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Parking on supercritical Galton-Watson trees

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Abstract. At each site of a supercritical Galton-Watson tree place a parking spot which can accommodate one car. Initially, an independent and identically distributed number of cars arrive at each vertex. Cars proceed towards the root in discrete time and park in the first available spot they arrive at. Let X be the total number of cars that arrive at the root. Goldschmidt and Przykucki proved that X undergoes a phase transition from being finite to infinite almost surely as the mean number of cars arriving to each vertex increases. We show that EX and $P(X = 0)$ are discontinuous at the critical threshold, describe the growth rate of EX above criticality, and prove that X stochastically increases as the initial car arrival distribution becomes less concentrated. We also provide a new characterization of the threshold with a generating function condition satisfied by the time of first arrival at the root. For the simple case that either 0 or 2 cars arrive at each vertex of a d-ary tree, we give improved bounds on the critical threshold and also prove that the location of the phase transition depends on more than just the mean number of cars arriving to each vertex.

1. Introduction

Parking, introduced over fifty years ago (Konheim and Weiss, 1966), is a stochastic process at the intersection of probability and combinatorics. The *parking process* on a tree $\mathcal{T}$ with root ρ begins with a parking spot at each vertex. Initially, η_v cars arrive at each vertex $v \in \mathcal{T}$ and move towards the root in discrete time steps. When a car arrives at an available spot, the car parks there and the spot becomes unavailable. If multiple cars arrive at the same available spot, then one is chosen uniformly at random to park there. The remaining cars continue moving towards the root. Let X be the total number of cars that arrive at ρ. This includes the η_ρ cars that initially arrive and all subsequent cars that arrive from further in the tree.

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There has been significant progress on understanding whether or not X is finite or infinite for a given η when $\mathcal{T}$ is a critical Galton-Watson tree (Goldschmidt and Przykucki, 2019; Chen and Goldschmidt, 2021; Curien and Hénard, 2019; Roulet and d'Aspremont, 2020). Less is known about the case that $\mathcal{T}$ is a supercritical Galton-Watson tree. The point of this article is to make some progress on this case and develop some machinery that might aid future work. We also provide a fairly detailed survey of parking from a probabilistic perspective in Section 1.1.

Suppose that the offspring distribution of the Galton-Watson tree $\mathcal{T}$ is described by the non-negative integer-valued random variable Z with $EZ = \lambda > 1$. Additionally, assume that the η_v are independent and identically distributed (i.i.d.) as $\eta(\alpha)$, which is a family of random variables $(\eta(\alpha))_{0\leq\alpha\leq 1}$ that is stochastically increasing in $\alpha = E\eta(\alpha)$. Stochastically increasing means that $P(\eta(\alpha) \geq x) \leq P(\eta(\alpha') \geq x)$ for all $x \geq 0$ and $\alpha \leq \alpha'$. Throughout this work we assume that $0 \leq \alpha \leq 1$. For this setting, Goldschmidt and Przykucki proved in Goldschmidt and Przykucki (2019, Theorem 3.4) that there exists $\alpha_c \in (0, 1)$ such that if $\alpha < \alpha_c$, then

$$EX = \frac{\lambda - \alpha - \lambda P(X = 0)}{\lambda - 1}$$

while if $\alpha > \alpha_c$, then, conditionally on the non-extinction of the tree, $X = \infty$ almost surely.

Unless stated otherwise, we let $\eta(\alpha)$ and α_c be as in Goldschmidt and Przykucki (2019, Theorem 3.4). What happens when $\alpha = \alpha_c$ was left open. Our first result shows that EX is finite at criticality.

Theorem 1.1. *For all $\alpha \leq \alpha_c$ it holds that*

$$EX = \frac{\lambda - \alpha - \lambda P(X = 0)}{\lambda - 1} \tag{1.1}$$

with $P(X = 0) > 0$.

Remark 1.2. After releasing our preprint, a simpler proof of (1.1) from Theorem 1.1 was pointed out to us by Olivier Hénard. This argument is given immediately following our proof of Theorem 1.1. We still include our original approach, since it develops machinery that we use to derive Theorem 1.3 and the upper bound in Proposition 1.4.

The behavior described in Theorem 1.1 for supercritcal trees is different than what occurs when $\mathcal{T}$ is instead a critical Galton-Watson tree conditioned to be infinite. In this setting $EX = \infty$ and $P(X = 0) = 0$ at criticality. See the discussion of results from Goldschmidt and Przykucki (2019); Chen and Goldschmidt (2021); Curien and Hénard (2019) following the statements of our results.

Determining the value of α_c remains an open problem. We introduce time to the process by viewing cars as arriving to their starting vertex at $t = 0$ and subsequently proceeding towards ρ at unit time increments $t = 1, 2, \ldots$. Let X_n be the number of cars that arrive at ρ up to time n. This is equivalent to the total number of cars that arrive at ρ (including those initially at ρ) in the process restricted to the first n levels of $\mathcal{T}$. One consequence of the proof of Theorem 1.1 is a formula for the growth rate of EX_n as well as a characterization for α_c in terms of the first time that a car arrives at the root.

Theorem 1.3. *If $\tau \in \{0, 1, \ldots\}$ is the time that the first car arrives at ρ, then*

$$\lim_{n\to\infty} \frac{EX_n}{\lambda^n} = \frac{\lambda}{\lambda - 1}(\alpha - E\lambda^{-\tau}). \tag{1.2}$$

Moreover,

$$\alpha_c = \sup\{\alpha \colon E\lambda^{-\tau} = \alpha\}. \tag{1.3}$$

Notice that taking $\lambda \downarrow 1$ in Theorem 1.3 gives $E\lambda^{-\tau} \to P(\tau < \infty) = P(X > 0)$. The value α_c in for a critical Galton-Watson a tree is thus the supremum of α such that $P(X > 0) = \alpha$ as observed in Curien and Hénard (2019).

