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Partial $*$-Algebras of Closable Operators II. States and Representations of Partial $*$-Algebras

By

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Abstract

This second paper on partial Op*-algebras is devoted to the theory of representations. A new definition of invariant positive sesquilinear forms on partial $*$-algebras is proposed, which enables to perform the familiar GNS construction. In order to get a better control of the corresponding representations, we introduce and study a restricted class of partial Op*-algebras, called partial GW*-algebras, which turn up naturally in a number of problems. As an example, we extend Powers' results about the standardness of GNS representations of abelian partial $*$-algebras.

§1. Introduction

In this second paper, we continue the theory of partial Op*-algebras. The main definitions have been given in the first paper [1] (to be denoted by I in the sequel) and we will use them freely, keeping consistently the same notation. As announced in I, the central topic of this paper is the theory of representations of partial $*$-algebras. We will define them in Section 2 below, together with several notions familiar in the case of representations of $*$-algebras [2]: extensions of representations, adjoint of a representation, commutants and bicommutants of various types, irreducibility.

Our interest in representations comes mainly from physical applications, such as Statistical Mechanics or Quantum Field Theory. Indeed a physical system is usually characterized by the algebra $\mathfrak{A}$ of its observables, usually an abstract $*$-algebra, and then each state on $\mathfrak{A}$ defines a representation of $\mathfrak{A}$ in some Hilbert space, via the Gel'fand-Naimark-Segal (GNS) construction

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[3]. Since, as argued in I, one should rather start from a *partial* $*$-algebra of observables, we have to generalize the GNS construction, and to begin with, the notion of state. In the case of a $*$-algebra $\mathfrak{A}$, a state is a normalized positive linear form on $\mathfrak{A}$. If $\mathfrak{A}$ is only a partial $*$-algebra, the positivity condition alone already requires the use of *sesquilinear* forms. For a $*$-algebra $\mathfrak{A}$, the GNS construction works only if the starting sesquilinear form ϕ on $\mathfrak{A} \times \mathfrak{A}$ is invariant, in the sense that $\phi(x^*y, z) = \phi(y, xz)$, for all $x, y, z \in \mathfrak{A}$. Clearly this definition is inapplicable for a partial $*$-algebra, since the products x^*y, xz need not exist. As an alternative, Antoine and Lassner [4, 5] have introduced the concept of *h-form*, that is, a positive sesquilinear form ϕ which is invariant in the following sense: if x is a left multiplier of z and x^* is a left multiplier of y, then $\phi(x^*y, z) = \phi(y, xz)$. Furthermore, a h-form ϕ is called *weakly GNS* if it satisfies the condition: for each $x \in \mathfrak{A}$, there is a sequence $\{a_n\}$ in $R(\mathfrak{A}) = \{a \in \mathfrak{A}; a$ is a right multiplier of each $x \in \mathfrak{A}\}$ such that $\lim_{n\to\infty} \phi(a_n - x, a_n - x) = 0$. This definition then leads to the following result ([5], Theorem 7.1): if $\mathfrak{A}$ is a semi-associative [1] partial $*$-algebra, every weakly GNS h-form ϕ on $\mathfrak{A}$ allows a GNS construction.

However, this proposal is not entirely satisfactory. First one should remove the assumption of semi-associativity of $\mathfrak{A}$, because partial Op*-algebras are not always semi-associative. Second, the definition of invariance of a positive sesquilinear form given above is too restrictive, because it excludes all non-zero vector forms on a non self-adjoint maximal weak partial Op*-algebra (see Section 3 below). We shall meet both objections at once by introducing (Definition 4.1) an alternative notion of invariant positive sesquilinear (i.p.s.) form on a partial $*$-algebra $\mathfrak{A}$. The new aspects here are : (i) the possible lack of (semi-) associativity of $\mathfrak{A}$ is explicitly taken into account ; (ii) the space $R(\mathfrak{A})$ of universal right multipliers is replaced by an arbitrary subspace $\mathfrak{B}$ of $R(\mathfrak{A})$, dense in a sense to be precised. This new definition is now flexible enough to allow all the vector forms we want, and moreover, any i.p.s. form defined in that way admits a GNS construction, as we will show in Section 3. Furthermore, another standard result remains true : the GNS representation π_ϕ associated to a state ϕ is irreducible iff the state ϕ is pure [36], that is, ϕ cannot be written as a convex combination of two other states (a state is simply a normalized i.p.s. form). By irreducibility of π_ϕ, we mean that its *bounded quasi-weak commutant* $\mathfrak{C}_{qw}(\pi_\phi)$ of π_ϕ is trivial.

However, when we try to implement the GNS construction with sesquilinear forms on partial Op*-algebras, we discover that very little can be said beyond the definitions, for this class is still too general. Moreover, there is no systematic way of constructing i.p.s. forms. At this point, the theory of topological quasi $*$-algebras [7] gives us a hint : there it is easy to obtain i.p.s. forms by taking limits of suitable linear forms defined on the dense subalgebra. Following this guide, we introduce in Section 3 a subclass, called *partial GW*-algebras*, that will allow more precise results. They are characterized by the fact that they contain many (a dense set of) bounded operators, and this allows to obtain i.p.s. forms by a limiting procedure, in particular the vector forms. Actually the class of partial GW*-algebras has an intrinsic mathematical interest, for they seem to be a natural generalization of von Neumann (W^*) algebras and of topological quasi $*$-algebras as well. In particular, if a partial GW*-algebra $\mathfrak{A}$ leaves the common domain invariant, (i.e. it is an Op*-algebra), then $\mathfrak{A}$ is an EW*-algebra [8], as it should. We will study partial GW*-algebras in detail in Section 4. In fact we give here a second definition, that turns out to be equivalent to the first one. This is a reminiscence of the two complementary approaches to von Neumann algebras, the algebraic one ($\mathfrak{A}'' = \mathfrak{A}$) and the topological one (closure in a suitable topology). We discuss also the interplay between partial Op*-algebras and topological quasi $*$-algebras (with respect to the strong* topology).

The real interest of partial GW*-algebras resides in that they arise naturally when one tries to generalize to partial Op*-algebras a number of properties known for von Neumann algebras or for Op*-algebras. For instance, vector forms on partial GW*-algebras are easily characterized, exactly as their counterparts for Op*-algebras [9]. Next, in the *abelian* case, Powers' criterion [2] for standardness of a self-adjoint representation extends naturally to partial GW*-algebras, as will be shown in Section 5. Finally, they allow a natural generalization to partial Op*-algebras of the Tomita-Takesaki theory [10] and of the theory of unbounded derivations [11].

§ 2. Representations of Partial $*$-Algebras

A *partial $*$-algebra* is a complex vector space $\mathfrak{A}$ with an involution $x \to x^*$ (i.e. $(x + \lambda y)^* = x^* = \bar{\lambda} y^*$, $x^{**} = x$) and a subset $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$ such that:

(*i*) $(x, y) \in \Gamma$ *iff* $(y^*, x^*) \in \Gamma$;

(*ii*) *if* $(x, y) \in \Gamma$ *and* $(x, z) \in \Gamma$, *then* $(x, \lambda y + \mu z) \in \Gamma$ *for all* $\lambda, \mu \in \mathbf{C}$;

(*iii*) *whenever* $(x, y) \in \Gamma$, *there exists an element* $xy \in \mathfrak{A}$ *with the usual properties of the multiplication*:

$x(y + \lambda z) = xy + \lambda(xz)$ *and* $(xy)^* = y^* x^*$, *for* (x, y), $(x, z) \in \Gamma$ *and* $\lambda \in \mathbf{C}$.

An element e of $\mathfrak{A}$ is said to be a *unit* if $e^* = e$, $(e, x) \in \Gamma$ and $ex = xe = x$ for every $x \in \mathfrak{A}$. Whenever $(x, y) \in \Gamma$, we say that x is a *left multiplier* of y and y a *right multiplier* of x, and write $x \in L(y)$ and $y \in R(x)$. By (ii), $L(x)$ and $R(x)$ are vector subspaces of $\mathfrak{A}$. For a subset $\mathfrak{B} \subset \mathfrak{A}$, we write

$$L(\mathfrak{B}) = \bigcap_{x \in \mathfrak{B}} L(x), \qquad R(\mathfrak{B}) = \bigcap_{x \in \mathfrak{B}} R(x).$$

Notice that the multiplication is not required to be associative, but it must be distributive with respect to the addition by (iii). A subspace $\mathfrak{B}$ of $\mathfrak{A}$ is said to be a *partial $*$-subalgebra* if $x^* \in \mathfrak{B}$ for all $x \in \mathfrak{B}$ and $x_1 x_2 \in \mathfrak{B}$ for any $x_1, x_2 \in \mathfrak{B}$ such that $x_1 \in L(x_2)$. A partial $*$-algebra $\mathfrak{A}$ is said to be *abelian* if $L(x) = R(x)$ for all $x \in \mathfrak{A}$ and $xy = yx$ for any $x \in \mathfrak{A}$ and $y \in R(x)$.

As usual, $\mathscr{D}$ denotes a dense subspace in a Hilbert space $\mathscr{H}$, and $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ is the set of all linear operators X such that $D(X) = \mathscr{D}$ and $D(X^*) \supset \mathscr{D}$. Then $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ is a partial $*$-algebra, denoted $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$, when equipped with the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger \equiv X^* \restriction \mathscr{D}$ and the weak partial multiplication $\square$: X_1 is a left multiplier of X_2 ($X_1 \in L^w(X_2)$ or $X_2 \in R^w(X_1)$) iff $X_2 \mathscr{D} \subset D(X_1^{\dagger *})$ and $X_1^\dagger \mathscr{D} \subset D(X_2^*)$ and then $X_1 \square X_2 = X_1^{\dagger *} X_2$. A *(weak) partial Op*-algebra* on $\mathscr{D}$ is a partial $*$-subalgebra $\mathfrak{M}$ of $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$, that is, $\mathfrak{M}$ is a subspace of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ such that $X^\dagger \in \mathfrak{M}$ for all $X \in \mathfrak{M}$ and $X_1 \square X_2 \in \mathfrak{M}$ for any $X_1, X_2 \in \mathfrak{M}$ such that $X_1 \in L^w(X_2)$. For any subset $\mathfrak{N}$ of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ and any $\xi \in \mathscr{D}$, we also define:

$$\mathfrak{N}^\xi = \{A \in \mathfrak{N};\ A\xi \in \mathscr{D}\},$$

$$\mathfrak{N}^{\mathscr{D}} = \{A \in \mathfrak{N};\ A\mathscr{D} \subset \mathscr{D}\}.$$

In particular, if $\mathfrak{M}$ is a partial Op*-algebra on $\mathscr{D}$, a special role will be played by the set $R^w(\mathfrak{M})$ of (universal) right multipliers of $\mathfrak{M}$ and the two related subsets $R^w(\mathfrak{M})^\xi$ ($\xi \in \mathscr{D}$) and $R^w(\mathfrak{M})^{\mathscr{D}}$.

