Pareto Analysis in Multiobjective Optimization Using the Colinearity Theorem and Scaling Method

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Abstract

This paper presents a method to predict the relative objective weighting scheme necessary to cause arbitrary members of a Pareto solution set to become optimal. First, a polynomial description of the Pareto set is constructed utilizing simulation and high performance computing. Then, using geometric relationships between the member of the Pareto set in question, the location of the utopia point and the polynomial coefficients, the weighting of the performance metrics which causes a particular member of the Pareto set to become optimal is determined. The use of this technique, termed the Scaling Method, is examined using a sample problem from the field of vehicle dynamics optimization. The Scaling Method is based on the Colinearity Theorem which is also presented in the paper.

Keywords

Multi-objective Optimization, Pareto Set, Objective Weighting

1 Motivation

It is widely recognized that design is a series of compromises. Compromises are made using tradeoffs between performance, cost, risk, and quality attributes. In a multiattribute design problem there are typically an infinite number of “optimal” solutions, based on the preferences and risk assessments of the designer(s). The final design will be a result of numerous tradeoffs, many times ad-hoc, each aimed at making a compromise decision between conflicting attributes. Viewing design formally as a collection of compromise decisions, however, is only a very recent evolution. Indeed, interest in decision theory can be traced back to the 1950’s (e.g., Luce and Raiffa, 1957), but the application of decision theory to multidisciplinary design is relatively recent and reviewed in Lewis and Mistree (1998).

In decision theory, and more specifically in decision-based design, there are two primary steps: generate the option space, and select the best option (Hazelrigg, 1996). The option space is the accumulation of all potential design solutions. Choosing from among this space is certainly not trivial but rather a function of tradeoffs and compromises. In this work, we are concerned with both of these steps. The decision making environment for this work is multiattribute design problems where there is more than one attribute under question. Also, we are dealing with mathematical representations of design problems and therefore are working with multiobjective optimization problems. This does not exclude the results of this paper from non-mathematic representations, such as look-up tables or logical rule-based models, but we focus on mathematical models specifically. We intend on providing decision support for determining the optimal solution, given a set of operating conditions, assumptions, risk profiles, and preferences. We do not intend to construct risk profiles or model preferences, but simply acknowledge that they must play a role in this kind of decision making problem.

In multiobjective optimization problems, there are two primary approaches to finding the preferred, if not optimal, design. The first involves determining relative importance of the attributes and aggregating the attributes into some kind of overall objective. Then, solving the optimization problem presumably would generate the optimal solution for a given set of attribute importances. The second approach involves populating a number of optimal solutions along the Pareto frontier and then selecting one based on the values of the attributes for a given solution. In both cases, there are complications. First, coming up with exact relative attribute weights is a daunting task with complicated ramifications (Messac, 2000). Only in the rarest of circumstances can this approach be taken and relied upon. Second, while selecting from a set of Pareto solutions may seem straightforward once a set of preferences are established, it may result in a non-optimal design for a certain design operating conditions. It is difficult to relate the choice of Pareto solution to any kind of optimal criteria. In this work, we develop a technique that can provide decision support for both of the approaches to multiobjective optimization problems. In the next section, we provide the necessary background about Pareto Sets for the development of the technique.

2 Pareto Set Background

Since multiobjective optimization problems are in question, we make use of the concept of Pareto sets, an efficient frontier of solutions in the performance space. There continue to be two principal challenges in Pareto sets: populating the Pareto set or finding Pareto solutions, and selecting from among the Pareto solutions. These challenges parallel the challenges of decision-based design. They are analogous to determining potential solutions and selecting from among the solutions. In Pareto analysis, there are added restrictions on the criteria for deciding upon a solution's inclusion in the set. A design vector x* is a Pareto optimum if and only if, for any x and i,

$$f_j(x) \le f_j(x^*), j = 1,...,m; j \ne i \Rightarrow f_i(x) \ge f_i(x^*) \tag{1}$$

We address both of these challenges in this work, although most of the paper is focused on the second challenge. In the next sections, we address both challenges and the current research in each area.

2.1 Population of the Pareto Set

The notion of an efficient frontier of solutions is not new (e.g., Luce, 1957), but methods of generating Pareto sets continue to be a topic of research. The weighted sum method of generating Pareto sets was shown to work well with convex problems decades ago by Geoffrion (1968) and while it is still a very popular method its deficiencies have been noted. Messac (2000) has effectively illustrated the problems associated with choosing weights for an aggregate objective function and has derived conditions that predict which Pareto solutions can be found using weights (Messac, *et al*. 2000). Dennis (1997) notes that, even with convex problems, taking an even spread of weights will not result in an even spread of points in the Pareto set, making some sections of the Pareto set difficult to populate. Koski (1985) emphasizes the weighted sum method's inadequacies when dealing with non-convex problems. Athan and Papalambros (1996) also look at using non-linear weights to better capture the non-convex Pareto set.

An alternative to these methods, compromise programming, was developed in the 1970's and it, too, is a subject of continuing research. This approach has the advantage of being able to generate many points on the efficient frontier. It works from a stationary utopia point and, through variation of a vector of weights, intersects points in the Pareto set. Chen, *et al.* (1999) use this method in an approach to robust design with reasonable success, although implementation of compromise programming is notably more difficult than the weighted sum method. Tappeta and Renaud (1999) use compromise programming to find an initial Pareto solution, construct local approximations to the Pareto surface and iteratively present the decision maker with a set of solution options to choose from.