One particularly simple choice for the distribution of $\eta(\alpha)$ is that it takes value 2 with probability $\alpha/2$ and otherwise is 0. After the initial cars that arrive park, the number of cars at each site is a Bernoulli random variable with parameter $\alpha/2$. Accordingly, we refer to the parking process with this distribution as *Bernoulli parking*. As a further simplification, we consider Bernoulli parking on $\mathbb{T}_d$ the infinite d-ary tree in which each vertex has d children. We denote the critical threshold for this specific setting by

$$\alpha_c(d) = \text{critical value for Bernoulli parking on } \mathbb{T}_d.$$

It was shown in Goldschmidt and Przykucki (2019, Theorem 3.5) that $0.03125 \leq \alpha_c(2) \leq 0.50$. We give an improved upper bound and also perform a small tweak to the proof of Goldschmidt and Przykucki (2019, Theorem 3.5) to slightly improve the lower bound.

Proposition 1.4. $0.03175 < \alpha_c(2) < 0.08698$

The calculation for the upper bound is computer-assisted which, theoretically, yields arbitrarily close upper bounds. However, runtime with exact precision quickly becomes an issue. Truncating the decimals in our calculations allows us to compute further and still have a rigorous bound, but at the cost of some accuracy. Nonetheless, we believe that the upper bound is very close to the correct value of $\alpha_c(2)$. Allowing for rounding error, the evidence suggests that $\alpha_c(2) \approx 0.0863$. See the proof of Proposition 1.4 for more details.

The lower bound comes from the observation that infinitely many visits to the root requires the existence of infinitely many connected subgraphs containing the root that at time 0 contain more cars than spots. A slight optimization to the union bound from Goldschmidt and Przykucki (2019) gives our improvement. Note that a similar approach is used in parts of Damron et al. (2019a) and Damron et al. (2021). Our lower bound is much further from what we believe to be the true value of $\alpha_c(2)$ than our upper bound. It would be interesting to find a method of dominating the number of root visits in parking on trees that yield better lower bounds. It is also natural to ask how $\alpha_c(d)$ changes as d is increased. A straightforward generalization of Goldschmidt and Przykucki (2019, Theorem 3.5), gives that $\alpha_c(d) \approx d^{-2}$.

Proposition 1.5. *It holds for all $d \geq 2$ that*

$$\frac{1}{2e^2} d^{-2} \leq \alpha_c(d) \leq 2d^{-2}. \tag{1.4}$$

Comparing the following result to Proposition 1.5, we see that the location of the phase transition on $\mathbb{T}_d$ depends on more than just the mean of the car arrival distribution.

Proposition 1.6. *Let $\alpha_c'(d)$ be the critical threshold for the parking process on $\mathbb{T}_d$ with $\eta(\alpha) = 3$ with probability $\alpha/3$ and 0 otherwise. It holds that*

$$\alpha_c'(d) \leq 3d^{-3}.$$

Combining this with (1.4) gives $\alpha_c'(d) < \alpha_c(d)$ for large enough d.

That the volatility of $\eta(\alpha)$ influences the location of the phase transition is part of a more general property of the parking process. Namely, that X increases when $\eta(\alpha)$ is replaced by a more volatile distribution. Note that this does not occur with transience and recurrence for branching random walk which only depends on the mean of the offspring distribution Athreya (2006). However, Hutchcroft recently proved that the global transience/recurrence behavior at criticality is sensitive to the distribution Hutchcroft (2020). On critical Galton-Watson trees with α fixed, Curien and Hénard proved in Curien and Hénard (2019) that α_c decreases linearly in $\text{var}(\eta)$ (see (1.9).) We prove a more general, albeit less precise, result.

Given random variables X and Y taking values in $[0, \infty)$, we say that Y dominates X in the *increasing convex* order (icx order) if for all increasing convex functions $\varphi\colon [0,\infty) \to \mathbb{R}$ it holds that $E\varphi(X) \leq E\varphi(Y)$. Denote this ordering by $X \preceq_{\text{icx}} Y$. Roughly speaking, the larger or less concentrated a distribution is, the larger it is in the increasing convex order. As the identity function is convex, when $X \preceq_{\text{icx}} Y$ we have $EX \le EY$. Moreover, if $EX = EY$, then, since x^2 is increasing and convex, we have $X \preceq_{\text{icx}} Y$ implies that $\text{var}(X) \le \text{var}(Y)$. See Shaked and Shanthikumar (2007) for a thorough survey of stochastic orders. We show for all Galton-Watson trees (not just supercritical) that X increases in the increasing convex order when η does. Consequently, so does EX.

Theorem 1.7. *Let X and X' denote the total number of cars that arrive at ρ for the parking process on a Galton-Watson tree with car arrival distributions η and η', respectively. If $\eta \preceq_{\text{icx}} \eta'$, then $X \preceq_{\text{icx}} X'$.*

An equivalent stochastic order is considered in Johnson and Junge (2018) for an interacting particle system known as the frog model. Unlike parking, the number of visits to the root in this process decreases if the initial particle distribution is replaced by one with the same mean that is larger in the increasing convex order. Like our Proposition 1.6, Johnson and Rolla proved that the location of the phase-transition for the frog model on trees is sensitive to the concentration of the initial configuration Johnson and Rolla (2019). An analogous effect occurs for the limiting shape in first passage percolation van den Berg and Kesten (1993); Marchand (2002).

In Remark 4.1 we explain how to generalize Theorem 1.7 to arbitrary locally finite trees. We do not give that argument since a significantly more general result concerning parking and the icx order was recently proven in Bahl et al. (2021). To close our line of pursuit, we provide a corollary which says that Bernoulli parking gives the maximal critical threshold among all arrival distributions whose supports do not include $\{1\}$. So, our estimates in Proposition 1.4 and Proposition 1.5 hold for a large family of arrival distributions. A more general version of this monotonicity result also appears in Bahl et al. (2021).

Corollary 1.8. *If α_c is the critical value for Bernoulli parking on a Galton-Watson tree $\mathcal{T}$, then $\alpha_c' \le \alpha_c$ with α_c' the critical value for parking on $\mathcal{T}$ with any other family $\eta'(\alpha)$ of car arrival distributions satisfying the hypotheses of Goldschmidt and Przykucki (2019, Theorem 3.4) and whose support does not include $\{1\}$.*

1.1. *Discussion.* Parking dynamics were introduced by Konheim and Weiss for $\mathcal{T} = [1, n]$ the path on n vertices and $\rho = 1$ Konheim and Weiss (1966). They fixed a parameter $\alpha \in (0, 1]$ and placed $\lceil \alpha n \rceil$ cars uniformly at random on $[1, n]$. Let A_n be the event that every car parks. Their main result was an asymptotic formula for the probability a given configuration is a parking function:

$$\lim_{n\to\infty} P(A_n) = (1 - \alpha)e^{\alpha}. \tag{1.5}$$

There has been significant followup study of the combinatorial structures that arise from parking functions. See the work of Stanley and Pitman (Stanley, 1997, 1998; Stanley and Pitman, 2002) as well as Diaconis and Hicks (2017). There are also connections to the Abelian sandpile and activated random walk (Chebikin and Pylyavskyy, 2005; Cabezas et al., 2014a).

