ and $Pv \in C^\infty(\mathbf{R}^n)$.*

We shall next prove microlocal versions of Theorems 1.6, 1.7 when

$$\Sigma = \{(x, \xi) \in T^*(\mathbf{R}^n) \setminus 0;\ x'' = \xi'' = 0\},\ \mathbf{R}^n \ni x = (x', x'') \in \mathbf{R}^{n-k} \times \mathbf{R}^k,$$

and $P \in L^2_1(\mathbf{R}^n)$ belongs to $L^{2,2}_c(\mathbf{R}^n, \Sigma)$ in a conic neighbourhood of

$$\varrho_0 = (0, (\xi_0', 0)) \in T^*(\mathbf{R}^n) \setminus 0.$$

Applying Taylor's formula, we get

$$P \equiv \sum_{|\alpha+\beta|=2} x''^\alpha a_{\alpha\beta}(x, D) D''^\beta + \lambda(x, D) \ \text{ near } \varrho_0, \tag{5.33}$$

where $a_{\alpha\beta} \in L^{(2+|\alpha|-|\beta|)/2}(\mathbf{R}^n)$, $\lambda \in L^1_1(\mathbf{R}^n)$ and the $a_{\alpha\beta}(x, \xi)$ are positively homogeneous of degree $(2 + |\alpha| - |\beta|)/2$ for $|\xi| \geq 1$.

THEOREM 5.9. *Suppose that $P \in L^2_1(\mathbf{R}^n)$ satisfies the conditions of Theorem 1.6 in a conic neighbourhood of ϱ_0, with Σ given by $x'' = \xi'' = 0$. Then there exists a linear operator $E\colon \mathcal{D}'(\mathbf{R}^n) \to \mathcal{D}'(\mathbf{R}^n)$ such that E is continuous $H^{\mathrm{loc}}_s(\mathbf{R}^n) \to H^{\mathrm{loc}}_{s+1}(\mathbf{R}^n)$ for all $s \in \mathbf{R}$, $WF(E) \subset \operatorname{diag}(T^*(\mathbf{R}^n) \setminus 0)$ and $PE \equiv EP \equiv I$ near ϱ_0.*

The technique of the proof is identical to the proofs above. The only difference is that we now consider vector valued pseudodifferential operators on $\mathbf{R}^{n-k}$ instead of $\mathbf{R}^{n-1}$, and that we have no "interior boundary" operators to worry about. We shall not repeat all the details. What we actually have to verify, is that

$$L_0(x, \xi', D'') = \sum_{|\alpha+\beta|=2} x''^\alpha a_{\alpha\beta}(x', 0, \xi', 0) D''^\beta + \lambda_1(x', 0, \xi', 0)\colon B^2_{\xi'}(\mathbf{R}^k) \to L^2(\mathbf{R}^k)$$

has a uniformly bounded inverse for (x', ξ') in a conic neighbourhood of $(0, \xi_0')$ when $|\xi'| \geq 1$. Here λ_1 is the positively homogeneous principal symbol of $\lambda(x, D)$. To prove this, we want to apply Theorem 3.5 or rather Remark 3.6 (with n replaced by k), so we write L_0 on a symmetric form:

$$\begin{aligned} L_0(x, \xi', D'') = \tfrac{1}{2} \sum_{|\alpha+\beta|=2} (x''^\alpha a_{\alpha\beta}(x', 0, \xi', 0) D''^\beta + D''^\beta a_{\alpha\beta}(x', 0, \xi', 0) x''^\alpha) + \\ + \left(\lambda_1(x', 0, \xi', 0) - \frac{1}{2i} \sum_{|\alpha|=1} a_{\alpha\alpha}(x', 0, \xi', 0)\right). \end{aligned}$$

Now we note:

1° $\lambda_1(x', 0, \xi', 0) - \frac{1}{2i} \sum_{|\alpha|=1} a_{\alpha\alpha}(x', 0, \xi', 0)$ is the subprincipal symbol of P at Σ.

2° $\sum x''^{\alpha} a_{\alpha\beta}(x', 0, \xi', 0)\xi''^{\beta}$ is precisely the quadratic form $a(t, t)$ in the introduction.

Thus the conditions of Theorem 1.6 imply that the conditions of Remark 3.6 are satisfied for the operator $L_0(x, \xi', D'')$, so this operator has a uniformly bounded inverse. Thus Theorem 5.9 is proved modulo details.

We shall finally give a microlocal version of Theorem 1.7. Thus assume that $P \in L^2_1(\mathbf{R}^n)$ belongs to $L^{2,2}_c(\mathbf{R}^n, \Sigma)$ in a conic neighbourhood of

$$\varrho_0 = (0, (\xi_0', 0)) \in T^*(\mathbf{R}^n) \setminus 0$$

and that $\operatorname{ind} p_m = 0$ on Σ when $\operatorname{codim} \Sigma = 2$. Σ is here given by $x'' = \xi'' = 0$. We assume that the condition (1.17) is not satisfied at ϱ_0. In view of Theorem 3.5 this means that the kernel and the cokernel of

$$L_0(0, x'', \xi_0', D''): B^2(\mathbf{R}^k) \to L^2(\mathbf{R}^k)$$

have the same dimension $r \neq 0$. As before we construct

$$R^+(x', D') \in L^0(\mathbf{R}^{n-k}; B^2_{\xi'}(\mathbf{R}^k), \mathbf{C}^r),$$

$$R^-(x', D') \in L^0(\mathbf{R}^{n-k}; \mathbf{C}^r, L^2(\mathbf{R}^k)),$$

so that Proposition 5.4 remains valid. More precisely there exists a map

$$\mathcal{G} = \begin{pmatrix} G & G^+ \\ G^- & G^{-+} \end{pmatrix} : C^\infty(\mathbf{R}^n) \oplus C^\infty(\mathbf{R}^{n-k}; \mathbf{C}^r) \to C^\infty(\mathbf{R}^n) \oplus C^\infty(\mathbf{R}^{n-k}; \mathbf{C}^r)$$

