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PARAMETRIZATION OF CONTRACTIVE BLOCK-OPERATOR MATRICES AND PASSIVE DISCRETE-TIME SYSTEMS

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ABSTRACT. Passive linear systems $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ have their transfer function $\Theta_\tau(\lambda) = D + \lambda C(I - \lambda A)^{-1}B$ in the Schur class $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$. Using a parametrization of contractive block operators the transfer function $\Theta_\tau(\lambda)$ is connected to the Sz.-Nagy – Foias characteristic function $\Phi_A(\lambda)$ of the contraction A . This gives a new aspect and some explicit formulas for studying the interplay between the system τ and the functions $\Theta_\tau(\lambda)$ and $\Phi_A(\lambda)$. The method leads to some new results for linear passive discrete-time systems. Also new proofs for some known facts in the theory of these systems are obtained.

1. INTRODUCTION

A bounded linear operator T acting from a Hilbert space \mathfrak{H}_1 into a Hilbert space \mathfrak{H}_2 is said to be

- (1) *contractive* if $\|T\| \leq 1$;
- (2) *isometric* if $\|Tf\| = \|f\|$ for all $f \in \mathfrak{H}_1 \iff T^*T = I_{\mathfrak{H}_1}$;
- (3) *co-isometric* if T^* is isometric $\iff TT^* = I_{\mathfrak{H}_2}$.

A linear system $\tau = (A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N})$ with bounded linear operators A, B, C, D and separable Hilbert spaces \mathfrak{H} (state space), \mathfrak{M} (incoming space), and \mathfrak{N} (outgoing space), of the form

$$(1.1) \quad \begin{cases} h_{k+1} = Ah_k + B\xi_k, \\ \sigma_k = Ch_k + D\xi_k, \end{cases} \quad k \geq 0,$$

where $\{h_k\} \subset \mathfrak{H}$, $\{\xi_k\} \subset \mathfrak{M}$, $\{\sigma_k\} \subset \mathfrak{N}$, is called a *discrete-time system*. The operators A, B, C , and D are called the *main operator*, the *control operator*, the *observation operator*, and the *feedthrough operator* of τ , respectively. If the linear operator $T_\tau : \mathfrak{H} \oplus \mathfrak{M} \rightarrow \mathfrak{H} \oplus \mathfrak{N}$ defined by the block form

$$(1.2) \quad T_\tau = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}$$

is contractive, then the corresponding discrete-time system is said to be *passive*. If the block-operator matrix T_τ is isometric (co-isometric, unitary), then the system is said to be isometric (co-isometric, conservative). Isometric and co-isometric systems were studied by L. de Branges and J. Rovnyak (see [21], [22]) and by T. Ando (see [2]), conservative systems have been investigated by B. Sz.-Nagy and C. Foias (see [33]) and M.S. Brodskiĭ (see [23]).

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Passive systems have been studied by D.Z. Arov et al (see [9], [10], [11], [12], [13], [14], [15]). The *transfer function*

$$(1.3) \quad \Theta_\tau(\lambda) := D + \lambda C(I_{\mathfrak{H}} - \lambda A)^{-1}B, \quad \lambda \in \mathbb{D},$$

of the passive system τ in (1.1) belongs to the *Schur class* $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$, i.e., $\Theta_\tau(\lambda)$ is holomorphic in the unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and its values are contractive linear operators from \mathfrak{M} into \mathfrak{N} . It is well known that a function $\Theta(\lambda)$ from the Schur class $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$ has almost everywhere non-tangential strong limit values $\Theta(\xi)$, $\xi \in \mathbb{T}$, where $\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$ stands for the unit circle; cf. [33]. The subspaces

$$(1.4) \quad \mathfrak{H}^c := \overline{\text{span}} \{A^n B \mathfrak{M} : n \in \mathbb{N}_0\} \text{ and } \mathfrak{H}^o := \overline{\text{span}} \{A^{*n} C^* \mathfrak{N} : n \in \mathbb{N}_0\}$$

are said to be the *controllable* and *observable* subspaces of the system τ , respectively. The notation \mathbb{N}_0 stands for the nonnegative integers; the positive integers will be denoted by \mathbb{N} . The system τ is said to be *controllable* (*observable*) if $\mathfrak{H}^c = \mathfrak{H}$ ($\mathfrak{H}^o = \mathfrak{H}$), and it is called *minimal* if τ is both controllable and observable. The system τ is said to be *simple* if $\mathfrak{H} = \text{clos} \{\mathfrak{H}^c + \mathfrak{H}^o\}$ (the closure of the span). It follows from (1.4) that

$$(1.5) \quad (\mathfrak{H}^c)^\perp = \bigcap_{n=0}^{\infty} \ker(B^* A^{*n}), \quad (\mathfrak{H}^o)^\perp = \bigcap_{n=0}^{\infty} \ker(CA^n),$$

and therefore there are the following alternative characterizations:

- (1) τ is controllable $\iff \bigcap_{n=0}^{\infty} \ker(B^* A^{*n}) = \{0\}$;
- (2) τ is observable $\iff \bigcap_{n=0}^{\infty} \ker(CA^n) = \{0\}$;
- (3) τ is simple $\iff \left(\bigcap_{n=0}^{\infty} \ker(B^* A^{*n}) \right) \cap \left(\bigcap_{n=0}^{\infty} \ker(CA^n) \right) = \{0\}$.

It is well known that every operator-valued function $\Theta(\lambda)$ from the Schur class $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$ can be realized as the transfer function of some passive system, which can be chosen as controllable isometric (observable co-isometric, simple conservative, minimal passive); cf. [22], [33], [2] [9], [11], [1]. Moreover, two controllable isometric (observable co-isometric, simple conservative) systems with the same transfer function are unitarily similar: two discrete-time systems

$$\tau_1 = \{A_1, B_1, C_1, D; \mathfrak{H}_1, \mathfrak{M}, \mathfrak{N}\} \quad \text{and} \quad \tau_2 = \{A_2, B_2, C_2, D; \mathfrak{H}_2, \mathfrak{M}, \mathfrak{N}\}$$

are said to be *unitarily similar* if there exists a unitary operator U from \mathfrak{H}_1 onto \mathfrak{H}_2 such that

$$A_1 = U^{-1}A_2U, \quad B_1 = U^{-1}B_2, \quad C_1 = C_2U;$$

cf. [21], [22], [2], [23], [1]. However, a result of D.Z. Arov [9] states that two minimal passive systems τ_1 and τ_2 with the same transfer function $\Theta(\lambda)$ are only *weakly similar*, i.e., there is a closed densely defined operator $Z : \mathfrak{H}_1 \rightarrow \mathfrak{H}_2$ such that Z is invertible, Z^{-1} is densely defined, and

$$ZA_1f = A_2Zf, \quad C_1f = C_2Zf, \quad f \in \text{dom } Z, \quad \text{and} \quad ZB_1 = B_2.$$

Weak similarity preserves neither the dynamical properties of the system nor the spectral properties of its main operator A . In [13], [14] necessary and sufficient conditions have been established for minimal passive systems with the same transfer function to be similar or to

be unitarily similar. These conditions involve additional operator-valued Schur functions $\varphi_\Theta(\lambda)$ and $\psi_\Theta(\lambda)$ which satisfy the inequalities

$$(1.6) \quad \varphi_\Theta^*(\xi)\varphi_\Theta(\xi) \leq I_{\mathfrak{M}} - \Theta^*(\xi)\Theta(\xi), \quad \psi_\Theta(\xi)\psi_\Theta^*(\xi) \leq I_{\mathfrak{N}} - \Theta(\xi)\Theta^*(\xi),$$

almost everywhere on \mathbb{T} , and they are uniquely (up to a constant unitary factor) determined by the following maximality property: if $\tilde{\varphi}(\lambda)$ and $\tilde{\psi}(\lambda)$ are operator-valued functions from the Schur class such that

$$(1.7) \quad \tilde{\varphi}^*(\xi)\tilde{\varphi}(\xi) \leq I_{\mathfrak{M}} - \Theta^*(\xi)\Theta(\xi), \quad \tilde{\psi}(\xi)\tilde{\psi}^*(\xi) \leq I_{\mathfrak{N}} - \Theta(\xi)\Theta^*(\xi),$$

then

$$(1.8) \quad \tilde{\varphi}^*(\xi)\tilde{\varphi}(\xi) \leq \varphi_\Theta^*(\xi)\varphi_\Theta(\xi), \quad \tilde{\psi}(\xi)\tilde{\psi}^*(\xi) \leq \psi_\Theta(\xi)\psi_\Theta^*(\xi),$$

almost everywhere on the unit circle \mathbb{T} . Here $\Theta(\xi)$, $\xi \in \mathbb{T}$, stands for the non-tangential strong limit value of $\Theta(\lambda)$ which exist almost everywhere on \mathbb{T} , cf. [33]. The functions $\varphi_\Theta(\lambda)$ and $\psi_\Theta(\lambda)$ are called the right and left *defect functions* (or the *spectral factors*), respectively, associated with $\Theta(\lambda)$; cf. [17], [18], [19], [20], [26].