Notice that (1.5) is never equal to zero. Lackner and Panholzer showed that there is a phase transition when $\mathcal{T}$ is a uniformly random tree on n vertices and $\lceil \alpha n \rceil$ cars are placed uniformly at random throughout the vertices. Again letting A_n be the event that every car parks, they proved that $P(A_n)$ has limiting behavior

$$\lim_{n\to\infty} P(A_n) = \begin{cases} \frac{\sqrt{1-2\alpha}}{1-\alpha}, & 0 \le \alpha \le 1/2 \\ 0, & \alpha \ge 1/2 \end{cases}. \tag{1.6}$$

Goldschmidt and Przykucki studied the natural limiting case of Lackner and Panholzer (2016) in Goldschmidt and Przykucki (2019). They let $\mathcal{T}$ be a Galton-Watson tree with a Poisson with mean 1 offspring distribution conditioned to be infinite. Each η_v is an independent Poisson random variable with mean α. With A the event that every car parks, they showed that $P(A)$ has the same formula as (1.6). Furthermore, they deduced the main theorem of Lackner and Panholzer (2016) as a corollary of their theorem on the infinite tree. Recently, Chen and Goldschmidt proved a similar result for when $\mathcal{T}$ is the limiting tree from a sequence of uniformly random rooted plane trees (Chen and Goldschmidt, 2021). In this case, the phase transition occurs at $\alpha = \sqrt{2} - 1 \approx 0.4142$, rather than 1/2.

The parking process has been studied from two alternative perspectives. Jones viewed parking as a model for runoff of rainfall in Jones (2019). Parking can also be thought of as an interacting particle system with mobile particle (cars) and stationary particles (spots) which mutually annihilate upon colliding. This was first studied on the integer lattice with cars performing simple random walk by Cabezas, Rolla, and Sidoravicius under the name *particle-hole model* (Cabezas et al., 2014b). Later Damron, Gravner, Junge, Lyu, and Sivakoff studied these dynamics on transitive unimodular graphs (Damron et al., 2019b). This is a special case of two-type diffusion-limited annihilating systems studied in the physics literature (Ovchinnikov and Zeldovich, 1978; Lee and Cardy, 1995) and also by mathematicians (Bramson and Lebowitz, 1991; Cabezas et al., 2018). More recently, Przykucki, Roberts, and Scott studied the parking process with cars performing simple random walk on the integers Przykucki et al. (2019). Johnson, Junge, Lyu and Sivakoff considered the process in continuous time on integer lattices and on bi-directed regular trees (Johnson et al., 2020). Such graphs with simple random walks are unimodular, so the phase transition happens when there is an equal initial density of cars and spots.

Returning our discussion to parking on Galton-Watson trees, both Goldschmidt and Przykucki (2019) and Chen and Goldschmidt (2021) rely on explicit formulas for the generating function of X when $\mathcal{T}$ is an unconditioned critical Galton-Watson tree. This is made possible through a renewal present in the parking process on Galton-Watson trees and then additional nice properties from the underlying offspring distributions (Poisson and Geometric with mean 1 in Goldschmidt and Przykucki, 2019 and Chen and Goldschmidt, 2021, respectively). Namely, if Z is the number of children of ρ, then

$$X = \eta_\rho + \sum_{i=1}^{Z} (X^{(i)} - 1)^+, \tag{1.7}$$

where the $X^{(i)}$ are i.i.d. copies of X and $x^+ = \max(0, x)$. Similar equations as (1.7):

$$Y_{n+1} = (Y_n^{(1)} + Y_n^{(2)} + \cdots + Y_n^{(Z)} - 1)^+, \tag{1.8}$$

model total arrivals at the root of a tree of height $n+1$ with the inital cars only arriving to the leaves of the tree. This setting is related to a spin glass model referred to as the Derrida-Retaux model. See Collet et al. (1984); Hu and Shi (2018); Hu et al. (2020). Aldous and Bandyopadhyay study many other similar recursive equations Aldous and Bandyopadhyay (2005).

How (1.8) grows from a starting distribution Y_0 is better understood than (1.7). For example, the analogue of Bernoulli parking on a binary tree is when: $P(Y_0 = 0) = 1 - p, P(Y_0 = 2) = p$, and $n = 2$. Collet et al. proved that $F_\infty = \lim_{n\to\infty} EY_n/2^n$ satisfies $F_\infty > 0$ for $p > 1/5$ and $F_\infty = 0$ for $p < 1/5$. The formula

$$p_c = \frac{1}{E[(Y_0 - 1)2^{Y0}] + 1}$$

was worked out for when Y_0 is integer valued Collet et al. (1984). Hu and Shi studied (1.8) with a random number of terms, which corresponds to a supercritical Galton-Watson tree Hu and Shi (2018). They studied how F_∞ behaves as $p \downarrow p_c$. It seems to us that the addition of the 'η_ρ'-term in (1.7) means that a different approach is needed than what has been used to analyze (1.8).