A *$*$-representation* of a partial $*$-algebra $\mathfrak{A}$ is a $*$-homomorphism of $\mathfrak{A}$ into $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$, for some pair $\mathscr{D} \subset \mathscr{H}$, that is, a linear map $\pi: \mathfrak{A} \to \mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$ such that : (i) $\pi(x^*) = \pi(x)^*$ *for every* $x \in \mathfrak{A}$; (ii) $x \in L(y)$ *in* $\mathfrak{A}$ *implies* $\pi(x) \in L(\pi(y))$ *and* $\pi(x) \square \pi(y) = \pi(xy)$. A priori one could also consider strong representations in $\mathscr{L}^\dagger_s(\mathscr{D}, \mathscr{H})$, but they seem of little interest, and we will not consider them in the sequel.

Extensions of $*$-representations are defined in a natural way. Let π_1 and π_2 be two $*$-representations of a partial $*$-algebra $\mathfrak{A}$ in $\mathscr{L}^\dagger_w(\mathscr{D}_1, \mathscr{H})$ and $\mathscr{L}^\dagger_w(\mathscr{D}_2, \mathscr{H})$ respectively. If $\pi_1(x) \subset \pi_2(x)$ for all $x \in \mathfrak{A}$, then π_2 is said to be an *extension* of π_1, and this is denoted by $\pi_1 \subset \pi_2$. Notice that the relation $\pi_1 \subset \pi_2$ is different from $\pi_1(\mathfrak{A}) \prec \pi_2(\mathfrak{A})$ or $\pi_1(\mathfrak{A}) \prec_w \pi_2(\mathfrak{A})$, where $\prec$ and $\prec_w$ denote the two notions of (algebraic) extension introduced in I.

As in the case of $*$-algebras, new representations may be obtained from a given one by extension and adjunction, with help of the extension theory for partial Op*-algebras developed in I. Let us recall the basic facts.

Let $\mathfrak{M}$ be a †-invariant subspace of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$. We denote by $t_{\mathfrak{M}}$ the topology on $\mathscr{D}$ defined by the family $\{\|\cdot\|_X;\ X \in \mathfrak{M}\}$ of seminorms: $\|\xi\|_X = \|\xi\| + \|X\xi\|$, $\xi \in \mathscr{D}$. We denote by $\tilde{\mathscr{D}}(t_{\mathfrak{M}})$ the completion of $\mathscr{D}$ relative to the topology $t_{\mathfrak{M}}$ and put

$$\tilde{X} = \bar{X} \restriction \tilde{\mathscr{D}}(t_{\mathfrak{M}}),\ X \in \mathfrak{M}.$$

Similarly, we put

$$\hat{D}(\mathfrak{M}) = \bigcap_{X \in \mathfrak{M}} D(\bar{X}), \ \hat{X} = \bar{X} \restriction \hat{\mathscr{D}}(\mathfrak{M}), \ X \in \mathfrak{M}.$$

We also define

$$\mathscr{D}^*(\mathfrak{M}) = \bigcap_{X \in \mathfrak{M}} D(X^*),$$

$$\mathscr{D}^{**}(\mathfrak{M}) = \bigcap_{X \in \mathfrak{M}} D(\imath^*(X)^*), \ \textit{where} \ \imath^*(X) \equiv X^{\dagger *} \restriction \mathscr{D}^*(\mathfrak{M}).$$

Then one has:

$$\mathscr{D} \subset \tilde{\mathscr{D}}(t_{\mathfrak{M}}) \subset \hat{\mathscr{D}}(\mathfrak{M}) \subset \mathscr{D}^{**}(\mathfrak{M}) \subset \mathscr{D}^*(\mathfrak{M}).$$

The set $\mathfrak{M}$ is said to be *closed* if $\mathscr{D} = \tilde{\mathscr{D}}(t_{\mathfrak{M}})$, *fully closed* if $\mathscr{D} = \hat{\mathscr{D}}(\mathfrak{M})$, *self-adjoint* if $\mathscr{D} = \mathscr{D}^*(\mathfrak{M})$, *algebraically self-adjoint* if $\mathscr{D}^{**}(\mathfrak{M}) = \mathscr{D}^*(\mathfrak{M})$.

Let π be a $*$-representation of a partial $*$-algebra $\mathfrak{A}$. If $\pi(\mathfrak{A})$ is closed (resp. fully closed) then π is called *closed* (resp. *fully closed*). We put

$$t_\pi = t_{\pi(\mathfrak{A})};$$

$$\mathscr{D}(\tilde{\pi}) = \tilde{\mathscr{D}}(t_\pi), \quad \tilde{\pi}(x) = \widetilde{\pi(x)}, \quad x \in \mathfrak{A};$$

$$\mathscr{D}(\hat{\pi}) = \hat{\mathscr{D}}(\pi(\mathfrak{A})), \quad \hat{\pi}(x) = \widehat{\pi(x)}, \quad x \in \mathfrak{A}.$$

Then $\tilde{\pi}$ is a closed $*$-representation of $\mathfrak{A}$ and $\hat{\pi}$ is a fully closed $*$-representation of $\mathfrak{A}$, and they satisfy the relation $\tilde{\pi} \subset \hat{\pi}$.

Next we define the adjoint of a representation as follows:

$$\mathscr{D}(\pi^*) = \bigcap_{x \in \mathfrak{A}} D(\pi(x)^*),$$

$$\pi^*(x) = \pi(x^*)^* \restriction \mathscr{D}(\pi^*), \ x \in \mathfrak{A};$$

and

$$\mathscr{D}(\pi^{**}) = \bigcap_{x \in \mathfrak{A}} D(\pi^*(x)^*),$$

$$\pi^{**}(x) = \pi^*(x^*)^* \restriction \mathscr{D}(\pi^{**}), \ x \in \mathfrak{A}.$$

Then π^{**} is a $*$-preserving linear map of $\mathfrak{A}$ into $\mathscr{L}^\dagger(\mathscr{D}(\pi^{**}), \mathscr{H})$ and one has:

$$\pi(x) \subset \tilde{\pi}(x) \subset \hat{\pi}(x) \subset \pi^{**}(x) \subset \pi^*(x) \tag{2.1}$$

for all $x \in \mathfrak{A}$.

In view of the inclusions (2.1), it is natural to extend to the present case the terminology used for representations of Op*-algebras [2, 12, 13]:

Definition 2.4. Let π be a $*$-representation of a partial $*$-algebra $\mathfrak{A}$ on $\mathscr{D}(\pi)$. Then π is said to be self-adjoint if $\mathscr{D}(\pi^*) = \mathscr{D}(\pi)$, essentially self-adjoint if $\mathscr{D}(\pi^*) = \mathscr{D}(\hat{\pi})$.

We remark that even if $\pi^* = \pi^{**}$, it is not necessarily a $*$-representation.

Next we define the weak commutants of a $*$-representation of a partial $*$-algebra. As for any †-invariant subset, we have the usual bounded commutants of $\pi(\mathfrak{A})$, namely $\pi(\mathfrak{A})'_w$ and $\pi(\mathfrak{A})'_{qw}$. In particular we have now:

$$\pi(\mathfrak{A})'_w = \{C \in \mathscr{B}(\mathscr{H}); (C\xi|\pi(x)\eta) = (\pi(x^*)\xi|C^*\eta), \textit{ for all } x \in \mathfrak{A} \textit{ and } \xi, \eta \in \mathscr{D}(\pi)\}.$$

In addition, we introduce a new one, specifically adapted to representations:

$$\mathfrak{C}_{qw}(\pi) = \{C \in \pi(\mathfrak{A})'_w; (C\pi(x_1^*)\xi|\pi(x_2)\eta) = (C\xi|\pi(x_1x_2)\eta), \text{ for all } x_1, x_2 \in \mathfrak{A} \textit{ such that } x_1 \in L(x_2) \textit{ and all } \xi, \eta \in \mathscr{D}(\pi)\}. \tag{2.2}$$

The point is that $\mathfrak{C}_{qw}(\pi)$ may differ from $\pi(\mathfrak{A})'_{qw}$ because $\pi(x_1) \square \pi(x_2)$ may exist, even if x_1x_2 does not.

These three bounded commutants are weakly closed $*$-invariant subspaces of $\mathscr{B}(\mathscr{H})$ and one has:

$$\pi(\mathfrak{A})'_{qw} \subset \mathfrak{C}_{qw}(\pi) \subset \pi(\mathfrak{A})'_w. \tag{2.3}$$

As for general †-invariant subsets, it is natural to compare the quasi-weak commutant $\mathfrak{C}_{qw}(\pi)$ of a representation π with that of its various extensions given in (2.1). The following result may be proved exactly as Lemma 3.8 of I.

Lemma 2.1. *Let π be a $*$-representation of a partial $*$-algebra. Then*:

$$\mathfrak{C}_{qw}(\hat{\pi}) = \mathfrak{C}_{qw}(\tilde{\pi}) = \mathfrak{C}_{qw}(\pi).$$

For an Op*-algebra $\mathfrak{A}$, additional extensions may be defined under the condition that $\mathfrak{U}'_w$ be an algebra [9, 14]. Similar results have been obtained in I, Theorem 3.9 for partial Op*-algebras, and quite naturally they extend to representations (with the same proof):

Lemma 2.2. *Let π be a $*$-representation of a partial $*$-algebra $\mathfrak{A}$ into $\mathscr{L}^\dagger_w(\mathscr{D}(\pi), \mathscr{H})$. Suppose $\mathfrak{C}_{qw}(\pi)$ is an algebra. Put*

$$\mathscr{D}(\pi_{qw}) = \{\textstyle\sum_{k=1}^n C_k\xi_k; C_k \in \mathfrak{C}_{qw}(\pi), \xi_k \in \mathscr{D}(\pi), k = 1, \dots, n, n \in \mathbb{N}\}$$

$$\pi_{qw}(x)(\textstyle\sum_{k=1}^n C_k\xi_k) = \sum_{k=1}^n C_k\pi(x)\xi_k, \; x \in \mathfrak{A}, \; \sum_{k=1}^n C_k\xi_k \in \mathscr{D}(\pi_{qw}).$$

Then π_{qw} is a $*$-representation of $\mathfrak{A}$ such that $\pi \subset \pi_{qw} \subset \pi^*$, $\pi_{qw}(\mathfrak{A})'_w = \mathfrak{C}_{qw}(\pi)$ and $\pi_{qw}(\mathfrak{A})'_w\mathscr{D}(\pi_{qw}) = \mathscr{D}(\pi_{qw})$.

Remarks. (1) Lemma 2.2 shows the usefulness of the commutant $\mathfrak{C}_{qw}(\pi)$. Indeed, suppose instead that $\pi(\mathfrak{A})'_w$ is an algebra and define:

$$\mathscr{D}(\pi_w) = \{\textstyle\sum_{k=1}^n C_k\xi_k; C_k \in \pi(\mathfrak{A})'_w, \xi_k \in \mathscr{D}(\pi), k = 1, \dots, n, n \in \mathbb{N}\}$$

$$\pi_w(x)(\textstyle\sum_{k=1}^n C_k\xi_k) = \sum_{k=1}^n C_k\pi(x)\xi_k, \; x \in \mathfrak{A}, \; \sum_{k=1}^n C_k\xi_k \in \mathscr{D}(\pi_w).$$

Then π_w is not necessarily a $*$-representation of $\mathfrak{A}$ (the situation is slightly better for strong $*$-representations).