so that Proposition 5.4 is valid with $m^+ = m^- = r$ and with $\mathbf{R}^{n-1}$ and $\mathbf{R}$ everywhere replaced by $\mathbf{R}^{n-k}$ and $\mathbf{R}^k$ in the formulas. The construction and the proofs remain unchanged, so in particular $G^+ = \chi E^+$, where $\chi \in L^0(\mathbf{R}^n)$ has its wavefront set close to ϱ_0 and $\chi \equiv I$ near ϱ_0 and $E^+ \in L^0(\mathbf{R}^{n-k}; \mathbf{C}^r, B^2_{\xi'}(\mathbf{R}^k))$. Proposition 5.3 remains valid, so G^{-+} $(= E^{-+})$ belongs to $L^0(\mathbf{R}^{n-k}; \mathbf{C}^r, \mathbf{C}^r)$ and has modulo $S^{-1/2}$ a symbol $G_0^{-+}(x', \xi')$ which is positively homogeneous of degree 0. Moreover $G_0^{-+}(0, \xi_0') = 0$. This non-ellipticity property implies that for all $\varepsilon > 0$ and $0 < \delta < 1/2$, there exists $v \in H^{\mathrm{comp}}_{-\delta}(\mathbf{R}^{n-k}) \setminus H^{\mathrm{comp}}_{-\delta+\varepsilon}(\mathbf{R}^{n-k})$ such that $WF(v) = \{(0, \lambda\varrho_0'); \lambda > 0\}$ and $G^{-+}(x', D')v \in H^{\mathrm{comp}}_0(\mathbf{R}^{n-k})$. This has been proved essentially by Hörmander [12] although the wavefront sets did not exist at that time.

With such a v we put $u = G^+v \in H^{\mathrm{comp}}_{1-\delta}(\mathbf{R}^n)$. Clearly $WF(u) \subset \{\lambda\varrho_0; \lambda > 0\}$. By (5.15) we have

$$PG^+v + R^-G^{-+}v \in C^\infty, \quad R^+G^+v \equiv v \bmod C^\infty.$$

From the second equation we see that $u = G^+v \in H^{\text{comp}}_{1-\delta} \setminus H^{\text{comp}}_{1-\delta+\varepsilon}$ and that $WF(u) = \{\lambda\varrho_0, \lambda > 0\}$. From the first equation we get $Pu \in H^{\text{loc}}_0(\mathbf{R}^n)$. To complete the proof of our microlocal Theorem 1.7 we have to prove that $Qu \in H^{\text{loc}}_{1-\delta+1/2}(\mathbf{R}^n)$ for all $Q \in L^0(\mathbf{R}^n)$ with positively homogeneous principal symbol, vanishing on Σ. Since $WF(u) = \{\lambda\varrho_0; \lambda > 0\}$ we can assume that $WF(Q)$ is close to ϱ_0. Since $\chi \equiv I$ near ϱ_0, we then have

$$Qu = QG^+v = Q\chi E^+v \equiv \chi QE^+v \mod C^\infty.$$

By Taylor's formula:

$$Q(x, D) = \sum_{|\alpha|=1} (A_\alpha(x, D)x''^\alpha + B_\alpha(x, D)D''^\alpha) + C(x, D)$$

where $A \in L^0, B \in L^{-1}, C \in L^{-1}$. Using the fact that $E^+ \in L^0(\mathbf{R}^{n-k}; \mathbf{C}^r, B^2_{\xi'})$, we obtain that $Qu \in H^{\text{loc}}_{1-\delta+1/2}(\mathbf{R}^n)$.

Summing up, we have proved

THEOREM 5.10. *Suppose that* $P \in L^2_1(\mathbf{R}^n)$ *belongs to* $L^{2,2}_c(\mathbf{R}^n, \Sigma)$ *in a conic neighbourhood of* $\varrho_0 = (0, (\xi'_0, 0))$, *where* Σ *is given by* $x'' = \xi'' = 0$. *If* (1.17) *is not valid at* ϱ_0, *then for every* $0 < \delta < 1/2$ *and* $\varepsilon > 0$ *there exists*

$$u \in H^{\text{comp}}_{1-\delta}(\mathbf{R}^n) \setminus H^{\text{comp}}_{1-\delta+\varepsilon}(\mathbf{R}^n)$$

such that $WF(u) = \{\lambda\varrho_0; \lambda > 0\}$, $Pu \in H^{\text{loc}}_0(\mathbf{R}^n)$ *and* $Qu \in H^{\text{loc}}_{1-\delta+1/2}(\mathbf{R}^n)$ *for all* $Q \in L^{0,1}(\mathbf{R}^n, \Sigma)$.

Remark 5.11. The idea to reduce the study of P to the study of G^{-+} is due to Grušin [9]. He has shown that the hypoellipticity of an operator on $\mathbf{R}^n$, of the type studied in this section, is sometimes equivalent to the hypoellipticity of another operator on $\mathbf{R}^{n-k}$. His general implicit results can be refined by working with wavefront sets. We let the reader verify (using Proposition 5.4) that several properties for P and G^{-+} are valid simultaneously; such as hypoellipticity, existence of parametrices or of a priori estimates. The problem of propagation of singularities for P can also be reduced to the same problem for G^{-+}. Everything is here microlocal of course.