In this paper passive discrete-time systems $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ of the form (1.1) are considered. Some new proofs and new formulas concerning these systems and their transfer functions $\Theta_\tau(\lambda)$ in (1.3) are presented. Also some new facts concerning the realization of operator-valued Schur functions $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ as transfer functions of passive systems τ are established. One of the main consequences of the approach used and developed in this paper can be formulated as follows:

Theorem. *Let $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ and assume that $\Theta(\lambda)$ is not a constant function.*

- (i) *Suppose that $\varphi_\Theta(\lambda) = 0$, $\psi_\Theta(\lambda) = 0$, and that $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ is a simple passive system with transfer function $\Theta(\lambda)$. Then τ is conservative and minimal. Furthermore, if $\Theta(\lambda)$ is bi-inner, then in addition $A \in C_{00}$.*
- (ii) *Suppose that $\varphi_\Theta(\lambda) = 0$ ($\psi_\Theta(\lambda) = 0$) and that $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ is a controllable (observable) passive system with transfer function $\Theta(\lambda)$. Then τ is isometric (co-isometric) and minimal. Furthermore, if $\Theta(\lambda)$ is inner (co-inner), then in addition $A \in C_0$. (C_0).*

The classes $C_{0\cdot}$, C_0 , and C_{00} are introduced in [33]; see also Section 6. The above theorem is very close to the following result established by D.Z. Arov, which was proved by means of the so-called *optimal* and **-optimal* realizations of Schur class functions (see [10], [11], [14]):

Theorem. ([10]) *Let $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$. Then:*

- (i) *if $\Theta(\lambda)$ is bi-inner and τ is a simple passive system with transfer function $\Theta(\lambda)$ then τ is conservative;*
- (ii) *if $\varphi_\Theta(\lambda) = 0$ or $\psi_\Theta(\lambda) = 0$ then all passive minimal systems with the same transfer function $\Theta(\lambda)$ are unitarily equivalent and if $\varphi_\Theta(\lambda) = 0$ and $\psi_\Theta(\lambda) = 0$ then they are in addition conservative.*

The arguments in the present paper use a parametrization of contractive block-operator matrices of the form

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{K} \\ \mathfrak{N} \end{pmatrix},$$

established in the papers [16], [24], and [32]; a new proof of the parametrization is presented. This parametrization leads to some explicit formulas for realizing operator-valued Schur functions as transfer functions of passive systems. In particular, the transfer function of a passive system is expressed in terms of the characteristic function of the main operator A of the system; cf. B. Sz.-Nagy and C. Foias [33]. The connection is used to study passive systems and their transfer functions via the Sz.-Nagy – Foias characteristic function. For instance, an exact form of the inner (co-inner, bi-inner) dilations for a passive system with a strongly stable (co-stable, bi-stable) main operator is established.

It follows the class of all continuous linear operators defined on a complex Hilbert space \mathfrak{H}_1 and taking values in a complex Hilbert space \mathfrak{H}_2 is denoted by $\mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ and $\mathbf{L}(\mathfrak{H}) := \mathbf{L}(\mathfrak{H}, \mathfrak{H})$. The domain, the range, and the null-space of a linear operator T are denoted by $\text{dom } T$, $\text{ran } T$, and $\ker T$, respectively. The set of all regular points of a closed operator T is denoted by $\rho(T)$.

2. THE MODEL OF SZ-NAGY AND FOIAS

For a contraction $A \in \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ the nonnegative square root $D_A = (I - A^*A)^{1/2}$ is said to be the *defect operator* of S and \mathfrak{D}_A stands for the closure of the range $\text{ran } D_A$. It is well known that the defect operators satisfy the following commutation relation:

$$(2.1) \quad AD_A = D_{A^*}A,$$

and that the block operator

$$(2.2) \quad \begin{pmatrix} A^* & D_A \\ D_{A^*} & -A \end{pmatrix} : \begin{pmatrix} \mathfrak{H}_2 \\ \mathfrak{D}_A \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{D}_{A^*} \end{pmatrix}$$

is unitary, cf. [33]. If $\mathfrak{H}_1 = \mathfrak{H}_2 = \mathfrak{H}$ then the transfer function of the conservative system

$$\{A^*, D_A, D_{A^*}, -A; \mathfrak{H}, \mathfrak{D}_A, \mathfrak{D}_{A^*}\}.$$

is given by

$$(2.3) \quad \Phi_A(\lambda) := (-A + \lambda D_{A^*}(I_{\mathfrak{H}} - \lambda A^*)^{-1}D_A) \upharpoonright \mathfrak{D}_A, \quad \lambda \in \mathbb{D}.$$

The function $\Phi_A(\lambda)$ is the Sz.-Nagy – Foias characteristic function of the contraction A and it belongs to the Schur class $\mathbf{S}(\mathfrak{D}_A, \mathfrak{D}_{A^*})$; cf. [33]. For the adjoint operator A^* the characteristic function takes the form

$$(2.4) \quad \Phi_{A^*}(\lambda) := (-A^* + \lambda D_A(I_{\mathfrak{H}} - \lambda A)^{-1}D_{A^*}) \upharpoonright \mathfrak{D}_{A^*} = \Phi_A(\bar{\lambda})^*.$$

Observe that $\Phi_{A^*}(\lambda)$ is the transfer function of the conservative system

$$(2.5) \quad \Sigma = \{A, D_{A^*}, D_A, -A^*; \mathfrak{H}, \mathfrak{D}_{A^*}, \mathfrak{D}_A\}.$$

The controllable and observable subspaces of the system Σ take the form

$$(2.6) \quad \mathfrak{H}_{\Sigma}^c = \overline{\text{span}} \{A^n D_{A^*} \mathfrak{D}_{A^*} : n \in \mathbb{N}_0\}, \quad \mathfrak{H}_{\Sigma}^o = \overline{\text{span}} \{A^{*n} D_A \mathfrak{D}_A : n \in \mathbb{N}_0\}.$$

It follows that

$$(2.7) \quad (\mathfrak{H}_{\Sigma}^c)^{\perp} = \bigcap_{n=0}^{\infty} \ker (D_{A^*} A^{*n}) = \bigcap_{n=1}^{\infty} \ker D_{A^{*n}},$$

and that

$$(2.8) \quad (\mathfrak{H}_\Sigma^o)^\perp = \bigcap_{n=0}^{\infty} \ker (D_A A^n) = \bigcap_{n=1}^{\infty} \ker D_{A^n}.$$

The subspace $(\mathfrak{H}_\Sigma^c)^\perp$ ($(\mathfrak{H}_\Sigma^o)^\perp$) is invariant under A^* (A , respectively) and the operator $A^* \upharpoonright (\mathfrak{H}_\Sigma^c)^\perp$ ($A \upharpoonright (\mathfrak{H}_\Sigma^o)^\perp$, respectively) is isometric. Clearly,

$$(\mathfrak{H}_\Sigma^c)^\perp \cap (\mathfrak{H}_\Sigma^o)^\perp = \{f \in \mathfrak{H} : \|f\| = \|A^n f\| = \|A^{*n} f\|, n \in \mathbb{N}\}.$$

This yields some basic facts, which are formulated in the next remark.

Remark 2.1. The conservative system Σ in (2.4) admits the following properties:

- (i) Σ is simple if and only if A is completely non-unitary; cf. [33, Theorem 3.2];
- (ii) if Σ is simple and $A^* (\mathfrak{H}_\Sigma^c)^\perp = (\mathfrak{H}_\Sigma^c)^\perp$, then $(\mathfrak{H}_\Sigma^c)^\perp = \{0\}$, i.e., Σ is controllable;
- (iii) if Σ is simple and $A (\mathfrak{H}_\Sigma^o)^\perp = (\mathfrak{H}_\Sigma^o)^\perp$, then $(\mathfrak{H}_\Sigma^o)^\perp = \{0\}$, i.e., Σ is observable.

3. AN IDENTITY FOR CONTRACTIONS

An identity is derived for a class of contractions. It is useful for the parametrization of contractions in block form and for the representation of transfer functions of passive systems.

Lemma 3.1. *Let \mathfrak{H} , \mathfrak{K} , \mathfrak{M} , and \mathfrak{N} be Hilbert spaces, and let the operator $F \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a contraction, let the operators $M \in \mathbf{L}(\mathfrak{M}, \mathfrak{D}_{F^*})$ and $K \in \mathbf{L}(\mathfrak{D}_F, \mathfrak{N})$ be contractions, and let the operator $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$ be a contraction. Then the operator G defined by*

$$(3.1) \quad G = KFM + D_{K^*} X D_M \in \mathbf{L}(\mathfrak{M}, \mathfrak{N})$$

satisfies the identity

$$(3.2) \quad \|h\|^2 - \|Gh\|^2 = \|D_F Mh\|^2 + \|D_X D_M h\|^2 + \|(D_K F M - K^* X D_M) h\|^2,$$

for all $h \in \mathfrak{M}$. In particular, G is a contraction.

Proof. From the definition of G in (3.1) one obtains

$$(3.3) \quad \begin{aligned} \|h\|^2 - \|Gh\|^2 &= \|h\|^2 - \|(KFM + D_{K^*} X D_M) h\|^2 \\ &= \|h\|^2 - \|KFMh\|^2 - \|D_{K^*} X D_M h\|^2 - 2\operatorname{Re} (KFMh, D_{K^*} X D_M h). \end{aligned}$$

Taking adjoints in (2.1) gives $K^* D_{K^*} = D_K K^*$, and hence

$$(KFMh, D_{K^*} X D_M h) = (D_K F Mh, K^* X D_M h).$$

The definition of D_{K^*} shows that

$$-\|KFMh\|^2 = \|D_K F Mh\|^2 - \|FMh\|^2,$$

and, likewise,

$$-\|D_{K^*} X D_M h\|^2 = -\|X D_M h\|^2 + \|K^* X D_M h\|^2.$$

Now the righthand side of (3.3) becomes

$$\begin{aligned} &\|h\|^2 - \|FMh\|^2 - \|X D_M h\|^2 \\ &\quad + \|D_K F Mh\|^2 + \|K^* X D_M h\|^2 - 2\operatorname{Re} (D_K F Mh, K^* X D_M h) \\ &= \|h\|^2 - \|FMh\|^2 - \|X D_M h\|^2 + \|(D_K F M - K^* X D_M) h\|^2. \end{aligned}$$

Finally, observe that

$$\|D_F Mh\|^2 = \|Mh\|^2 - \|FMh\|^2, \quad \|D_X D_M h\|^2 = \|h\|^2 - \|Mh\|^2 - \|XD_M h\|^2.$$

Hence the proof of (3.2) is complete. \square

4. CONTRACTIVE BLOCK OPERATORS

The following theorem goes back to [16], [24], [32]; other proofs of the theorem can be found in [30], [31], [6], and an equivalent parametrization is given in [28]. The present proof is based on an approximation procedure and is along the lines of the proof in [8] for the parametrization of all quasi-selfadjoint extensions of a symmetric contraction.