(2) Partial $*$-algebras may be tricky. For instance, let π be a $*$-representation of a partial *$*$-algeba* $\mathfrak{A}$. If $\mathfrak{A}$ is a $*$-algebra, then $\pi(\mathfrak{A})$ is an Op^*-algebra, but when $\mathfrak{A}$ is only a partial $*$-algebra, $\pi(\mathfrak{A})$ is not necessarily a partial Op^*-algebra, and we have to consider the minimal partial Op*-algebra $\mathfrak{M}_w[\pi(\mathfrak{A})]$ containing $\pi(\mathfrak{A})$, as defined in I.

The last notion we want to introduce is that of irreducibility. Let $\mathfrak{A}$ be a partial $*$-algebra, π a $*$-representation of $\mathfrak{A}$ in $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$. Then we will say that π is *irreducible* iff its bounded quasi-weak commutant $\mathfrak{C}_{qw}(\pi)$ of π contains only multiples of the identity, $\mathfrak{C}_{qw}(\pi) = \{\lambda I, \lambda \in \boldsymbol{C}\}$. This definition is admittedly conservative, but for GNS representations it leads to the expected correspondence between pure states and irreducible representations (see Section 4). The same result does not hold, in general, if we replace the quasi-weak commutant $\mathfrak{C}_{qw}(\pi)$ by the weak bounded commutant $\pi(\mathfrak{A})'_w$, and a fortiori by the weak *unbounded* one $\pi(\mathfrak{A})'_\sigma$.

Let us recall here, for future use, that the weak unbounded commutant $\mathfrak{N}'_\sigma$ of a †-invariant subset $\mathfrak{N}$ of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ is defined [15, 16] as :

$$\mathfrak{N}'_\sigma = \{Y \in \mathscr{L}^\dagger(\mathscr{D}, \mathscr{H});\ (X\xi | Y\eta) = (Y^\dagger\xi | X^\dagger\eta), \quad \text{for all } \xi, \eta \in \mathscr{D} \text{ and } X \in \mathfrak{N}\}.$$

It is a weakly closed, †-invariant subspace of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$, and its bounded part equals $\mathfrak{N}'_w | \mathscr{D}$.

§3. Invariant Positive Sesquilinear Forms on Partial $*$-Algebras

As always [3], the crucial question is how to build concrete representations. For $*$-algebras, the GNS construction is usually the answer. In order to extend it to partial $*$-algebras, we introduce a notion of invariant sesquilinear form for which the GNS construction is always possible, and we give some examples. This concept is slightly more general than an earlier one of h-form [4, 5].

Let $\mathfrak{A}$ be a partial $*$-algebra. A sesquilinear form φ on $\mathfrak{A} \times \mathfrak{A}$ is called *positive if* $\varphi(x, x) \geq 0$, $\forall x \in \mathfrak{A}$. When $\mathfrak{A}$ has a unit e, a positive sesquilinear form φ on $\mathfrak{A} \times \mathfrak{A}$ is called a *state* if $\varphi(e, e) = 1$. For each positive sesquilinear form φ on $\mathfrak{A} \times \mathfrak{A}$, we have

$$\varphi(x, y) = \overline{\varphi(y, x)}, \quad \forall x, y \in \mathfrak{A} \tag{3.1}$$

$$|\varphi(x, y)|^2 \leq \varphi(x, x)\, \varphi(y, y), \quad \forall x, y \in \mathfrak{A} \tag{3.2}$$

and hence

$$\mathfrak{N}_\varphi \equiv \{x \in \mathfrak{A};\ \varphi(x, x) = 0\} = \{x \in \mathfrak{A};\ \varphi(x, y) = 0 \quad \textit{for all } y \in \mathfrak{A}\},$$

and so $\mathfrak{N}_\varphi$ is a subspace of $\mathfrak{A}$. For each $x\in\mathfrak{A}$ we denote by $\lambda_\varphi(x)$ the coset of $\mathfrak{A}/\mathfrak{N}_\varphi$ which contains x, and define an inner product $(.|.)$ on $\lambda_\varphi(\mathfrak{A})$ by

$$(\lambda_\varphi(x)|\lambda_\varphi(y))=\varphi(x, y), \quad x, y\in\mathfrak{A}.$$

We denote by $\mathscr{H}_\varphi$ the Hilbert space obtained by the completion of the pre-Hilbert space $\lambda_\varphi(\mathfrak{A})$. We are now ready to introduce our notion of invariance.

Definition 3.1. Let φ be a positive sesquilinear form on $\mathfrak{A}\times\mathfrak{A}$. If there exists a subspace $\mathfrak{B}$ of $R(\mathfrak{A})$ such that:

(1) $\lambda_\varphi(\mathfrak{B})$ is dense in $\mathscr{H}_\varphi$;

(2) $\varphi(xb_1, b_2)=\varphi(b_1, x^*b_2)$, $\forall x\in\mathfrak{A}$, $\forall b_1, b_2\in\mathfrak{B}$;

(3) $\varphi(x_1^*b_1, x_2b_2)=\varphi(b_1, (x_1x_2)b_2)$, $\forall x_1\in L(x_2)$, $\forall b_1, b_2\in\mathfrak{B}$;

(4) if $\mathfrak{A}$ has a unit e, then $e\in\mathfrak{B}$,

then φ is said to be $\mathfrak{B}$-invariant. If φ is $R(\mathfrak{A})$-invariant, it is simply called invariant.

We denote by $\mathscr{F}_\varphi$ the family of subspaces $\mathfrak{B}$ satisfying the conditions (1)–(4) of Definition 3.1. Given $\mathfrak{B}\in\mathscr{F}_\varphi$, there exists a maximal subspace in $\mathscr{F}_\varphi$ containing $\mathfrak{B}$; we denote it by $[\mathfrak{B}]$.

We will show below that the GNS-construction is possible for every $\mathfrak{B}$-invariant positive sesquilinear form on $\mathfrak{A}\times\mathfrak{A}$. Before that, let us comment briefly the definition of invariance just given. As compared to the earlier notion of h-form [4, 5], two new aspects are crucial:

(*i*) *Condition* (3) *takes explicitly into account the possible lack of* (*semi*)-*associativity of* $\mathfrak{A}$;

(*ii*) *Conditions* (1)–(4) *are not imposed to the whole set* $R(\mathfrak{A})$, *but only to the dense subspace* $\mathfrak{B}$. *The reason is that* $R(\mathfrak{A})$ *may be too large or difficult to characterize completely, whereas it is often easy to find a suitable subspace* $\mathfrak{B}$.

These two modifications together make Definition 3.1 more flexible than the earlier one of h-form. We will illustrate this by concrete examples at the end of this section.

Proposition 3.2. *Let* φ *be a positive sesquilinear form on* $\mathfrak{A}\times\mathfrak{A}$. *Then the following statements hold.*

(1) *Suppose* φ *is* $\mathfrak{B}$-*invariant. Put*

$$\pi_\varphi^{\mathfrak{B}}(x)\lambda_\varphi(b)=\lambda_\varphi(xb), \quad x\in\mathfrak{A}, \quad b\in\mathfrak{B}.$$

Then $\pi_\varphi^{\mathfrak{B}}$ *is a* $*$-*representation of* $\mathfrak{A}$ *into* $\mathscr{L}_w^\dagger(\lambda_\varphi(\mathfrak{B}), \mathscr{H}_\varphi)$. *If* $\mathfrak{A}$ *has a unit e, the vector* $\Omega_\varphi=\lambda_\varphi(e)$ *is cyclic for* $\pi_\varphi^{\mathfrak{B}}$. *If* φ *is invariant, we denote* $\pi_\varphi^{R(\mathfrak{A})}$ *simply by* π_φ.

(2) $\pi_\varphi^{\mathfrak{B}}\subset\pi_\varphi^{[\mathfrak{B}]}$ *for each* $\mathfrak{B}\in\mathscr{F}_\varphi$ *and* $\pi_\varphi^{\mathfrak{B}}\neq\pi_\varphi^{[\mathfrak{B}]}$ *in general.*

(3) *Let* $\mathfrak{B}_1$, $\mathfrak{B}_2\in\mathscr{F}_\varphi$. *Then* $[\mathfrak{B}_1]\neq[\mathfrak{B}_2]$ *if and only if* $\widehat{\pi_\varphi^{[\mathfrak{B}_1]}}\neq\widehat{\pi_\varphi^{[\mathfrak{B}_2]}}$.

Proof. (1) It follows from (2) in Definition 3.1 that $\pi_\varphi^{\mathfrak{B}}(x)\in\mathscr{L}^\dagger(\lambda_\varphi(\mathfrak{B}), \mathscr{H}_\varphi)$

and $\pi_\varphi^{\mathfrak{B}}(x)^\dagger \equiv \pi_\varphi^{\mathfrak{B}}(x)^* \upharpoonright \lambda_\varphi(\mathfrak{B}) = \pi_\varphi^{\mathfrak{B}}(x^*)$ for each $x \in \mathfrak{A}$. It is clear that $\pi_\varphi^{\mathfrak{B}}$ is a linear *-invariant map of $\mathfrak{A}$ into $\mathscr{L}_w^\dagger(\lambda_\varphi(\mathfrak{B}), \mathscr{H}_\varphi)$. Furthermore, by (3) in Definition 3.1, we have

$$(\pi_\varphi^{\mathfrak{B}}(x_1)^\dagger \lambda_\varphi(b_1) | \pi_\varphi^{\mathfrak{B}}(x_2) \lambda_\varphi(b_2)) = \varphi(x_1^* b_1, x_2 b_2) = (\lambda_\varphi(b_1) | \pi_\varphi^{\mathfrak{B}}(x_1 x_2) \lambda_\varphi(b_2))$$

$$(\pi_\varphi^{\mathfrak{B}}(x_2) \lambda_\varphi(b_1) | \pi_\varphi^{\mathfrak{B}}(x_1)^\dagger \lambda_\varphi(b_2)) = (\lambda_\varphi(b_1) | \pi_\varphi^{\mathfrak{B}}((x_1 x_2)^*) \lambda_\varphi(b_2)),$$

for each $x_1 \in L(x_2)$ and $b_1, b_2 \in \mathfrak{B}$. Hence, $\pi_\varphi^{\mathfrak{B}}(x_1) \in L^w(\pi_\varphi^{\mathfrak{B}}(x_2))$ and $\pi_\varphi^{\mathfrak{B}}(x_1) \,\square\, \pi_\varphi^{\mathfrak{B}}(x_2) = \pi_\varphi^{\mathfrak{B}}(x_1 x_2)$ for each $x_1 \in L(x_2)$. Thus $\pi_\varphi^{\mathfrak{B}}$ is a *-representation of $\mathfrak{A}$ into $\mathscr{L}_w^\dagger(\lambda_\varphi(\mathfrak{B}), \mathscr{H}_\varphi)$.

(2) The inclusion $\widehat{\pi_\varphi^{\mathfrak{B}}} \subset \widehat{\pi_\varphi^{[\mathfrak{B}]}}$ is obvious, and the example given in Corollary 3.8 (3) below shows that $\widehat{\pi_\varphi^{\mathfrak{B}}} \neq \widehat{\pi_\varphi^{[\mathfrak{B}]}}$ in general.

