Parametrization of the Driven Betatron Oscillation

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An AC dipole is used to excite driven transverse motions of a beam for synchrotron diagnosis. The driven oscillation excited by an AC dipole differs from the free oscillation and the difference could affect linear optics measurements more than 6% in the LHC, RHIC, and Tevatron. An AC dipole changes the amplitude function of the betatron motion like a thin quadrupole magnet. By introducing a proper amplitude function for the driven motion, the difference between the free and driven oscillation becomes clearer and the data interpretation of the driven oscillation is simplified.

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I. INTRODUCTION

An AC dipole produces an oscillating dipole magnetic field and excites transverse motions of a beam in a synchrotron for machine diagnosis (Fig 1). Unlike conventional single turn kicker/pinger magnets, it drives a beam close to the betatron frequency typically for several thousands revolutions. If its magnetic field is adiabatically changed, it can create large coherent oscillations without decoherence and emittance growth [1]. This property of an AC dipole make it a useful diagnosis tool for a synchrotron, especially when it is used with a good beam position monitor (BPM) system.

AC dipoles have been used in the BNL RHIC [2] and was also tested in the BNL AGS [1] and CERN SPS [3]. For the FNAL Tevatron a vertical AC dipole has been in operations [4–6]. A motivation to use AC dipoles for the Tevatron is its recently recently upgraded BPM system whose resolution is about 20 μm [7]. There is also an ongoing project to develop AC dipoles for LHC too [8].

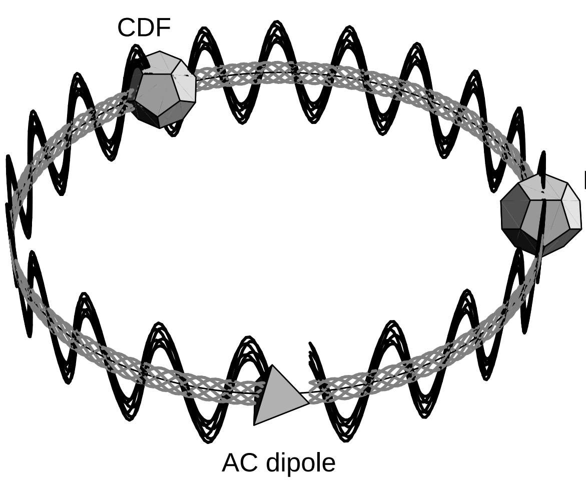

FIG. 1: An image of the incoherent free oscillations (gray) and coherent driven oscillations (black) excited by the Tevatron AC dipole. Since free betatron oscillations of individual particles are incoherent, coherent oscillations must be excited to observe oscillations and diagnose an accelerator. An AC dipole is one of such tool to excited coherent transverse motions in a synchrotron like a pinger/kicker magnet.

The driven oscillation excited by an AC dipole is different from the free oscillation and the difference can be clearly seen with the Tevatron BPM system. If the difference is simply ignored, linear optics measurements could be affected more than 6% in the LHC and RHIC and 12% in the Tevatron. This paper presents a model of the driven betatron oscillation which clarify the difference between the free and driven oscillation. It also helps data interpretation of the driven oscillation and shows an analogy between an oscillating dipole field and gradient error.

II. A MODEL OF THE DRIVEN OSCILLATION

An oscillating dipole field of an AC dipole creates two driving term. The existence of two driving terms make the difference between the free and driven oscillation. This section present a new expression of the driven oscillation which is convenient to treat two driving terms at the same time. It also shows the difference of the free and driven oscillations can be explained as the difference of their amplitude functions.

A. Two Driving Terms of an Oscillating Dipole Field

The tune of an AC dipole $\nu_{\rm acd}$ is the the ratio between the frequencies of the AC dipole $f_{\rm acd}$ and the beam revolution $f_{\rm rev}$: $\nu_{\rm acd} = f_{\rm acd}/f_{\rm rev}$. In the following, for any tunes, only their fractional parts are considered. For instance, if $f_{\rm acd}/f_{\rm rev}$ is larger than one, the tune of the AC dipole $\nu_{\rm acd}$ is the fractional part of $f_{\rm acd}/f_{\rm rev}$. From the Nypquist sampling theorem [**?**], $1 - \nu_{\rm acd}$ is also the tune of the AC dipole for a circulat-

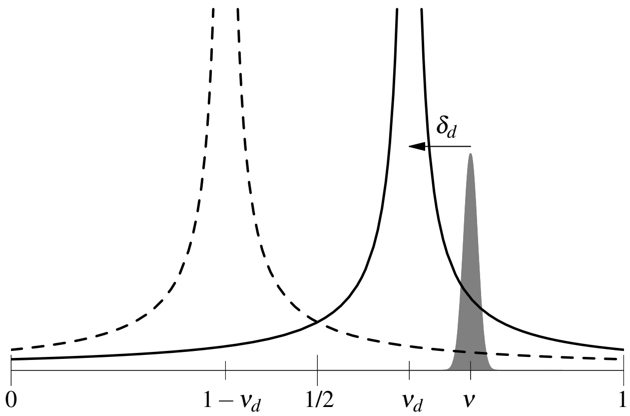

FIG. 2: The tune spread and resonance curves of two driving terms created by an oscillating dipole field of an AC dipole. The shaded area represents the tune spread of a beam. The solid and dashed lines are resonance curves of the two driving terms. In typical operations of an AC dipole, two driving terms are outside of the tune spread to prevent beam losses. For the LHC, RHIC, and Tevatron, the minimum $|\delta_d|$ is about 0.01. The secondary driving term at $1-\nu_d$ causes the difference between the free and driven oscillations. When the machine tune gets closer to the half integer, the secondary driving term also gets closer to the machine tune and its effect on a beam gets larger. Since the machine tunes of the LHC, RHIC, and Tevatron are about 0.3, 0.7, and 0.58, the secondary driving term has larger impacts in the Tevatron than the LHC and RHIC.