6. Application of Fourier integral operators

We shall first prove Theorem 1.2. On many essential points the proof will be the same as in [4]. It is well known and easy to prove that a submanifold $\Sigma \subset T^*(\Omega) \setminus 0$ of codimension 2 is non-involutive if and only if Σ can be locally given by $p_1(x, \xi) = p_2(x, \xi) = 0$ where p_1 and p_2 are smooth real valued functions such that $\{p_1, p_2\} \neq 0$. In the case when Σ is conic, we can choose p_1 and p_2 positively homogeneous of degree 0. It is also well known that if Σ is such a surface, then locally there exists a homogeneous canonical transformation

$$\mathcal{H}\colon T^*(\Omega) \setminus 0 \to T^*(\mathbf{R}^n) \setminus 0$$

such that Σ is mapped into the plane $x_n = \xi_n = 0$. See for instance [4]. This observation makes it possible to apply Fourier integral operators and Theorem 1.2 will follow from Theorem 5.6. As the main step we shall prove the following microlocal result.

PROPOSITION 6.1. *Let* $P \in L^{m,M}(\Omega, \Sigma)$ *be as in Theorem* 1.2. *Then for every* $\varrho \in T^*(\Omega) \setminus 0$, *there exists a conic open neighbourhood* $V_\varrho \subset T^*(\Omega) \setminus 0$ *and properly supported operators* $F_\varrho, F_\varrho^+, F_\varrho^-\colon \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ *satisfying* (1.7)—(1.9) *in* V_ϱ *and satisfying* (1.6). *Moreover*

(6.1) $(\varrho, \varrho) \in WF'(F_\varrho) \subset \operatorname{diag}(T^*(\Omega) \setminus 0)$,

(6.2) $WF'(F_\varrho^\pm) \subset \operatorname{diag}(\Sigma^\pm)$ *and if* $\varrho \in \Sigma^\pm$ *we have* $(\varrho, \varrho) \in WF'(F^\pm)$.

Proof. If $\varrho \notin \Sigma$, we can take $F_\varrho \in L^{-m}(\Omega)$ being a microlocal parametrix of P and we can take $F_\varrho^\pm = 0$. Since the result for $\varrho \in \Sigma^-$ can be obtained by duality, once we have settled the case $\varrho \in \Sigma^+$, we see that it suffices to study the case $\varrho \in \Sigma^+$. Then of course we shall take $F_\varrho^- = 0$ and after multiplying P to the left by an elliptic operator of suitable order we can assume that $m = M$.

We now let $\mathcal{H}\colon T^*(\Omega) \setminus 0 \to T^*(\mathbf{R}^n) \setminus 0$ be a homogeneous canonical transformation, defined in a conic neighbourhood of ϱ and mapping Σ into

$$\tilde{\Sigma} = \{(x, \xi) \in T^*(\mathbf{R}^n) \setminus 0;\, x_n = \xi_n = 0\}.$$

Put $\varrho_0 = \mathcal{H}(\varrho)$ and let Γ be a closed conic subset of the graph of $\mathcal{H}$, containing (ϱ_0, ϱ) as an interior point. Put

$$\Gamma' = \{(\mu, -\nu);\, (\mu, \nu) \in \Gamma\}.$$

To $\mathcal{H}$ we associate a properly supported Fourier integral operator

$$A \in I^0(\mathbf{R}^n \times \Omega;\, \Gamma')$$

with positively homogeneous principal symbol of degree 0 non-vanishing at $(\varrho_0, -\varrho)$. (Here we use Hörmander's notations [11].) Then $A^*A \in L^0(\Omega)$ is non-characteristic at ϱ and applying Proposition 2.2.2 in Hörmander [11] we see that there exists $B \in \mathrm{L}^0(\Omega)$ non-characteristic at ϱ such that $B^*B \equiv A^*A$ near ϱ. Let $B' \in L^0(\Omega)$ be a properly supported parametrix of B near ϱ and put

$$U = AB' \in I^0(\mathbf{R}^n \times \Omega,\, \Gamma'). \tag{6.3}$$

Then

$$U^*U \equiv I \text{ near } \varrho, \quad UU^* \equiv I \text{ near } \varrho_0. \tag{6.4}$$

One of the fundamental results of [11] now tells us that for every $Q \in L^k(\Omega)$ with principal symbol q we have $UQU^* \in L^k(\mathbf{R}^n)$ and the principal symbol is $q \circ \mathcal{H}^{-1}$ in a neighbourhood of $\varrho_0 = \mathcal{H}(\varrho)$. It also follows from the proofs in [11], that if A (and U) are choosen with symbols which are asymptotic sums of positively homogeneous symbols of suitable orders, for every admissible choice of phase functions and local coordinates, then UQU^* is a classical pseudodifferential operator if Q is. (Q is called a classical pseudodifferential operator of order k if the symbol is an asymptotic sum of positively homogeneous symbols of orders $k, k-1, k-2, \ldots$)

LEMMA 6.2. *$\tilde{P} = UPU^* \in L^M(\mathbf{R}^n)$ belongs to $L^{M,M}(\mathbf{R}^n, \tilde{\Sigma})$ in a conic neighbourhood of ϱ_0 and $\operatorname{ind} \tilde{p}_M = M$ at ϱ_0 if $\tilde{p}_M$ is the positively homogeneous principal symbol of $\tilde{P}$.*

Proof. Since $\tilde{p}_M = p_M \circ \mathcal{H}^{-1}$ near ϱ_0 it is clear that $\tilde{p}_M$ satisfies (1.2a) near ϱ_0 with respect to $\tilde{\Sigma}$ and that $\operatorname{ind} \tilde{p}_M = M$ at ϱ_0. The problem is to handle the lower order terms. Take $q_1, q_2 \in C^\infty(T^*(\Omega) \setminus 0)$ positively homogeneous of degree 0 so that Σ is given by $q_1(x, \xi) = q_2(x, \xi) = 0$ in a neighbourhood of ϱ and $\{q_1, q_2\} \neq 0$ near ϱ. Let $Q_1, Q_2 \in L^0(\Omega)$ be classical properly supported operators with principal symbols q_1, q_2. Then using Taylor's formula, (1.2b) and (1.3) it is easy to see that

$$P \equiv \sum_{j+k \leq M} A_{jk}(x, D) Q_1^j Q_2^k + B \text{ near } \varrho, \tag{6.5}$$

where $A_{jk}(x, D) \in L^{(M+j+k)/2}(\Omega)$ and $B(x, D) \in L^{(M-1)/2}(\Omega)$ are classical pseudodifferential operators. Conversely if (6.5) is valid with such A_{jk} and B, we see that (1.2b) is valid. Putting $\tilde{A}_{jk} = UA_{jk}U^*$, $\tilde{Q}_\nu = UQ_\nu U^*$, $\tilde{B} = UBU^*$ and using (6.4), we see that

$$\tilde{P} \equiv \sum_{j+k \leq M} \tilde{A}_{jk} \tilde{Q}_1^j \tilde{Q}_2^k + \tilde{B} \text{ near } \varrho_0. \tag{6.6}$$