(3) Suppose that $\widehat{\pi_\varphi^{[\mathfrak{B}_1]}} = \widehat{\pi_\varphi^{[\mathfrak{B}_2]}}$. For every $b \in [\mathfrak{B}_2]$ and $x_1 \in \mathfrak{A}$, there exists a sequence $\{a_n\}$ in $[\mathfrak{B}_1]$ such that $\lim_{n\to\infty} \lambda_\varphi(a_n) = \lambda_\varphi(b)$ and $\lim_{n\to\infty} \lambda_\varphi(x_1^* a_n) = \lambda_\varphi(x_1^* b)$. Then we have

$$\begin{aligned} \varphi(x_1^* b, x_2 a) &= \lim_{n\to\infty} (\lambda_\varphi(x_1^* a_n) | \lambda_\varphi(x_2 a)) \\ &= \lim_{n\to\infty} \varphi(x_1^* a_n, x_2 a) \\ &= \lim_{n\to\infty} \varphi(a_n, (x_1 x_2) a) \\ &= \varphi(b, (x_1 x_2) a) \end{aligned}$$

for every $x_2 \in R(x_1)$ and $a \in [\mathfrak{B}_1]$. Similarly,

$$\varphi(x_1^* a, x_2 b) = \varphi(a, (x_1 x_2) b)$$

for every $x_1 \in L(x_2)$, $a \in [\mathfrak{B}_1]$ and $b \in [\mathfrak{B}_2]$. Hence, $[\mathfrak{B}_1] + \boldsymbol{C} b \in \mathscr{F}_\varphi$. By the maximality of $[\mathfrak{B}_1]$, we have $b \in [\mathfrak{B}_1]$. Therefore, $[\mathfrak{B}_1] = [\mathfrak{B}_2]$. This completes the proof.

We call the triple $(\pi_\varphi^{\mathfrak{B}}, \lambda_\varphi, \mathscr{H}_\varphi)$ the *GNS-construction* for φ, based on $\mathfrak{B}$. When φ is invariant, the triple $(\pi_\varphi, \lambda_\varphi, \mathscr{H}_\varphi)$ is called simply the GNS-construction for φ. If $\mathscr{F}_\varphi \neq \emptyset$, then φ is said to be *GNS-representable.*

By construction, the vector $\Omega_\varphi = \lambda_\varphi(e)$ is cyclic for $\pi_\varphi^{\mathfrak{B}}$, and it is even strongly cyclic. This concept, familiar for Op*-algebras [2], may be adapted to the present situation, but as expected it ramifies into several different notions (see also [5]). These are studied in detail in a separate paper [17], together with the applications to the GNS construction.

We assume now that $\mathfrak{A}$ has a unit e, and consider a $\mathfrak{B}$-invariant state φ on $\mathfrak{A} \times \mathfrak{A}$. As usual, we say that the state φ is *pure* if it cannot be written as a convex combination of two $\mathfrak{B}$-invariant states φ_1, φ_2:

$$\varphi \neq \lambda \varphi_1 + (1-\lambda)\varphi_2, \ 0 < \lambda < 1.$$

The interest of this concept is that the equivalence between the purity of a state φ and the irreducibility of its GNS representation $\pi_\varphi^{\mathfrak{B}}$ extends to partial $*$-algebras.

Proposition 3.3. *Let $\mathfrak{A}$ be a partial $*$-algebra with unit and φ a $\mathfrak{B}$-invariant state on $\mathfrak{A} \times \mathfrak{A}$. Then the GNS representation $\pi_\varphi^{\mathfrak{B}}$ is irreducible, in the sense that $\mathfrak{C}_{qw}(\pi_\varphi^{\mathfrak{B}}) = CI$, if and only if φ is pure.*

Proof. We follow essentially Powers [2]. Assume that φ is not pure: $\varphi = \lambda \varphi_1 + (1-\lambda)\varphi_2$, $0 < \lambda < 1$, $\varphi_1 \neq \varphi_2$. On the dense domain $\lambda_\varphi(\mathfrak{B})$ consider the sesquilinear form

$$(\lambda_\varphi(a) \mid \lambda_\varphi(b))^{(1)} = \lambda \varphi_1(a, b), \quad a, b \in \mathfrak{B}.$$

Since

$$\lambda \varphi_1(a, a) \leq \varphi(a, a) = \| \lambda_\varphi(a) \|^2, \quad a \in \mathfrak{B},$$

this form is bounded. Thus it extends to the whole space $\mathscr{H}_\varphi$ and defines a unique bounded operator A, $0 < A < I$, $A \neq \lambda I$, such that

$$(\lambda_\varphi(a) \mid \lambda_\varphi(b))^{(1)} = (\lambda_\varphi(a) \mid A \lambda_\varphi(b)).$$

Now φ_1 is a state, in particular it is $\mathfrak{B}$-invariant; this implies, by condition (2) of Definition 3.1, that $A \in \mathfrak{C}_{qw}(\pi_\varphi^{\mathfrak{B}})$.

Conversely, suppose $\mathfrak{C}_{qw}(\pi_\varphi^{\mathfrak{B}}) \neq CI$. Although $\mathfrak{C}_{qw}(\pi_\varphi^{\mathfrak{B}})$ is not an algebra, it is easily shown [2] that there exists an element $B \in \mathfrak{C}_{qw}(\pi_\varphi^{\mathfrak{B}})$ such that $0 < B < I$. Then $(\Omega_\varphi \mid B\Omega_\varphi) > 0$, where $\Omega_\varphi = \lambda_\varphi(e)$ is cyclic for $\pi_\varphi^{\mathfrak{B}}$. Otherwise, we would have, for all $a, b \in \mathfrak{B}$:

$$\begin{aligned}
|(\lambda_\varphi(a) \mid B \lambda_\varphi(b))| &= |(\lambda_\varphi(a) \mid B \pi_\varphi^{\mathfrak{B}}(b) \Omega_\varphi)| \\
&= |(\pi_\varphi^{\mathfrak{B}}(b^*) \lambda_\varphi(a) \mid B \Omega_\varphi)| \\
&= |(\pi_\varphi^{\mathfrak{B}}(b^* a) \Omega_\varphi \mid B \Omega_\varphi)| \\
&= |(B^{1/2} \pi_\varphi^{\mathfrak{B}}(b^* a) \Omega_\varphi \mid B^{1/2} \Omega_\varphi)| \\
&\leq \| B^{1/2} \lambda_\varphi(b^* a) \| . \ (\Omega_\varphi \mid B \Omega_\varphi) = 0,
\end{aligned}$$

and this would imply $B = 0$. Similarly $(\Omega_\varphi \mid (1-B)\Omega_\varphi) > 0$. We define now two sesquilinear forms on $\mathfrak{A} \times \mathfrak{A}$:

$$\varphi_1(x, y) = (\lambda_\varphi(x) \mid B \lambda_\varphi(y)) . (\Omega_\varphi \mid B \Omega_\varphi)^{-1},$$

$$\varphi_2(x, y) = (\lambda_\varphi(x) \mid (1-B) \lambda_\varphi(y)) . (\Omega_\varphi \mid (1-B) \Omega_\varphi)^{-1}.$$

These two forms are positive and $\mathfrak{B}$-invariant; for instance:

$$\begin{aligned}\varphi_1(xb_1, b_2) &= (\pi_\varphi^{\mathfrak{B}}(x)\lambda_\varphi(b_1) \mid B\lambda_\varphi(b_2)).(\Omega_\varphi \mid B\Omega_\varphi)^{-1} \\ &= (\lambda_\varphi(b_1) \mid B\pi_\varphi^{\mathfrak{B}}(x^*)\lambda_\varphi(b_2)).(\Omega_\varphi \mid B\Omega_\varphi)^{-1} \\ &= \varphi_1(b_1, x^*b_2).\end{aligned}$$

Similarly, for $x_1 \in L(x_2)$:

$$\begin{aligned}\varphi_1(x_1^*b_1, x_2b_2) &= (\pi_\varphi^{\mathfrak{B}}(x_1^*)\lambda_\varphi(b_1) \mid B\pi_\varphi^{\mathfrak{B}}(x_2)\lambda_\varphi(b_2)).(\Omega_\varphi \mid B\Omega_\varphi)^{-1} \\ &= (\lambda_\varphi(b_1) \mid B\pi_\varphi^{\mathfrak{B}}(x_1x_2)\lambda_\varphi(b_2)).(\Omega_\varphi \mid B\Omega_\varphi)^{-1} \\ &= \varphi_1(b_1, (x_1x_2)b_2).\end{aligned}$$

Finally, φ_1 and φ_2 are states and $\varphi = \lambda\varphi_1 + (1-\lambda)\varphi_2$, with $\lambda = (\Omega_\varphi \mid B\Omega_\varphi)$, which concludes the proof.

Notice that, if we define the irreducibility of $\pi_\varphi^{\mathfrak{B}}$ by the condition $\pi_\varphi^{\mathfrak{B}}(\mathfrak{A})'_w = \boldsymbol{C}I$, then the positive sesquilinear form φ_1 on $\mathfrak{A} \times \mathfrak{A}$ defined in the proof of Proposition 3.3 need not verify the equality:

$$\varphi_1(x_1^*b_1, x_2b_2) = \varphi_1(b_1, (x_1x_2)b_2),$$

so that φ_1 is not $\mathfrak{B}$-invariant. This also shows that, for a partial $*$-algebra, the quasi-weak commutant $\mathfrak{C}_{qw}(\pi)$ of a $*$-representation π plays more important role than its weak commutant $\pi(\mathfrak{A})'_w$.

Let us given now some examples of invariant positive sesquilinear forms on partial $*$-algebras.

(i) *Invariant positive sesquilinear forms on topological quasi $*$-algebras*: Let $\mathfrak{A}$ be a topological quasi $*$-algebra [7]; that is, $\mathfrak{A}$ is the completion $[\mathfrak{A}_0]^\tau$ of a topological $*$-algebra $\mathfrak{A}_0$ with topology τ. We denote by $P(\mathfrak{A}_0, \tau\times\tau)$ the set of all positive linear functionals f on $\mathfrak{A}_0$ such that $(a, b) \to f(b^*a) \in \boldsymbol{C}$ is continuous on $\mathfrak{A}_0 \times \mathfrak{A}_0$, and denote by $IP(\mathfrak{A}\times\mathfrak{A}, \tau\times\tau)$ the set of all (jointly) continuous invariant positive sesquilinear forms on $\mathfrak{A}\times\mathfrak{A}$. For each $f \in P(\mathfrak{A}_0, \tau\times\tau)$ we put:

$$\overline{f^0}(x, y) = \lim_{\alpha,\beta} f(b_\beta^* a_\alpha), \quad x, y \in \mathfrak{A},$$

where $\{a_\alpha\}$ and $\{b_\beta\}$ are nets in $\mathfrak{A}_0$ such that $a_\alpha \xrightarrow[\tau]{} x$ and $b_\beta \xrightarrow[\tau]{} y$. Then the map $f \to \overline{f^0}$ is an injection of $P(\mathfrak{A}_0, \tau\times\tau)$ into $IP(\mathfrak{A}\times\mathfrak{A}, \tau\times\tau)$; if $\mathfrak{A}$ has a unit, then the map is a bijection.