Theorem 4.1. *Let $A \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, $B \in \mathbf{L}(\mathfrak{M}, \mathfrak{K})$, $C \in \mathbf{L}(\mathfrak{H}, \mathfrak{N})$, and $D \in \mathbf{L}(\mathfrak{M}, \mathfrak{N})$. The operator matrix*

$$(4.1) \quad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{K} \\ \mathfrak{N} \end{pmatrix}$$

is a contraction if and only if T is of the form

$$(4.2) \quad T = \begin{pmatrix} A & D_{A^*} M \\ KD_A & -KA^* M + D_{K^*} X D_M \end{pmatrix},$$

where $A \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, $M \in \mathbf{L}(\mathfrak{M}, \mathfrak{D}_{A^*})$, $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$, and $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$ are contractions, all uniquely determined by T . Furthermore, the following equality holds for all $f \in \mathfrak{H}$, $h \in \mathfrak{M}$:

$$(4.3) \quad \left\| \begin{pmatrix} f \\ h \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} A & D_{A^*} M \\ KD_A & -KA^* M + D_{K^*} X D_M \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix} \right\|^2 \\ = \|D_K(D_A f - A^* Mh) - K^* X D_M h\|^2 + \|D_X D_M h\|^2 \geq 0.$$

Proof. Assume that T is of the form (4.2), where $A \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, $M \in \mathbf{L}(\mathfrak{M}, \mathfrak{D}_{A^*})$, $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$, and $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$ are contractions. Then T can be written in the form (4.6). By applying Lemma 3.1 to (4.6) one obtains (4.3) from (3.2). Thus, T is a contraction.

Conversely, assume that $T \in \mathbf{L}(\mathfrak{H} \oplus \mathfrak{M}, \mathfrak{K} \oplus \mathfrak{N})$ in (4.1) is a contraction. Denote by $P_{\mathfrak{K}}$ and $P_{\mathfrak{N}}$ the orthogonal projections in the Hilbert space $\mathfrak{K} \oplus \mathfrak{N}$ onto \mathfrak{K} and \mathfrak{N} , respectively, so that $A = P_{\mathfrak{K}} T \upharpoonright \mathfrak{H}$, $B = P_{\mathfrak{K}} T \upharpoonright \mathfrak{M}$, $C = P_{\mathfrak{N}} T \upharpoonright \mathfrak{H}$, and $D = P_{\mathfrak{N}} T \upharpoonright \mathfrak{M}$. Since $T \upharpoonright \mathfrak{H}$ is a contraction, one has

$$\|Cf\|^2 \leq \|f\|^2 - \|Af\|^2 \quad \text{for all } f \in \mathfrak{H}.$$

It follows that $C = KD_A$, where $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$ is a contraction, which is uniquely determined by A and C . The operators T^* and $T^* \upharpoonright \mathfrak{K}$ are also contractions. Therefore, one concludes that $B^* = ND_{A^*}$, where $N \in \mathbf{L}(\mathfrak{D}_{A^*}, \mathfrak{M})$ is a contraction, uniquely determined by A and B . Let $M := N^* \in \mathbf{L}(\mathfrak{M}, \mathfrak{D}_{A^*})$. Contractivity of T and the relation (2.1) imply

$$(4.4) \quad \begin{aligned} 0 &\leq \|f\|^2 + \|h\|^2 - \|Af + D_{A^*} Mh\|^2 - \|KD_A f + Dh\|^2 \\ &= \|f\|^2 + \|h\|^2 - \|Af\|^2 - \|D_{A^*} Mh\|^2 - 2\operatorname{Re}(Af, D_{A^*} Mh) - \|KD_A f + Dh\|^2 \\ &= \|D_A f\|^2 + \|A^* Mh\|^2 - 2\operatorname{Re}(D_A f, A^* Mh) + \|D_M h\|^2 - \|KD_A f + Dh\|^2 \\ &= \|D_A f - A^* Mh\|^2 + \|D_M h\|^2 - \|KD_A f + Dh\|^2, \end{aligned}$$

for all $f \in \mathfrak{H}$ and $h \in \mathfrak{M}$. Since $\text{ran } M \subset \mathfrak{D}_{A^*}$ and $A^*\mathfrak{D}_{A^*} \subset \mathfrak{D}_A$, there exists a sequence $\{f_n\}_{n=1}^\infty \subset \mathfrak{D}_A$ such that for a given vector $h \in \mathfrak{M}$ the equality

$$\lim_{n \rightarrow \infty} D_A f_n = A^* M h$$

is satisfied. Hence (4.4) implies that

$$\|K A^* M h + D h\|^2 \leq \|D_M h\|^2, \quad h \in \mathfrak{M}.$$

Similarly taking into account that T^* is a contraction one gets

$$\|M^* A K^* g + D^* g\|^2 \leq \|D_{K^*} g\|^2, \quad g \in \mathfrak{N}.$$

The last inequality yields that there exists a contraction $Z \in \mathbf{L}(\mathfrak{D}_{K^*}, \mathfrak{M})$ such that

$$M^* A K^* + D^* = Z D_{K^*},$$

and Z is uniquely determined by M , A , K , and D ; thus, in particular, by T . Substituting $D = -K A^* M + D_{K^*} Z^*$ into (4.4) shows that for all $f \in \mathfrak{H}$, $h \in \mathfrak{M}$,

$$\begin{aligned} & \|D_A f - A^* M h\|^2 + \|D_M h\|^2 - \|K D_A f - K A^* M h + D_{K^*} Z^* h\|^2 \\ &= \|D_A f - A^* M h\|^2 + \|D_M h\|^2 - \|K(D_A f - A^* M h)\|^2 + \|K^* Z^* h\|^2 - \|Z^* h\|^2 \\ &\quad - 2\text{Re}(D_K(D_A f - A^* M h), K^* Z^* h) \\ &= \|D_K(D_A f - A^* M h) - K^* Z^* h\|^2 + \|D_M h\|^2 - \|Z^* h\|^2 \geq 0. \end{aligned}$$

Finally, choose a sequence $\{f_n\}_{n=1}^\infty \subset \mathfrak{D}_A$ such that for a given vector $h \in \mathfrak{M}$ the equality

$$\lim_{n \rightarrow \infty} D_K D_A f_n = D_K A^* M h + K^* Z^* h$$

is satisfied. This yields $\|Z^* h\| \leq \|D_M h\|$ for all $h \in \mathfrak{M}$. Therefore there exists a contraction $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$, uniquely determined by Z and M , such that $Z^* = X D_M$. Thus

$$(4.5) \quad D = -K A^* M + D_{K^*} X D_M$$

and here all the contractions are uniquely determined by T . This completes the proof. \square

Observe that if T is given by (4.2), then it can be rewritten in the form

$$T = \begin{pmatrix} I_{\mathfrak{K}} & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} A & D_{A^*} \\ D_A & -A^* \end{pmatrix} \begin{pmatrix} I_{\mathfrak{H}} & 0 \\ 0 & M \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & D_{K^*} X D_M \end{pmatrix},$$

where the operators

$$\mathcal{K} = \begin{pmatrix} I_{\mathfrak{K}} & 0 \\ 0 & K \end{pmatrix} : \begin{pmatrix} \mathfrak{K} \\ \mathfrak{D}_A \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{K} \\ \mathfrak{N} \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} I_{\mathfrak{H}} & 0 \\ 0 & M \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_{A^*} \end{pmatrix}$$

are contractions and the operator

$$\mathcal{U} = \begin{pmatrix} A & D_{A^*} \\ D_A & -A^* \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_{A^*} \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{K} \\ \mathfrak{D}_A \end{pmatrix}$$

is unitary. Introduce the contraction \mathcal{X} by

$$\mathcal{X} = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_M \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{K} \\ \mathfrak{D}_{K^*} \end{pmatrix}.$$

Since $D_{\mathcal{K}^*} = 0 \oplus D_{K^*} \in \mathbf{L}(\mathfrak{K} \oplus \mathfrak{N})$ and $D_{\mathcal{M}} = 0 \oplus D_M \in \mathbf{L}(\mathfrak{H} \oplus \mathfrak{M})$ one can write

$$(4.6) \quad T = \mathcal{K} \mathcal{U} \mathcal{M} + D_{\mathcal{K}^*} \mathcal{X} D_{\mathcal{M}}.$$

Corollary 4.2. *Let $A \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$ be a contraction. Assume that $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{N})$, $M \in \mathbf{L}(\mathfrak{M}, \mathfrak{D}_{A^*})$, and $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$ are contractions. Then the operator T in (4.2) is:*

- (i) *isometric if and only if $D_X D_M = 0$ and $D_K D_A = 0$;*
- (ii) *co-isometric if and only if $D_{X^*} D_{K^*} = 0$ and $D_{M^*} D_{A^*} = 0$.*

Proof. By symmetry it suffices to prove statement (i). Suppose that $D_X D_M = 0$. If, in addition, A is isometric, i.e., if $D_A = 0$, then $\mathfrak{D}_A = \{0\}$, $A^* \upharpoonright \mathfrak{D}_{A^*} = 0$, and $\text{dom } K = \text{dom } D_K = \{0\}$, so that $K^* = 0 \in \mathbf{L}(\mathfrak{N}, \{0\})$. Now the identity (4.3) in Theorem 4.1 shows that T is isometric. On the other hand, if A is not isometric but $D_K D_A = 0$, then $D_K = 0$, i.e., K is isometric, since $\text{dom } D_K = \mathfrak{D}_A$. In this case $K^* D_{K^*} = D_K K^* = 0$ and since $\text{ran } X \subset \mathfrak{D}_{K^*}$, one has also $K^* X = 0$. Thus, again (4.3) shows that T is isometric.

Conversely, assume that T is isometric. Then from (4.3) it is clear that $D_X D_M = 0$. Moreover, taking $h = 0$ in (4.3) one obtains $D_K D_A = 0$. \square

As the proof shows the equality $D_K D_A = 0$ means that there are two cases:

- (1) $D_A = 0$, i.e. $\mathfrak{D}_A = \{0\}$;
- (2) $\mathfrak{D}_A \neq \{0\}$ and $D_K = 0$.

In the case (1) A is isometric. In the case (2) the operator K is isometric. Likewise, one can interpret the equality $D_X D_M = 0$: either M is isometric, or M is not isometric, in which case X is isometric.