Yoo and Hajela (1999) effectively use an immune-based genetic algorithm (GA) to generate the Pareto-Edgeworth solutions in one run of the GA. Narayanan and Azarm (1999) improve multiobjective GA's using filtering, mating restrictions, and the idea of objective constraints in order to detect Pareto solutions in the non-convex region of the Pareto set.

This work will use yet another method to determine the Pareto set: a grid search of the design space. Efficient points from the grid search are fit by a polynomial to approximate the Pareto set, resulting in several advantages over other methods. This is pursued in Section 3. Once the Pareto set has been generated, the next challenge is choosing the best solution from among the set–the subject of the next section.

2.2 Selection from among the Pareto Solutions

Just as several methods exist to determine what solutions compose the Pareto set of efficient solutions there are several techniques to determine which member of the Pareto set is the optimal solution. Similar to decision-based design, the simplest method is to choose a solution based on the values of the objectives and how well they match the preferred values. This method is used in (Nelson, *et al*. 1999) when manipulating multiple Pareto solution sets in product platform design. Das (1999) introduces the concept of "order of efficiency" as an attempt to create a meta-metric, stronger than the Pareto conditions, to rank order Pareto solutions. Hazelrigg (1996) argues that the selection from a set of solutions should be guided by a "meta-objective" of maximizing profit. Designer preferences can be accommodated by using utility theory to establish a rank-ordering of solutions based on uncertain and changing individual and group preferences (Callaghan and Lewis, 2000). Horn, *et al*. (1994) use Multi-Attribute Utility Analysis to select the preferred solution from the Pareto set of solutions. Eschenauer, *et al.* (1990) describes the most widely used method: L_p norms. This technique minimizes the distance from the Pareto set to an ideal solution (i.e. utopia point) to find the optimal solution according to the following formula:

$$\text{Minimize } \left(\sum_{i=1}^{m} \left(f_i(x) - f_i^* \right)^p \right)^{\frac{1}{p}} \tag{2}$$

Typical applications of the L_p norm are the L_1, L_2 and L_∞ norms (where p = 1, 2 and ∞, respectively). A fine summary of solution techniques is provided by Azarm, *et al*. (1998) should the reader desire a more in-depth exploration of the subject.

In this work, we utilize the L_2 norm (p = 2) to derive a technique for finding the optimal attribute weights necessary to make any member of the Pareto set optimal. Said another way, we determine what design conditions are required to make a certain Pareto solution the preferred solution. In Section 3, we present a theorem, the Colinearity Theorem, in a general sense and then apply it in a method called the Scaling Method in Section 4. This is followed by the use of these techniques in a vehicle dynamics design example. First, however, it is instructive to step back and examine the performance space in more detail. This is done in the next section.

2.3 A Conceptual Look at the Performance Space and Axis Scaling

Pareto Set analysis typically takes place in the performance space. This is the space created by considering the design objectives as coordinate axes. On these axes the performance of each possible design is plotted, one point per design. Each design has associated with it a specific performance or outcome on each objective. In general, the plot of possible outcomes appears as a region in the performance space, as depicted in a general two objective problem in Figure 1.

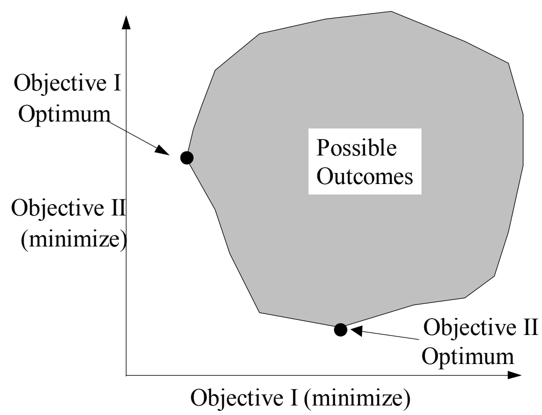

Figure 1: A Generic Performance Space

Figure 1 notes that the intention is to minimize both objectives. The individual optimum for each objective is also noted. These correspond to the best possible performance for each given objective. Unfortunately, two distinct points appear–one for each objective. Since both objectives cannot be simultaneously optimized, any attempt to choose a single design to perform well across both objectives will necessarily be a compromise design. Optimal performance on any one objective implies sub-optimal performance on the other.

Figure 2 shows how a certain distance measure, the L_2 norm, determines the optimal compromise design. First, the range of possible designs is narrowed significantly through the concept of a Pareto set, a subset of the set of possible outcomes shown in Figure 1. The Pareto set has the property that, for any point in the Pareto set, there does not exist another point in the set of possible outcomes with a better performance on both objective axes simultaneously. As such, the

Pareto set is sometimes referred to as the "efficient frontier" of the performance space. Whatever compromise design is chosen, it must be a member of the Pareto set–the remainder of possible outcomes can be ignored. They represent design points that give sub-optimal performance on both axes simultaneously.