Let us give a concrete example [5]. Take $\mathfrak{A}_0 = C^\circ(\Delta)$, the space of continuous functions on a finite interval $\Delta \subset \boldsymbol{R}$, with pointwise multiplication and an L^p-norm $(1 \le p < \infty)$. Then $\mathfrak{A} = L^p(\Delta, dt)$ is an abelian topological quasi-$*$-algebra, and therefore $L(\mathfrak{A}) = R(\mathfrak{A}) = \mathfrak{A}_0$. It is semi-associative, but not associative [18]. If $2 < p < \infty$, any positive function $\rho \in L^{p/p-2}(\Delta, dt)$

defines a positive sesquilinear form φ_ρ on $\mathfrak{A}$ by the following relation:

$$\varphi_\rho(f, g) = \int_\Delta f(t)\overline{g(t)}\ \rho(t)\,dt,$$

$\left(\text{and a state if } \int_\Delta \rho(t)\,dt = 1\right)$. The form φ_ρ is jointly continuous in the L^p-norm of $\mathfrak{A}$. Furthermore, $\mathfrak{N}_{\varphi_\rho} = \{0\}$, $\mathscr{H}_{\varphi_\rho} = L^2(\Delta, \rho dt)$ and the form φ_ρ is invariant. Hence we may build the GNS representation: $\pi_{\varphi_\rho}(f)$ is simply the operator of multiplication by $f \in L^p$ on the domain $\mathscr{D}(\pi_{\varphi_\rho}) = C^\circ(\Delta)$, and so is its full closure $\widehat{\pi_{\varphi_\rho}}(f)$ on the domain $\mathscr{D}(\widehat{\pi_{\varphi_\rho}}) = L^\infty(\Delta, \rho dt)$. This representation is of course highly reducible. Its commutants [1, 19] consist also of multiplication operators, namely:

$$[\pi_{\varphi_\rho}(\mathfrak{A})]'_\sigma = L^p(\Delta, \rho dt),$$

$$[\pi_{\varphi_\rho}(\mathfrak{A})]'_\square = [\pi_{\varphi_\rho}(\mathfrak{A})]'_{\square b} = [\pi_{\varphi_\rho}(\mathfrak{A})]'_w = \mathfrak{C}_{qw}(\pi_{\varphi_\rho}) = L^\infty(\Delta, \rho dt).$$

(ii) *Invariant positive sesquilinear forms on partial Op*-algebras*: Let $\mathfrak{M}$ be an Op*-algebra on $\mathscr{D}$. A *vector form* on $\mathfrak{M}$ is a positive sesquilinear form $\sum_{k=1}^n \omega_{\xi_k}^o$ on $\mathfrak{M} \times \mathfrak{M}$ of the following type:

$$(\textstyle\sum_{k=1}^n \omega_{\xi_k}^o)(X, Y) = \sum_{k=1}^n \omega_{\xi_k}^o(X, Y) = \sum_{k=1}^n (X\xi_k | Y\xi_k),\ X, Y \in \mathfrak{M},\ \{\xi_k\} \subset \mathscr{D},$$

and every such form is invariant. However, this need not be the case if $(\mathfrak{M}, \mathscr{D})$ is a partial Op*-algebra. So the question arises: *when is a vector form on a partial Op*-algebra invariant*? Let $\mathfrak{M}$ be a partial Op*-algebra on $\mathscr{D}$, and $R^w(\mathfrak{M})^\xi$ $(\xi \in \mathscr{D})$, $R^w(\mathfrak{M})^{\mathscr{D}}$ the sets defined in Section 2. We remark that if $\mathfrak{M}$ is self-adjoint, then $R^w(\mathfrak{M})^{\mathscr{D}} = R^w(\mathfrak{M})$. The following result is easy:

Proposition 3.4. *Let $\mathfrak{M}$ be a partial Op*-algebra on $\mathscr{D}$. Then the following statements hold.*

(1) *Let ξ be an element of $\mathscr{D}$ such that $\overline{R^w(\mathfrak{M})^\xi \xi} = \overline{\mathfrak{M}\xi}$. Then ω_ξ^0 is an $R^w(\mathfrak{M})^\xi$-invariant positive sesquilinear form on $\mathfrak{M} \times \mathfrak{M}$. In particular, if $\mathfrak{M}$ is algebraically self-adjoint, then ω_ξ^0 is invariant.*

(2) *Suppose $R^w(\mathfrak{M})^{\mathscr{D}}$ is t_s-dense in $\mathfrak{M}$. Then every vector form on $\mathfrak{M} \times \mathfrak{M}$ is $R^w(\mathfrak{M})^{\mathscr{D}}$-invariant. In particular, if $\mathfrak{M}$ is algebraically self-adjoint, then ω_ξ^0 is invariant.*

However, this general result is not very constructive, since it is often difficult in practice to characterize the whole set $R^w(\mathfrak{M})$ for an arbitrary partial Op*-algebra $\mathfrak{M}$. But the case of quasi-*algebra discussed in (i) above gives us a hint. In that case it was easy to obtain invariant positive sesquilinear forms on $\mathfrak{A}$ by taking limits of positive linear forms on $\mathfrak{A}_0$, which is dense in $\mathfrak{A}$. This is the key fact: what we need is a class of partial Op*-algebras containing a subset of bounded operators, dense in a suitable topology. The most natural choice for the latter is the strong*-topology t_{s^*}. Then the first question is whether there are partial Op*-algebras which are topological quasi-*algebras for t_{s^*}. We will examine this point in Section 4 below, but the answer is not very encouraging. The next step is to find a type of partial Op*-algebras with the required properties. It turns out that the following definition yields a useful class.

Definition 3.5. Let $\mathscr{D}$ be a dense subspace of $\mathscr{H}$ and $\mathfrak{M}_0$ a von Neumann algebra such that $\mathfrak{M}'_0\mathscr{D} = \mathscr{D}$. Then a weak partial Op*-algebra $\mathfrak{M}$ on $\mathscr{D}$ is called a partial GW*-algebra over $\mathfrak{M}_0$ if it is fully closed and equals the t_{s^*}-closure $[\mathfrak{M}_0 \upharpoonright \mathscr{D}]^{s^*}$ of $\mathfrak{M}_0|\mathscr{D}$ in $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$.

We will study those partial GW*-algebras in detail in Section 4, and in particular we will give another, equivalent, definition. The simplest nontrivial example is the following. Let $\mathscr{D}$ be such that $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ is fully closed. Then $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$ is a partial GW*-algebra over $\mathscr{B}(\mathscr{H})$, since $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}) = [\mathscr{B}(\mathscr{H}) \upharpoonright \mathscr{D}]^{s^*}$.

We first investigate under what conditions a positive sesquilinear form on a partial GW*-algebra is a vector form. The following results are proved as in Ref. 9.

Lemma 3.6. *Let $\mathfrak{M}$ be a partial GW*-algebra on $\mathscr{D}$ over $\mathfrak{M}_0$ and φ be a positive sesquilinear form on $\mathfrak{M} \times \mathfrak{M}$. Then,*

(1) *The following statements are equivalent:*

$(1)_1$ *φ is a vector form on $\mathfrak{M} \times \mathfrak{M}$.*

$(1)_2$ *φ is t_s-continuous and $\varphi \upharpoonright \mathfrak{M}_0 \times \mathfrak{M}_0$ is an invariant positive sesquilinear form on $\mathfrak{M}_0 \times \mathfrak{M}_0$.*

$(1)_3$ *There exists finite subsets $\{\xi_k\}$ and $\{\eta_k\}$ of $\mathscr{D}$ such that $|\varphi(X, Y)| \leq |\sum_{k=1}^n (X\xi_k|Y\eta_k)|$ for all $X, Y \in \mathfrak{M}$ and $\varphi|\mathfrak{M}_0 \times \mathfrak{M}_0$ is invariant.*

$(1)_4$ *$\varphi(X, Y) = \overline{f^0}(X, Y) \equiv \lim_{\alpha,\beta} f(B_\beta^* A_\alpha)$, $X, Y \in \mathfrak{M}$ for some t_w-continuous positive linear functional f on $\mathfrak{M}_0$, where $\{A_\alpha\}$ and $\{B_\beta\}$ are nets in $\mathfrak{M}_0$ such that $A_\alpha \to X$ and $B_\beta \to Y$ in t_{s^*}.*

(2) *The following statements are equivalent:*

$(2)_1$ *$\varphi = \sum_{n=1}^\infty \omega^0_{\xi_n}$ for some $\{\xi_n\} \in \mathscr{D}^\infty(\mathfrak{M})$.*

$(2)_2$ *φ is $t^{\mathfrak{M}}_{\sigma s}$-continuous and $\varphi|\mathfrak{M}_0 \times \mathfrak{M}_0$ is invariant.*

$(2)_3$ *There exists elements $\{\xi_n\}, \{\eta_n\}$ of $\mathscr{D}^\infty(\mathfrak{M})$ such that $|\varphi(X, Y)| \leq |\sum_{n=1}^\infty (X\xi_n|Y\eta_n)|$ for all $X, Y \in \mathfrak{M}$ and $\varphi|\mathfrak{M}_0 \times \mathfrak{M}_0$ is invariant.*

$(2)_4$ *$\varphi(X, Y) = \overline{f^0}(X, Y) \equiv \lim_{\alpha,\beta} f(B_\beta^* A_\alpha)$, $X, Y \in \mathfrak{M}$ for some $t^{\mathfrak{M}}_{\sigma w}$-continuous positive linear functional f on $\mathfrak{M}_0$, where $\{A_\alpha\}$ and $\{B_\beta\}$ are nets in $\mathfrak{M}_0$ such that $A_\alpha \to X$ and $B_\beta \to Y$ in $t^{\mathfrak{M}}_{\sigma s^*}$.*

In this case, if $\mathfrak{M}_0$ has a separating vector, then $\varphi = \omega^0_\xi$ for some $\xi \in \mathscr{D}$ and if $\mathfrak{M}_0$ has a cyclic and separating vector ξ_0, then $\varphi = \omega^0_{\xi_\varphi}$ for a unique vector ξ_φ in $\mathscr{P}^\sharp_{\xi_0} \cap \mathscr{D}$, where $\mathscr{P}^\sharp_{\xi_0}$ is the natural positive cone associated with the achieved left

Hilbert algebra $\mathfrak{M}_0\xi_0$[20].

Proof.