5. TRANSFER FUNCTIONS OF PASSIVE SYSTEMS

Let $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ be a passive linear system with the corresponding block representation

$$(5.1) \quad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}.$$

The next theorem gives an expression of the transfer function $\Theta_\tau(\lambda)$ of τ by means of the characteristic function of the main operator A and the parameters of the block representation of the operator T in (4.2). For this purpose, define the following operator-valued holomorphic functions

$$(5.2) \quad \varphi(\lambda) := \begin{pmatrix} D_K \Phi_{A^*}(\lambda) M - K^* X D_M \\ -D_X D_M \end{pmatrix} : \mathfrak{M} \rightarrow \begin{pmatrix} \mathfrak{D}_K \\ \mathfrak{D}_M \end{pmatrix}, \quad \lambda \in \mathbb{D},$$

and

$$(5.3) \quad \psi(\lambda) := \begin{pmatrix} K \Phi_{A^*}(\lambda) D_{M^*} - D_{K^*} X M^* & D_{K^*} D_{X^*} \end{pmatrix} : \begin{pmatrix} \mathfrak{D}_{M^*} \\ \mathfrak{D}_{K^*} \end{pmatrix} \rightarrow \mathfrak{N}, \quad \lambda \in \mathbb{D}.$$

Theorem 5.1. *Let $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ be a passive linear system and let (4.2) be the representation of the block operator T in (5.1). Then the transfer function $\Theta_\tau(\lambda)$ of τ and the characteristic function $\Phi_{A^*}(\lambda)$ of A^* in (2.4) are connected via*

$$(5.4) \quad \Theta_\tau(\lambda) = K \Phi_{A^*}(\lambda) M + D_{K^*} X D_M, \quad \lambda \in \mathbb{D};$$

in particular, $\Theta_\tau(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$. In addition, the identities

$$(5.5) \quad \|D_{\Theta_\tau(\lambda)} h\|^2 = \|D_{\Phi_{A^*}(\lambda)} M h\|^2 + \|\varphi(\lambda) h\|^2, \quad h \in \mathfrak{M},$$

$$(5.6) \quad \|D_{\Theta_\tau(\lambda)}g\|^2 = \|D_{\Phi_A(\bar{\lambda})}K^*g\|^2 + \|\psi(\lambda)^*g\|^2, \quad g \in \mathfrak{N},$$

hold and the functions $\varphi(\lambda)$ and $\psi(\lambda)$ in (5.2) and (5.3) are Schur functions.

Proof. Using (4.2) the equalities (1.3) and (2.4) yield (5.4). It is clear that $\Phi_{A^*}(\lambda)$ is a Schur function. Hence, by Lemma 3.1 $\Theta_\tau(\lambda)$ is a Schur function, too. The relations

$$(5.7) \quad \|D_{\Theta_\tau(\lambda)}h\|^2 = \|D_{\Phi_{A^*}(\lambda)}Mh\|^2 + \|D_X D_M h\|^2 + \|(D_K \Phi_{A^*}(\lambda)M - K^* X D_M)h\|^2, \quad h \in \mathfrak{M},$$

$$(5.8) \quad \|D_{\Theta_\tau(\lambda)}g\|^2 = \|D_{\Phi_A(\bar{\lambda})}K^*g\|^2 + \|D_{X^*} D_{K^*} g\|^2 + \|(D_{M^*} \Phi_A(\bar{\lambda})K^* - M X^* D_{K^*})g\|^2, \quad g \in \mathfrak{N},$$

follow from (3.2) and (2.4). Furthermore, the definitions (5.2) and (5.3) show that

$$(5.9) \quad \|\varphi(\lambda)h\|^2 = \|D_X D_M h\|^2 + \|(D_K \Phi_{A^*}(\lambda)M - K^* X D_M)h\|^2, \quad h \in \mathfrak{M},$$

and

$$(5.10) \quad \|\psi^*(\lambda)g\|^2 = \|D_{X^*} D_{K^*} g\|^2 + \|(D_{M^*} \Phi_A(\bar{\lambda})K^* - M X^* D_{K^*})g\|^2, \quad g \in \mathfrak{N}.$$

Now (5.7) and (5.8), together with (5.9) and (5.10) yield (5.5) and (5.6). It is clear from these identities that the values of $\varphi(\lambda)$ and $\psi(\lambda)$, $\lambda \in \mathbb{D}$, are contractive operators and, hence, they are Schur functions. \square

Proposition 5.2. *Let $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ be a passive linear system and let Σ be the conservative system in (2.5) induced by the contraction A . Then the controllable and observable subspaces of the systems τ and Σ satisfy the inclusions*

$$(5.11) \quad \mathfrak{H}^c \subset \mathfrak{H}_\Sigma^c \quad \text{and} \quad \mathfrak{H}^o \subset \mathfrak{H}_\Sigma^o.$$

In particular, if the system τ is controllable (observable, minimal, simple), then so is the system Σ . Moreover, if τ is isometric (co-isometric), then the equality $\mathfrak{H}^o = \mathfrak{H}_\Sigma^o$ ($\mathfrak{H}^c = \mathfrak{H}_\Sigma^c$) holds.

Proof. The block representation (4.2) in Theorem 4.1 shows that $B = D_{A^*}M$ and $C = K D_A$. Hence the controllable and observable subspaces (1.4) for τ can be rewritten as

$$(5.12) \quad \mathfrak{H}^c = \overline{\text{span}} \{A^n D_{A^*} M \mathfrak{M} : n \in \mathbb{N}_0\}, \quad \mathfrak{H}^o = \overline{\text{span}} \{A^{*n} D_A K^* \mathfrak{N} : n \in \mathbb{N}_0\}.$$

Since $\text{ran } M \subset \mathfrak{D}_{A^*}$ and $\text{ran } K^* \subset \mathfrak{D}_A$ the inclusions (5.11) follow directly from the representations of \mathfrak{H}_Σ^c and \mathfrak{H}_Σ^o in (2.6).

If τ is isometric then $D_K D_A = 0$ by Corollary 4.2. Here either $D_A = 0$, or $D_A \neq 0$ in which case $D_K = 0$. If $D_K = 0$, i.e. K is isometric, then from (1.5) and (2.8) one obtains

$$(\mathfrak{H}^o)^\perp = \bigcap_{n=0}^{\infty} \ker (K D_A A^n) = \bigcap_{n=0}^{\infty} \ker (D_A A^n) = (\mathfrak{H}_\Sigma^o)^\perp.$$

If $D_A = 0$ then clearly $\mathfrak{H}^o = \mathfrak{H}_\Sigma^o = \{0\}$. Thus in both cases the equality $\mathfrak{H}^o = \mathfrak{H}_\Sigma^o$ holds.

If τ is co-isometric then $D_{M^*} D_{A^*} = 0$ by Corollary 4.2. If here $D_{M^*} = 0$ then (1.5) and (2.7) imply

$$(\mathfrak{H}^c)^\perp = \bigcap_{n=0}^{\infty} \ker (M^* D_{A^*} A^{*n}) = \bigcap_{n=0}^{\infty} \ker (D_{A^*} A^{*n}) = (\mathfrak{H}_\Sigma^c)^\perp.$$

In the case that $D_{A^*} = 0$ one has $\mathfrak{H}^c = \mathfrak{H}_\Sigma^c = \{0\}$. Therefore $\mathfrak{H}^c = \mathfrak{H}_\Sigma^c$. \square

Corollary 5.3. *Let $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ be a passive linear system and let (4.2) be the representation of the block operator T in (5.1). Then:*

(i) *If τ is isometric, then $\varphi(\lambda) = 0$. In this case*

$$\|D_{\Theta_\tau(\lambda)}h\| = \|D_{\Phi_{A^*}(\lambda)}Mh\|, \quad h \in \mathfrak{M}.$$

Conversely, if $\varphi(\lambda) = 0$ and τ is controllable, then τ is isometric.

(ii) *If τ is co-isometric, then $\psi(\lambda) = 0$. In this case*

$$\|D_{\Theta_\tau^*(\lambda)}g\| = \|D_{\Phi_A(\bar{\lambda})}K^*g\|, \quad g \in \mathfrak{N}.$$

Conversely, if $\psi(\lambda) = 0$ and τ is observable, then τ is co-isometric.

Proof. (i) Assume that τ is isometric. According to Corollary 4.2 $D_X D_M = 0$ and $D_K D_A = 0$. Here either $D_A = 0$, so that $\text{dom } K = \text{dom } D_K = \{0\}$ and $K^* = 0$, or $D_K = 0$ and then $K^* X = 0$. In each case the definition (5.2) shows that $\varphi(\lambda) = 0$.

Conversely, assume that $\varphi(\lambda) = 0$ and that τ is controllable. In view of (5.2) the condition $\varphi(\lambda) = 0$ means that

$$(5.13) \quad D_X D_M = 0, \quad D_K \Phi_{A^*}(\lambda) M = K^* X D_M, \quad \lambda \in \mathbb{D}.$$

The definition (2.4) of $\Phi_{A^*}(\lambda)$ implies the power series representation

$$(5.14) \quad \Phi_{A^*}(\lambda) = -A^* + \sum_{n=0}^{\infty} \lambda^{n+1} D_A A^n D_{A^*},$$

which together with the second identity in (5.13) gives

$$(5.15) \quad -D_K A^* M = K^* X D_M$$

and

$$(5.16) \quad D_K D_A A^n D_{A^*} M = 0, \quad n \in \mathbb{N}_0.$$

Since τ is controllable, (5.16) combined with (5.12) yields $D_K D_A = 0$. By Corollary 4.2 τ is isometric and (i) is proved.

The proof of (ii) is similar. For later use we only mention that $\psi(\lambda) = 0$ is equivalent to

$$(5.17) \quad D_{X^*} D_{K^*} = 0, \quad D_{M^*} \Phi_A(\lambda) K^* = M X^* D_{K^*}, \quad \lambda \in \mathbb{D},$$

where $\Phi_A(\lambda) = \Phi_{A^*}(\bar{\lambda})^*$ is the characteristic function of the contraction A ; see (2.3). \square

6. ISOMETRIC, CO-ISOMETRIC, AND CONSERVATIVE SYSTEMS

A function $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ is said to be *inner* if $\Theta^*(\xi)\Theta(\xi) = I_{\mathfrak{M}}$ for almost all $\xi \in \mathbb{T}$, and is said to be *co-inner* if $\Theta(\xi)\Theta^*(\xi) = I_{\mathfrak{N}}$ for almost all $\xi \in \mathbb{T}$. A function $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ is said to be *bi-inner* if it is both inner and co-inner. A contraction A in a Hilbert space \mathfrak{H} belongs to the classes C_0 , or $C_{\cdot 0}$ if

$$s - \lim_{n \rightarrow \infty} A^n = 0 \quad \text{or} \quad s - \lim_{n \rightarrow \infty} A^{*n} = 0,$$

respectively. By definition, $C_{00} := C_0 \cap C_{\cdot 0}$. The completely non-unitary part of a contraction A belongs to the class $C_{\cdot 0}$, C_0 , or C_{00} if and only if its characteristic function $\Phi_A(\lambda)$ in (2.3) is inner, co-inner, or bi-inner, respectively; cf. [33].