ing beam. Hence, an oscillating dipole field creates a pair of driving terms at $\nu_{\rm acd}$ and $1-\nu_{\rm acd}$. Obviously, the driving term closer to the machine tune ν (which is also $0<\nu<1$) has bigger effects on a beam. In the following, the driving term closer to ν is called the primary and the other is called the secondary. A symbol ν_d is used for the primary driving tune:

$$\nu_d \equiv \begin{cases} \nu_{\rm acd} & \text{when} \quad |\nu_{\rm acd}-\nu| < |(1-\nu_{\rm acd})-\nu| \\ 1-\nu_{\rm acd} & \text{when} \quad |(1-\nu_{\rm acd})-\nu| < |\nu_{\rm acd}-\nu| \, . \end{cases} \tag{1}$$

For example, frequencies of the AC dipole and beam revolution in the Tevatron are $f_{\rm acd} \simeq 20.5$ kHz and $f_{\rm rev} \simeq 47.7$ kHz and hence the tune of the AC dipole is $\nu_{\rm acd} = 20.5/47.7 \simeq 0.43$. Since the machine tune of the Tevatron is $\nu \simeq 0.58$, $1-\nu_{\rm acd} \simeq 0.57$ is the primary driving tune and $\nu_{\rm acd} \simeq 0.43$ is secondary in this case.

The distance from the primary driving term to the machine tune $\delta_d \equiv \nu_d - \nu$ is an important parameter of the driven betatron oscillation. As seen in later, the existence of two driving terms makes a difference between the free and driven oscillations and affects linear optics measurements. Ideally, if a beam is driven very close to a machine tune ν ($\delta_d \to 0$), the effect of the primary driving term becomes dominant and the secondary driving term can be simply ignored. In reality, however, the finite tune spread of a beam causes beam losses if $|\delta_d|$ is too small and there is always a limit for $|\delta_d|$ (Fig 2). The AC dipoles are currently used in the RHIC and Tevatron and will be used in the LHC. In these synchrotrons, the minimum $|\delta_d|$ is about 0.01 to prevent beam losses. To measure the linear optics of these synchrotrons better than 6% accuracy, $|\delta_d| = 0.01$ is not small enough to ignore the secondary driving.

In usual operations of an AC dipole, its magnetic field is first adiabatically ramped up and kept at the maximum during a measurement. When the field is at the maximum, the position of the driven beam x_d is given by [1, 10, 11]

$$\begin{aligned} x_d(nC+\Delta s) \simeq\ & \frac{\theta_{\rm acd}\sqrt{\beta_{\rm acd}}}{4\sin[\pi(\nu_{\rm acd}-\nu)]}\sqrt{\beta(\Delta s)}\cos[2\pi\nu_{\rm acd}n + \psi(\Delta s) + \pi(\nu_{\rm acd}-\nu) + \chi_{\rm acd}] \\ & + \frac{\theta_{\rm acd}\sqrt{\beta_{\rm acd}}}{4\sin[\pi((1-\nu_{\rm acd})-\nu)]}\sqrt{\beta(\Delta s)}\cos[2\pi(1-\nu_{\rm acd})n + \psi(\Delta s) + \pi((1-\nu_{\rm acd})-\nu) - \chi_{\rm acd}] \, , \end{aligned} \tag{2}$$

where C is the circumference of a ring, Δs ($0<\Delta s<C$) is the longitudinal position measured from the location of the AC dipole, $\theta_{\rm acd}$ is the maximum kick angle of an AC dipole which depends on its integrated field strength and the magnetic rigidity of the beam, $\beta_{\rm acd}$ is the amplitude function at the location of the AC dipole, ψ is the phase advance of the free oscillation measured from the location of the AC dipole, and $\chi_{\rm acd}$ is the initial phase of the AC dipole. Two terms in Eq 2 are completely symmetric and represent the effects from two driving terms [**?**]. Define a parameter λ_d which describes the ratio between the primary and secondary modes:

$$\lambda_d \equiv \frac{\sin(\pi\delta_d)}{\sin(2\pi\nu+\pi\delta_d)} = \frac{\sin[\pi(\nu_d-\nu)]}{\sin[\pi((1-\nu_d)-\nu)]} \, . \tag{3}$$

As discussed previously, the minimum $|\delta_d|$ is about 0.01 for the hadron synchrotrons where AC dipoles are used. When $|\delta_d| = 0.01$, $|\lambda_d| \simeq 0.06$ for the Tevatron with $\nu \simeq 0.58$ and $|\lambda_d| \simeq 0.03$ for the LHC and RHIC with $\nu \simeq 0.3$ or 0.7. This is the effect of the secondary driving term on the beam motion.