- $(1)_1 \Rightarrow (1)_3$: This is trivial.
- $(1)_3 \Rightarrow (1)_4$: Define $f(A)=\varphi(A, I)$, $A\in\mathfrak{M}_0$. By $(1)_3$, f *is a* t_w-continuous positive linear functional on $\mathfrak{M}_0$; then the relation $\varphi=\overline{f^0}$ is immediate.
- $(1)_4 \Rightarrow (1)_2$: Obvious.
- $(1)_2 \Rightarrow (1)_1$: Since φ is t_s-continuous, it follows that

$$|\varphi(X, Y)| \leq (\textstyle\sum_{k=1}^{l} \|X\xi_k\|)(\sum_{j=1}^{m} \|Y\eta_j\|), \quad X, Y\in\mathfrak{M} \tag{3.4}$$

for some $\{\xi_k\}$, $\{\eta_k\}\subset\mathscr{D}$. Then

$$\|\lambda_\varphi(X)\| \leq \textstyle\sum_{k=1}^{n} \|X\xi_k\|, \quad X\in\mathfrak{M} \tag{3.5}$$

for some $\{\xi_k\}\subset\mathscr{D}$. We first show the statement $(1)_1$ in the case where

$$\|\lambda_\varphi(X)\| \leq \|X\zeta\|, \quad X\in\mathfrak{M}$$

for some $\zeta\in\mathscr{D}$. Then it is easily shown that $C=(KP_\zeta)^*(KP_\zeta)\in\mathfrak{M}_0'$, where $P_\zeta = \mathrm{Proj}\,\overline{\mathfrak{M}_0\zeta}$ and K is a bounded linear map of $P_\zeta\mathscr{H}$ onto $\mathscr{H}_\varphi$ which is an extension of the map $A\zeta\to\lambda_\varphi(A)\in\lambda_\varphi(\mathfrak{M}_0)$, and

$$\varphi(A, B)=(AC^{1/2}\zeta\,|\,BC^{1/2}\zeta), \quad A, B\in\mathfrak{M}_0.$$

Since $\mathfrak{M}_0'\mathscr{D}=\mathscr{D}$, we have

$$\varphi(A, B)=(A\xi\,|\,B\xi), \quad A, B\in\mathfrak{M}_0$$

for some $\xi\in\mathscr{D}$. Considering

$$[\mathfrak{M}_0]_n=\{[A]=\begin{bmatrix} A & 0 & \dots \\ 0 & A & \dots \\ \dots & \dots & A \end{bmatrix}; A\in\mathfrak{M}_0\},$$

$$\zeta=\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \dots \\ \zeta_n \end{bmatrix},$$

$$[\varphi]([A],[B])=\varphi(A, B), \quad A, B\in\mathfrak{M}_0$$

in the general case (3.5), we can prove

$$\varphi(A, B)=\textstyle\sum_{k=1}^{n}(A\xi_k\,|\,B\xi_k), \quad A, B\in\mathfrak{M}_0$$

for some $\{\xi_k\}\subset\mathscr{D}$, which implies by (3.4) and $[\mathfrak{M}_0]^{s^*}=\mathfrak{M}$ that $\varphi=\sum_{k=1}^{n}\omega_{\xi_k}^0$ for $\{\xi_k\}\subset\mathscr{D}$. This proves the equivalence of $(1)_1$-$(1)_4$.

A similar proof yields the equivalence of the statements $(2)_1$-$(2)_4$, replacing $\mathfrak{M}$, ζ by

$$[\mathfrak{M}]_\infty = \{[X] = \begin{bmatrix} X & 0 & \dots\dots \\ 0 & X & 0 \dots \\ \dots & 0 & X \quad 0 \\ \dots & \dots & \dots\dots \end{bmatrix}; X \in \mathfrak{M}\},$$

$$\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \dots \end{bmatrix} \in \mathscr{D}^\infty(\mathfrak{M}).$$

The remaining assertions are proved in the same way as in Ref. 9.

The following concrete example [5] illustrates the situation described in Lemma 3.6. Take a Hilbert space $\mathscr{H}$ and a self-adjoint operator H in $\mathscr{H}$ such that $\exp(-\beta H)$ is nuclear for every $\beta > 0$. Let $\{\phi_n, \lambda_n\}$ be the eigenvectors and the corresponding eigenvalues of H. Define

$$\mathscr{D} = \mathscr{D}^\infty(e^{\beta H}) = \textstyle\bigcap_{\beta > 0} \mathscr{D}(e^{\beta H}).$$

Then $\mathfrak{M} = \mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$ is self-adjoint, hence it is a partial GW*-algebra over $\mathscr{B}(\mathscr{H})$. Consider now the following positive sesquilinear form over $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$:

$$\varphi_\beta(A, B) = Z^{-1} \textstyle\sum_{n=1}^\infty (A\phi_n | B\phi_n) e^{-\beta\lambda_n}, \quad Z = \mathrm{Tr}(e^{-\beta H}).$$

The restriction of φ_β to $\mathscr{B}(\mathscr{H})$ is the familiar state $Z^{-1}\mathrm{Tr}(A^*B\, e^{-\beta H})$. Clearly this form verifies all the conditions of statement (2) of Lemma 3.6, in particular it is invariant. It is a h-form in the sense of [4, 5], but it is not a vector form, since $\mathfrak{M}_0 = \mathscr{B}(\mathscr{H})$ has no separating vector. This form φ_β was used by Bouziane and Martin [21] in their proof of the Bogoliubov inequality for unbounded operators.

Let us come back to the general discussion. With help of Lemma 3.6, we obtain now a characterization of vector forms on partial GW*-algebras which illustrates the parallelism of the latter with topological quasi-*algebras. Indeed, using the von Neumann density theorem, Proposition 3.4 (2) and Lemma 3.6, we may prove the following

Proposition 3.7. *Let $\mathfrak{M}$ be a partial GW*-algebra on $\mathscr{D}$ over $\mathfrak{M}_0$ such that $\mathfrak{M}_0^{\mathscr{D}} \cap (\mathfrak{M}_0^{\mathscr{D}})^* = \mathfrak{M}_0 \cap \mathscr{L}^\dagger(\mathscr{D})$ is a nondegenerate *-subalgebra of $\mathfrak{M}_0$. Then the following statements are equivalent:*

(1) *φ is a t_s-continuous, $(\mathfrak{M}_0 \cap \mathscr{L}^\dagger(\mathscr{D}))$-invariant, positive sesquilinear form on $\mathfrak{M} \times \mathfrak{M}$.*

(2) *φ is a t_s-continuous positive sesquilinear form on $\mathfrak{M} \times \mathfrak{M}$ and $\varphi \restriction \mathfrak{M}_0 \times \mathfrak{M}_0$ is invariant.*

(3) *φ is a vector form on $\mathfrak{M} \times \mathfrak{M}$.*

(4) *$\varphi = \overline{f^0}$ for some t_w-continuous positive linear functional f on $\mathfrak{M}_0$.*

As an example, we take again a fully closed $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$, which is a partial GW*-algebra on $\mathscr{D}$ over $\mathscr{B}(\mathscr{H})$. Since $\{\xi \otimes \bar{\eta}; \xi, \eta \in \mathscr{D}\} \subset \mathscr{B}(\mathscr{H})^{\mathscr{D}} \cap (\mathscr{B}(\mathscr{H})^{\mathscr{D}})^*$, where $(\xi \otimes \bar{\eta})\zeta = (\zeta | \eta)\xi$ for $\zeta \in \mathscr{H}$, it follows that $\mathscr{B}(\mathscr{H})^{\mathscr{D}} \cap (\mathscr{B}(\mathscr{H})^{\mathscr{D}})^*$ is a nondegenerate $*$-subalgebra of $\mathscr{B}(\mathscr{H})$. Then we have the following result:

Corollary 3.8. (1) Every vector form on $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$ is GNS-representable.
(2) No non-zero vector form on $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$ is a h-form.
(3) Suppose $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$ is algebraically self-adjoint. Then every vector form on $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$ is invariant and, for each $\xi \neq 0 \in \mathscr{D}$, $\pi^{\mathfrak{B}}_{\omega^0_\xi}(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$ (resp. $\pi_{\omega^0_\xi}(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$) is unitarily equivalent to $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$ (resp. $\iota^*(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$, where $\mathfrak{B}$ denotes the linear span of $\{\eta \otimes \bar{\zeta}; \eta, \zeta \in \mathscr{D}\}$. Hence, if $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$ is not self-adjoint, then $\pi^{\mathfrak{B}}_{\omega^0_\xi} \subsetneqq \pi_{\omega^0_\xi}$.
(4) Suppose $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$ is not algebraically self-adjoint. Then, for each $\xi \neq 0 \in \mathscr{D}$, there exist $A_1, A_2 \in R^w(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$ and $X \in \mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$ such that $\omega^0_\xi(X \,\square\, A_1, A_2) \neq \omega^0_\xi(A_1, X^\dagger \,\square\, A_2)$.

Proof. (1) This follows from Proposition 3.7.

(2) Suppose ω^0_ξ is a h-form for some $\xi \neq 0 \in \mathscr{D}$. Let $X \in \mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$ be such that $X^\dagger = X$ and $\bar{X} \neq X^*$. Since $\eta \otimes \bar{\xi} \in R^w(X)$ for each $\eta \in D(X^*)$, we have

$$
\begin{aligned}
(X^*\eta | \zeta) &= \|\xi\|^{-4}\, \omega^0_\xi(X \,\square\, (\eta \otimes \bar{\xi}), \zeta \otimes \bar{\xi}) \\
&= \|\xi\|^{-4}\, \omega^0_\xi(\eta \otimes \bar{\xi}, X^\dagger \,\square\, (\zeta \otimes \bar{\xi})) \\
&= (\eta | X^*\zeta)
\end{aligned}
$$

for each $\eta, \zeta \in D(X^*)$, which implies $\bar{X} = X^*$. This is a contradiction.

(3) Clearly $\mathfrak{B}\xi = \mathscr{D}$. Hence $\pi^{\mathfrak{B}}_{\omega^0_\xi}(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$ is unitarily equivalent to $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$. We show that $R^w(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))\xi = \mathscr{D}^*(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$. Clearly, $R^w(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))\xi \subset \mathscr{D}^*(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$. Conversely, take an arbitrary $\eta \in \mathscr{D}^*(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$. Then $\eta \otimes \bar{\xi} \in R^w(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$ and $\eta = \|\xi\|^{-2}\, (\eta \otimes \bar{\xi})\xi \in R^w(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))\xi$. Therefore it follows that $R^w(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))\xi = \mathscr{D}^*(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$, which implies that $\pi_{\omega^0_\xi}(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$ is unitarily equivalent to $\iota^*(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$.

(4) Suppose $\omega^0_\xi(X \,\square\, A_1, A_2) = \omega^0_\xi(A_1, X^\dagger \,\square\, A_2)$ for each $X \in \mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$ and $A_1, A_2 \in R^w(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$. Since $\eta \otimes \bar{\xi} = R^w(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$ for each $\eta \in D^*(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$, we have

$$
\begin{aligned}
(X^{\dagger *}\eta | \zeta) &= \|\xi\|^{-4}\, \omega^o_\xi(X \,\square\, (\eta \otimes \bar{\xi}), \zeta \otimes \bar{\xi}) \\
&= \|\xi\|^{-4}\, \omega^o_\xi(\eta \otimes \bar{\xi}, X^\dagger \,\square\, (\zeta \otimes \bar{\xi})) \\
&= (\eta | X^*\zeta)
\end{aligned}
$$

for each $X \in \mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$ and $\eta, \zeta \in \mathscr{D}^*(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$, and hence $\mathscr{D}^{**}(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})) = \mathscr{D}^*(\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H}))$, which contradicts the assumption that $\mathscr{L}^\dagger_w(\mathscr{D}, \mathscr{H})$ is not algebraically self-adjoint. This completes the proof.

Remark 3.9. (1) Statements (1) and (2) of Corollary 3.8 show that the invariant positive sesquilinear forms used in this paper are more general that the h-forms introduced in Refs. 4,5. Since the latter do not contain any vector form, for the example considered here, the concept of h-form is probably too restrictive.

(2) Statements (3) and (4) explain why we have not imposed conditions (1)–(4) of Definition 3.1 to the whole set $R(\mathfrak{A})$, but only to some appropriate subspace $\mathfrak{B}$: if $R(\mathfrak{A})$ happens to be too large, as in the present case, the earlier definition of h-form becomes unduly restrictive. On the other hand, $R(\mathfrak{A})$ may also be defficult to characterize completely, although it is in general easy to find a subspace $\mathfrak{B}$.