Lemma 6.1. *Let $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ be a passive system with transfer function $\Theta_\tau(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$. If $\Theta_\tau(\lambda)$ is inner, then the restriction $A \upharpoonright \mathfrak{H}^c$ belongs to the class C_0 .. If $\Theta_\tau(\lambda)$ is co-inner, then the restriction $A^* \upharpoonright \mathfrak{H}^o$ belongs to the class C_0 ..*

Proof. If $\Theta_\tau(\lambda)$ is inner, then (5.5) in Theorem 5.1 implies that $D_{\Phi_{A^*}}(\xi)M = 0$ for almost all $\xi \in \mathbb{T}$, i.e.,

$$\|\Phi_{A^*}(\xi)Mh\|^2 = \|Mh\|^2 \quad \text{for almost all } \xi \in \mathbb{T}, \quad h \in \mathfrak{M}.$$

Therefore, the norm of the vector-function $\Phi_{A^*}(\xi)Mh$ in the Hardy space $H^2(\mathfrak{D}_A)$ equals $\|Mh\|$; cf. [33]. From (2.4) one obtains

$$\|Mh\|^2 = \|\Phi_{A^*}(\xi)Mh\|_{H^2(\mathfrak{D}_A)}^2 = \sum_{n=0}^{\infty} \|D_A A^n D_{A^*} Mh\|^2 + \|A^* Mh\|^2, \quad h \in \mathfrak{M}.$$

This implies that

$$\|D_{A^*} Mh\|^2 - \lim_{m \rightarrow \infty} \|A^m D_{A^*} Mh\|^2 = \|D_{A^*} Mh\|^2, \quad h \in \mathfrak{M},$$

and, consequently,

$$\lim_{m \rightarrow \infty} A^m D_{A^*} Mh = 0, \quad h \in \mathfrak{M}.$$

Now for every $n \in \mathbb{N}_0$

$$(6.1) \quad \lim_{m \rightarrow \infty} A^m (A^n D_{A^*} Mh) = A^n \left(\lim_{m \rightarrow \infty} A^m D_{A^*} Mh \right) = 0, \quad h \in \mathfrak{M}.$$

Since $\mathfrak{H}^c = \overline{\text{span}} \{A^n D_{A^*} M\mathfrak{M} : n \in \mathbb{N}_0\}$ and A is contractive, the identity (6.1) implies that $\lim_{m \rightarrow \infty} A^m k = 0$ for all $k \in \mathfrak{H}^c$, i.e., the restriction $A \upharpoonright \mathfrak{H}^c$ belongs to the class C_0 ..

Similarly one can prove the other statement. \square

The following result from [33] is needed in the sequel.

Theorem 6.2. ([33]) *Let \mathfrak{M} be a separable Hilbert space and let $N(\xi)$, $\xi \in \mathbb{T}$, be an $\mathbf{L}(\mathfrak{M})$ -valued measurable function such that $0 \leq N(\xi) \leq I_{\mathfrak{M}}$. Then there exist a Hilbert space \mathfrak{K} and an outer function $\varphi(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{K})$ satisfying the following conditions:*

- (i) $\varphi^*(\xi)\varphi(\xi) \leq N^2(\xi)$ almost everywhere on \mathbb{T} ;
- (ii) if $\tilde{\mathfrak{K}}$ is a Hilbert space and $\tilde{\varphi}(\lambda) \in \mathbf{S}(\mathfrak{M}, \tilde{\mathfrak{K}})$ is such that $\tilde{\varphi}^*(\xi)\tilde{\varphi}(\xi) \leq N^2(\xi)$ almost everywhere on \mathbb{T} , then $\tilde{\varphi}^*(\xi)\tilde{\varphi}(\xi) \leq \varphi^*(\xi)\varphi(\xi)$ almost everywhere on \mathbb{T} .

Moreover, the function $\varphi(\lambda)$ is uniquely defined up to a left constant unitary factor.

Assume that $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ and denote by $\varphi_\Theta(\xi)$ and $\psi_\Theta(\xi)$, $\xi \in \mathbb{T}$, the functions which are described in (1.6), (1.7), and (1.8). Their existence is guaranteed by Theorem 6.2 with $N^2(\xi) = I_{\mathfrak{M}} - \Theta^*(\xi)\Theta(\xi)$ and $N^2(\bar{\xi}) = I_{\mathfrak{N}} - \Theta(\bar{\xi})\Theta^*(\bar{\xi})$, respectively. Clearly, if $\Theta(\lambda)$ is inner or co-inner, then $\varphi_\Theta = 0$ or $\psi_\Theta = 0$, respectively. In the case that the system τ is simple and conservative the following result has been established in [10], [11], [14], [18], [19], [20].

Theorem 6.3. *Let $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ be a simple conservative system with transfer function $\Theta_\tau(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$. Then:*

- (i) *the subspace $(\mathfrak{H}^o)^\perp$ ($(\mathfrak{H}^c)^\perp$) is invariant under A (A^*) and the restriction $A \upharpoonright (\mathfrak{H}^o)^\perp$ ($A^* \upharpoonright (\mathfrak{H}^c)^\perp$) is a unilateral shift;*

(ii) the functions $\varphi_{\Theta}(\lambda)$ and $\psi_{\Theta}(\lambda)$ take the form

$$(6.2) \quad \varphi_{\Theta}(\lambda) = P_{\Omega}(I_{\mathfrak{H}} - \lambda A)^{-1}B, \quad \psi_{\Theta}(\lambda) = C(I_{\mathfrak{H}} - \lambda A)^{-1}\upharpoonright_{\Omega_*},$$

where

$$(6.3) \quad \Omega = (\mathfrak{H}^o)^{\perp} \ominus A(\mathfrak{H}^o)^{\perp}, \quad \Omega_* = (\mathfrak{H}^c)^{\perp} \ominus A^*(\mathfrak{H}^c)^{\perp},$$

and P_{Ω} is the orthogonal projector in \mathfrak{H} onto Ω .

By Theorem 5.1 the functions $\varphi(\lambda)$ and $\psi(\lambda)$ defined by (5.2) and (5.3) satisfy

$$(6.4) \quad \varphi^*(\lambda)\varphi(\lambda) \leq I_{\mathfrak{M}} - \Theta_{\tau}^*(\lambda)\Theta_{\tau}(\lambda), \quad \psi(\lambda)\psi^*(\lambda) \leq I_{\mathfrak{N}} - \Theta_{\tau}(\lambda)\Theta_{\tau}^*(\lambda),$$

for all $\lambda \in \mathbb{D}$; see (5.5) and (5.6). Since all the functions involved in these inequalities have limiting values almost everywhere on \mathbb{T} , it follows from (6.4) that

$$(6.5) \quad \varphi^*(\xi)\varphi(\xi) \leq I_{\mathfrak{M}} - \Theta_{\tau}^*(\xi)\Theta_{\tau}(\xi), \quad \psi(\xi)\psi^*(\xi) \leq I_{\mathfrak{N}} - \Theta_{\tau}(\xi)\Theta_{\tau}^*(\xi),$$

for almost all $\xi \in \mathbb{T}$. Hence, by Theorem 6.2, the functions $\varphi(\lambda)$ and $\psi(\lambda)$ satisfy the inequalities

$$(6.6) \quad \varphi(\xi)^*\varphi(\xi) \leq \varphi_{\Theta}^*(\xi)\varphi_{\Theta}(\xi), \quad \psi(\xi)\psi^*(\xi) \leq \psi_{\Theta}(\xi)\psi_{\Theta}^*(\xi),$$

for almost all $\xi \in \mathbb{T}$. In particular, (6.6) shows that if $\varphi_{\Theta}(\xi) = 0$, then $\varphi(\xi) = 0$ and if $\psi_{\Theta}(\xi) = 0$, then $\psi(\xi) = 0$.

For a proof of Theorem 6.3 see [10], [11], [14]; the proof is based on the notions of *optimal* and **-optimal* passive systems. In the sequel the representations of the functions $\varphi_{\Theta}(\lambda)$ and $\psi_{\Theta}(\lambda)$ as given in Theorem 6.3 are needed. Furthermore, the connections between the system τ and the system Σ in (2.5) will be used; cf. Theorem 5.1.

Corollary 6.4. *If the system $\tau = \{A, B, C, D, \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ is simple and conservative then $\varphi_{\Theta}(\lambda) = 0$ ($\psi_{\Theta}(\lambda) = 0$) if and only if the system τ is observable (controllable).*

Proof. Let $\varphi_{\Theta}(\lambda) = 0$ for all $\lambda \in \mathbb{D}$. In view of (6.2) this means that $P_{\Omega}(I_{\mathfrak{H}} - \lambda A)^{-1}B = 0$ for all $\lambda \in \mathbb{D}$. Therefore, $P_{\Omega}A^n B f = 0$ for all $f \in \mathfrak{M}$ and $n = 0, 1, \dots$. This is equivalent to the equality $P_{\Omega}\mathfrak{H}^c = 0$, i.e., $\Omega \subset (\mathfrak{H}^c)^{\perp}$. On the other hand, (6.3) shows that $\Omega \subset (\mathfrak{H}^o)^{\perp}$. Thus, $\Omega \subset (\mathfrak{H}^c)^{\perp} \cap (\mathfrak{H}^o)^{\perp}$ and, because the system τ is simple, this gives $\Omega = \{0\}$, i.e., $A(\mathfrak{H}^o)^{\perp} = (\mathfrak{H}^o)^{\perp}$. Since τ is isometric, the equality $\mathfrak{H}^o = \mathfrak{H}_{\Sigma}^o$ holds by Proposition 5.2 and hence by Remark 2.1 $(\mathfrak{H}_{\Sigma}^o)^{\perp} = \{0\}$, i.e., the systems Σ and τ are observable.