In Curien and Hénard (2019), Curien and Hénard confirmed a conjecture from Goldschmidt and Przykucki (2019) by generalizing the phase transition results from Goldschmidt and Przykucki

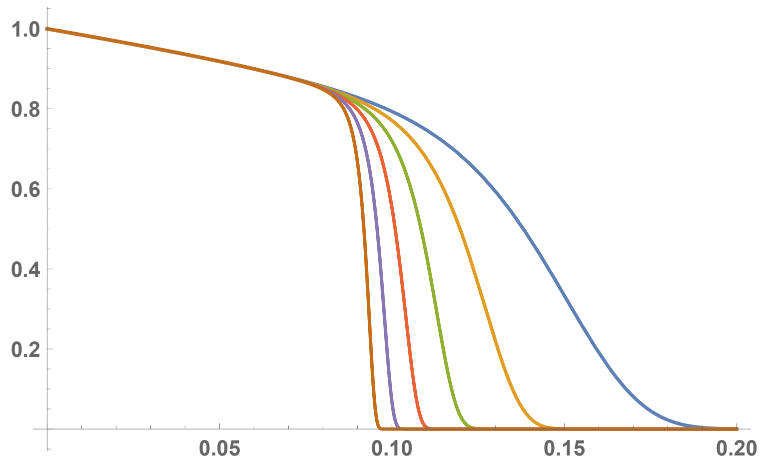

FIGURE 1.1. Plots of $P(X_n = 0)$ in Bernoulli parking on $\mathbb{T}_2$ for $n = 10, 15, 20, 30, 35, 40$ (arranged right to left) and $\alpha \in [0, 0.2]$. The plotted values possibly have small floating point inaccuracies for large n. We do not have an explicit formula for $P(X = 0)$, but these curves are increasingly accurate approximations. The fact that $P(X = 0)$ at $\alpha_c(2)$ is seen in the steepening curves.

(2019) and Chen and Goldschmidt (2021) to arbitrary Galton-Watson trees whose offspring distributions have mean 1 and finite variance Σ^2. They proved that when the η_v are i.i.d. with mean α and variance σ^2 and $\mathcal{T}$ is such a Galton-Watson tree conditioned to be infinite, a phase transition for EX occurs when

$$\theta := (1 - \alpha)^2 - \Sigma^2(\sigma^2 + \alpha^2 - \alpha) = 0. \tag{1.9}$$

For example, if the offspring distribution is Poisson with mean 1 and arrival distribution is Poisson with mean α, solving (1.9) gives $\alpha = 1/2$ as in Goldschmidt and Przykucki (2019).

What happens at criticality? For the setting in Goldschmidt and Przykucki (2019), Goldschmidt and Przykucki proved that EX undergoes a discontinuous phase transition on the critical Poisson Galton-Watson tree:

$$EX = \begin{cases} 1 - \sqrt{1 - 2\alpha}, & \alpha \leq 1/2 \\ \infty, & \alpha > 1/2 \end{cases}.$$

In particular, $EX = 1$ when $\alpha = 1/2$. Similar behavior was observed for EX in the setting in Chen and Goldschmidt (2021). All of this is covered by the main theorem of Curien and Hénard (2019) which implies, among other things, that

$$E(X-1)^+ = \begin{cases} \dfrac{1 - \sqrt{\theta} + \alpha}{\Sigma^2}, & \theta \geq 0 \\ \infty, & \theta < 0 \end{cases}$$

with θ defined at (1.9). Contat generalized some results from Curien and Hénard (2019) to the setting in which the arrival distribution depends on the degree of the vertex Roulet and d'Aspremont (2020). Subsequently, Contant and Curien described the critical window for the parking process on Cayley trees Contat and Curien (2021).

As remarked earlier, less is known about the phase transition on supercritical trees. It would be good to have an exact formula for $\alpha_c(d)$, and more generally for the critical threshold on supercritical Galton-Watson trees analogous to the main theorem of Curien and Hénard (2019). Even a heuristic for where the threshold should be would be nice to have. It is unclear to us if, like for critical Galton-Watson trees, the threshold only depends on the mean and variance of η. One clear difference from the formula for α_c at (1.9), is that even if the variance of Z is 0, i.e. $Z \equiv d$, then the critical threshold can change. So it seems likely that, if a closed form for α_c exists on supercritical Galton-Watson trees, it is more involved than (1.9).

Although we prove that $P(X = 0) > 0$ for $\alpha \leq \alpha_c(d)$, we do not have a closed formula for $P(X = 0)$, nor for EX. Nor do we have a conjecture. In Figure 1.1 we give a few plots of $P(X_n = 0)$ for Bernoulli parking on $\mathbb{T}_2$, where X_n is the number of cars that arrive at ρ up to and including time n.

1.2. *Organization.* In Section 2 we prove Theorem 1.1, and Theorem 1.3. Section 3 has the results for Bernoulli parking: Proposition 1.4, Proposition 1.5, and Proposition 1.6. Section 4 has the stochastic comparison results: Theorem 1.7 and Corollary 1.8.

2. Critical behavior

Recall that X is the total number of cars that arrive at ρ when $\mathcal{T}$ is a Galton-Watson tree with offspring distribution Z satisfying $EZ = \lambda > 1$. The number of cars arriving to the site v is η_v which has distribution $\eta := \eta(\alpha)$. Let X_n be the number of cars that arrive at ρ up to time n. We let $X_0 = \eta_\rho$. Let $q_n = P(X_n = 0)$.

Our starting point is a formula for EX_{n+1}. Define the functions

$$G_n(\alpha) = \sum_{i=0}^{n} \lambda^{-i} q_i; \quad F(\alpha) = \frac{\lambda(1-\alpha)}{\lambda - 1}; \quad C(\alpha) = \frac{1-\alpha}{\lambda-1}. \tag{2.1}$$

Also, let $G(\alpha) = \lim_{n\to\infty} G_n(\alpha)$.

Proposition 2.1. *Let G_n, F and C be as in (2.1). It holds for all $n \geq 0$ that*

$$EX_{n+1} = (G_n(\alpha) - F(\alpha))\,\lambda^{n+1} + C(\alpha). \tag{2.2}$$

Proof: The truncated analogue of (1.7) is

$$X_{n+1} = \eta_\rho + \sum_{i=1}^{Z} (X_n - 1)^+,$$

which follows from renewal properties of $\mathcal{T}$. Taking the expected value of both sides gives

$$EX_{n+1} = \alpha + \lambda(EX_n - P(X_n > 0))$$

Iterating the recursion yields

$$EX_{n+1} = \lambda^{n+1} EX_0 + \sum_{i=0}^{n} \lambda^i \alpha - \sum_{i=1}^{n+1} \lambda^i (1 - q_{n-i+1})),$$

which simplifies to (2.2) after expanding the $\sum \lambda^{-i}$ terms and factoring out λ^{n+1}. $\square$

Proposition 2.1 gives us a necessary and sufficient condition to have $EX < \infty$.

Lemma 2.2. $G(\alpha) - F(\alpha) = 0$ *if and only if* $EX < \infty$.