Hence $\tilde{P} \in L^{M,M}(\mathbf{R}^n, \tilde{\Sigma})$ near ϱ_0.

We now apply Theorem 5.6 to $\tilde{P}$ and let $\tilde{F}, \tilde{F}^+$ be the corresponding parametrix operators, satisfying (5.20)—(5.24). Putting $F_\varrho = U^*\tilde{F}U$, $F_\varrho^+ = U^*\tilde{F}^+U$ it is immediate that F_ϱ, F_ϱ^+ have all the properties stated in Proposition 6.1 if we note that $WF'(U)$ and $WF'(U^*)$ are contained in the graphs of $\mathcal{H}$ and $\mathcal{H}^{-1}$ respectively.

PROPOSITION 6.3. *Let* $V_\varrho, F_\varrho, F_\varrho^\pm$ *be as in Proposition* 6.1. *Then* $F_\varrho \equiv F_{\varrho'}$ *and* $F_\varrho^\pm \equiv F_{\varrho'}^\pm$ *in* $V_\varrho \cap V_{\varrho'}$ *for all* $\varrho, \varrho' \in T^*(\Omega) \setminus 0$.

We refer to [4, Section 2] for the proof.

Now let $\chi_j \in L^0(\Omega), j \in J$ be a locally finite collection of properly supported pseudodifferential operators such that for corresponding $\varrho_j \in T^*(\Omega) \setminus 0$ we have

$$WF(\chi_j) \subset V_{\varrho_j}, \quad \sum_{j \in J} \chi_j \equiv I. \tag{6.7}$$

Define

$$F = \sum \chi_j F_{\varrho_j}, \quad F^\pm = \sum \chi_j F_{\varrho_j}^\pm. \tag{6.8}$$

Clearly $WF'(F) \subset \operatorname{diag}(T^*(\Omega) \setminus 0)$, $WF'(F^\pm) \subset \operatorname{diag}(\Sigma^\pm)$, so after adding operators with C^∞ kernels, we can assume that $F, F^\pm$ are properly supported and have the H_s-continuity properties as in Theorem 1.2. It follows from Proposition 6.3 that $F \equiv F_\varrho, F^\pm \equiv F_\varrho^\pm$ in V_ϱ for all $\varrho \in T^*(\Omega) \setminus 0$. Thus all the properties in Theorem 1.2 follow from Proposition 6.1. Apart from Lemma 6.2 the proof above has been taken from [4]. (Similar partitions of unity and applications of Fourier integral operators have been made by Duistermaat and Hörmander [3].)

Theorem 1.4 follows from Theorem 5.8 by a much simpler application of Fourier integral operators and we omit the details.

We next want to prove Theorems 1.6 and 1.7. First a more or less wellknown lemma:

LEMMA 6.4. *Let* $\Sigma \subset T^*(\Omega) \setminus 0$ *be a closed conic non-involutive submanifold and let* $\varrho \in \Sigma$. *Then there exists a homogeneous canonical transformation* $\mathcal{H}: T^*(\Omega) \setminus 0 \to T^*(\mathbf{R}^n) \setminus 0$ *defined in a conic neighbourhood of* ϱ *and mapping* Σ *into* $\tilde{\Sigma} = \{(x, \xi) \in T^*(\mathbf{R}^n) \setminus 0; x'' = \xi'' = 0\}$. *Here* $x'', \xi'' \in \mathbf{R}^k$, $2k = \operatorname{codim}(\Sigma)$.

Proof. We know already that this is true when $\operatorname{codim} \Sigma = 2$ and we shall make an induction over $\operatorname{codim} \Sigma$. In a neighbourhood of ϱ we take a real valued $q_1 \in C^\infty$, positively homogeneous of degree $1/2$ such that $q_1 = 0$ on Σ and $dq_1 \neq 0$. Since Σ is non-involutive, the Hamilton field H_{q_1} is not tangential to Σ and there exists therefore a real valued C^∞ function q_2, positively homogeneous of degree $1/2$ such that $H_{q_1} q_2 = \{q_1, q_2\} = 1$ near ϱ and $q_2 = 0$ on Σ.

Σ_0 defined by $q_1(x, \xi) = q_2(x, \xi) = 0$ is then a non-involutive submanifold of codimension 2 so there exists a homogeneous canonical transformation

$$\mathcal{H}_0: T^*(\Omega) \setminus 0 \to T^*(\mathbf{R}^n) \setminus 0,$$

defined in a conic neighbourhood of ϱ and mapping Σ_0 into the plane $x_n = \xi_n = 0$. Since $\Sigma \subset \Sigma_0$ near ϱ, we see that

$$\mathcal{H}_0(\Sigma) = \Sigma' \times 0 \subset (T^*(\mathbf{R}^{n-1}) \setminus 0) \times (T^*(\mathbf{R})),$$

where $\Sigma' \subset T^*(\mathbf{R}^{n-1}) \setminus 0$ is non-involutive, conic and $\operatorname{codim} \Sigma' = \operatorname{codim}(\Sigma) - 2$. Thus if the lemma is true for Σ', it must be true for Σ and the lemma follows by induction.