(3) The case of topological quasi-$*$-algebras is discussed in a recent paper by one of us [22], where a notion of state is introduced in such a way that the GNS construction becomes possible. The approach of that paper is, however, quite different from the present one.

§4. Partial GW*-Algebras

Some generalizations of von Neumann algebras have been studied in [8, 15, 23–27]. We describe them here briefly. Let $\mathfrak{M}_0$ be a von Neumann algebra on a Hilbert space $\mathscr{H}$ and $\mathscr{D}$ a dense subspace in $\mathscr{H}$. Suppose (i) $\mathfrak{M}_0'\mathscr{D} \subset \mathscr{D}$ and (ii) $\mathfrak{M}_0\mathscr{D} \subset \mathscr{D}$. Then the t_s^*-closure $\mathfrak{M} \equiv [\mathfrak{M}_0 \restriction \mathscr{D}]^{s^*} \cap \mathscr{L}^\dagger(\mathscr{D})$ of $\mathfrak{M}_0 \restriction \mathscr{D}$ into $\mathscr{L}^\dagger(\mathscr{D})$ is an Op*-algebra on $\mathscr{D}$ such that the bounded part $\mathfrak{M}_b$ equals $\mathfrak{M}_0 \restriction \mathscr{D}$ and $\bar{X}$ is affiliated with $\mathfrak{M}_0$ for each $X \in \mathfrak{M}$. A $*$-subalgebra of $\mathfrak{M}$ containing $\mathfrak{M}_0 \restriction \mathscr{D}$ is called an *EW*-algebra on $\mathscr{D}$ over* $\mathfrak{M}_0$. Such algebras have been studied in [9, 23–26]. But, this class is too restrictive for most of the interesting Op*-algebras. For instance, if $\mathfrak{M}_0$ is purely infinite, then $\mathfrak{M}_b = \mathfrak{M}_0 \restriction \mathscr{D}$ [26].

Suppose that only the condition (i) $\mathfrak{M}_0'\mathscr{D} \subset \mathscr{D}$ holds. Then $\mathfrak{M} \equiv [\mathfrak{M}_0 \restriction \mathscr{D}]^{s^*} \cap \mathscr{L}^\dagger(\mathscr{D})$ is an Op*-algebra on $\mathscr{D}$ such that $(\mathfrak{M}_w')' = \mathfrak{M}_0$ and $\mathfrak{M} = (\mathfrak{M}_0 \restriction \mathscr{D})''_{w\sigma} \cap \mathscr{L}^\dagger(\mathscr{D}) = \{X \in \mathscr{L}^\dagger(\mathscr{D}); \bar{X}$ is affiliated with $\mathfrak{M}_0\}$. This *Op*-algebra* is said to be a *generalized von Neumann algebra* (or a *GW*-algebra*) *on $\mathscr{D}$ over* $\mathfrak{M}_0$, and it plays an important role for the study of the unbounded Tomita-Takesaki theory [25]. However, the bounded part $\mathfrak{M}_b$ of $\mathfrak{M}$ does not necessarily equal $\mathfrak{M}_0 \restriction \mathscr{D}$. Hence, it seems meaningful to study the t_s^*-closure $[\mathfrak{M}_0 \restriction \mathscr{D}]^{s^*}$ itself. The t_s^*-closure $[\mathfrak{M}_0 | \mathscr{D}]^{s^*}$ is a weak partial Op*-algebra on $\mathscr{D}$ with bounded part $\mathfrak{M}_0 \restriction \mathscr{D}$, and so its full closure is a partial GW*-algebra over $\mathfrak{M}_0$, as introduced in Section 3.

In this section we analyze in detail the partial GW*-algebras introduced in Section 3. They present a common feature with topological quasi $*$-algebras: both contain a dense $*$-algebra. In the case of partial GW*-algebras, this distinguished subset is a von Neumann algebra, which is dense for the strong* topology of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$.

We begin our discussion on the connection between these two structures in $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ with a natural question: *given a $*$-algebra $\mathfrak{M} \subset \mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ (possibly an Op*-algebra), is $[\mathfrak{M}]^{s^*}$ a partial Op*-algebra or a topological quasi $*$-algebra?* Clearly, for $[\mathfrak{M}]^{s^*}$ to be a topological quasi $*$-algebra, it is necessary that the multiplications $A \mapsto AB$, $A \mapsto BA$, for $B \in \mathfrak{M}$, be continuous maps from $\mathfrak{M}$ into itself. Unfortunately this does not occur frequently.

Before examining the question whether $[\mathfrak{M}]^{s^*}$ is a partial Op*-algebra, we recall that if $\mathfrak{N}$ is a †-invariant subset of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$, various unbounded commutants of $\mathfrak{N}$ may be defined [15, 16, 18, 19], but we need only the weak one, $\mathfrak{N}'_\sigma$, and also several unbounded bicommutants of $\mathfrak{N}$, namely:

$$\mathfrak{N}''_{w\sigma} \equiv (\mathfrak{N}'_w)'_\sigma,\ \mathfrak{N}''_{qw\sigma} \equiv (\mathfrak{N}'_{qw})'_\sigma,\ \mathfrak{N}''_{\sigma\sigma} \equiv (\mathfrak{N}'_\sigma)'_\sigma .$$

Then $\mathfrak{N}''_{w\sigma}$, $\mathfrak{N}''_{qw\sigma}$ and $\mathfrak{N}''_{\sigma\sigma}$ are three weakly closed, t_{s*}-complete, †-invariant subspaces of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ such that $\mathfrak{N} \subset \mathfrak{N}''_{\sigma\sigma} \subset \mathfrak{N}''_{w\sigma} \subset \mathfrak{N}''_{qw\sigma}$. (*Remark*: *the set* $\mathfrak{N}$ *is called an SV*-set if* $\mathfrak{N} = \mathfrak{N}''_{w\sigma}$[19]).

Now if $\mathfrak{M}$ is a $*$-algebra in $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ and $[\mathfrak{M}]^{s^*}$ is its t_{s*}-closure, Proposition 3.3 of [19] provides a criterion for $[\mathfrak{M}]^{s^*}$ to be a (weak) partial Op*-algebra: we have only to require that $\mathfrak{M}'_w\mathscr{D} = \mathscr{D}$ and $[\mathfrak{M}]^{s^*} = \mathfrak{M}''_{w\sigma}$. This fact motivates the alternative definition of partial GW*-algebra given below, where precisely these conditions will appear.

The next proposition is, in a sense, a warning: if we ask the behavior of the objects in consideration to be too reasonable, we fall into an almost trivial case!

Proposition 4.1. *If* $(\mathfrak{M}, \mathfrak{M}_0)[t_{s*}]$ *is a topological quasi $*$-algebra with* $\mathfrak{M}_0$ *a closed Op*-algebra, then* $\mathfrak{M}$ *is an Op*-algebra.*

Proof. Let $(\mathfrak{M}, \mathfrak{M}_0)[t_{s*}]$ be a topological quasi $*$-algebra and $X \in \mathfrak{M}$. Thus there is a net $\{X_\alpha\} \subset \mathfrak{M}_0$ such that $X_\alpha \to X$ in t_{s*}; let now $B \in \mathfrak{M}_0$, then $\{BX_\alpha f\}$ is a Cauchy net in the norm or, equivalently, $\{X_\alpha f\}$ is a Cauchy net in $t_{\mathfrak{M}_0}$. Hence there is $\psi \in \tilde{\mathscr{D}}(\mathfrak{M}_0)$ such that $X_\alpha f \to \psi$ in $t_{\mathfrak{M}_0}$ and thus $\psi = \bar{X} f$. This in turn implies that $Xf \in \mathscr{D} = \tilde{\mathscr{D}}(\mathfrak{M}_0)$.

The situation described in the previous discussion becomes particularly clear when $\mathfrak{M}$ consists of bounded operators only. In this case, first of all, the problem posed by Proposition 4.1 cannot occur, since a bounded operator algebra is never closed in $\mathscr{D}$, if $\mathscr{D} \neq \mathscr{H}$. We have in fact:

Proposition 4.2. *Let* $\mathfrak{M}_0$ *be a *-algebra with unit in* $\mathscr{B}(\mathscr{H}) \subset \mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ *and*

$\mathfrak{M} = [\mathfrak{M}_0]^{s^*}$ *its* t_{s^*}*-closure. If* $(\mathfrak{M}_0)'_w \mathscr{D} = \mathscr{D}$, *then* $\mathfrak{M}$ *is a weak partial Op*-algebra and it is stable under the strong multiplication. Moreover, if* $\mathfrak{M}_0 \mathscr{D} \subset \mathscr{D}$, *then* $(\mathfrak{M}, \mathfrak{M}_0)[t_{s^*}]$ *is a topological quasi *-algebra.*

Proof. If $\mathfrak{M}_0$ is a *-algebra with unit in $\mathscr{B}(\mathscr{H})$, then by Proposition 3.3.1 of [16], $\mathfrak{M} = (\mathfrak{M}_0)''_{w\sigma}$ and therefore, by Proposition 3.3 of [19], $\mathfrak{M}$ is a weak partial Op*-algebra. Since it is the unbounded commutant of $(\mathfrak{M}_0)'_w$, it is also stable under the strong multiplication. If $\mathfrak{M}_0 \mathscr{D} \subset \mathscr{D}$, then we have only to remark that, since $\mathfrak{M}_0$ consists only of bounded operators, both the right and the left multiplication are continuous.

For the sequel of our discussion, we need more information on the nature of the bicommutants. First notice that, for any †-invariant subset $\mathfrak{N} \subset \mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$, the following relations hold true:

$$\mathfrak{N}'_w = (\mathfrak{N}''_{w\sigma})'_w \subset (\mathfrak{N}'_w)'', \ \mathfrak{N}'_{qw} = (\mathfrak{N}''_{qw\sigma})'_w \subset (\mathfrak{N}'_{qw})'', \tag{4.1}$$

which implies

$$(\mathfrak{N}''_{w\sigma})''_{w\sigma} = \mathfrak{N}''_{w\sigma}, \ (\mathfrak{N}''_{qw\sigma})''_{w\sigma} = \mathfrak{N}''_{qw\sigma}. \tag{4.2}$$

Then we have:

Lemma 4.3. *Let* $\mathfrak{N}$ *be a* †*-invariant subset of* $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$. *Then the following statements hold.*

(1) $\mathfrak{N}'_w$ *is a von Neumann algebra if and only if* $\mathfrak{N}''_{w\sigma} = [(\mathfrak{N}'_w)']^{s^*}$.

(2) $\mathfrak{N}'_{qw}$ *is a von Neumann algebra if and only if* $\mathfrak{N}''_{qw\sigma} = [(\mathfrak{N}'_{qw})']^{s^*}$.

