

**Mathematical
Surveys
and
Monographs**
Volume 132

Parametrized Homotopy Theory

**J. P. May
J. Sigurdsson**



American Mathematical Society

Contents

Prologue	1
Part I. Point-set topology, change functors, and proper actions	11
Introduction	13
Chapter 1. The point-set topology of parametrized spaces	15
Introduction	15
1.1. Convenient categories of topological spaces	15
1.2. Topologically bicomplete categories and ex-objects	16
1.3. Convenient categories of ex-spaces	19
1.4. Convenient categories of ex- G -spaces	22
1.5. Philosophical comments on the point-set topology	23
1.6. Technical point-set topological lemmas	24
1.7. Appendix: nonassociativity of smash products in $\mathcal{T}op_*$	26
Chapter 2. Change functors and compatibility relations	29
Introduction	29
2.1. The base change functors $f_!$, f^* , and f_*	30
2.2. Compatibility relations	32
2.3. Change of group and restriction to fibers	35
2.4. Normal subgroups and quotient groups	37
2.5. The closed symmetric monoidal category of retracts	40
Chapter 3. Proper actions, equivariant bundles and fibrations	43
Introduction	43
3.1. Proper actions of locally compact groups	43
3.2. Proper actions and equivariant bundles	47
3.3. The bundle construction	48
3.4. Spaces of the homotopy types of G -CW complexes	51
3.5. Some classical theorems about fibrations	53
3.6. Quasifibrations	54
Part II. Model categories and parametrized spaces	57
Introduction	59
Chapter 4. Topologically bicomplete model categories	61
Introduction	61
4.1. Model theoretic philosophy: h , q , and m -model structures	62
4.2. Strong Hurewicz cofibrations and fibrations	63

4.3. Towards classical model structures in topological categories	66
4.4. Classical model structures in general and in \mathcal{K} and \mathcal{U}	69
4.5. Compactly generated q -type model structures	72
Chapter 5. Well-grounded topological model categories	77
Introduction	77
5.1. Over and under model structures	78
5.2. The specialization to over and under categories of spaces	82
5.3. Well-grounded topologically bicomplete categories	85
5.4. Well-grounded categories of weak equivalences	87
5.5. Well-grounded compactly generated model structures	90
5.6. Properties of well-grounded model categories	91
Chapter 6. The qf -model structure on \mathcal{K}_B	97
Introduction	97
6.1. Some of the dangers in the parametrized world	98
6.2. The qf model structure on the category \mathcal{K}/B	100
6.3. Statements and proofs of the thickening lemmas	102
6.4. The compatibility condition for the qf -model structure	105
6.5. The quasifibration and right properness properties	107
Chapter 7. Equivariant qf -type model structures	109
Introduction	109
7.1. Families and non-compact Lie groups	110
7.2. The equivariant q and qf -model structures	111
7.3. External smash product and base change adjunctions	115
7.4. Change of group adjunctions	118
7.5. Fiber adjunctions and Brown representability	122
Chapter 8. Ex-fibrations and ex-quasifibrations	127
8.1. Ex-fibrations	128
8.2. Preservation properties of ex-fibrations	129
8.3. The ex-fibrant approximation functor	131
8.4. Preservation properties of ex-fibrant approximation	133
8.5. Quasifibrant ex-spaces and ex-quasifibrations	135
Chapter 9. The equivalence between $\text{Ho } G\mathcal{K}_B$ and $hG\mathcal{W}_B$	137
Introduction	137
9.1. The equivalence of $\text{Ho } G\mathcal{K}_B$ and $hG\mathcal{W}_B$	138
9.2. Derived functors on homotopy categories	139
9.3. The functors f_* and F_B on homotopy categories	140
9.4. Compatibility relations for smash products and base change	142
Part III. Parametrized equivariant stable homotopy theory	147
Introduction	149
Chapter 10. Enriched categories and G -categories	151
Introduction	151
10.1. Parametrized enriched categories	151
10.2. Equivariant parametrized enriched categories	153

10.3. G -topological model G -categories	155
Chapter 11. The category of orthogonal G -spectra over B	159
Introduction	159
11.1. The category of \mathcal{S}_G -spaces over B	159
11.2. The category of orthogonal G -spectra over B	163
11.3. Orthogonal G -spectra as diagram ex- G -spaces	166
11.4. The base change functors f^* , $f_!$, and f_*	167
11.5. Change of groups and restriction to fibers	170
11.6. Some problems concerning non-compact Lie groups	172
Chapter 12. Model structures for parametrized G -spectra	175
Introduction	175
12.1. The level model structure on $G\mathcal{S}_B$	176
12.2. Some Quillen adjoint pairs relating level model structures	179
12.3. The stable model structure on $G\mathcal{S}_B$	180
12.4. Cofiber sequences and π_* -isomorphisms	183
12.5. Proofs of the model axioms	186
12.6. Some Quillen adjoint pairs relating stable model structures	190
Chapter 13. Adjunctions and compatibility relations	195
Introduction	195
13.1. Brown representability and the functors f_* and F_B	196
13.2. The category $G\mathcal{E}_B$ of excellent prespectra over B	200
13.3. The level ex-fibrant approximation functor P on prespectra	202
13.4. The auxiliary approximation functors K and E	205
13.5. The equivalence between $\mathrm{Ho} G\mathcal{P}_B$ and $hG\mathcal{E}_B$	207
13.6. Derived functors on homotopy categories	208
13.7. Compatibility relations for smash products and base change	209
Chapter 14. Module categories, change of universe, and change of groups	215
Introduction	215
14.1. Parametrized module G -spectra	215
14.2. Change of universe	219
14.3. Restriction to subgroups	223
14.4. Normal subgroups and quotient groups	226
Part IV. Parametrized duality theory	229
Introduction	231
Chapter 15. Fiberwise duality and transfer maps	233
Introduction	233
15.1. The fiberwise duality theorem	234
15.2. Duality and trace maps in symmetric monoidal categories	236
15.3. Transfer maps of Hurewicz fibrations	238
15.4. The bundle construction on parametrized spectra	240
15.5. Π -free parametrized Γ -spectra	242
15.6. The fiberwise transfer for $(\Pi; \Gamma)$ -bundles	244
Chapter 16. Closed symmetric bicategories	247