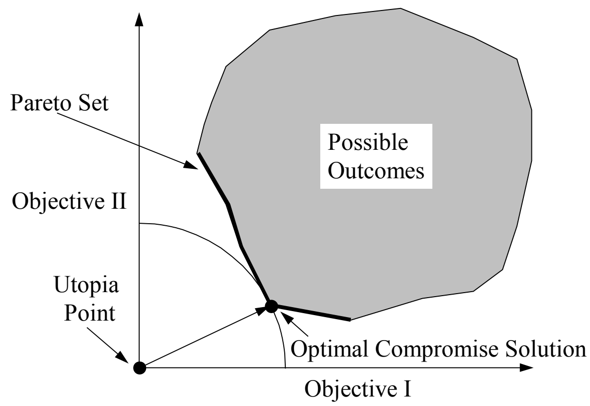

Figure 2: Optimal Compromise Solution Using the L_2 Norm

The concept of utopia point is important when using distance functions, such as the L_2 norm. It is the theoretical best performance point that can be achieved. While more than one method for locating the utopia point exists (e.g. Miettinen, (1999)), this discussion presents the definition which agrees with the vehicle dynamics problem presented later. In this case, the utopia point is taken to reside at the origin of the coordinate system. While we present the results of the example using this utopia point location, the derivation in Section 4 is general. The development can be used with other utopia point definitions.

Now, according to the L_2 norm method (Eschenauer *et al*. 1990), the optimal compromise design is the member of the Pareto set which lies geometrically closest to the utopia point, calculated in terms of vector distance in the performance space. Figure 2 locates the optimal point using the L_2 norm method. A vector is drawn from the utopia point to the optimal point on the Pareto set. To emphasize that this is, indeed, the closest point, a circle centered on the utopia point with the radius of the vector is included. Since no other points on the Pareto set appear within this circle the point shown must be the closest to the utopia point.

This sample problem has been investigated with an arbitrary baseline weighting of the performance objectives. That is, a certain relative importance (ratio of objective weights) is assumed to produce Figures 1 and 2. At this point the precise ratio used is not important. What is important, however, is how Figure 2 changes if the relative importance of the weights changes. Suppose the scenario in Figures 1 and 2 is generated by considering the two objectives to have equal importance. If the importance placed on objective I doubles (such that objective I is twice as important as objective II) a different compromise design may be the optimal solution.

Figure 3 shows how this change in relative importance of the objectives is handled. Compared with Figure 2, the objective I axis in Figure 3 is stretched to twice its original length, reflecting the doubling of the importance on that axis. This reshapes or "rescales" the set of possible outcomes and the Pareto set. As a result, the member of the Pareto set which is closest to the utopia point is different than that in Figure 2. A new optimal compromise design is located using the L_2 norm in Figure 3 for the case when objective I is twice as important as objective II.

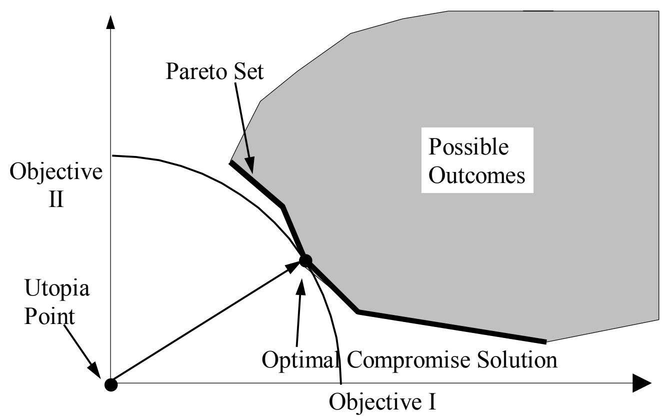

Figure 3: Rescaled Performance Space Doubling the Importance of Objective I

It is important to note that the design points composing the Pareto Set are no different in Figure 3 than they were with the baseline weighting of the objectives. By definition Pareto Set points are independent of the relative importance of the design objectives. The only difference is the way these points are plotted in the performance space. In Section 4 this geometric interpretation of objective weighting is applied in the Scaling Method.

Through the use of this generic example, a conceptual summary of performance space analysis has been presented. In the next section the Colinearity Theorem is introduced and a proof is given.

3 Mathematical Basis: The Colinearity Theorem

When analyzing problems involving compromise decisions, the construction of a Pareto set is useful in identifying all possible "good" solutions from the set of possible designs. Each member of the Pareto set is potentially the optimal solution to the problem at hand, depending on the relative weights of the objectives. Deciding which Pareto set member is optimal requires these weights to be known so that existing methods, such as the L_p-norm and utopia point concept, can be applied—thereby determining the optimal solution. Definitions of these decision theory terms and techniques are summarized by Palli, *et al.* (1998), many of which follow from the early game theory and compromise decision work described by Luce and Raiffa (1957).

A substantial number of analysis techniques exist, as reflected by Chen, *et al.* (1999). Every technique studying compromise decisions which strives to identify an optimum point is constrained by a given set of performance metrics. That is, had the relative objective weighting been different, the point chosen from the Pareto set as optimal would likely also have been different. While the designs composing the Pareto Set are, by definition, independent of any specific objective weighting, the optimal Pareto Set solution is necessarily a function of the particular objective weighting under discussion.

This section states and proves a new theorem termed the Colinearity Theorem. It has been developed as an extension of the L_2 norm in the performance space. The L_2 norm locates the best point in a Pareto set as the one which lies geometrically closest to the utopia point. This is visualized as a circle, centered at the utopia point, which determines the optimum point as the first design point encountered as the radius of the circle is increased. Instead, the Colinearity Theorem relates the best point in a Pareto set to the utopia point in terms of another geometric construct—the shape of the Pareto set itself. It is the inclusion of information regarding the shape of the Pareto set which gives the Colinearity Theorem its value. While the L_2 norm and Colinearity Theorem are in agreement in that they always determine the same Pareto point to be the optimal solution, the Colinearity Theorem lends itself more readily to expansion and generalization. Indeed, the Colinearity Theorem is the underlying mathematical premise for the Scaling Method, developed in Section 5.