B. A New Parametrization of the Driven Betatron Oscillation

As seen later, this 6% effect of the secondary driving term can be clearly observed for the driven oscillation in the Tevatron. This section presents another expression of the driven motion to get better understanding of the secondary driving term effect. Eq 2 can be actually written in the following compact form which includes the effects from both driving terms:

$$x_d(s;\delta_d) = A_d(\delta_d)\sqrt{\beta_d(s;\delta_d)}\cos(\psi_d(s;\delta_d) \pm \chi_{\rm acd}) \ , \qquad (4)$$

where A_d is a quantity with dimensions of (length)$^{1/2}$:

$$A_d(\delta_d) \equiv \frac{\theta_{\rm acd}}{4\sin(\pi\delta_d)}\sqrt{(1-\lambda_d^2)\beta_{\rm acd}} \ , \qquad (5)$$

β_d is the amplitude function of the driven oscillation:

$$\beta_d(s;\delta_d) \equiv \frac{1+\lambda_d^2 - 2\lambda_d\cos(2\psi(s)-2\pi\nu)}{1-\lambda_d^2}\beta(s) \ , \qquad (6)$$

ψ_d is the phase advance of the driven oscillation measured from the location of the AC dipole:

$$\psi_d(s;\delta_d) \equiv \int_0^s \frac{d\bar{s}}{\beta_d(\bar{s};\delta_d)} \ , \qquad (7)$$

and the sign in front of $\chi_{\rm acd}$ is positive when $\nu_d = \nu_{\rm acd}$ and negative when $\nu_d = 1-\nu_{\rm acd}$. Hence, the position of the driven oscillation can be written in the same form as the free oscillation even when the effect of the both driving term is included. Since A_d is a constant of motion, the difference between the free and driven oscillations comes from the amplitude function β_d and phase advance ψ_d. As discussed previously, in the limit $\nu_d \to \nu$, the primary driving term becomes dominant and the secondary driving term can be ignored. Since $\lambda_d \to 0$ in this limit, β_d and ψ_d converge to β and ψ. Therefore, the changes of the amplitude function and phase advance due to the secondary driving term.

A relation between the phase advances of free and driven oscillations, ψ and ψ_d, is given by

$$\begin{aligned}\tan(\psi_d - \pi\nu_d) &= \frac{1+\lambda_d}{1-\lambda_d}\tan(\psi-\pi\nu) \\ &= \frac{\tan(\pi\nu_d)}{\tan(\pi\nu)}\tan(\psi-\pi\nu) \ . \end{aligned} \qquad (8)$$

For the free oscillation, the phase advance in a single turn is $\psi(s+C)-\psi(s) = 2\pi\nu$ (mod 2π). From the equation above, the phase advance of the driven oscillation in a single turn is $\psi_d(s+C)-\psi_d(s) = 2\pi\nu_d$ (mod 2π).

III. PROPERTIES OF THE AMPLITUDE FUNCTION β_d

As seen in the previous section, the difference between the free and driven oscillations lies in the difference of their amplitude functions. Understanding properties of the amplitude function for the driven oscillation β_d is crucial when an AC dipole is used for synchrotron diagnosis.

A. Review of a Gradient Error

If a synchrotron has a gradient error, its machine tune ν and amplitude function β change [9]. Since tune and amplitude function are coupled in betatron oscillations, a change in tune involves a change in amplitude function and vice versa. This is true for the driven betatron oscillation too. When a beam is driven by an AC dipole, its tune is the primary driving tune ν_d instead of the machine tune ν. As seen in the previous section, the amplitude function also changes to β_d from β. As a matter of fact, the relation between the changes of the tune and amplitude function is the same for a gradient error and an oscillating dipole field. Hence, reviewing the effects of a gradient error is helpful to understand the driven oscillation.

Suppose a synchrotron has a gradient error with the strength $q_{\rm error} = B'\ell/(B\rho)$ at the longitudinal position $s=0$. Then, the equation of motion is given by

$$x'' + k(s)x = -q_{\rm error}\left[\sum_{n=-\infty}^{\infty}\delta(s-Cn)\right]x \ , \qquad (9)$$

where the prime denotes the derivative by the longitudinal coordinate s, k is the spring constant, and δ is the Dirac's delta function.

By comparing the single turn transfer matrices with and without the gradient error, the new tune ν_q and amplitude function β_q satisfy the following two equations [9]:

$$q_{\rm error} = 2\frac{\cos(2\pi\nu)-\cos(2\pi\nu_q)}{\beta_{\rm error}\sin(2\pi\nu)} \qquad (10)$$

$$\frac{\beta_q}{\beta} = \frac{\sin(2\pi\nu)}{\sin(2\pi\nu_q)} - q\beta_{\rm error}\frac{\sin\psi\sin(2\pi\nu-\psi)}{\sin(2\pi\nu_q)} \ , \qquad (11)$$

where $\beta_{\rm error}$ is the amplitude function at the gradient error and ψ is the phase advance measured from the gradient error. By substituting the first equation into the second, the ratio between the new and original amplitude functions β_q/β is given by

$$\frac{\beta_q}{\beta} = \frac{1+\lambda_q^2 - 2\lambda_q\cos(2\psi-2\pi\nu)}{1-\lambda_q^2} \ . \qquad (12)$$

Here, λ_q is a similar parameter to λ_d in Eq 3. It is defined with the tune shift $\delta_q \equiv \nu_q - \nu$: $\lambda_q \equiv \sin(\pi\delta_q)/\sin(2\pi\nu+\pi\delta_q)$. When the gradient error $q_{\rm error}$ is small, the new and original amplitude functions satisfy

$$\frac{\beta_q - \beta}{\beta} \simeq -2\lambda_q\cos(2\psi-2\pi\nu) \ . \qquad (13)$$

This quantity behaving like a standing wave in a synchrotron is the β-beat. The amplitude of the β-beat is $2|\lambda_q|$.

B. Analogy to a Gradient Error

As seen in Eqs 6 and 12, the relation between β_d and δ_d for an oscillation dipole field is the same as the relation between β_q and δ_q for a gradient error. The following argument gives an insight why an oscillating dipole field changes the amplitude function like a gradient error.