Conversely, if τ is observable then $(\mathfrak{H}_{\Sigma}^o)^{\perp} = (\mathfrak{H}_{\tau}^o)^{\perp} = \{0\}$, so that $\Omega = \{0\}$ and $\varphi_{\Theta}(\lambda) = 0$.

Similarly it is seen that $\psi_{\Theta}(\lambda) = 0$ if and only if τ is controllable. \square

Theorem 6.5. *Let $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ be a passive system with transfer function $\Theta_{\tau}(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$. Assume that $\Theta_{\tau}(\lambda)$ is not constant. Then:*

- (i) *If τ is controllable and $\varphi_{\Theta}(\lambda) = 0$, then τ is isometric and minimal. Moreover, if $\Theta_{\tau}(\lambda)$ is inner, then $A \in C_{0..}$.*
- (ii) *If τ is observable and $\psi_{\Theta}(\lambda) = 0$, then τ is co-isometric and minimal. Moreover, if $\Theta_{\tau}(\lambda)$ is co-inner, then $A \in C_{.0}$.*
- (iii) *If τ is simple, $\varphi_{\Theta}(\lambda) = 0$, and $\psi_{\Theta}(\lambda) = 0$, then τ is conservative and minimal. Moreover, if $\Theta_{\tau}(\lambda)$ is bi-inner, then $A \in C_{00}$.*

Proof. (i) & (ii) It suffices to prove (i), as the proof of (ii) is completely similar. Therefore, assume that τ is controllable and that $\varphi_\Theta(\lambda) = 0$. Then (6.6) implies that $\varphi(\lambda) = 0$ and hence τ is isometric by Corollary 5.3. By Corollary 4.2 this means that $D_X D_M = 0$ and $D_K D_A = 0$. If $D_A = 0$, i.e., A is isometric, then $\mathfrak{D}_A = \{0\}$, $D_{K^*} = I_{\mathfrak{N}}$, and (5.4) in Theorem 5.1 shows that $\Theta_\tau(\lambda) = X D_M$ for all $\lambda \in \mathbb{D}$, which is impossible as $\Theta_\tau(\lambda)$ is not constant. Therefore, $D_A \neq 0$ and then $D_K = 0$, i.e., K is isometric. Next it is shown that for the conservative system Σ in (2.5) one has $\varphi_\Sigma(\lambda) = 0$. Since τ is controllable, also Σ is controllable and in particular simple; see Proposition 5.2. By (6.2) in Theorem 6.3

$$\varphi_\Sigma(\lambda) = P_\Omega(I_{\mathfrak{H}} - \lambda A)^{-1} D_{A^*} = P_\Omega \left(\sum_{n=0}^{\infty} \lambda^n A^n D_{A^*} \right), \quad \lambda \in \mathbb{D},$$

where P_Ω is the orthogonal projection from \mathfrak{H} onto $\Omega := (\mathfrak{H}_\Sigma^c)^\perp \ominus A(\mathfrak{H}_\Sigma^c)^\perp$, see (2.8). From the definition of the function $\varphi_\Sigma(\lambda)$ and (5.5) one obtains

$$\|\varphi_\Sigma(\lambda) M h\|^2 \leq \|D_{\Phi_{A^*}(\lambda)} M h\|^2 \leq \|D_{\Theta_\tau(\lambda)} h\|^2, \quad h \in \mathfrak{M}.$$

Now the assumption $\varphi_\Theta(\lambda) = 0$ and Theorem 6.2 imply that $\varphi_\Sigma(\lambda) M = 0$, $\lambda \in \mathbb{D}$. Hence

$$P_\Omega A^n D_{A^*} M = 0, \quad n \in \mathbb{N}_0,$$

and thus $P_\Omega \mathfrak{H}^c = \{0\}$; see (5.12). Since τ is controllable, one has $P_\Omega = 0$. This shows that $\varphi_\Sigma(\lambda) = 0$ and hence by Corollary 6.4 Σ is also observable, i.e., $\mathfrak{H}_\Sigma^c = \mathfrak{H}$. Since τ is isometric, Proposition 5.2 shows that also τ is observable. Thus, τ is minimal.

Now assume that $\Theta_\tau(\lambda)$ is inner. Since τ is controllable one has $\mathfrak{H}^c = \mathfrak{H}$ and thus $A \in C_0$ by Lemma 6.1.

(iii) Let τ be simple and assume that $\varphi_\Theta(\lambda) = 0$ and $\psi_\Theta(\lambda) = 0$. Then (6.6) implies that $\varphi(\lambda) = 0$ and $\psi(\lambda) = 0$, and hence the inequalities in (5.13) and (5.17) hold. Therefore, see (5.15) and (5.16), one obtains

$$(6.7) \quad -D_K A^* M = K^* X D_M, \quad -D_{M^*} A K^* = M X^* D_{K^*},$$

and

$$(6.8) \quad D_K D_A A^n D_{A^*} M = 0, \quad D_{M^*} D_{A^*} A^{*n} D_A K^* = 0, \quad n \in \mathbb{N}_0.$$

Let $f \in \mathfrak{M}$. The equality $-D_K A^* M f = K^* X D_M f$ and

$$\text{ran } D_K \cap \text{ran } K^* = \text{ran } D_K K^* = \text{ran } K^* D_{K^*}$$

(cf. [7]), imply that $K^*(X D_M f - D_{K^*} v) = 0$ for some $v \in \mathfrak{N}$. Since $\ker K^* \subset \text{ran } D_{K^*}^2$, one has $X D_M f = D_{K^*} h_1$ for some $h_1 \in \mathfrak{N}$. Then $D_K(A^* M f + K^* h_1) = 0$ so that $g_0 := A^* M f + K^* h_1 \in \ker D_K = \ker D_K^2$, i.e., $g_0 = K^* K g_0$, and here $h_0 := K g_0 \in \ker D_{K^*}$. Hence, $A^* M f = -K^* h_1 + K^* h_0 = -K^* h$ with $h = h_1 - h_0$ and, moreover, $D_{K^*} h = D_{K^*} h_1 = X D_M f$. Now $-D_{M^*} A K^* h = M X^* D_{K^*} h$ gives

$$D_{M^*} A A^* M f = M X^* X D_M f.$$

Taking into account the equality $D_X D_M = 0$ one obtains

$$D_{M^*} A A^* M f = M X^* X D_M f = M D_M f = D_{M^*} M f$$

for every $f \in \mathfrak{M}$. Hence $D_{M^*} D_{A^*}^2 M = 0$. It follows that

$$0 = D_{M^*} D_{A^*}^2 M D_M = (D_{A^*} D_{M^*})^* (D_{A^*} D_{M^*}) M$$

and hence $D_{A^*}D_{M^*}M = 0$. Since $M^* \in \mathbf{L}(\mathfrak{D}_{A^*}, \mathfrak{M})$, one has $\mathfrak{D}_{M^*} \subset \mathfrak{D}_{A^*}$. Therefore

$$D_{M^*}M = MD_M = 0.$$

This means that the operator M^* is a partial isometry. Similarly it can be proved that the operator K is a partial isometry. It follows that

$$\begin{aligned} \mathfrak{D}_A &= \text{ran } K^* \oplus \ker K, & D_K \upharpoonright \text{ran } K^* &= 0, & D_K \upharpoonright \ker K &= I_{\ker K}, \\ \mathfrak{D}_{A^*} &= \text{ran } M \oplus \ker M^*, & D_{M^*} \upharpoonright \text{ran } M &= 0, & D_{M^*} \upharpoonright \ker M^* &= I_{\ker M^*}. \end{aligned}$$

Since $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$, (6.7) gives $D_K A^* M = 0$ and $D_{M^*} A K^* = 0$. This means that

$$A^* : \text{ran } M \rightarrow \text{ran } K^*, \quad A : \text{ran } K^* \rightarrow \text{ran } M,$$

and consequently

$$A : \ker K \rightarrow \ker M^*, \quad A^* : \ker M^* \rightarrow \ker K.$$

Therefore

$$D_{A^*}^2 : \ker M^* \rightarrow \ker M^*, \quad D_A^2 : \ker K \rightarrow \ker K,$$

so that

$$M^* D_{A^*}^2 \varphi = 0 \text{ for all } \varphi \in \ker M^*, \quad K D_A^2 \psi = 0 \text{ for all } \psi \in \ker K.$$

The equalities (6.8) yield

$$M^* D_{A^*} A^{*n} D_A D_K = 0, \quad K D_A A^n D_{A^*} D_{M^*} = 0, \quad n \in \mathbb{N}_0.$$

Now, let $\psi \in \ker K$. Then $\varphi = A\psi \in \ker M^*$ and $D_{M^*} A\psi = A\psi$, so that

$$0 = K D_A A^n D_{A^*} A\psi = K D_A A^{n+1} D_A \psi \quad \text{for all } n \in \mathbb{N}_0.$$

Since $K D_A^2 \psi = 0$, one has in fact

$$K D_A A^m D_A \psi = 0, \quad m \in \mathbb{N}_0.$$

Similarly, $M^* D_{A^*} A^{*m} D_A \psi = 0$, $m \in \mathbb{N}_0$. This means that the vector $D_A \psi$ belongs to $(\mathfrak{H}^c)^\perp \cap (\mathfrak{H}^o)^\perp$. Since τ is simple, it follows that $D_A \psi = 0$ and thus $\psi = 0$, i.e., $\ker K = \{0\}$. Similarly $\ker M^* = \{0\}$. Thus, the operators K and M^* are isometries. In addition $D_X D_M = 0$ and $D_{X^*} D_{K^*} = 0$; see (5.13), (5.17). Hence, by Corollary 4.2 the operator T in (4.2) is unitary, i.e., τ is conservative. Furthermore, minimality of τ follows from Corollary 6.4.