3.1 Development of the Colinearity Theorem

For the time being assume that the Pareto Set is described by a line ℓ in the performance space whose equation or mathematical representation is known. A technique for determining this representation of the Pareto Set is given in Section 3.2. With this information the Colinearity Theorem can be defined as follows:

<u>The Colinearity Theorem</u>:

An internal point B of a Pareto set ℓ is an optimal point if and only if the utopia point, point B and the instantaneous center of curvature of the Pareto Set ℓ at point B are collinear, provided the instantaneous center of curvature of the Pareto Set at point B does not lie between point B and the utopia point.

Exception: If the Pareto set does not have a continuous slope at B or if B is an endpoint (i.e. "non-internal" point) of the Pareto set then this condition need not be met for point B to be optimal.

As an aid to the proof to this theorem, consider Figure 4 where

ℓ is a curve representing a Pareto set

A is the utopia point

B is the point on ℓ closest to the utopia point

C is the instantaneous center of curvature of ℓ at B

T is the tangent to ℓ at B.

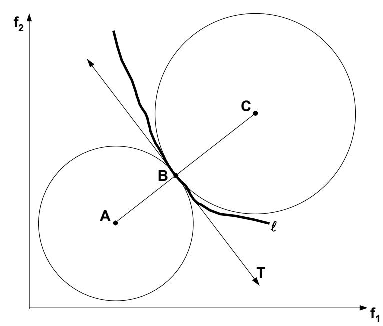

Figure 4: Geometry Associated with the Colinearity Theorem

A proof of the Colinearity Theorem is now presented. Three points are said to be collinear if they are points on the same line, according to elementary geometry. Additionally, a line drawn between any two points on a line will have the same slope as the original line. If A, B and C in Figure 4 are colinear then line segment AB and line segment BC must have the same slope as line segment AC. By verifying this is the case the colinearity of A, B and C can be proven.

Proof

Because C is the instantaneous center of curvature of ℓ at point B the line segment BC is, by definition, perpendicular to the tangent of ℓ at B. The tangent to ℓ at point B is denoted T in Figure 4.

By the definition of optimal point using the L_2 norm, point B is known to be the closest member of ℓ to the utopia point, A. Thus, a circle drawn about A with radius AB will touch line ℓ only at point B.

Since the circle and the line at point B have continuous slopes the circle centered at A must be tangent to ℓ at B. The tangent to ℓ at B is T. A circle's radius is always perpendicular to a tangent on the circumference, so line segment AB must be perpendicular to T.

Both AB and BC are perpendicular to T and therefore parallel to each other. Parallel line segments, by definition, have the same slope. Since point B is common to both line segments we know that A, B and C must be members of the same line. This completes the proof of the Colinearity Theorem for internal Pareto set points.

At points where the slope of the Pareto set is discontinuous or where the optimal point is a Pareto set endpoint the Colinearity Theorem does not need to be satisfied for optimality to exist—thus the reference to "internal points" of the Pareto set. Non-internal points can be corners, jumps or endpoints of the Pareto set. These points do not have a continuous derivative. As such, these points do not have an instantaneous center of curvature and cannot satisfy the Colinearity Theorem for optimality. Non-internal points are typically few and can easily be checked individually. Through the use of the Scaling Method they can often be readily examined without additional analysis.

This proof has assumed that the objectives are being minimized. This is not a requirement, but presented for illustrative purposes.

3.2 Polynomial Description of the Pareto Set

The approach of the Colinearity Theorem is based upon knowledge of the shape of the Pareto Set. It is the local curvature of the Pareto Set which determines what relative objective weighting will cause a local Pareto Set member to become optimal. Section 2.1 presented various techniques to populate the Pareto Set. Our approach centers on a grid search of the design space followed by a polynomial fit in the performance space. First, a discretization of the design space is conducted. Each combination of design variables is evaluated to determine the corresponding performance with respect to the design objectives. The results are then plotted in the performance space and, in a manner similar to that shown in Figure 2, the points which satisfy the criteria for the Pareto Set are identified.

Despite the fact that most of the design points chosen will not be members of the efficient frontier there are distinct advantages to the grid search approach. Perhaps key among these are the fact that the entire Pareto Set, both convex and non-convex regions, is guaranteed to be located given a fine enough discretization of the design space. This overcomes a principal shortcoming of any of the techniques of Section 2.1. Furthermore, advanced grid search techniques may be employed to enhance the efficiency of the search, although this has not been addressed by the authors.

With a representative number of Pareto Set points determined by the grid search, the process of polynomial fitting can begin. In a two-objective problem, one objective is represented as a function of the other and a polynomial fit applied in the performance space. Since a polynomial has two fewer inflection points than the order of the polynomial, the use of a high order is important to capture both the convex and non-convex regions in the approximation. The result is a simple, continuous description of the Pareto Set.

The fitting of a polynomial allows information about the shape of the Pareto Set to be easily accessed as required by the Colinearity Theorem. While it has not been pursued, a series of splines–done properly–would also satisfactorily represent the Pareto set for use with the Colinearity Theorem. Also, when dealing with three or more objectives the Pareto set is no longer a line but rather a hyperplane. In such problems a response surface would be fit to the Pareto Set. This, too, is the subject of future research.