Proof: First note that $F(\alpha), C(\alpha) > 0$ for $\alpha < 1$. For such α, we must have $G(\alpha) - F(\alpha) \geq 0$. Otherwise, since $G_n \uparrow G$, we would have $G_n(\alpha) - F(\alpha) < -\delta$ for some $\delta > 0$, which gives the contradiction that $EX_n \downarrow -\infty$. If $G(\alpha) - F(\alpha) = 0$, then (2.2) implies that $EX \leq C(\alpha)$ for all $n \geq 1$. The monotone convergence theorem implies that $EX_n \uparrow EX_n \leq C(\alpha) < \infty$.

If $F(\alpha) - G(\alpha) > 0$ then, since $G_n \uparrow G$ is strictly increasing in n, we have $G_N(\alpha) - F(\alpha) = \delta$ for some $\delta > 0$ and large enough N. The formula at (2.2) implies that $EX_n \geq \delta d^n$ for all $n \geq N$, and thus $EX = \infty$. $\square$

To describe what happens at $\alpha = \alpha_c$ we require continuity of G, which relies on continuity of q_n in α. First we prove that the distribution of $\eta(\alpha)$ is continuous in α.

Lemma 2.3. *Suppose that $(\eta(\alpha))$ is a stochastically increasing family of random variables supported on the nonnegative integers with $E\eta(\alpha) = \alpha$. It holds for all $k \geq 0$ that $P(\eta(\alpha) = k)$ is a continuous function in α.*

Proof: Let $\alpha' < \alpha$. Since $\alpha = E\eta(\alpha) = \sum_{m\geq 0} P(\eta(\alpha) > m)$, we write

$$\alpha - \alpha' = E\eta(\alpha) - E\eta(\alpha') = \sum_{m=0}^{\infty}[P(\eta(\alpha) > m) - P(\eta(\alpha') > m)].$$

Because $(\eta(\alpha))$ is stochastically increasing, each summand is positive. Thus, for any $m \geq 0$ we have $P(\eta(\alpha) > m) - P(\eta(\alpha') > m) \leq \alpha - \alpha'$, which can be made arbitrarily small. It follows that $P(\eta(\alpha) > m)$ is continuous in α for all $m \geq 0$. This implies that $1 - P(\eta(\alpha) > m) = \sum_{k=0}^{m} P(\eta(\alpha) = k)$ is also continuous. Iteratively applying this fact for $m = 0, 1, \ldots$ gives that $P(\eta(\alpha) = k)$ is continuous for all $k \geq 0$. □

Lemma 2.4. *q_n is continuous in α for all $n \geq 0$.*

Proof: Let $\mathcal{T}_n$ denote the subset of $\mathcal{T}$ containing all vertices within distance n of ρ. Fix $N > 0$ and partition

$$\begin{aligned} q_n &= P(X_n = 0, |\mathcal{T}_n| \leq N) + P(X_n = 0, |\mathcal{T}_n| > N) \\ &\leq P(X_n = 0, |\mathcal{T}_n| \leq N) + P(|\mathcal{T}_n| > N). \end{aligned} \tag{2.3}$$

Using Markov's inequality we have $P(|\mathcal{T}_n| > N) \leq \lambda^{n+1}/N$, and can be made arbitrarily small for fixed n.

Observe that there are finitely many trees $\mathcal{T}_n \leq N$, and the event $X_n = 0$ requires that all $\eta_v \leq N$ with $v \in \mathcal{T}_n$, otherwise ρ would be visited. Thus, $P(X_n = 0, |\mathcal{T}_n| \leq N)$ is a finite sum involving only products of the probabilities $P(\eta = k)$ for $k \leq N$. By Lemma 2.3, this is continuous. Hence, for $\alpha \in (0, 1)$ and any $\epsilon > 0$, we may choose N so that $P(|\mathcal{T}_n| > N) < \epsilon/3$ and δ so that

$$|P(X_n(\alpha) = 0, |T_n| \leq N) - P(X_n(\alpha') = 0), |T_n| \leq N)| < \epsilon/3$$

for all $|\alpha - \alpha'| < \delta$. Here $X_n(\alpha)$ signifies the dependence of X_n on α. Applying this to (2.3) gives for all $|\alpha - \alpha'| < \delta$ we have $|q_n(\alpha) - q_n(\alpha')| < \epsilon$, and so q_n is continuous at α. □

Now we can prove that EX is finite when $\alpha = \alpha_c$.

Proof of Theorem 1.1: By Lemma 2.2, it suffices to prove that $G(\alpha_c) - F(\alpha_c) = 0$. We claim that $G(\alpha)$ is continuous for all $\alpha \in (0, 1)$. By our hypothesis that $P(\eta = k)$ is continuous in α and Lemma 2.4, the q_n are continuous functions of α. It follows that each $G_n = \sum_{i=0}^{n} \lambda^{-i} q_i$ is continuous. Moreover, the convergence $G_n \uparrow G$ is uniform since

$$G(\alpha) - G_n(\alpha) = \sum_{i>n} \lambda^{-i} q_i \leq \sum_{i>n} \lambda^{-i}$$

which can be made arbitrarily small for all sufficiently large n. A uniformly convergent sequence of continuous functions is continuous, so G is continuous. As F is also continuous, it follows that $G(\alpha) - F(\alpha)$ is continuous for all $\alpha \in (0, 1)$.

Goldschmidt and Przykucki (2019, Theorem 3.4) tells us that $\alpha < \alpha_c$ implies $EX < \infty$. Thus, Lemma 2.2 gives that $G(\alpha) - F(\alpha) = 0$ for all $\alpha < \alpha_c$. Continuity of $G - F$ implies that

$$G(\alpha_c) - F(\alpha_c) = \lim_{\alpha \to \alpha_c^-} [G(\alpha) - F(\alpha)] = 0.$$

Thus, $EX < \infty$ when $\alpha = \alpha_c$. The explicit formula (1.1) follows from taking expectation in (1.7) and solving for EX, which is valid whenever $EX < \infty$.