The last assertion is now obtained directly from (i) and (ii). Also, if $\Theta_\tau(\lambda)$ is bi-inner then $D_{\Phi^*(\xi)} M = 0$ and $D_{\Phi(\xi)} K^* = 0$ almost everywhere on \mathbb{T} . Since $\text{ran } M = \mathfrak{D}_{A^*}$ and $\text{ran } K^* = \mathfrak{D}_A$, the characteristic function $\Phi_{A^*}(\lambda)$ is bi-inner by Corollary 5.3. Since τ and hence also Σ is simple, the operator A is completely non-unitary; see Remark 2.1. Therefore, A belongs to the class C_{00} . \square

Since every two controllable isometric (observable co-isometric) realizations of an operator-valued function from the Schur class are unitarily similar (see [2], [1]), the following theorem is a corollary of Theorem 6.5; cf. [10], [11], [14].

Theorem 6.6. *Let $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$. Then:*

- (i) *if $\Theta(\lambda)$ is bi-inner and τ is a simple passive system with transfer function $\Theta(\lambda)$, then τ is conservative;*
- (ii) *if $\varphi_\Theta(\lambda) = 0$ or $\psi_\Theta(\lambda) = 0$, then all passive minimal systems with transfer function $\Theta(\lambda)$ are unitarily equivalent, and if $\varphi_\Theta(\lambda) = 0$ and $\psi_\Theta(\lambda) = 0$, then they are in addition conservative.*

7. BI-STABLE PASSIVE SYSTEMS AND BI-INNER DILATIONS OF THEIR TRANSFER FUNCTIONS

Let $\Theta(\lambda)$ be a function from the Schur class $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$. Following [15] the function $\Theta(\lambda)$ is said to have an *inner dilation* if there exists a function $\Theta_r(\lambda)$ such that

$$\Theta(\lambda) = \begin{pmatrix} \Theta(\lambda) \\ \Theta_r(\lambda) \end{pmatrix} \in \mathbf{S}(\mathfrak{M}, \mathfrak{N} \oplus \mathfrak{L})$$

is inner. The function $\Theta(\lambda)$ is said to have a *co-inner dilation* if there exists a function $\Theta_l(\lambda)$ such that

$$\Theta(\lambda) = \begin{pmatrix} \Theta_l(\lambda) & \Theta(\lambda) \end{pmatrix} \in \mathbf{S}(\mathfrak{K} \oplus \mathfrak{M}, \mathfrak{N})$$

is co-inner. The function $\Theta(\lambda)$ is said to have a *bi-inner dilation* if there exist functions $\Theta_{11}(\lambda)$, $\Theta_{22}(\lambda)$, and $\Theta_{21}(\lambda)$ such that

$$\Theta(\lambda) = \begin{pmatrix} \Theta_{11}(\lambda) & \Theta(\lambda) \\ \Theta_{21}(\lambda) & \Theta_{22}(\lambda) \end{pmatrix} \in \mathbf{S}(\mathfrak{K} \oplus \mathfrak{M}, \mathfrak{N} \oplus \mathfrak{L})$$

is bi-inner.

Recall that a system $\tau = \{A, B, C, D, \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ is said to be *strongly stable* (*strongly co-stable*) if the operator A belongs to the class C_0 . (C_0); cf. [10], [15]. The following result is well known; cf. [10]. The present proof is based on the parametrization in Theorem 4.1 and the relations between the transfer function $\Theta_\tau(\lambda)$ and the characteristic function $\Phi_{A^*}(\lambda)$ established in Theorem 5.1.

Proposition 7.1. (cf. [10]) *Let $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ be a passive system with transfer function $\Theta_\tau(\lambda)$. Then:*

- (i) *if τ is strongly stable then $\Theta_\tau(\lambda)$ has an inner dilation;*
- (ii) *if τ is strongly co-stable then $\Theta_\tau(\lambda)$ has a co-inner dilation;*
- (iii) *if τ is strongly stable and strongly co-stable then $\Theta_\tau(\lambda)$ has a bi-inner dilation.*

Proof. (i) Let τ be strongly stable. Then the characteristic function Φ_{A^*} is an inner function, i.e. $\Phi_{A^*}(\xi)^* \Phi_{A^*}(\xi) = I_{\mathfrak{D}_{A^*}}$ for almost all $\xi \in \mathbb{T}$. It follows from (5.5) that

$$I_{\mathfrak{N}} - \Theta_\tau(\xi)^* \Theta_\tau(\xi) = \varphi(\xi)^* \varphi(\xi),$$

for almost all $\xi \in \mathbb{T}$. In other words, the function

$$\Theta(\lambda) := \begin{pmatrix} \Theta_\tau(\lambda) \\ \varphi(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{D},$$

is an inner dilation of Θ_τ .

(ii) Let τ be strongly co-stable. Then the characteristic function $\Phi_A(\lambda) = \Phi_{A^*}(\bar{\lambda})^*$ is an inner function, i.e., $\Phi_A(\xi)^* \Phi_A(\xi) = I_{\mathfrak{D}_A}$ for almost all $\xi \in \mathbb{T}$. Now it follows from (5.6) that

$$I_{\mathfrak{N}} - \Theta_\tau(\xi) \Theta_\tau(\xi)^* = \psi(\xi) \psi(\xi)^*,$$

for almost all $\xi \in \mathbb{T}$. In other words, the function

$$\Theta(\lambda) := \begin{pmatrix} \psi(\lambda) & \Theta_\tau(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{D},$$

is a co-inner dilation of $\Theta(\lambda)$.

(iii) Let τ be strongly stable and strongly co-stable. Define

$$\Theta_{21}(\lambda) = \begin{pmatrix} K^* X M^* + D_K \Phi_{A^*}(\lambda) D_{M^*} & -K^* D_{X^*} \\ D_X M^* & X^* \end{pmatrix}, \quad \lambda \in \mathbb{D}.$$

Using the formulas in Theorem 5.1 it can be checked with a straightforward calculation that the function

$$\Theta(\lambda) := \begin{pmatrix} \psi(\lambda) & \Theta_\tau(\lambda) \\ \Theta_{21}(\lambda) & \varphi(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{D},$$

satisfies the following two identities:

$$(7.1) \quad I - \Theta(\lambda)^* \Theta(\lambda) = M_1^* D_{\Phi_{A^*}(\lambda)}^2 M_1, \quad I - \Theta(\lambda) \Theta(\lambda)^* = K_1^* D_{\Phi_A(\bar{\lambda})}^2 K_1,$$

where $M_1 = \begin{pmatrix} D_{M^*} & 0 & M \end{pmatrix}$ and $K_1 = \begin{pmatrix} K^* & D_K & 0 \end{pmatrix}$. Since the characteristic function $\Phi_{A^*}(\lambda)$ is bi-inner, (7.1) shows that the function $\Theta(\lambda)$ is a bi-inner dilation of $\Theta_\tau(\lambda)$. \square

8. OPERATORS OF THE CLASS $C(\alpha)$ AND CORRESPONDING PASSIVE SYSTEMS

A bounded operator A on a Hilbert space \mathfrak{H} is said to belong to the class $C(\alpha)$, $\alpha \in (0, \pi/2)$, if

$$(8.1) \quad \|A \sin \alpha \pm i \cos \alpha I\| \leq 1,$$

cf. [4]. Let $A_R = (A + A^*)/2$ and $A_I = (A - A^*)/2i$ be the real and imaginary parts of A . Then the condition (8.1) is equivalent to

$$(8.2) \quad |(A_I f, f)| \leq \frac{\tan \alpha}{2} \|D_A f\|^2 \quad \text{for all } f \in \mathfrak{H},$$

cf. [5]. In particular (8.2) shows that the operators in $C(\alpha)$ are contractive. The inequality (8.2) also implies that it is natural to define the class $C(0)$ as the set of all selfadjoint contractions. Let

$$\tilde{C} = \bigcup \{ C(\alpha) : \alpha \in [0, \pi/2) \}.$$

The class \tilde{C} was studied in [4], [5]. In particular, it was proved in [4] that if $A \in \tilde{C}$, then

- (i) $\text{ran } D_{A^n} = \text{ran } D_{A^{*n}} = \text{ran } D_{A_R}$ for all $n \in \mathbb{N}$;
- (ii) the subspace $\mathfrak{D}_A = \mathfrak{D}_{A^*}$ reduces the operator A and, moreover, $A \upharpoonright \mathfrak{D}_A$ is a completely non-unitary contraction of the class C_{00} , while $A \upharpoonright \ker D_A$ is selfadjoint and unitary.

Let A belong to the class \tilde{C} and let $\Phi_A(\lambda)$ in (2.3) be its characteristic function. Then $\Phi_A(\lambda)$ is bi-inner (see [33]) and there exist unitary non-tangential strong limit values

$$\Phi_A(\pm 1) = s - \lim_{\lambda \rightarrow \pm 1} \Phi_A(\lambda);$$

cf. [4]. Observe that if A is a selfadjoint contraction (i.e. belongs to the class $C(0)$) then

$$\Phi_A(\pm 1) = \pm I_{\mathfrak{D}_A}.$$

Define the sets

$$P_+(\alpha) := \{ \lambda : |\lambda \sin \alpha + i \cos \alpha| < 1 \}, \quad P_-(\alpha) := \{ \lambda : |\lambda \sin \alpha - i \cos \alpha| < 1 \}.$$

Theorem 8.1. ([5]) *Let $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{N}, \mathfrak{N}\}$ be a passive linear system. If the operator T in (5.1) belongs to the class $C(\alpha)$ for $\alpha \in [0, \pi/2)$, then the transfer function $\Theta_\tau(\lambda)$ has the following properties:*

(i) $\Theta_\tau(\lambda)$ is holomorphic on the domain

$$P(\alpha) = P_+(\alpha) \cup P_-(\alpha);$$

(ii) the following implications hold for all $\beta \in [\alpha, \pi/2]$:

$$\lambda \in P_+(\beta) \Rightarrow \|\Theta_\tau(\lambda) \sin \beta + i \cos \beta I\|_{\mathfrak{H}} \leq 1,$$

and

$$\lambda \in P_-(\beta) \Rightarrow \|\Theta_\tau(\lambda) \sin \beta - i \cos \beta I\|_{\mathfrak{H}} \leq 1;$$

(iii) the non-tangential limit values $\Theta_\tau(\pm 1)$ exist and they belong to the class $C(\alpha)$ in the Hilbert space \mathfrak{H} ;

(iv) the coefficients $\{G_n\}$ of the Taylor expansion

$$\Theta_\tau(\lambda) = \sum_{n=0}^{\infty} \lambda^n G_n, \quad |\lambda| < 1,$$

belong to the class $C(\alpha)$ in the Hilbert space \mathfrak{H} .