4 Application of the Colinearity Theorem: The Scaling Method

Now that the Colinearity Theorem is stated and proven, it is developed into a method to predict the scaling of the objective weights needed to cause a member of the Pareto set to become the optimal solution. This procedure, termed the Scaling Method, has been developed to realize the information the Colinearity Theorem makes available. The goal is to predict the relative objective weighting required to cause any member of the Pareto set to become the optimal solution on the basis of the information contained in the shape of the Pareto set.

Figure 5a presents the performance space for a general compromise decision problem. The point (f_1*, f_2*) is the optimal solution for a baseline weighting of the objectives on the abscissa and ordinate. This baseline weighting of objectives f_1 and f_2 is arbitrary, although for simplicity it is convenient to weight the objectives equally. Other important points on this figure are the utopia point, denoted (f_{1u}, f_{2u}), the instantaneous center of curvature at the optimal point, denoted (f_{1c}*, f_{2c}*), and a non-optimal point on the Pareto set, (f_1, f_2).

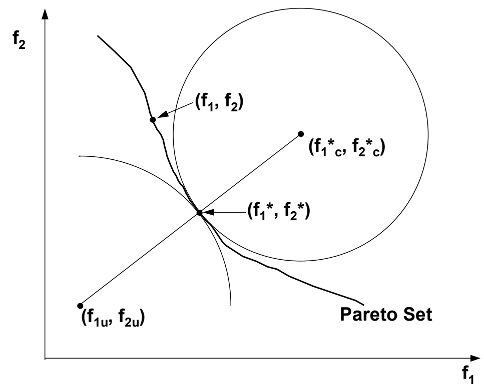

Figure 5a: Relating The Colinearity Theorem to Pareto Set Terminology

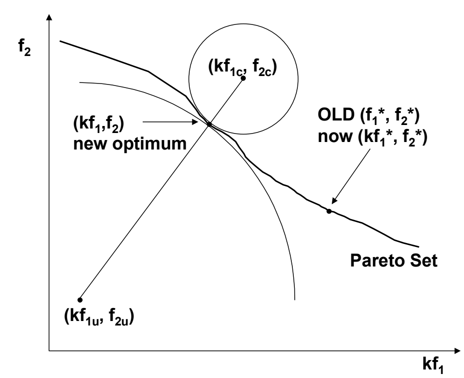

Figure 5b: Rescaled Abscissa by an Amount k

Stretching or "rescaling" the f_1-axis by an appropriate amount k will, in general terms, cause Figure 5a to look like Figure 5b. The point (f_1, f_2) is now (kf_1, f_2). Because of the value of k selected the formerly sub-optimal point (f_1, f_2) is now the optimal point (kf_1, f_2). The previously optimal point is now sub-optimal.

The Colinearity Theorem allows the appropriate value of k to be determined. Three points—the utopia point, the point we wish to make optimal (f_1, f_2) and the center of curvature of the Pareto set at (f_1, f_2) from Figure 5a—must be colinear for optimality to occur. Equating slopes between the first pair of points and the second pair of points (refer to Figure 5b) and considering objective f_2 to be a function of objective f_1, gives the relationship:

$$\frac{f_2 - f_{2u}}{kf_1 - kf_{1u}} = -\frac{1}{f_2'} \tag{3}$$

See the Appendix for a detailed explanation of how relationships change when the abscissa is rescaled. Solving for k, the only unknown, gives:

$$k = -\frac{(f_2 - f_{2u})f_2'}{f_1 - f_{1u}} \tag{4}$$

Through the use of (4) the change in relative objective weights, k, needed to change a sub-optimal member of the Pareto Set (f_1, f_2) into the optimal point can be determined. Implementation of (4) to predict under what preferences a member of the efficient frontier becomes the optimal solution is termed the Scaling Method. It is an extension of the Colinearity

Theorem and it predicts the change in relative weighting of the objectives, k, from the baseline condition as a function of:

1. The original (unscaled) location of the point of interest on the Pareto Set (f_1, f_2)
2. The original slope, f_2', of the Pareto set at the point of interest (f_1, f_2), and
3. The original location of the utopia point (f_{1u}, f_{2u})

This approach is illustrated in the vehicle design problem that follows.

5 Case Study: Vehicle Dynamics Design

Whether it is NASCAR, CART, Formula One, or even local racetracks, the difference between winning a race and not winning comes down to the ability of a driver to get the most out of his or her racecar. While having a talented driver is always desirable, even the most talented driver can do nothing more than realize the full potential of the vehicle. The core vehicle design, how it is "set-up" and the "tuning" done by a race team are aimed at an optimal compromise that allows the driver to repeatedly turn fast lap times at a particular racetrack. Any advantage gained though vehicle design and tuning, however small, increases the vehicle's potential and, with a talented driver, will translate into an increase in on-track performance. Vehicle simulations are now used not only prior to and during a race weekend to guide tuning of the race car, but also in the design phase where parameters which are not adjustable must be set and optimized.

The scope of vehicle simulation continues to grow. Advances in the memory and speed of computers have led to the use of increasingly complex vehicle models with an ever expanding scope. The modeling of performance around a single corner has developed into full lap analysis. Even further, it is now possible to analyze all the tracks on the season schedule when designing a new car and attempt to optimize core design parameters—those which are not easily changed once the car is built—before the car even exists. An example of one such variable is the longitudinal (fore/aft) center of gravity location. The range of possible adjustment on this variable throughout a season is very small. It must be optimized in the design stage before the car is constructed. Further importance is placed on the use of vehicle simulations as racing sanctioning bodies impose tighter restrictions on the amount of on-track testing teams are allowed to do.