To establish that $P(X = 0) > 0$ whenever $\alpha \le \alpha_c$, we proceed by contradiction. If $P(X = 0) = 0$, then we claim that there almost surely exists an infinite sequence of cars that eventually arrive at ρ. Indeed, let v_1 be the location of the first car to arrive at ρ from a vertex distinct from ρ. The existence of v_1 is guaranteed by the fact that $P(X = 0) = P(X = 0 \mid \eta_\rho = 0)P(\eta_\rho = 0) = 0$, hence a car will almost surely arrive at ρ even after ignoring any arrivals from η_ρ. If there are several vertices, break the tie in an arbitrary manner. We then apply the same reasoning to v_1 to obtain a vertex v_2 initially housing the first car to arrive at v_1 from a vertex distinct from v_2. Since a car originating at v_1 reached ρ, the car from v_2 will also visit ρ. Iterating gives a sequence of infinitely many distinct vertices with a car that visits ρ, hence $X = \infty$ almost surely. However, this contradicts our just-established result that $\alpha \le \alpha_c$ implies X is almost surely finite. □

Alternate proof of Theorem 1.1: Let $X_n(\alpha)$ be the number of cars that arrive at ρ up to time n with $\eta(\alpha)$ cars arriving initially to each site. Let $X(\alpha) = X_\infty(\alpha)$. Since the $\eta(\alpha)$ are stochastically increasing, so are the $X_n(\alpha)$. We claim that the functions

$$f_n\colon [0, \alpha_c] \to [0, \infty]\colon \alpha \to EX_n(\alpha)$$

are left-continuous. Indeed, for any $0 \le \alpha_1 < \alpha_2 < \alpha_c$ we have

$$E[X_n(\alpha_2)] - E[X_n(\alpha_1)] \le \sum_{k=0}^{n} \lambda^k [E\eta(\alpha_2) - E\eta(\alpha_1)] = (\alpha_2 - \alpha_1) \sum_{k=0}^{n} \lambda^k.$$

The first inequality comes from the observation that the amount of extra arrivals at the root can be no greater than the total amount of arrivals at the entire tree. Since the f_n are left-continuous and non-decreasing, it follows that

$$f_n \nearrow f := EX(\alpha)$$

is also left-continuous. Left-continuity along with the uniform bound on $f(\alpha)$ for $0 \le \alpha < \alpha_c$ from Goldschmidt and Przykucki (2019, Theorem 3.4) imply the bound at (1.1) holds for $f(\alpha_c)$. □

Proof of Theorem 1.3: It follows from Lemma 2.2 that

$$\alpha_c = \sup\{\alpha\colon G(\alpha) - F(\alpha) = 0\}.$$

Notice that $q_n = P(X_n = 0) = P(\tau > n)$. We then have

$$G(\alpha) = \sum_{i=0}^{\infty} \lambda^{-i} P(\tau > i) = \sum_{i=0}^{\infty} \sum_{m>i} \lambda^{-i} P(\tau = m) + \sum_{i=0}^{\infty} \lambda^{-i} P(\tau = \infty)].$$

Apply Fubini's theorem to the double sum to write this as

$$\begin{aligned} G(\alpha) &= \sum_{m>0} \sum_{i=0}^{m-1} \lambda^{-i} P(\tau = m) + \sum_{i=0}^{\infty} \lambda^{-i} P(\tau = \infty) \\ &= \sum_{m>0} \frac{\lambda - \lambda^{-m+1}}{\lambda - 1} P(\tau = m) + \frac{\lambda}{\lambda - 1} P(\tau = \infty) \\ &= \frac{\lambda}{\lambda - 1} \left(P(\tau = \infty) + \sum_{m>0} [P(\tau = m) - \lambda^{-m} P(\tau = m)] \right). \end{aligned}$$

After grouping the $P(\tau = m)$ terms and the $\lambda^{-m} P(\tau = m)$ terms and accounting for the fact that both are missing $P(\tau = 0)$, this simplifies to

$$G(\alpha) = \frac{\lambda}{\lambda - 1} \left(1 - E\lambda^{-\tau} \right).$$

Now, subtracting $F(\alpha) = \lambda(\lambda-1)^{-1}(1-\alpha)$ and simplifying a bit gives

$$G(\alpha) - F(\alpha) = \frac{\lambda}{\lambda - 1}\left(\alpha - E\lambda^{-\tau}\right). \tag{2.4}$$

Proposition 2.1 tells us that

$$\frac{EX_{n+1}}{\lambda^n} = (G_n(\alpha) - F(\alpha)) + C(\alpha)\lambda^{-n}.$$

Taking the limit of the above and applying the equality at (2.4) gives (1.2). As for (1.3), by Lemma 2.2 we have EX is finite if and only if $\alpha - E\lambda^{-\tau} = 0$, thus α_c is the largest solution to this equation.

□

3. Bernoulli parking on $\mathbb{T}_d$

3.1. *Bounds for* $\alpha_c(2)$. The idea for improving previous estimates on $\alpha_c(2)$ is to generate closed forms for $q_n := P(X_n = 0)$ using a recursive relationship. Let $p = \alpha/2$. For $n \geq 1$ define $V_n = (X_n - \eta_\rho)^+$ to be the number of cars that arrive at ρ between time 1 and n. Let $r_{n,j} = P(V_n = j)$. Notice we have the simple relationship

$$q_n = (1-p)r_{n,0}. \tag{3.1}$$

The following lemma describes a recursion satisfied by the $r_{n,j}$.

Lemma 3.1. *Let $n \geq 1$. Set $r_{n,j} = 0$ for $j < 0$ and $j > 2^{n+1} - 2$. For $n = 0$,*

$$r_{1,0} = (1-p)^2, \quad r_{1,1} = 2p(1-p), \quad r_{1,2} = p^2.$$

When $j = 0$ we have

$$r_{n+1,0} = (1-p)^2(r_{n,0} + r_{n,1})^2.$$

It holds for all $0 < j \leq 2^{n+1} - 2$ that

$$\begin{aligned} r_{n+1,j} = p^2 &\left(\sum_{k=0}^{j-2} r_{n,k} r_{n,j-k-2}\right) \\ &+ 2p(1-p)\left(r_{n,0}r_{n,j-1} + \sum_{k=0}^{j-1} r_{n,k} r_{n,j-n}\right) \\ &+ (1-p)^2\left(2r_{n,0}r_{n,j+1} + \sum_{k=1}^{j+1} r_{n,k} r_{n,j-k+2}\right). \end{aligned}$$