Observe, that $\sigma(T) \subset \overline{P_+(\alpha)} \cap \overline{P_-(\alpha)}$, where $\overline{P_\pm(\alpha)} := \{\lambda : |\lambda \sin \alpha \pm i \cos \alpha| \leq 1\}$. It follows from (ii) that for $\beta \in [\alpha, \pi/2]$ the values $\Theta_\tau(\lambda)$ with $\lambda \in \overline{P_+(\beta)} \cap \overline{P_-(\beta)}$ belong to the class $C(\beta)$ in the Hilbert space \mathfrak{H} .

The next proposition, when combined with Proposition 7.1, shows that if the operator T in (5.1) corresponding to the passive system $\tau = \{A, B, C, D, \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ belongs to the class \tilde{C} , then the transfer function of τ admits a bi-inner dilation.

Proposition 8.2. *Let $\tau = \{A, B, C, D; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ be a passive linear system and let T in (5.1) belong to \tilde{C} . If τ is controllable (observable), then τ is strongly stable and strongly co-stable.*

Proof. According to Theorem 4.1 the operator T in (5.1) takes the form (4.2), where $A \in \mathbf{L}(\mathfrak{H})$, $K \in \mathbf{L}(\mathfrak{D}_A, \mathfrak{M})$, $M \in \mathbf{L}(\mathfrak{M}, \mathfrak{D}_{A^*})$, and $X \in \mathbf{L}(\mathfrak{D}_M, \mathfrak{D}_{K^*})$ are contractions. Suppose that T belongs to $C(\alpha)$ for some $\alpha \in [0, \pi/2]$, i.e.,

$$\|T \sin \alpha \pm i \cos \alpha I\| \leq 1.$$

Then

$$\|A \sin \alpha \pm i \cos \alpha I\| = \|(P_{\mathfrak{H}} T \upharpoonright \mathfrak{H}) \sin \alpha \pm i \cos \alpha I_{\mathfrak{H}}\| \leq 1.$$

This means that the operator A belongs to the class $C(\alpha)$ in the subspace \mathfrak{H} . It follows that $\text{ran } D_A = \text{ran } D_{A^*}$. As a consequence of Douglas theorem [25] it is seen that there exists a bounded and boundedly invertible operator L in the subspace \mathfrak{H} such that

$$D_A = D_{A^*} L.$$

It follows by induction from the equalities $AD_A = D_{A^*} A$ and $A^* D_{A^*} = D_A A^*$ that

$$(8.3) \quad A^n D_{A^*} = D_{A^*} (A L^{-1})^n, \quad A^{*n} D_A = D_A (A^* L)^n, \quad n \in \mathbb{N}.$$

Suppose that the system τ is controllable, so that

$$\mathfrak{H}^c = \overline{\text{span}} \{ \text{ran } A^n D_{A^*} M : n \in \mathbb{N}_0 \} = \mathfrak{H}.$$

Then the first identities in (8.3) imply $\mathfrak{H}^c \subset \mathfrak{D}_{A^*}$ and hence $\mathfrak{D}_{A^*} = \mathfrak{H}$. Because $A \in C(\alpha)$, one has $\mathfrak{D}_A = \mathfrak{D}_{A^*}$ and therefore A belongs to the class C_{00} , i.e., the system τ is strongly stable and strongly co-stable.

Suppose that the system τ is observable. Then

$$\mathfrak{H}^o = \overline{\text{span}} \{ \text{ran } A^{*n} D_A K^* : n \in \mathbb{N}_0 \} = \mathfrak{H}$$

and from the second identities in (8.3) one obtains $\mathfrak{H}^o \subset \mathfrak{D}_A$, so that $\mathfrak{D}_A = \mathfrak{H}$. Since $A \in C(\alpha)$, one obtains once again that A belongs to the class C_{00} , i.e. the system τ is strongly stable and strongly co-stable. \square

REFERENCES

- [1] D. Alpay, A. Dijksma, J. Rovnyak, and H.S.V. de Snoo, *Schur functions, operator colligations, and Pontryagin spaces*, Oper. Theory: Adv. Appl., 96, Birkhäuser Verlag, Basel-Boston, 1997.
- [2] T. Ando, *De Branges spaces and analytic operator functions*, Division of Applied Mathematics, Research Institute of Applied Electricity, Hokkaido University, Sapporo, Japan, 1990.
- [3] Yu.M. Arlinskiĭ, Contractive extensions of a dual pair of contractions and their resolvents, Ukrain. Math. J., 37 no. 2 (1985), 247–250.
- [4] Yu.M. Arlinskiĭ, A class of contractions in a Hilbert space, Ukrain. Math. J., 39 no. 6 (1987), 691–696.
- [5] Yu.M. Arlinskiĭ, Characteristic functions of operators of the class $C(\alpha)$, Izv. Vyssh. Uchebn. Zaved. Mat., no. 2 (1991), 13–21.
- [6] Yu.M. Arlinskiĭ, Extremal extensions of a $C(\alpha)$ -suboperator and their representations, Operator Theory: Adv. Appl., 162 (2006), 47–69.
- [7] Yu.M. Arlinskiĭ, S. Hassi, and H.S.V. de Snoo, Q -functions of Hermitian contractions of Kreĭn–Ovcharenko type, Integral Equat. Oper. Theory, 53 no. 2 (2005), 153–189.
- [8] Yu.M. Arlinskiĭ, S. Hassi, and H.S.V. de Snoo, Q -functions of quasi-selfadjoint contractions, Operator Theory: Adv. Appl., 163 (2006), 23–54.
- [9] D.Z. Arov, Passive linear stationary dynamical systems, Sibirsk. Math. J., 20 no. 2 (1979), 211–228 (Russian).
- [10] D.Z. Arov, Stable dissipative linear stationary dynamical scattering systems, J. Operator Theory, 1 (1979), 95–126 (Russian).
- [11] D.Z. Arov, M.A. Kaashoek, and D.P. Pik, Minimal and optimal linear discrete time-invariant dissipative scattering systems, Integr. Equat. Oper. Theory, 29 (1997), 127–154.
- [12] D.Z. Arov, M.A. Kaashoek, and D.P. Pik, The Kalman – Yakubovich – Popov inequality for infinite dimensional discrete time dissipative system, J. Operator Theory, 55 no. 2 (2006), 393–438.
- [13] D.Z. Arov and M.A. Nudel'man, A criterion for the unitarily similarity of minimal passive systems of scattering with a given transfer function, Ukrain. Math. J., 52 no. 2 (2000), 161–172.
- [14] D.Z. Arov and M.A. Nudel'man, Conditions for the similarity of all minimal passive realizations of a given transfer function (scattering and resistance matrices), Mat. Sb., 153 no. 6 (2002), 3–24.
- [15] D.Z. Arov and O.J. Staffans, Bi-inner dilations and bi-stable passive scattering realizations of Schur class operator-valued functions, Int. Equat. Oper. Theory (to appear).
- [16] Gr. Arsene and A. Gheondea, Completing matrix contractions, J. Operator Theory, 7 (1982), 179–189.
- [17] M. Bakonyi and T. Constantinescu, *Schur's algorithm and several applications*, Pitman Research Notes in Mathematics Series, Longman House, Harlow, UK, 1992.
- [18] S.S. Boiko and V.K. Dubovoj, On some extremal problem connected with the suboperator of scattering through inner channels of the system, Report of National Academy of Sciences of Ukraine, no. 4 (1997), 7–11.
- [19] S.S. Boiko, V.K. Dubovoj, B. Fritzsche, and B. Kirstein, Contractive operators, defect functions and scattering theory, Ukrain. Math. J., 49 no. 4 (1997), 481–489.
- [20] S.S. Boiko, V.K. Dubovoj, B. Fritzsche, and B. Kirstein, Models of contractions constructed from the defect function of their characteristic function, Operator Theory: Adv. Appl., 123 (2001), 67–87.
- [21] L. de Branges and J. Rovnyak, *Square summable power series*, Holt, Rinehart and Winston, New-York, 1966.
- [22] L. de Branges and J. Rovnyak, Appendix on square summable power series, Canonical models in quantum scattering theory, in: *Perturbation Theory and its Applications in Quantum Mechanics* (ed. C.H. Wilcox), New-York, 1966, pp. 295–392.

- [23] M.S. Brodskii, Unitary operator colligations and their characteristic functions, *Uspekhi Math. Nauk*, 33 (4) (1978), 141–168 (Russian) [English transl.: *Russian Math. Surveys*, 33 no. 4 (1978), 159–191].
- [24] Ch. Davis, W.M. Kahan, and H.F. Weinberger, Norm preserving dilations and their applications to optimal error bounds, *SIAM J. Numer. Anal.*, 19 no. 3 (1982), 445–469.
- [25] R.G. Douglas, On majorization, factorization and range inclusion of operators in Hilbert space, *Proc. Amer. Math. Soc.*, 17 (1966), 413–416.
- [26] V.K. Dubovoj and R.K. Mohhamed, Defect functions of holomorphic contractive matrix functions, regular extensions and open systems, *Math. Nachr.*, 160 (1993), 69–110.
- [27] P.A. Fillmore and J.P. Williams, On operator ranges, *Adv. Math.*, 7 (1971), 254–281.
- [28] B. Fritzsche, B. Kirstein, and J. Lorenz, On an extension problem for contractive block Hankel operator matrices, *Analysis*, 25 (2005), 23–54.
- [29] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, Heidelberg, 1995.
- [30] V.U. Kolmanovich and M.M. Malamud, Extensions of sectorial operators and dual pairs of contractions, Manuscript No. 4428-85, Deposited at VINITI, (1985), 1–57 (Russian).
- [31] M.M. Malamud, On some classes of extensions of sectorial operators and dual pair of contractions, *Operator Theory: Adv. Appl.*, 124 (2001), 401–448.
- [32] Yu.L. Shmul'yan and R.N. Yanovskaya, Blocks of a contractive operator matrix, *Izv. Vyssh. Uchebn. Zaved. Mat.*, no. 7 (1981), 72–75.
- [33] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, North-Holland, New York, 1970.

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