Vehicle design is considered an excellent example of a multidisciplinary design optimization problem. There are vehicle dynamicists, aerodynamicists, tire designers, engine builders, shock absorber specialists, mechanics, a driver–all of whom have different outlooks and control over the performance of the car. Furthermore, during a lap at a typical race, a driver sees a number of different types of corners and straights. In Figure 6, the configuration for the Indianapolis Motor Speedway Roadcourse, site of the Formula One United States Grand Prix in September 2000 is shown. There are many turns and straight-aways, each with its own optimal vehicle characteristics. The optimal car for the sweeping final corner, an 840 foot radius turn on the lower right, is different than the tight 114 foot radius turn in the upper center of the track.

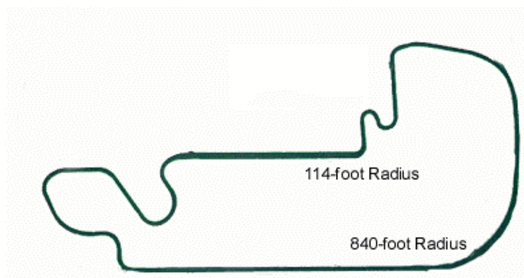

Figure 6: Indianapolis Motor Speedway Roadcourse Configuration

Designing a car to perform well across turns of all radii on a single track involves a set of complicating tradeoffs. Simulating these tradeoffs is a difficult task, as a complete vehicle simulation is very complex and computationally challenging to run efficiently. In this paper we use a simplified vehicle model to illustrate the use of the techniques introduced in Sections 3 and 4 to this type of problem and to predict under what race conditions (combination of radii) a particular vehicle design will be optimal.

The vehicle model is based on the classic Bicycle Model in Milliken and Milliken (1995) which has been expanded to include four individual wheels. Equations of motion are written for lateral acceleration, longitudinal acceleration and yaw acceleration. The tires, which may be different front and rear, are modeled using tabular tire data measured on a state-of-the-art tire testing machine and represent numerous tire non-linearities such as load sensitivity and slip angle saturation. Wheel loads are calculated based on static load, aerodynamic downforce and lateral load transfer.

Through computer simulation, the vehicle is entered into a special race for which the compromises involved can be easily studied. This race is defined as the time to complete "x laps on a small radius (100 foot radius) circle and y laps on a large (400 foot radius) radius circle", each at best steady-state speeds. This abstraction of a true racetrack is especially well-suited to highlighting how the objectives have competing optimal designs. Since the vehicle cannot be optimized for both radii simultaneously due to vehicle aerodynamics, tire performance and other speed dependent vehicle behaviors a compromise design is needed. This compromise design is one which allows the vehicle to perform optimally for the race, not just an individual circle. The ratio of x to y, denoted k, can be viewed as the weighting of two design objectives, the elapsed times on each individual radius. The shortest time to complete the total distance around both radii will win the race.

For this study, the vehicle has two design variables. These variables, roll stiffness distribution (K') and weight distribution (a'), are two of the three "magic numbers" Wright (1998) identifies as being fundamental to race car design and performance. They are used by the vehicle designer and vehicle dynamicist to "tune" the car's handling in an attempt to optimize performance. The analysis begins with a grid search of the design space from which the performance space is plotted and the Pareto set identified. The design space has been discretized into 546 (a', K') pairs. Each design pair results in a certain level of performance on each of the two radii. Figure 7 plots the performance space for the vehicle for a baseline race defined as one lap on each radius (k = 1). This gives each radius equal importance. Thus, the axes are the time required to complete one lap on each of the 100 foot radius and the 400 foot radius.

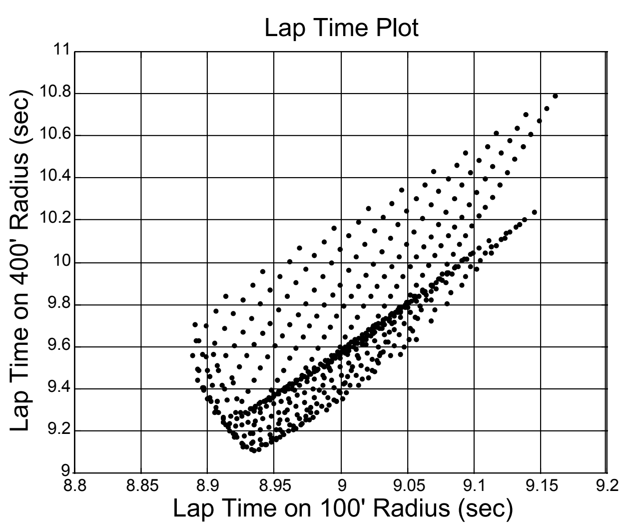

Figure 7: Grid Search Results in the Performance Space

From Figure 7 the optimum lap times for each radius are easily located. For the 100 foot radius the leftmost point on this figure gives the optimum where the vehicle can do no better than approximately an 8.88 second lap. Likewise, for the 400 foot radius the bottommost point in the plot gives the optimum. This point indicates approximately a 9.1 second lap as the best possible. These two points are consistent with the detailed vehicle simulations in Kasprzak (1998). The two optimums do not occur at the same point so the race definition, combining laps on each radius, will need a compromise solution to arrive at the optimal design and performance.