Proof: This follows from (1.7). Label the two children of the root as x and y. The formulas for $r_{1,j}$ come from the fact that $V_1 = \mathbf{1}\{\eta_x = 2\} + \mathbf{1}\{\eta_y = 2\}$ is a Binomial random variable. The formula for $r_{n+1,0}$ comes from the requirement that $\eta_x, \eta_y = 0$ and that no more than one car visits each of x and y, respectively. Clearly, $r_{n,j} = 0$ for $j < 0$, and since the number of vertices in $\mathbb{T}_2$ up to distance n from ρ is $2^{n+1} - 1$. The formula for $r_{n+1,j}$ comes from conditioning on η_x, η_y and then partitioning on k cars arriving at x. Special considerations need to be made when $k = j+1$ since, in this case, either 0 or 1 cars can arrive at y. Similarly for when $k = 0$. □

Proof of Proposition 1.4: We start with the upper bound. It follows from Lemma 2.2 that $G(\alpha) - F(\alpha) > 0$ if and only if $\alpha > \alpha_c$. Since $G_n \uparrow G$, if

$$G_n(\alpha) - F(\alpha) > 0 \tag{3.2}$$

then $G(\alpha) - F(\alpha) > 0$. Thus, if we can find a pair n, α satisfying (3.2), then α is an upper bound on α_c.

Using Lemma 3.1 and (3.1), it is not so taxing to write out q_n for small values of n by hand, but this quickly becomes intractable. A computer can calculate q_n for much larger values of n, but still is limited, since the number of terms and degree of the polynomials grow exponentially. Rounding error makes any estimates obtained with floating point calculations non-rigorous.

We avoid this issue by truncating all numbers (at the 200th decimal place). Let g_n be the analogue of G_n, but with all decimals truncated. Since every summand in the formula for q_n from Lemma 3.1 and (3.1) are positive, truncating gives a lower bound: $g_n(\alpha) \leq G_n(\alpha)$ for all α. It only takes a few seconds to show that for $\alpha_0 = 0.08698$ we have

$$g_{50}(\alpha_0) - F(\alpha_0) > 0,$$

and thus $G(\alpha_0) - F(\alpha_0)$ is also positive. Thus, $\alpha_c < 0.08698$.

The improvement to the lower bound uses a similar union bound approach as in Goldschmidt and Przykucki (2019, Theorem 3.5). The authors show that if $X = \infty$ then there is an infinite sequence (H_n) of connected subgraphs containing the root with

(a) $|H_n| = n$ and
(b) there are at least $\lceil n/2 \rceil$ vertices in $v \in H_n$ with $\eta_v = 2$.

The number of such subgraphs of size n is counted by the nth Catalan number which is bounded by 4^n.

The number of vertices with $\eta_v = 2$ in a subgraph of size n is a Binomial random variable with parameters n and $p = \alpha/2$. Thus, the probability of such a subgraph containing $k > n/2$ vertices with cars initially arriving to them is

$$\sum_{k>n/2} \binom{n}{k} p^k (1-p)^{n-k} \leq 2^n \sum_{k>n/2} p^k (1-p)^{n-k}.$$

Above we use the fact that $\binom{n}{k} \leq \binom{n}{\lceil n/2 \rceil}$ for all $k \geq \lceil n/2 \rceil$. If $p < 1/2$, then

$$\sum_{k>n/2} p^k (1-p)^{n-k} = p^{\lceil n/2 \rceil} (1-p)^{n-\lceil n/2 \rceil} \sum_{k=0}^{n/2} \left(\frac{p}{1-p} \right)^k \leq 2Cp^{n/2}(1-p)^{n/2}$$

for $C = \sum_{k=0}^{\infty} (p/(1-p))^k = (1-p)/(1-2p)$. The '2' coefficient is to correct for the periodicity coming from the $(1-p)^{n-\lceil n/2 \rceil}$ term.

Applying a union bound tells us that the probability that the sequence (H_n) exists is bounded by

$$\sum_{n=1}^{\infty} 4^n 2^n (2Cp^{n/2}(1-p)^{n/2}) = 2C \sum_{n=1}^{\infty} (64p(1-p))^{n/2}.$$

For $p < \frac{1}{2} - \frac{\sqrt{15}}{8} \approx 0.015877$ the term $64p(1-p) < 1$ and the series is summable. It follows from the Borel-Cantelli lemma that there is no infinite sequence (H_n) with the required properties, and thus $P(X = \infty) = 0$. Switching back to $\alpha = 2p$, this gives $\alpha_c > 2(0.015877) = 0.031754$.

$\square$

Remark 3.2. Regarding the truncation used with g_n to obtain upper bounds on α_c, there is some loss of accuracy from truncation. However, being able to compute for larger n is better than computing with perfect accuracy. For example, it took several hours to show that $G_{23}(112/1000) - F(112/100) > 0$ working with rational numbers. For $n = 23$ most of the fractions have millions of digits in the denominator and numerator. Still, with enough computing power, proceeding with exact calculations would give arbitrarily close upper bounds on α_c.

Remark 3.3. The improvement to the lower bound in Goldschmidt and Przykucki (2019, Theorem 3.5) is the $(1-p)^{n/2}$ term. The authors did not optimize to include it and instead of finding p satisfying $64p(1-p) < 1$, they required that $64p < 1$. This gives their lower bound of $1/64$.

3.2. *Asymptotic behavior of* $p_c(d)$.

Proof of Proposition 1.5*:* Both bounds are straightforward generalizations of Theorem 3.5 in Goldschmidt and Przykucki (2019). For the upper bound, the idea is to compare to percolation that considers the d^2 vertices at distance two from the root. If one of these vertices has $\eta_v = 2$, then it and its ancestor will be parked at. When $\alpha/2 > d^{-2}$, basic percolation theory tells us that there is almost surely an infinite connected path of occupied parking spots. As observed by Goldschmidt and Przykucki, the odd generations of the tree along this path almost surely have infinitely many cars arrive at them, of which infinitely many will reach the root since the spots on the path are parked at and the start of the path is some finite distance from ρ. It follows that $X = \infty$ almost surely. Thus, $\alpha_c(d) \le 2d^{-2}$.