The lemma gives us the existence of a suitable canonical transformation in order to apply the Fourier integral operators. We must also check that the condition (1.17) is stable under conjugation with such operators:

LEMMA 6.5. *Suppose that* $P = AQ_1Q_2 + B$, *where* $A \in L^m(\Omega)$, $Q_1, Q_2 \in L^0(\Omega)$, $B \in L^{m-1}(\Omega)$ *are properly supported classical pseudodifferential operators with principal symbols* a, q_1, q_2, b. *Let* $\varrho \in T^*(\Omega) \setminus 0$ *be a point where* $q_1 = q_2 = 0$. *If we write the symbol of* P *in local coordinates as*

$$p(x, \xi) = p_m(x, \xi) + p_{m-1}(x, \xi) \mod S^{m-2}$$

where p_m *and* p_{m-1} *are positively homogeneous of degree* m *and* $m-1$, *then the subprincipal symbol*

$$S_P(x, \xi) = p_{m-1}(x, \xi) - (2i)^{-1} \sum_1^n \frac{\partial^2 p_m(x, \xi)}{\partial x_j \partial \xi_j}$$

is invariantly defined at ϱ *and is in fact given by the formula*

$$S_P(\varrho) = b(\varrho) + a(\varrho)(2i)^{-1}\{q_1, q_2\}(\varrho).$$

The proof is evident. We now let $P \in L_c^{m,2}(\Omega, \Sigma)$ satisfy the assumptions of Theorem 1.6. Let $\varrho \in \Sigma$. Near ϱ, Σ is given by $q_1 = q_2 = \ldots = q_{2k} = 0$ where q are real valued positively homogeneous C^∞ functions of degree 0 and $dq_1, dq_2, \ldots, dq_{2k}$ are linearly independent. If $Q_\nu \in L^0(\Omega)$ are properly supported classical pseudodifferential operators with principal symbols q_ν, we can apply Taylor's formula and write

$$P \equiv \sum\sum A_{jk} Q_j Q_k + B \quad \text{near } \varrho, \tag{6.9}$$

where $A_{jk} \in L^m$, $B \in L^{m-1}$, are classical pseudodifferential operators. We denote the principal symbols by a_{jk} and b. It follows from Lemma 6.5 that S_P is invariantly defined on Σ and given by the formula

$$S_P = (2i)^{-1} \sum\sum a_{jk}\{q_j, q_k\} + b. \tag{6.10}$$

It follows from (6.9), (6.10) that the condition (1.17) is invariant under multiplication of P with elliptic operators. Thus we can assume from now on that $P \in L^{2,2}(\Omega, \Sigma)$.

Let $\mathcal{H}\colon T^*(\Omega) \setminus 0 \to T^*(\mathbf{R}^n) \setminus 0$ be the homogeneous canonical transformation in Lemma 6.4, mapping Σ into $\tilde{\Sigma}$, given by $x'' = \xi'' = 0$. Let $U \in I^0(\mathbf{R}^n \times \Omega, \Gamma')$ be an associated Fourier integral operator, properly supported which is non-characteristic at $(\mathcal{H}(\varrho), \varrho) = (\varrho_0, \varrho)$. Take $U' \in I^0(\Omega \times \mathbf{R}^n, (\Gamma^{-1})')$ such that $U'U \equiv I$ near ϱ and $UU' \equiv I$ near ϱ_0. Put $\tilde{P} = UPU' \in L^2(\mathbf{R}^n)$. Then $\tilde{P} \in L^{2,2}(\mathbf{R}^n, \tilde{\Sigma})$ in a conic neighbourhood of ϱ_0 in view of (6.9) if U is suitably chosen so that the homogeneity in the lower order symbols is preserved. Moreover $S_{\tilde{P}} \circ \mathcal{H} = S_P$ at Σ in a conic neighbourhood of ϱ, in view of (6.10). If $p_2, \tilde{p}_2$ are the positively homogeneous principal symbols of P and $\tilde{P}$ respectively, then $\tilde{p}_2 \circ \mathcal{H} = p_2$ near ϱ. If $A_\varrho, \tilde{A}_{\varrho_0}$ are the corresponding matrices defined by (1.16), we obtain therefore that $A_\varrho = (d\mathcal{H})^{-1}\tilde{A}_{\varrho_0} d\mathcal{H}$ so A_ϱ and $\tilde{A}_{\varrho_0}$ have the same eigenvalues with the same algebraic multiplicities.

We have now shown that the condition (1.17) is satisfied for P at ϱ if and only if it is satisfied for $\tilde{P}$ at ϱ_0. Thus an application of Fourier integral operators and Theorem 5.9 gives us a microlocal version of Theorem 1.6 analogous to Proposition 6.1. By a partition of unity as above we obtain Theorem 1.6 globally. In the same way we get Theorem 1.7 from Theorem 5.10.

We shall end this section by giving an example of Theorems 1.6 and 1.7. Let $P, Q \in L^{m/2}(\Omega)$ be classical pseudodifferential operators, with principal symbols p, q, positively homogeneous of degree $m/2$. Suppose that P and Q have the same characteristic set $\emptyset \neq \Sigma \subset T^*(\Omega) \setminus 0$ and that $-i\{p, \bar{p}\} > 0$, $-i\{q, \bar{q}\} < 0$ on Σ. Then it is well known that QP is neither locally solvable nor hypoelliptic so it can not satisfy the conditions of Theorem 1.6. (See for instance [4].) However PQ does and we are going to verify that.