Introduction	247
16.1. Recollections about bicategories	248
16.2. The definition of symmetric bicategories	249
16.3. The definition of closed symmetric bicategories	252
16.4. Duality in closed symmetric bicategories	255
16.5. Composites and naturality of dualities	259
16.6. A quick review of triangulated categories	261
16.7. Compatibly triangulated symmetric bicategories	262
16.8. Duality in triangulated symmetric bicategories	266
Chapter 17. The closed symmetric bicategory of parametrized spectra	269
Introduction	269
17.1. The definition of the bicategory $\mathcal{E}x$	269
17.2. Base change spectra	273
17.3. Duality of base change spectra	277
17.4. Using $\mathcal{E}x$ to encode relations between $\mathrm{Ho}G\mathcal{S}_B$ and $\mathrm{Ho}G\mathcal{S}$	278
17.5. Sketch proofs of the compatible triangulation axioms	280
Chapter 18. Costenoble-Waner duality	285
Introduction	285
18.1. The two notions of duality in $\mathrm{Ho}G\mathcal{S}_B$	286
18.2. Costenoble-Waner dualizability of finite cell spectra	288
18.3. Costenoble-Waner V -duality	290
18.4. Preliminaries on unreduced relative mapping cones	292
18.5. V -duality of G -ENRs	295
18.6. Parametrized Atiyah duality for closed manifolds	296
18.7. Parametrized Atiyah duality for manifolds with boundary	300
18.8. The proof of the Costenoble-Waner duality theorem	302
Chapter 19. Fiberwise Costenoble-Waner duality	311
Introduction	311
19.1. Costenoble-Waner duality and homotopical Poincaré duality	312
19.2. The bicategories $\mathcal{E}x_B$	314
19.3. Comparisons of bicategories	316
19.4. The bundle construction pseudo-functor	319
19.5. The fiberwise Costenoble-Waner duality theorem	320
19.6. Fiberwise Poincaré duality	324
19.7. The Adams isomorphism	326
19.8. Some background and comparisons	328
Part V. Homology and cohomology, Thom spectra, and addenda	333
Introduction	335
Chapter 20. Parametrized homology and cohomology theories	337
Introduction	337
20.1. Axioms for parametrized homology and cohomology theories	338
20.2. Represented homology and cohomology theories	341
20.3. Coefficient systems and restriction maps	343
20.4. The Serre spectral sequence	344

20.5. Poincaré duality and the Thom isomorphism	347
20.6. Relative Poincaré duality	350
20.7. Products in parametrized homology and cohomology	350
20.8. The representability of homology theories	353
Chapter 21. Equivariant parametrized homology and cohomology	357
Introduction	357
21.1. Equivariant homology and cohomology theories	358
21.2. Represented equivariant theories	360
21.3. Change of base and equivariant coefficient systems	361
21.4. Duality theorems and orientations	363
21.5. Products and the representability of homology	366
21.6. Fiberwise parametrized homology and cohomology	367
21.7. Fiberwise Poincaré duality and orientations	369
Chapter 22. Twisted theories and spectral sequences	373
Introduction	373
22.1. Twisted homology and cohomology theories	374
22.2. Automorphism monoids of spectra and $GL_1(k)$	375
22.3. Twisted K -theory	378
22.4. The simplicial spectral sequence	380
22.5. Čech type spectral sequences	384
22.6. The twisted Rothenberg–Steenrod spectral sequence	386
22.7. The parametrized Künneth spectral sequence	388
Chapter 23. Parametrized FSP's and generalized Thom spectra	393
Introduction	393
23.1. \mathcal{D} -functors with products in symmetric monoidal categories	395
23.2. The specialization of \mathcal{D} -FP's to spaces and ex-spaces	397
23.3. Group, monoid, and module FCP's; examples	399
23.4. The two-sided bar construction on FCP's	402
23.5. Examples: iterated Thom spectra	403
23.6. \mathcal{S}_c -FCP's and \mathcal{L} -spaces	405
23.7. Universal spherical fibration spectra	407
23.8. Some historical background	408
Chapter 24. Epilogue: cellular philosophy and alternative approaches	411
Introduction	411
24.1. CW spaces over B	412
24.2. CW spectra and stable homotopy categories	415
24.3. Structured spectra and well-grounded model categories	418
24.4. The stable category of parametrized spectra	420
24.5. Towards parametrized S_G -modules	423
Bibliography	425
Index	433
Index of notation	439