When the magnetic field of an AC dipole is maximum, the equation of motion is given by

$$x'' + k(s)x = -\sum_n \theta_{\rm acd} \cos(2\pi\nu_d n \pm \chi_{\rm acd})\delta(s - Cn) \ . \quad (14)$$

The right-hand-side of this equation describes the kicks by the AC dipole at $s = 0$. The summation is for the time period when the magnetic field of the AC dipole field is maximum and the sign in front of the initial phase $\chi_{\rm acd}$ is the same convention as Eq 4. Eq 4 is the particular solution of this inhomogeneous Hill's equation when the field of the AC dipole is adiabatically ramped up to the maximum. Since the phase of the driven oscillation ψ_d increases by $2\pi\nu_d$ (mod 2π) in one revolution, the position of the driven oscillation at the location of the AC dipole $s = Cn$ is

$$x_d(Cn;\delta_d) = A_d(\delta_d)\sqrt{\beta_d(0;\delta_d)}\cos(2\pi\nu_d n \pm \chi_{\rm acd}) \ . \quad (15)$$

Notice the phases of the driven oscillation x_d and the AC dipole in Eq 14 are both $2\pi\nu_d n \pm \chi_{\rm acd}$ at the location of the AC dipole. Hence, when the beam passes the AC dipole, its magnetic field is proportional to the position of the driven oscillation x_d like a quadrupole magnet. This is the physical reason why an oscillating dipole field changes the amplitude function like a gradient error. The phases of the driven oscillation and the AC dipole are synchronized like this only when the magnetic field of the AC dipole is maximum after the adiabatic ramp. The proportional constant of the driven motion and the AC dipole field, $q_{\rm acd}$, is given by

$$q_{\rm acd} = \frac{\theta_{\rm acd}}{A_d\sqrt{\beta_d(0)}} = 2\frac{\cos(2\pi\nu) - \cos(2\pi\nu_d)}{\beta_{\rm acd}\sin(2\pi\nu)} \ . \quad (16)$$

The relation among $q_{\rm acd}$, ν_d, and $\beta_{\rm acd}$ is the same as Eq 10. Formally, the solution of Eq 14 x_d satisfies the following equation

$$x_d'' + k(s)x_d = -q_{\rm acd}\left[\sum_n \delta(s - Cn)\right] x_d \ . \quad (17)$$

This equation is exactly the same as the Hill's equation with an gradient error Eq 9. By comparing Eqs 9, 10, 16, and 17, it is trivial that the relation of β_d and δ_d is the same as the relation of β_q and δ_q.

C. Ring-wide Behavior of β_d

From the previous two sections, the amplitude function β_d behaves as if there is a gradient error. Hence, β_d is beating relative to β and the beating amplitude is about $2|\lambda_d|$ from Eq 13. Remember the effect of the secondary driving term on the beam motion is the order of λ_d. Since the amplitude function is proportional to the square of the position, its effect on the amplitude function is the order of $2|\lambda_d|$.

As discussed before, the minimum difference between the primary driving tune and machine tune δ_d is about 0.01 for the LHC, RHIC, and Tevatron. Then, the difference of the amplitude functions β and β_d satisfy $|\beta_d - \beta|/\beta \lesssim 2|\lambda_d| \simeq 0.06/\sin(2\pi\nu)$. Table I shows the estimated beating amplitude of β_d relative to β in the LHC, RHIC, and Tevatron when $\delta_d = 0.01$. Since the magnitude of $\sin(2\pi\nu)$ cannot be larger than one, the beating amplitude $2|\lambda_d|$ cannot be smaller than 6% when $\delta_d = 0.01$. As seen in Fig 2, the effect of the secondary driving term gets larger when the machine tune gets closer to the half integer. This is why the beating amplitude for the Tevatron is larger compared to the LHC and RHIC.

TABLE I: Difference between amplitude functions of the free and driven betatron oscillations β and β_d in the LHC, RHIC, and Tevatron. The last line is the beating amplitude of β_d relative to β.

Parameter	LHC	RHIC	Tevatron
ν	.3	.7	.58
$\lvert\delta_d\rvert$	.01	.01	.01
$2\lvert\lambda_d\rvert$	6-7%	6-7%	12-13%

When turn-by-turn beam positions at all BPMs are given for the free oscillation, the relative β-function can be computed by simply comparing the square of the oscillation amplitude at each BPM. If the same analysis is applied to the turn-by-turn data of the driven oscillation, what is calculated is β_d instead of the β. If the difference between β_d and β is simply ignored and β is determined in this way, the error is as large as $2|\lambda_d|$. Even worse, since the beating of β_d cannot be distinguished from the real β-beat caused by gradient errors, the real β-beat cannot be measured without depending on a machine model in this way.

To calculate β from turn-by-turn data of the driven oscillation without depending on a machine model, multiple sets of data are necessary [5]. Fig 3 shows amplitude functions of the free and driven oscillations β and $\beta_d(\delta_d = -0.01)$. They are both measured from data of the driven oscillation. Multiple

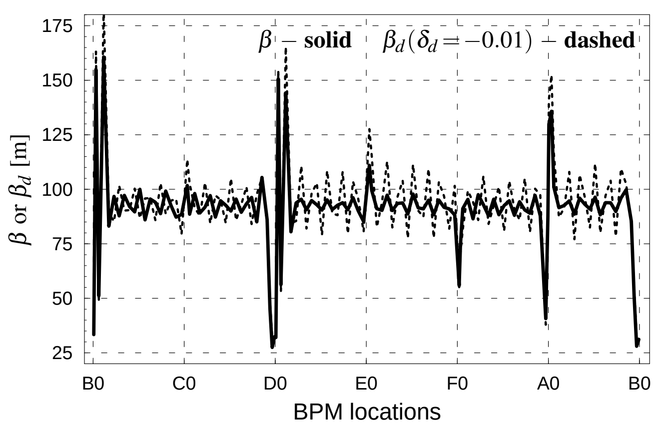

FIG. 3: The amplitude functions of the free and driven oscillations: β (solid) and β_d when $\delta_d = -0.01$ (dashed). Both β and β_d is calculated from turn-by-turn data of the driven oscillation. Multiple data sets are used for β [5] and the amplitude square is simply compared at each BPM for β_d. As expected, β_d shows the 10-15% beating relative to β. If the difference of β and β_d is simply ignored, the β measurement has this much error and the real β-beat cannot be distinguished from the beating of β_d.