First of all we see that Σ must be a closed conic non-involutive submanifold of codimension 2, since Σ is given by the equations $\operatorname{Re} p = \operatorname{Im} p = 0$ and $\{\operatorname{Re} p, \operatorname{Im} p\} \neq 0$ on Σ. It is easy to see that $P \circ Q \in L^{m,2}(\Omega, \Sigma)$ and that $\operatorname{ind}(pq) = 0$ on Σ. (See Proposition 1.3.) By Lemma 6.5 we have

$$S_{P \circ Q} = (2i)^{-1}\{p, q\} \text{ at } \Sigma. \tag{6.11}$$

For $\varrho \in \Sigma$, let $a_\varrho(t, t)$ be the quadratic form defined in the introduction, so that in our case we have

$$a_\varrho(t, t) = \langle t, dp\rangle\langle t, dq\rangle,\ t \in T_\varrho(T^*(\Omega)).$$

Let A_ϱ be defined by (1.16). It is natural to guess that the complex Hamilton vectors H_p and H_q are the eigenvectors of A_ϱ with non-zero eigenvalues. An easy check shows that this is true and that the corresponding eigenvalues are $\frac{1}{2}\{p, q\}$ and $-\frac{1}{2}\{p, q\}$.

It is easy to verify that K_ϱ, defined in the introduction, is the set of eigenvalues of A_ϱ for which the corresponding eigenvectors v satisfy $-i\,\sigma(v, \bar{v}) > 0$. (Cf. Lemma 3.8 and its proof.) In particular in our example K_ϱ contains $\frac{1}{2}\{p, q\}$ as its only element and the condition (1.17) takes the form

$$\frac{1}{2i}\{p, q\} + \frac{1}{i}(\tfrac{1}{2}\{p, q\} + k\{p, q\}) \neq 0, \quad k \in \mathbf{Z}^+ \cup \{0\}, \quad \text{at } \Sigma,$$

which is trivially satisfied. On the other hand we see that for the operator $Q \circ P$ the condition (1.17) is not satisfied anywhere on Σ.

Remark 6.6. Boutet de Monvel and Trèves [1] have independently studied the case when $\operatorname{codim}(\Sigma) = 2, M = 2$. In this case it is always possible to write the operator in the form $P \circ Q + R$ with P and Q as above and with $R \in L^{m-1}(\Omega)$ having a positively homogeneous principal symbol r of degree $m - 1$. The condition (1.17) takes the form:

$$r(\varrho) + \frac{1}{i} k\{p, q\}(\varrho) \neq 0 \text{ for all } k \in \mathbf{Z}^+ \text{ and } \varrho \in \Sigma. \tag{6.12}$$

Boutet de Monvel-Trèves have shown the equivalence between (6.12) and the estimate (1.18).

We now generalize our example and take k couples (P_ν, Q_ν), $1 \leq \nu \leq k$ as above. Let the principal symbols be p_ν, q_ν, and let Σ be defined by

$$p_1 = p_2 = \ldots = p_k = 0.$$

We assume that

1° $d \operatorname{Re} p_1, d \operatorname{Im} p_1, \ldots, d \operatorname{Re} p_k, d \operatorname{Im} p_k$ are linearly independent on Σ,

2° $\{p_\nu, p_\mu\} = \{q_\nu, q_\mu\} = 0$ on Σ for all ν, μ,

3° $\{p_\nu, q_\mu\} = 0$ on Σ when $\nu \neq \mu$.

The manifold Σ is then non-involutive of codimension $2k$ and it may happen that $\sum_1^k P_\nu \circ Q_\nu \in L^{m,2}(\Omega, \Sigma)$ (for instance if the q_ν do not differ too much from the $\bar{p}_\nu$). In that case the condition (1.17) takes the form:

$$\sum_1^\nu \left(\frac{1}{2i}\{p_\nu, q_\nu\} + \frac{1}{i}(\tfrac{1}{2}\{p_\nu, q_\nu\} + a_\nu\{p_\nu, q_\nu\})\right) \neq 0 \text{ on } \Sigma \text{ for all } a_\nu \in \mathbf{Z}^+ \cup \{0\}.$$

This condition is trivially satisfied, because the $\frac{1}{2}\{p_\nu, q_\nu\}$ belong to some common open half-plane in $\mathbf{C}$. On the other hand the operator $\sum_1^\nu Q_\nu P_\nu$ does not satisfy (1.17).

Appendix

In order to prove Lemma 5.7, we shall prove that any $p \in L^M(\mathbf{R}^n; \mathbf{C}^m, \mathbf{C}^k)$ can be multiplied to the left and to the right by elliptic factors, so that the symbol becomes asymptotically equal to a matrix (a_{jk}) with $a_{jk} = 0$ for $j \neq k$ and $a_{jk} = 0$ or 1 when $j = k$. This can hold of course only in a suitable subset of $T^*(\mathbf{R}^n) \setminus 0$ which we now define.

Definition A.1. Let $\delta \geq 0$, $0 < C < 1/2$ and suppose that

$$K = \{(x_\nu, \xi_\nu) \in T^*(\mathbf{R}^n) \setminus 0;\ \nu = 1, 2, \ldots\},$$

where $\{x_\nu\}$ is bounded and $2 \leq 2|\xi_\nu| \leq |\xi_{\nu+1}|$ for all ν. We write

$$V = M(K, C, \delta)$$

for the disjoint union:

$$V = \bigcup_{\nu=1}^{\infty} \{(x, \xi) \in T^*(\mathbf{R}^n) \setminus 0;\ |x - x_\nu| < C|\xi_\nu|^{-\delta},\ |\xi - \xi_\nu| \leq C|\xi_\nu|^{1-\delta}\}$$

and we say that V is a δ-set. If $W = M(K, C', \delta)$ is another δ-set, we write $W = (C'/C)V$. For any subset Ω of V, we write $\Omega \prec V$ if $\Omega \subset \varrho V$ for some ϱ with $0 < \varrho < 1$.