Figure 8a shows the grid search points which are members of the Pareto set along with the results of fitting a seventh-order polynomial via least-squares criteria. The polynomial gives a representative description of the Pareto set throughout the range of the Pareto set, except at the far right end where the fit acquires a positive slope. As a result, analysis of this Pareto set for cases where the 400 foot radius design dominates the compromise design can be expected to produce slightly inaccurate results. While the use of an eighth order fit resolves this inaccuracy, the seventh-order fit is presented here to highlight potential errors caused by a fit which does not fully represent the Pareto set.

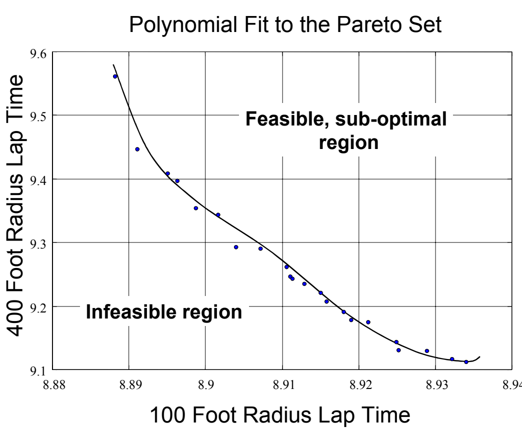

Figure 8a: The Pareto Set with Seventh-Order Polynomial

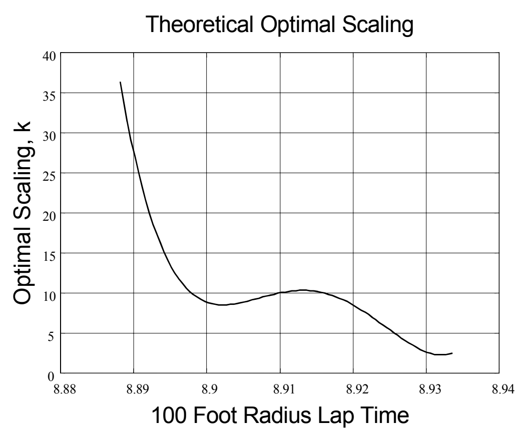

Figure 8b: Corresponding Optimal Scaling Predicted by (4)

Figure 8b presents the outcome of applying the Scaling Method (4) to the seventh-order fit Pareto set. This figure associates the optimal race definition k with the vehicle performance on the 100 foot radius. Note that since the origin (0,0) is being used as the utopia point, (4) reduces to a simple form. Since Figure 8a shows a one-to-one correspondence of performance on the 100 foot radius to that on the 400 foot radius, Figure 8b also specifies the optimal 400 foot radius performance for any race definition k. Furthermore, since each point on the Pareto set is the result of a unique combination of design variables, a relationship between race definition and vehicle design is also acquired for the entire range of k values.

Consider the case of a race definition k = 15. That is, the race will contain 15 laps on the 100 foot radius and one lap on the 400 foot radius, each at best steady-state performance. Figure 8b predicts an optimal performance on the 100 foot radius circle of 8.894 seconds per lap or 133.4117 seconds total time on the 100 foot radius. If the abscissa of Figure 8a is rescaled by an amount equal to k (15) then Figure 9a is generated. It shows the shape of the Pareto set in terms of the elapsed time on each radius for an k = 15 race. A circle drawn about the utopia point (0,0), representative of the L_2 norm technique for locating an optimum point, identifies 113.4117 seconds as the optimum performance on the 100 foot radius, identical to that predicted by the Scaling Method. The L_2 norm and the Scaling Method each predict the same optimal performance, as desired.

Close examination of Figure 8b reveals a region where the scaling k is associated with more than one performance criteria on the 100 foot radius. One such value is k = 8.4. For this race definition three optimal 100 foot radius performance points are predicted: 8.9198, 8.9114 and 8.9031 seconds per lap on the 100 foot radius. These are denoted by the dashed lines in Figure 9b. Rescaling the original Figure 8a by k (8.4) produces Figure 9b. In this figure the circle about the utopia point touches only two of the three points. The middle point is not an optimum point. For the central, non-optimal point, the center of curvature of the Pareto set lies between the Pareto set and the utopia point. This point does not satisfy the Colinearity Theorem and is therefore not a candidate to be an optimal point, even though (4) produces this point as a solution. Thus, the Scaling Method and the Colinearity Theorem are best used in tandem to arrive at optimal solutions in the Pareto set. Calculation of the center of curvature in addition to the use of (4) gives a more complete picture of the Pareto set's behavior.