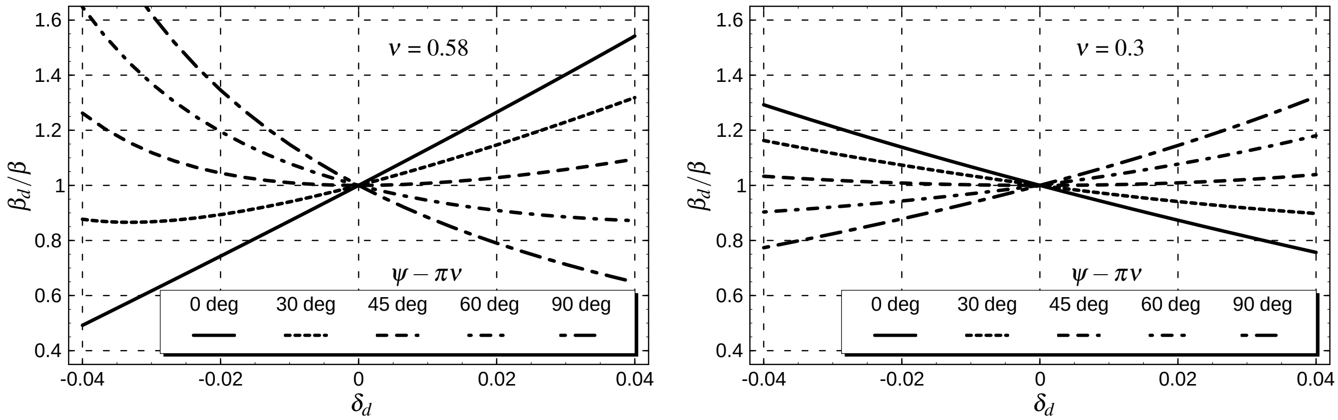

FIG. 4: The relation between the amplitude functions of the free and driven betatron oscillations β and β_d. The ratio β_d/β is numerically calculated from Eq 6 by changing the difference between the primary driving tune and machine tune δ_d and the phase advance ψ. The left plot is for the machine tune 0.58 like the Tevatron and the right is for 0.3 like the LHC and RHIC. Since the secondary driving term gets closer and λ_d gets larger when the machine tune is closer to the half integer, β_d/β is larger and the nonlinearity is stronger in the left plot. The nonlinearity gets larger when $\psi - \pi\nu$ gets closer to 45 deg and $\cos(2\psi - 2\pi\nu)$ gets closer to zero.

data sets are used to calculate β as described in [5] and β_d is calculated by comparing the square of the amplitude at each BPM. In the arc of the Tevatron, β is about 100 m and β_d is showing the beta-beat like structure of 10-15% ($\pm$10-15 m) as expected in the table I.

D. Relation between β_d and δ_d

In the previous section, only the lowest order is considered in the relation between β and β_d. From Eq 6, the relation becomes nonlinear when λ_d is large or the phase term $\cos(2\psi - 2\pi\nu)$ is close to zero. As already seen in the previous section, the difference of β and β_d has not a small impact on the linear optics measurement. Hence, it is important to understand the properties of Eq 6 in wide ranges of parameters. Fig 4 shows the numerical simulations of β_d/β based on Eq 6. Two plots are for two different machine tunes: $\nu = 0.58$ like the Tevatron and $\nu = 0.3$ like the LHC and RHIC. Since λ_d with $\nu = 0.58$ is almost twice of λ_d with $\nu = 0.3$ for the same δ_d, the nonlinearity grows much faster with δ_d in the left plot with $\nu = 0.58$. It is also clear in the left plot that the nonlinearity gets larger when $|\cos(2\psi - 2\pi\nu)|$ gets closer to zero. Such a nonlinear relation between β_d and δ_d can be actually seen for the driven oscillation excited in the Tevatron. An example is shown in the next section.

IV. EVIDENCES OF THE SECONDARY DRIVING TERM

This section presents two more properties of the driven betatron oscillation measured in the Tevatron. The model including the secondary driving term (or the different amplitude function for the driven oscillation) fits well to both of the examples. These examples support the importance of the secondary driving term in the driven oscillation.

A. Rotation of the Phase Space Ellipse

Parameters corresponding to the other Courant-Snyder parameters α and γ can be also defined from the amplitude function of the driven oscillation β_d as the free oscillation:

$$\alpha_d(s;\delta_d) \equiv -\frac{1}{2}\frac{d\beta_d(s;\delta_d)}{ds} \tag{18}$$

$$\gamma_d(s;\delta_d) \equiv \frac{1+\alpha_d(s;\delta_d)^2}{\beta_d(s;\delta_d)} \,. \tag{19}$$

The explicit forms of these parameters are given by

$$\alpha_d = \frac{1+\lambda_d^2 - 2\lambda_d\cos(2\psi - 2\pi\nu)}{1-\lambda_d^2}\,\alpha - \frac{2\lambda_d\sin(2\psi-2\pi\nu)}{1-\lambda_d^2} \tag{20}$$

$$\gamma_d = \frac{1+\lambda_d^2 + 2\lambda_d\cos(2\psi + 2\arctan\alpha - 2\pi\nu)}{1-\lambda_d^2}\,\gamma \,. \tag{21}$$