If $\delta' \geq \delta \geq 0$ and V is a δ-set we define $S^m_{1-\delta'}(\bar{V})$ to be the space of restrictions to $\bar{V}$ of elements in $S^m_{1-\delta'}(\mathbf{R}^n \times \mathbf{R}^n)$. We define $S^m_{1-\delta'}(V)$ as the projective limit of all $S^m_{1-\delta'}(\varrho\bar{V})$ with $0 < \varrho < 1$. It is easy to verify that $S^m_{1-\delta'}(V)$ is the set of $p(x, \xi) \in C^\infty(V)$ satisfying the usual estimates for the $D^\alpha_x D^\beta_\xi p$ uniformly in all the $\varrho\bar{V}, 0 < \varrho < 1$. The notion of asymptotic convergence is immediately defined in $S^m_{1-\delta'}(\bar{V})$ and in $S^m_{1-\delta'}(V)$. In particular if $p_j \in S^{m_j}_{1-\delta'}(V)$, $j = 1, 2$, $0 \leq \delta \leq \delta' < 1/2$ we can define $p_1 \circ p_2 \in S^{m_1+m_2}_{1-\delta'}(V)$ modulo $S^{-\infty}(V)$ by the usual composition formula

$$p_1 \circ p_2 \sim \sum p_1^{(\alpha)} p_{2(\alpha)}/i^{|\alpha|}\alpha!,\quad p_{(\alpha)} = \partial^\alpha p/\partial x^\alpha \quad p^{(\alpha)} = \partial^\alpha p/\partial \xi^\alpha.$$

THEOREM A.2. *Let $0 \leq \delta < \delta' < 1/2$, $m \in \mathbf{R}$ and let $p \in S^m_{1-\delta}(V; \mathbf{C}^{m_1}, \mathbf{C}^{m_2})$ where V is a δ-set. Then there exists a δ'-set $W \prec V$ and $p_j, p_j' \in S^M_{1-\delta'}(W; \mathbf{C}^{m_j}, \mathbf{C}^{m_j})$ $j = 1, 2$ for some $M \in \mathbf{R}$ such that*

$$p_j \circ p_j' \sim p_j' \circ p_j \sim \textit{identity in } \mathbf{C}^{m_j},\ j = 1, 2$$

and

$$p_2 \circ p \circ p_1 \sim (a_{jk})_{\substack{1 \leq j \leq m_2 \\ 1 \leq k \leq m_1}} \textit{ in } W$$

where $a_{jk} = 0$ for $j \neq k$ and $a_{jk} = 0$ or 1 for $j = k$.

In general, if $p, q \in S^m_{1-\delta'}(W; \mathbf{C}^{m_1}, \mathbf{C}^{m_2})$ and $p_2 \circ p \circ p_1 \sim q$ in W with W, p_1, p_2 as in the theorem, we shall say that p and q are equivalent in W. For the proof of the theorem we need a definition and two lemmas.

Definition A.3. Let V be a δ-set and let $p \in S^M_{1-\delta'}(\bar{V})$. Then we write $\deg_{\bar{V}} p$ for the infimum of all $m \in \mathbf{R}$ such that $\sup_{(x,\xi)\in\bar{V}} |p(x, \xi)|/(1 + |\xi|^m) < +\infty$. For $p \in S^M_{1-\delta'}(V)$ we define $\deg_V p = \sup_{0<\varrho<1} \deg_{\overline{\varrho V}} p$. Clearly $\deg_V p \leq M$.

LEMMA A.4. *Let* V *be a* δ*-set and let* $p \in S^M_{1-\delta'}(V)$*, where* $\delta' \geq \delta \geq 0$*. Then* $\deg_V (D^\alpha_x D^\beta_\xi p) \leq \deg_V (p) + |\alpha|\delta' - |\beta|(1 - \delta')$ *for all multiindices* α *and* β*. In particular* $p \in S^{\deg(p)+\varepsilon}_{1-\delta'}(V)$ *for all* $\varepsilon > 0$.

Proof. Suppose that $V = \bigcup_1^\infty V_\nu$ as in Definition A.1, where λV_ν, $\lambda > 0$ (and $V_\nu = 1 \cdot V_\nu$) are given by

$$|x - x_\nu| \leq C\lambda|\xi_\nu|^{-\delta},\ |\xi - \xi_\nu| \leq C\lambda|\xi_\nu|^{1-\delta}$$

with some positive constant $C < 1/2$. As in the proof of Theorem 2.9 in Hörmander [13], we use the elementary inequality

$$|v'(0)| \leq (2 \sup |v| \cdot \sup |v''|)^{1/2} + 2a^{-1} \sup |v|,\ v \in C^2([-a, a]).$$

For given λ, μ with $0 < \lambda < \mu < 1$ we obtain with some constant C

$$\sup_{\lambda V_\nu} |D_{x_j} p| \leq C((\sup_{\mu V_\nu} |p| \sup_{\mu V_\nu} |D^2_{x_j} p|)^{1/2} + |\xi_\nu|^\delta \sup_{\mu V_\nu} |p|). \tag{A.1}$$

In fact, for every $(x, \xi) \in \lambda V_\nu$ it suffices to consider the restriction of p to the straight line through (x, ξ) parallel to the x_j-axis. From (A.1) we obtain

$$\deg (D_{x_j} p) \leq \max (\deg (p)/2 + \deg (D^2_{x_j} p)/2,\ \delta + \deg (p)),$$

where the degrees are taken in V. In this inequality we can of course replace p by $D^{\nu-1}_{x_j} p$ for any $\nu \geq 1$. Thus if we put $a_\nu = \deg (D^\nu_{x_j} p) - \nu\delta$, we obtain

$$a_\nu \leq \max ((a_{\nu-1} + a_{\nu+1})/2,\ a_{\nu-1}). \tag{A.2}$$

The fact that $p \in S^M_{1-\delta'}(V)$ implies

$$a_\nu \leq M + \nu(\delta' - \delta). \tag{A.3}$$

With a simple geometric (convexity) argument we obtain from (A.2) and (A.3) that

$$a_\nu \leq \deg (p) + \nu(\delta' - \delta),$$

and in particular that $\deg (D_{x_j} p) \leq \deg (p) + \delta'$, $1 \leq j \leq n$. Similarly we have $\deg (D_{\xi_j} p) \leq \deg (p) + \delta' - 1$, $1 \leq j \leq n$ and by iteration we get our lemma.