The lower bound follows the argument in the proof of Proposition 1.4 concerning the existence of a sequence of subgraphs (H_n). The only modification needed is that on $\mathbb{T}_d$ the number of connected subgraphs containing the root with n vertices is equal to the generalized Catalan number

$$C_{d,n} = \frac{1}{(d-1)n+1}\binom{dn}{n} \le \binom{dn}{n}$$

See Hilton and Pedersen (1991). Using a standard upper bound on binomial coefficients (see Cormen et al., 2001), we have

$$\binom{dn}{n} \le \left(e\frac{dn}{n}\right)^n = (ed)^n.$$

Thus, we can replace 4^n with $(ed)^n$ when applying a union bound. The rest of the quantities are unchanged. So, again letting $p = \alpha/2$, we have $X < \infty$ almost surely whenever

$$(de)^n 2^n p^{n/2}(1-p)^{n/2} < 1.$$

Bounding the $(1-p)^{n/2}$ term by 1, solving for p, then making the replacement $\alpha/2 = p$ gives the claimed lower bound. □

We conclude this section by showing that the value of α_c depends on how concentrated the arrival distribution is.

Proof of Proposition 1.6*:* As in the proof of the upper bound in Proposition 1.5 we compare to percolation. Now that $\eta(\alpha) = 3$ with probability $\alpha/3$ we can consider the vertices at distance three from the root. Whenever $\alpha/3 \ge d^{-3}$ there is almost surely an infinite connected path of spots that are parked at, which, by similar reasoning as before, implies that $X = \infty$ almost surely. Thus, $\alpha_c' \le 3d^{-3}$. □

4. The increasing convex order

Proof of Theorem 1.7*:* Let X_n and X_n' be the number of arrivals at ρ up to time n for parking with arrival distributions η and η' as in Proposition 2.1. We claim that it suffices to prove that

$$X_n \preceq_{\text{icx}} X_n' \text{ for all } n \ge 0. \tag{4.1}$$

Suppose we show (4.1). It follows from the closure under sequences property Shaked and Shanthikumar (2007, Theorem 4.A.8.(c)) that, whenever EX and EX' are finite, we have $X \preceq_{\text{icx}} X'$. On the other hand, if EX is infinite, then (4.1) implies that $EX_n \le EX_n'$ and thus EX' is also infinite. By Goldschmidt and Przykucki (2019, Theorem 3.4), this implies that $P(X_n = \infty) = 1 = P(X_n' = \infty)$. So trivially we have $X \preceq_{\text{icx}} X'$.

To establish (4.1), we proceed inductively. By hypothesis we have

$$X_0 = \eta_\rho \preceq_{\text{icx}} \eta_\rho' = X_0'.$$

Now, supposing that $X_n \preceq_{\text{icx}} X'_n$. Since $\varphi(x) = (x-1)^+$ is an increasing convex function on $[0, \infty)$, it follows from our inductive hypothesis and Shaked and Shanthikumar (2007, Theorem 4.A.8.(a)) that

$$(X_n - 1)^+ \preceq_{\text{icx}} (X'_n - 1)^+. \tag{4.2}$$

Applying (4.2) along with Shaked and Shanthikumar (2007, Theorem 4.A.9.) for random sums of i.i.d. random variables whose respective summands are dominated in the increasing convex order gives

$$\sum_{i=1}^{Z} (X_n^{(i)} - 1)^+ \preceq_{\text{icx}} \sum_{i=1}^{Z} ((X_n^{(i)})' - 1)^+. \tag{4.3}$$

Also, since $\eta_\rho \preceq \eta'_\rho$, Shaked and Shanthikumar (2007, Theorem 4.A.8.(d)) and (4.3) imply that

$$\eta_\rho + \sum_{i=1}^{Z} (X_n^{(i)} - 1)^+ \preceq_{\text{icx}} \eta'_\rho + \sum_{i=1}^{Z} ((X_n^{(i)})' - 1)^+. \tag{4.4}$$

The left and right formulas in (4.4) are exactly the recursive equations for X_{n+1} and X'_{n+1} as in (2.2). This gives (4.1), which concludes the argument. $\square$

Remark 4.1. The proof of Theorem 1.7 could be generalized so that the result holds for parking on an arbitrary locally finite tree. We would take as inductive hypothesis that total arrivals at the root respects the icx order for all such trees of height n or less. As in our argument for Galton-Watson trees, we can obtain an i.i.d. decomposition for the total number of visits at the root from distance $n+1$ in terms of trees of height no larger than n. The inductive hypothesis along with the machinery from Shaked and Shanthikumar (2007) could then be applied in a similar manner.

Proof of Corollary 1.8: Let $\eta := \eta(\alpha)$ be the car arrival distribution for Bernoulli parking. By Theorem 1.7, it suffices to prove for fixed α that $\eta \preceq_{\text{icx}} \eta'(\alpha) := \eta'$. This follows from a straightforward adaptation of Johnson and Junge (2018, Proposition 15 (b)).

Let ψ be an increasing convex function on $[0, \infty)$. It is more convenient to work with $\varphi(x) = \psi(x) - \psi(0)$ so that $\varphi(0) = 0$. This is without loss of generality since linearity of expectation ensures that $E[\varphi(\eta)] \leq E[\varphi(\eta')]$ implies that $E[\psi(\eta)] \leq E[\psi(\eta')]$. First, we have

$$E[\varphi(\eta)] = (\alpha/2)\varphi(2). \tag{4.5}$$

As for η', let $a = E[\eta' \mid \eta' \geq 2]$. Since $\varphi(0) = 0$ and $P(\eta' = 1) = 0$, we can condition and apply Jensen's inequality

$$\begin{aligned} E[\varphi(\eta')] &= E[\varphi(\eta') \mid \eta' \geq 2] P(\eta' \geq 2) \\ &\geq \varphi(a) P(\eta' \geq 2). \end{aligned} \tag{4.6}$$

As $a \geq 2$ and φ is convex, the point $(a, \psi(a))$ lies above the secant line connecting $(0,0)$ and $(2, \varphi(2))$. It follows that $a\varphi(2)/2 \leq \varphi(a)$. Applying this to (4.6) gives

$$E[\varphi(\eta')] \geq a \frac{\varphi(2)}{2} P(\eta' \geq 1).$$

Notice that $aP(\eta' \geq 1) = E\eta' = \alpha$, so we have $E[\varphi(\eta')] \geq (\alpha/2)\varphi(2) = E[\varphi(\eta)]$ by (4.5). Thus, $\eta \preceq_{\text{icx}} \eta'$. $\square$

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