When β_d, α_d, γ_d, and A_d are defined this way, they satisfy a relation like Courant-Snyder invariance:

$$A_d^2 = \gamma_d x_d^2 + 2\alpha_d x_d x_d' + \beta_d x_d'^2 \,. \tag{22}$$

Therefore, the turn-by-turn position and angle of the driven oscillation also form an ellipse on the phase space. like the free oscillation. Since not only A_d but also the Courant-Snyder like parameters β_d, α_d, and γ_d depend on the difference between the primary driving tune ν_d and the machine tune ν: δ_d, both the area and shape of the phase space ellipse changes with δ_d for the driven oscillation. In two collision straight sections of the Tevatron, B0 and D0, there are pairs of BPMs with no magnet in-between. Hence a beam runs straight between a pair of BPMs and both position and angle at any location between a pair of BPMs can be directly measured. Fig 5 shows

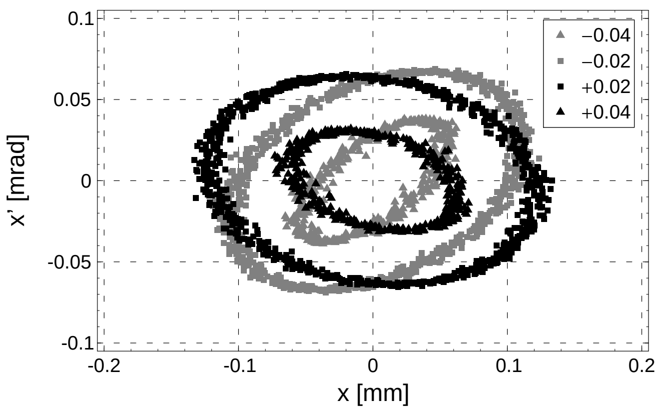

FIG. 5: Phase space ellipses of the driven oscillations when $\delta_d = \pm 0.02$ and ± 0.04. Here, δ_d is the difference between the primary driving term and the machine tune. Since the Courant-Snyder like parameters of the driven oscillation β_d, α_d, and γ_d depend on δ_d, not only the area but also the shape of the ellipse changes along with δ_d. The figure is the phase space at one of the low-β location (B0) of the Tevatron where the derivative of the amplitude function α is zero by design.

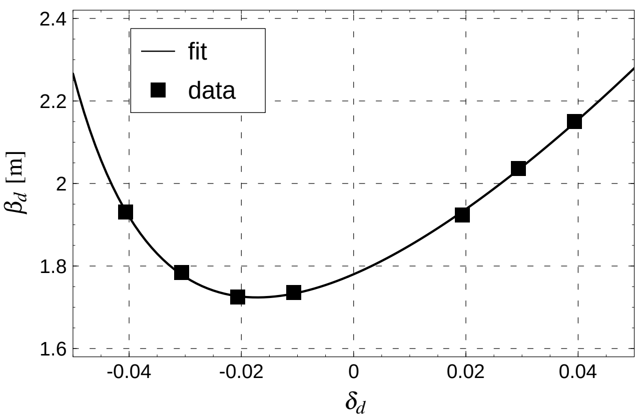

FIG. 6: The relation between the amplitude function of the driven oscillation β_d and the difference between the primary driving tune to the machine tune δ_d. The location is the same low-β point of the Tevatron (B0) as Fig 5. The amplitude function β_d at each data point is determined from the shape of an ellipse in Fig 5. The line is the fit of Eq 6 to the data points. Despite the strong nonlinearity, Eq 6 is fitting well. In the figure, $\beta_d(\delta_d = 0)$ corresponds to β at the location.

the measured phase ellipses of the driven oscillations by using a pair of such BPMs. When the frequency of the AC dipole was changed to adjust δ_d to ± 0.04 and ± 0.02, the kick angle of the AC dipole θ_{acd} was kept the same. As expected, the shape of the phase space ellipse changes with δ_d. Since δ_d dependence of β_d, α_d, and γ_d comes from the secondary driving term, the rotation of the phase space ellipse is its qualitative evidence.

By fitting Eq (22) to an ellipse in Fig 5, its area πA_d^2, β_d, α_d, and γ_d can be determined. Fig 6 shows measured β_d from the fits to ellipses in Fig 5 (and three more). The line in the figure is the fit of Eq 6 to the data with parameters β and ψ. The model of Eq 6 is fitting well to the data even though the nonlinearity is strong in the relation between β_d and δ_d at the location. The β-function at the location can be calculated as one of the fit parameter. In the figure, $\beta = \beta_d(\delta_d = 0)$.

B. Asymmetric Oscillation Amplitude around the Tune

If the secondary driving term is negligible, by ignoring the smaller term of Eq 2 or taking the limit of $\lambda_d \to 0$ in Eqs 4, 5, and 6, the amplitude of the driven oscillation can be approximated by

$$a_d^{(0)}(s;\delta_d) \equiv \frac{\theta_{\text{acd}}\sqrt{\beta_{\text{acd}}\beta(s)}}{4|\sin(\pi\delta_d)|} . \tag{23}$$

Remember δ_d is the difference between the primary driving tune and the machine tune: $\nu_d - \nu$. In this case, the amplitude of the driven oscillation depends on the primary driving tune ν_d only through $|\sin(\pi\delta_d)|$ and is symmetric around the tune ν. From Eqs 4, 5, and 6, the amplitude including the effect of the secondary driving term $a_d(s;\delta_d)$ is given by

$$\begin{aligned} &a_d(s;\delta_d) \\ &= a_d^{(0)}(s;\delta_d)\sqrt{1+\lambda_d^2-2\lambda_d\cos(2\psi(s)-2\pi\nu)} . \end{aligned} \tag{24}$$