LEMMA A.5. *Suppose that* $p \in S^M_{1-\delta}(V)$*, where* V *is a* δ*-set,* $0 \leq \delta < 1/2$*. If* $\deg_V p = 0$ and $\delta < \delta' < 1/2$*, there exists a* δ'*-set* $W \prec V$ *and a symbol* $p' \in S^{(\delta'-\delta)/2}_{1-\delta'}(W)$ *such that* $p \circ p' \sim p' \circ p \sim 1$ *in* W.

Proof. Since $\deg_V p = 0$, there is a set

$$K = \{(x_\nu, \xi_\nu);\ \nu = 1, 2, \ldots\} \prec V$$

such that $2 \leq 2|\xi_\nu| \leq |\xi_{\nu+1}|$ for all ν and such that

$$|p(x_\nu, \xi_\nu)| \geq 2|\xi_\nu|^{-(\delta'-\delta)/2}. \tag{A.4}$$

Put $W = M(K, 1/3, \delta')$. After having taken away finitely many (x_ν, ξ_ν) from K, we may assume that $W \prec V$. Using that $p \in S^{(\delta'-\delta)/3}_{1-\delta}(V)$ by Lemma A.4, we see that there is a constant C such that

$$|p(x, \xi) - p(x_\nu, \xi_\nu)| \leq C|\xi_\nu|^{-\delta'+(\delta'-\delta)/3+\delta} = C|\xi_\nu|^{-2(\delta'-\delta)/3} \tag{A.5}$$

for all (x, ξ) satisfying

$$|x - x_\nu| < |\xi_\nu|^{-\delta'}/3, \quad |\xi - \xi_\nu| < |\xi_\nu|^{1-\delta'}/3.$$

After having taken away finitely many (x_ν, ξ_ν) from K we get from (A.4) and (A.5) that

$$|p(x, \xi)| \geq |\xi|^{-(\delta'-\delta)/2} \text{ in } W. \tag{A.6}$$

This inequality and the fact that $p \in S^{(\delta'-\delta)/2}_{1-\delta}(V)$ implies that $1/p \in S^{(\delta'-\delta)/2}_{1-\delta'}(W)$. (Cf. Hörmander [13] p. 166.) The usual construction of a pseudodifferential parametrix of an elliptic operator, now gives the desired $p' \in S^{(\delta'-\delta)/2}_{1-\delta'}(W)$, having $1/p$ as a principal part.

The proof of Theorem A.2 is now easy. Let p and V be as in the theorem. If $\deg_V p = -\infty$ there is nothing to prove. After multiplication with a suitable elliptic factor, we can assume that $\deg_V p = 0$. Permuting the columns or the rows of p does not change the equivalence class. We can therefore assume that $\deg_V p_{11} = 0$ where p is written in matrix form: $p = (p_{jk})$. Let δ and δ' be as in the theorem and take δ'' such that $\delta < \delta'' < \delta'$. By Lemma A.5, there exists a δ''-set $V'' \prec V$ and a $q \in S^{(\delta''-\delta)/2}_{1-\delta''}(V'')$ such that $q \circ p_{11} \sim p_{11} \circ q \sim 1$. Composing p to the right with the $m_1 \times m_1$-matrix

$$\begin{pmatrix} q & -qp_{12} & -qp_{13} & \cdots & -qp_{1m_1} \\ 0 & q & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & q \end{pmatrix}$$

we see that p is equivalent in V'' to a matrix where the first row is of the form $(1, 0, \ldots, 0)$. A similar multiplication of the obtained matrix to the left shows that p is equivalent to a matrix of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

where $A \in S^m_{1-\delta''}(V''; \mathbf{C}^{m_1-1}, \mathbf{C}^{m_2-1})$ for some m. Iterating this process we get the desired equivalent form after at most $\min(m_1, m_2)$ steps.

We can now prove Lemma 5.7. Let $Q \in L^0(\mathbf{R}^n; \mathbf{C}^m, \mathbf{C}^k)$, $m > k$ and let $\varrho \in T^*(\mathbf{R}^n) \setminus 0$. Let $V = M(K, 1/3, \delta)$, where $0 < \delta < 1/2$ and K is a subset of the half ray through ϱ, satisfying the assumptions of Definition A.1. Take δ' so that $\delta < \delta' < 1/2$. By Theorem A.2 there exists a δ'-set $W \prec V$ such that $Q(x, \xi)$ is equivalent in W to a matrix symbol, having only zeros in the last $m-k$ columns. This implies that there exists a symbol $q \in S^0_{1-\delta'}(W; \mathbf{C}^m, \mathbf{C}^m)$ such that $Q \circ q \sim 0$ and such that the degree of q in $(1/2)W$ is $> -\infty$. Now it is easy to construct a symbol $\chi \in S^0_{1-\delta'}(W)$ with support in $(2/3)W$ and equal to 1 in $(1/2)W$. If $A \in L^0(\mathbf{R}^n; \mathbf{C}^m, \mathbf{C}^m)$ has symbol $\sim q \circ \chi$ and is properly supported it follows that $QA \equiv 0$ and $WF(A) = \{\lambda\varrho; \lambda > 0\}$. It is clear that there is some $u \in H_0^{\mathrm{comp}}(\mathbf{R}^n)$ with $WF(u) = \{\lambda\varrho; \lambda > 0\}$ in the image of A and since $Qu \in C^\infty$, the lemma is proved.

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