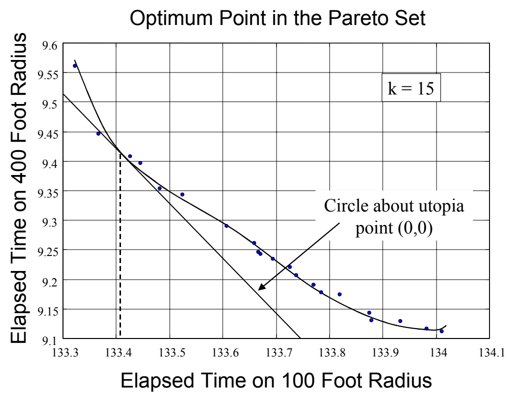

Figure 9a: Verification of Theoretical Optimal Scaling when k = 15

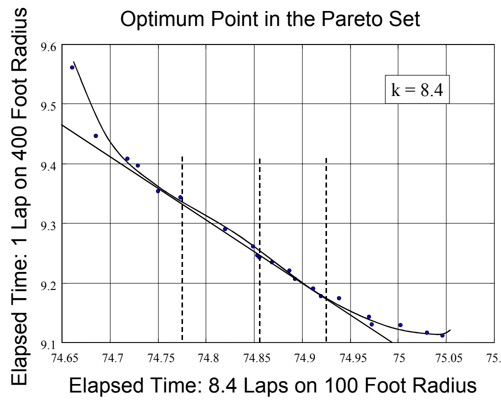

Figure 9b: Theoretical Optimal Scaling when k = 8.4

The other two points identified in this k = 8.4 example are indeed optimal design points and produce identical race times to one another. The leftmost optimum results in a race time of (8.4×8.9031) + 9.1688 = 83.9545 seconds while the rightmost optimum gives (8.4×8.9198) + 9.0282 = 83.9545 seconds. These are identical, optimal race times. In contrast, the center point gives a race time of (8.4×8.9114) + 9.2459 = 84.102 seconds, nearly 0.15 seconds slower than the other two design points. As predicted by the Colinearity Theorem, because the center of curvature of the Pareto set is between the utopia point and the Pareto set, this point is not optimal.

These two race definitions serve to show the validity and usefulness of the Scaling Method and Colinearity Theorem. A more detailed discussion of this example is provided in Kasprzak, *et al.* (1999). Additional comments about these techniques are provided in the next section.

6 Comments on the Scaling Method

When using the Scaling Method, arriving at a polynomial which describes the Pareto set accurately is critical. Use of a high-order polynomial is desirable so that both the concave and convex regions of the Pareto set can be represented. A polynomial curve has two less inflection points than the order of the polynomial and these inflection points may not all fall within the range of the Pareto set. It is very important that the polynomial chosen meets the criteria used to specify the Pareto set. As noted earlier, the seventh-order fit presented above violates a Pareto set criterion by reversing the sign of its slope near the right end of the fit. This results in negative scaling values being predicted for this region, seen at the right end of the trace in Figure 8b. These negative values have no physical meaning and lead to solution inaccuracies. The use of an eighth-order fit with this example, while not presented here, provides a better representation of the Pareto set.

The polynomial fit can only be as good as the data to which it is applied. A sufficiently fine grid search of the design space is required to populate the Pareto set. While this method has the distinct advantage of finding the entire Pareto set, both concave and convex regions, the search increment may be very small and require a large number of points to be calculated. Since most points calculated will not be Pareto set points this method may, at first glance, be deemed inefficient. Still, the grid search technique is guaranteed to find the entire Pareto set, given a fine enough discretization. This alone is a key point. The combination of a grid search and polynomial to describe the Pareto set has the ability to provide an accurate, smooth and continuous description of the whole Pareto set, regardless of the existence of both convex and non-convex regions.

The Colinearity Theorem and Scaling Method exploit the use of this grid search and polynomial description of the Pareto set. These techniques pull a vast amount of information out of the polynomial fit. The optimal objective performances can be calculated for any and all ratios of objective weights with ease via (4). To be of real use to the designer, relating these performance results to their corresponding design variables is desirable. While these results appear in the performance space, another method has been developed, termed the Contour Method, which relates these results to the design space. This method, presented in Kasprzak and Lewis (2000), further enhances the utility of the Scaling Method and Colinearity Theorem.

7 Conclusions

Starting with a grid search of the design space, the Pareto set is populated and then approximated using a least-squares fit polynomial. This Pareto set is based on a baseline goal definition with equally weighted objectives, referred to as $k = 1$. The Colinearity Theorem states that any internal member of the Pareto set is optimal only if the utopia point, the point of interest in the Pareto set and the instantaneous center of curvature of the Pareto set at the point of interest are colinear. From this the Scaling Method is derived, which allows the relative objective weighting to be predicted for any point on the Pareto set, based on the polynomial fit. Alternatively, the optimal performance for all ratios of objective weights can be readily determined. Thus, from a grid search of the design space the shape of the entire Pareto set can be determined, as well as the entire mapping of the optimum Pareto set solution to the relative objective weighting. This provides a vast amount of information and practical insight to the problem under consideration.

Verification of this method against results obtained using an L_2 norm shows the new methods produce results complementary to the established technique.

While the method presented here addresses only two-objective compromise decision making, the merits of extending the Colinearity Theorem and the Scaling Method to multiple dimensions are clear. Describing the Pareto set as a hyperplane in n-dimensions and applying the Colinearity Theorem in n-dimensional space appear to be possible. While the concepts involved remain unchanged, the mathematical expression of these ideas in n-dimensions will be more complicated than in two dimensions.

Acknowledgements

We would like to thank Milliken Research Associates, Inc. for their support of this research. We would also like to thank NSF Grant DMI-9875706 for their support.

Appendix

Table 1 supports the development in Section 4. It shows how various Pareto Set parameters change with rescaling of the abscissa in the performance space by an amount k.

Table 1. Effects of Abscissa Scaling on Pareto Set Attributes

	Before rescaling	After rescaling by k
Sub-optimal Point of Interest on Pareto Set	(x_1, y_1)	(kx_1, y_1)
Instantaneous slope of Pareto Set at Point of Interest	m_1	m_1/k
Equation of Line Tangent to Pareto Set at Point of Interest	$y_1 = m_1 x_1 + b_1$	$y_1 = \frac{m_1}{k}(kx_1) + b_1$ $y_1 = m_1 x_1 + b_1$
Slope of Line Normal to Pareto Set at Point of Interest	$-1/m_1$	$-1/m_1$

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