Now, the amplitude depends on ν_d through the factor $[1+\lambda_d^2-2\lambda_d\cos(2\psi(s)-2\pi\nu)]^{1/2}$ too. Up to the first order of δ_d, the amplitude is approximated by

$$a_d(s;\delta_d) \simeq a_d^{(0)}(s;\delta_d)\left[1-\frac{\pi\cos(2\psi(s)-2\pi\nu)}{\sin(2\pi\nu)}\,\delta_d\right] . \tag{25}$$

Hence, the secondary driving term makes the ν_d dependence of the amplitude asymmetric around the machine tune ν. The magnitude of this asymmetry at each location is determined by the factor $\cos(2\psi - 2\pi\nu)$ [**?**].

Fig 7 shows the relation between the amplitude of the driven oscillation and ν_d at three BPM locations in the Tevatron. In the measurements, only the frequency of the AC dipole was changed and its kick angle θ_{acd} was kept the same. The dashed and solid lines represent the fits of Eq 23 and Eq 24 to the data. The fit parameters are $\theta_{\text{acd}}[\beta_{\text{acd}}\beta(s)]^{1/2}$ and ν for Eq 23 and $\theta_{\text{acd}}[\beta_{\text{acd}}\beta(s)]^{1/2}$, ν, and ψ for Eq 24 [**?**]. At two locations where $|\cos(2\psi - 2\pi\nu)|$ is close to one, the asymmetry around the machine tune ($\nu \simeq 0.5785$) is clear and the fit without the secondary driving term (Eq 23) is off the data.

Although the effect of the secondary driven term is clear in Fig **??**, there is even a better evidence that Eq 24 with the secondary driving term fits the data better. From the fits in Fig 7, the machine tune ν is determined at each BPM location. Fig 8 shows machine tunes determined at all BPM locations from the fits of the amplitude versus ν_d. The dashed and solid lines represent machine tunes from the fits of Eqs 23 and 24. Since the machine tune ν is a constant of a synchrotron, variations of the measured machine tune over BPMs show the inaccuracy of the measurement. From the figure, it is clear the model including the secondary driving fits to the data mush better. This is another qualitative evidence of the secondary driving term effect in the driven oscillation.

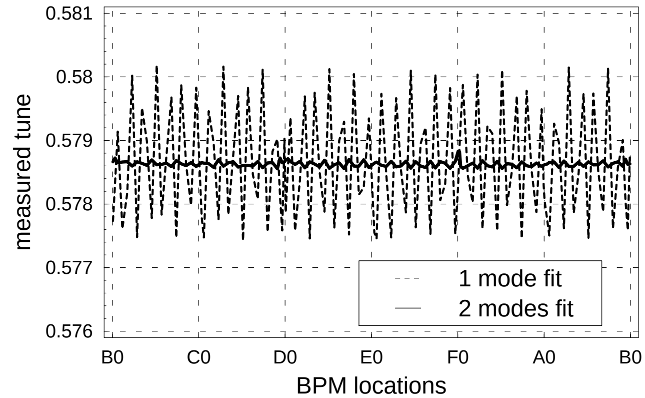

FIG. 8: The measured machine tune at all BPM locations from the fits of the amplitude versus the primary driving tune ν_d in Fig 7. The solid line includes the effect of the secondary driving term and the dashed line does not. Since the tune is a constant of a synchrotron, the equation with the secondary driving term is the better model of the driven oscillation. The beating of the dashed line is caused by ignoring the beating of β_d relative to β.

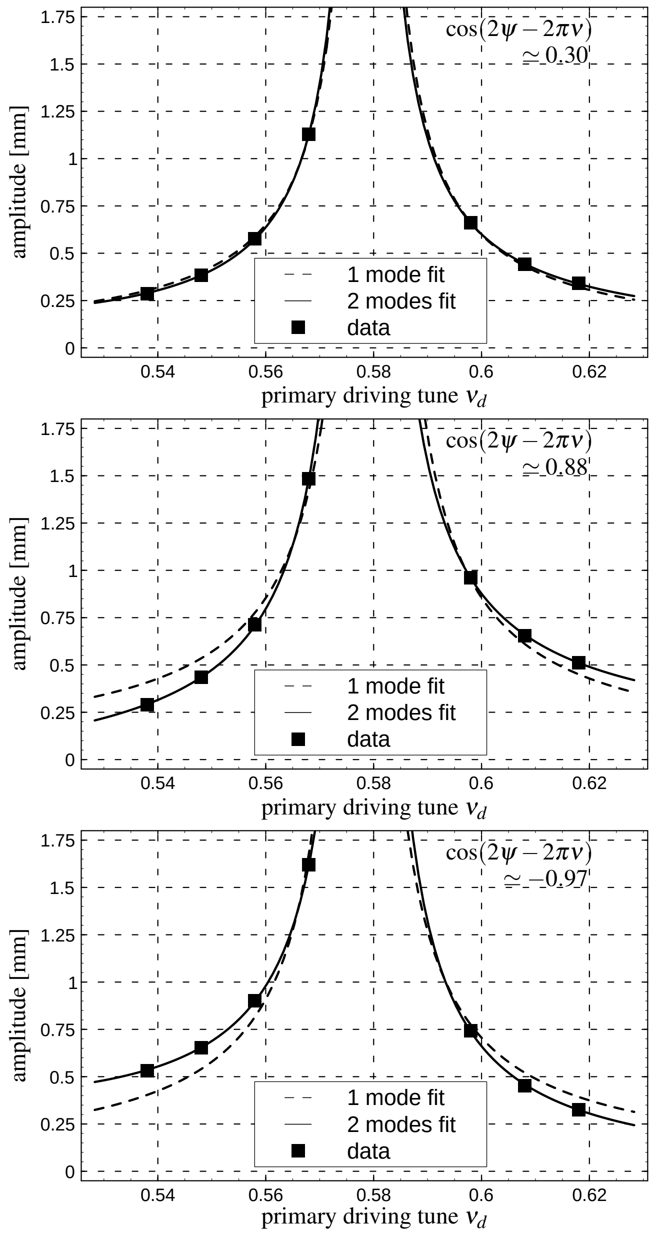

FIG. 7: The relation between the amplitude of the driven oscillation and the primary driving tune ν_d at three BPM locations in the Tevatron. In the measurements, only ν_d was changed. The solid and dashed lines are fits with and without the effect of the secondary driving term. The asymmetry around the tune $\nu \simeq 0.5785$ becomes clear when $|\cos(2\psi - 2\pi\nu)| \simeq 1$. In the second plot where $\cos(2\psi - 2\pi\nu) > 1$, the amplitude is larger in the region $\nu_d > \nu$. As predicted from Eq 25, the relation flips in the third plot where the sign of $\cos(2\psi - 2\pi\nu)$ becomes negative.

V. CONCLUSION

An oscillating field of an AC dipole creates two driving terms for a circulating beam: primary and secondary. The secondary driving term actually has important roles in the driven oscillation. If it is simply ignored, linear optics measurements using an AC dipole could be affected 6-7% in the LHC and RHIC and 12-13% in the Tevatron. For the driven oscillation in the Tevatron, the effects of the secondary driving term could seen in the data. Examples are the change of the amplitude function and the rotation of the phase space ellipse. The paper presented a model of the driven oscillation which includes the effects from both of the driving terms. The model fitted well to the data of the driven oscillation in the Tevatron.

[1] M. Bai *et al.*, Phys. Rev. E **56**, p. 6002 (1997).
[2] M. Bai *et al.*, in *Proceedings of the 8th European Particle Accelerator Conference*, Paris, 2002, edited by T. Garvey *et al.* (EPS-IGA and CERN, Geneva, 2002), p. 1115.
[3] O. Berrig *et al.*, in *Proceedings of the 5th European Workshop on Beam Diagnostics and Instrumentation for Particle Accelerators*, Grenoble, 2001, edited by K. Scheidt *et al.* (ESRF, Grenoble, 2001), p. 82.
[4] R. Miyamoto *et al.*, in *Proceedings of the 12th Beam Instrumentation Workshop*, Batavia, IL, 2006, edited by T. S. Meyer

and R. C. Webber (AIP, Melville, New York, 2006), p. 402.

[5] R. Miyamoto *et al*., in *Proceedings of the 22nd Particle Accelerator Conference*, Albuquerque, NM, 2007, edited by S. Hardage *et al*., (IEEE, Piscataway, NJ, 2007), p. 0000.

[6] R. Miyamoto *et al*., in *Proceedings of the 22nd Particle Accelerator Conference*, Albuquerque, NM, 2007, edited by S. Hardage *et al*., (IEEE, Piscataway, NJ, 2007), p. 0000.

[7] S. Wolbers *et al*., in *Proceedings of the 21st Particle Accelerator Conference*, Knoxville, TN, 2005, edited by C. Horak *et al*. (IEEE, Piscataway, NJ, 2005), p. 410.

[8] R. Miyamoto *et al*., in *Proceedings of the 22nd Particle Accelerator Conference*, Albuquerque, NM, 2007, edited by S. Hardage *et al*., (IEEE, Piscataway, NJ, 2007), p. 0000.

[9] See textbooks of the accelerator physics such as D. A. Edwards and M. J. Syphers, *An Introduction to the Physics of High Energy Accelerators*, (John Wiley & Sons, Inc., New York, 1993) or S. Y. Lee, *Accelerator Physics* (*Second Edition*), (World Scientific, Singapore, 2004).

[10] S. Peggs, in *Proceedings of the 18th Particle Accelerator Conference*, New York, NY, 1999, edited by A. Luccio and W. MacKay (IEEE, Piscataway, NJ, 2005), p. 1572.

[11] R. Tomas, Phys. Rev. ST Accel. Beams **8**, 024401 (2005).

[12] M. J. Syphers and R. Miyamoto, in *Proceedings of the 22nd Particle Accelerator Conference*, Albuquerque, NM, 2007, edited by S. Hardage *et al*., (IEEE, Piscataway, NJ, 2007), p. 0000.

[] The most common example of the sampling theorem in accelerator physics is two peaks of the tune spectrum on a Schottky monitor.

[] The exact expression of x_d includes other modes which are inversely proportional to the ramp up time and oscillate with the machine tune ν. If the ramp up is slow enough, all of these modes are very small and even decohere before the end of the ramp up. Hence these ignored modes do not affect the motion of the beam centroid but they affect the beam size[11].

[] The origin of this behavior is β_d. As seen in Fig 4, the relation between β_d and δ_d is quite nonlinear in the Tevatron and so Eq 25 is not a good approximation for the quantitative argument. It is good only for the qualitative argument.

[] The ring-wide β function is determined from the fit of Eq $a_d(s;\delta_d)$ up to a constant $\theta_{\rm acd}\sqrt{\beta_{\rm acd}}$. The constant can be determined from the analysis using a pair of BPMs in the collision straight sections. See [5] for details. The phase advance ψ can be also determined for the fit.