Proof. (1) Suppose $\mathfrak{N}'_w$ is a von Neumann algebra. By (4.1), $(\mathfrak{N}''_{w\sigma})'_w$ is a von Neumann algebra. Hence it follows from Theorem 3.9 of I that $\varepsilon_w(\mathfrak{N}''_{w\sigma})$ is a †-invariant subspace of $\mathscr{L}^\dagger(\mathscr{D}_w(\mathfrak{N}), \mathscr{H})$ such that

$$\varepsilon_w(\mathfrak{N}''_{w\sigma}) \succ \mathfrak{N}''_{w\sigma}, \ \varepsilon_w(\mathfrak{N}''_{w\sigma})'_w = \mathfrak{N}'_w, \ \varepsilon_w(\mathfrak{N}''_{w\sigma})'_w \mathscr{D}_w(\mathfrak{N}) = \mathscr{D}_w(\mathfrak{N}),$$

which implies

$$C\, \varepsilon_w(X) \subset \varepsilon_w(X)\, C$$

for each $X \in \mathfrak{N}''_{w\sigma}$ and $C \in \mathfrak{N}'_w$. Hence $\overline{\varepsilon_w(X)}$ is affiliated with $(\mathfrak{N}'_w)'$, and so there exists a sequence $\{A_n\}$ in $(\mathfrak{N}'_w)'$ such that

$$\lim_{n \to \infty} A_n \xi = \varepsilon_w(X)\xi \text{ and } \lim_{n \to \infty} A_n^* \xi = \varepsilon_w(X)^\dagger \xi$$

for each $\xi \in \mathscr{D}_w(\mathfrak{N})$. In particular, we have

$$\lim_{n \to \infty} A_n \xi = X\xi \text{ and } \lim_{n \to \infty} A_n^* \xi = X^\dagger \xi$$

for each $\xi \in \mathscr{D}$, and hence $X \in [(\mathfrak{N}'_w)']^{s^*}$. Thus we have $\mathfrak{N}''_{w\sigma} \subset [(\mathfrak{N}'_w)']^{s^*}$. The converse inclusion follows from $(\mathfrak{N}'_w)' \upharpoonright \mathscr{D} \subset \mathfrak{N}''_{w\sigma}$ and the strong*-closedness of $\mathfrak{N}''_{w\sigma}$.

Conversely suppose that $\mathfrak{N}''_{w\sigma} = [(\mathfrak{N}'_w)']^{s^*}$. Take arbitrary $C_1, C_2 \in \mathfrak{N}'_w$ and $X \in \mathfrak{N}$. Since $\mathfrak{N} \subset \mathfrak{N}''_{w\sigma} = [(\mathfrak{N}'_w)']^{s^*}$, there exists a net $\{A_\alpha\}$ in $(\mathfrak{N}'_w)'$ such that $\lim_\alpha A_\alpha \xi = X\xi$ and $\lim_\alpha A^*_\alpha \xi = X^\dagger \xi$ for each $\xi \in \mathscr{D}$. Then we have

$$\begin{aligned}(C_1 C_2 X\xi \mid \eta) &= \lim_\alpha (C_1 C_2 A_\alpha \xi \mid \eta)\\ &= \lim_\alpha (C_1 C_2 \xi \mid A^*_\alpha \eta)\\ &= (C_1 C_2 \xi \mid X^\dagger \eta)\end{aligned}$$

for each ξ, $\eta \in \mathscr{D}$, and hence $C_1 C_2 \in \mathfrak{N}'_w$, which implies that $\mathfrak{N}'_w$ is a von Neumann algebra.

Using Theorem 3.9 (2) of I and Eq. (4.1), we can prove the statement (2) in a similar way.

Proposition 4.4. *Let $\mathfrak{M}$ be a $\dagger$-invariant subset of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ satisfying the following conditions:*

(1) *$\mathfrak{M}_b$ is a von Neumann algebra;*

(2) *$\mathfrak{M}'_b \mathscr{D} = \mathscr{D}$.*

Then $(\mathfrak{M}_b)'_\sigma[t_{s^*}]$ is a topological quasi $*$-algebra over $\mathfrak{M}'_b$.

Proof. Since $\mathfrak{M}_b$ is a von Neumann algebra, then $\mathfrak{M}_b = (\mathfrak{M}'_b)'_w$. By Lemma 4.3, we have

$$(\mathfrak{M}'_b)''_{w\sigma} = [(\mathfrak{M}'_b)''_w]^{s^*},$$

or, in other words, $(\mathfrak{M}_b)'_\sigma = [\mathfrak{M}'_b]^{s^*}$. The statement follows then from the previous proposition.

The two conditions of Proposition 4.4 are typical for the partial GW*-algebras as defined up to now. In particular, if $\mathfrak{M}$ is a partial GW*-algebra over $\mathfrak{M}_b$, then $\mathfrak{M}_b$ is t_{s^*}-dense in $\mathfrak{M}$; hence we get

$$\mathfrak{M}'_\sigma = (\mathfrak{M}_b)'_\sigma \text{ and } \mathfrak{M}'_w = \mathfrak{M}'_b,$$

and so,

Corollary 4.5. *If $\mathfrak{M}$ is a partial GW*-algebra over $\mathfrak{M}_b$, then $\mathfrak{M}'_\sigma[t_{s^*}]$ is a topological quasi $*$-algebra over $\mathfrak{M}'_w$.*

Remark. If $\mathfrak{M}_b \mathscr{D} = \mathscr{D}$, then $\mathfrak{M}'_\sigma$ is also a partial Op*-algebra, thus it is almost a partial GW*-algebra: the only property that may fail is fully-closedness, and it is not clear that the full closure $\mathfrak{M}'_\sigma$ still verifies the conditions for a partial Op*-algebra.

Now we are ready to introduce an alternative definition for partial GW*-

algebras. Of course, we will show later that the two are indeed equivalent.

Definition 4.6. A fully closed weak partial Op*-algebra $\mathfrak{M}$ on $\mathscr{D}$ is said to be a partial GW*-algebra on $\mathscr{D}$ if $\mathfrak{M}'_w\mathscr{D} = \mathscr{D}$ and $\mathfrak{M}''_{w\sigma} = \mathfrak{M}$.

We first remark that for any properly infinite von Neumann algebra $\mathfrak{M}_0$ on $\mathscr{H}$ there exists a partial GW*-algebra $\mathfrak{M}$ on $\mathscr{D}$ such that $\mathfrak{M} \neq \mathfrak{M}_b$ and $(\mathfrak{M}'_w)' = \mathfrak{M}_0$. In fact, let H be a self-adjoint unbounded operator in $\mathscr{H}$ affiliated with $\mathfrak{M}_0$ and $\mathscr{D} \equiv \bigcap_{n\in N} \mathscr{D}(H^n)$. Then

$$\mathfrak{M} = \{X \in \mathscr{L}^\dagger(\mathscr{D}, \mathscr{H});\ \bar{X} \textit{ is affiliated with } \mathfrak{M}_0\}$$

is a partial GW*-algebra on $\mathscr{D}$ containing $\{H^n \restriction \mathscr{D};\, n \in N\}$ such that $(\mathfrak{M}'_w)' = \mathfrak{M}_0$.

The main problem is, of course, how to construct partial GW*-algebras. The examples above give us a hint. Let $\mathfrak{N}$ be a †-invariant subset of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$; then we are led naturally to the following questions:

(A) *When is* $\mathfrak{N}''_{w\sigma}$ *a partial GW*-algebra?*

(B) *When does one have* $\mathfrak{N}''_{w\sigma} = [(\mathfrak{N}'_w)']^{s^*}$?

Question (B) has obtained an implicit answer by means of Eq. (4.2) and by Lemma 4.3.

In order to solve Question (A), we need a new concept. Given a *-invariant set $\mathfrak{B}$ of bounded operators, we define:

$$\mathscr{R}^p(\mathfrak{B}', \mathscr{D}) = \{X \in \mathscr{L}^\dagger(\mathscr{D}, \mathscr{H});\quad \bar{X} \textit{ is affiliated with } \mathfrak{B}'\}$$

(this object was introduced in [19] where it was denoted $\mathfrak{B}_\eta$), it generalizes the corresponding subset of $\mathscr{L}^\dagger(\mathscr{D})$ introduced in [25]. The following relations are obvious:

$$\mathfrak{B}' \subset \mathscr{R}^p(\mathfrak{B}', \mathscr{D}) \subset [\mathfrak{B}']^{s^*} \subset \mathfrak{B}'_\sigma. \tag{4.3}$$

Then we have the following result.

Theorem 4.7. *Let* $\mathfrak{N}$ *be a* †-invariant subset of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$. *Then the following statements are equivalent*:

(1) $\widehat{\mathfrak{N}''_{w\sigma}}$ *is a partial GW*-algebra on* $\hat{\mathscr{D}}(\mathfrak{N}''_{w\sigma})$.

(2) $\mathfrak{N}''_{w\sigma}$ *is *-symmetric* [19]: *that is,* $(I + X^*\bar{X})^{-1} \in (\mathfrak{N}'_w)'$ *for each* $X \in \mathfrak{N}''_{w\sigma}$.

(3) $\mathfrak{N}''_{w\sigma} = [(\mathfrak{N}'_w)']^{s^*} = \mathscr{R}^p((\mathfrak{N}'_w)', \mathscr{D})$.

(4) $\mathfrak{N}'_w \hat{\mathscr{D}}(\mathfrak{N}''_{w\sigma}) = \hat{\mathscr{D}}(\mathfrak{N}''_{w\sigma})$

Proof.

- $(1) \Rightarrow (4)$: This follows from (4.1).
- $(4) \Rightarrow (3)$: Since $\mathfrak{N}'_w \hat{\mathscr{D}}(\mathfrak{N}''_{w\sigma}) = \hat{\mathscr{D}}(\mathfrak{N}''_{w\sigma})$, it follows that $\bar{X}$ is affiliated with $(\mathfrak{N}'_w)'$ for each $X \in \mathfrak{N}''_{w\sigma}$, which implies $\mathfrak{N}''_{w\sigma} \subset \mathscr{R}^p((\mathfrak{N}'_w)', \mathscr{D})$. By (4.3) we have

$$\mathfrak{N}''_{w\sigma} = [(\mathfrak{N}'_w)']^{s^*} = \mathscr{R}^p((\mathfrak{N}'_w)', \mathscr{D}).$$

∘ (3) ⇒ (2) : This is trivial.

∘ (2) ⇒ (1) : Since $\mathfrak{N}''_{w\sigma}$ is $*$-symmetric, it follows that $\mathfrak{N}'_w \hat{\mathscr{D}}(\mathfrak{N}''_{w\sigma}) = \hat{\mathscr{D}}(\mathfrak{N}''_{w\sigma})$, which implies that

$$\begin{aligned}(C(X_1 \square X_2)\xi|\eta) &= (X_1^{\dagger *} X_2\xi | C^*\eta) \\ &= (X_2\xi | \overline{X_1^\dagger}\, C^*\eta) \\ &= (X_2\xi | C^* X_1^\dagger \eta) \\ &= (\hat{X}_2 C\xi | X_1^\dagger \eta) \\ &= (C\xi | (X_1 \square X_2)^\dagger \eta)\end{aligned}$$

for $X_1, X_2 \in \mathfrak{N}''_{w\sigma}$, $X_1 \in L^w(X_2)$, $C \in \mathfrak{N}'_w$ and $\xi, \eta \in \mathscr{D}$. Hence, $X_1 \square X_2 \in \mathfrak{N}''_{w\sigma}$, and so $\mathfrak{N}''_{w\sigma}$ is a weak partial Op*-algebra on $\mathscr{D}$. As stated in I, Section 3, $\widehat{\mathfrak{N}''_{w\sigma}}$ is a fully closed weak partial Op*-algebra on $\hat{\mathscr{D}}(\mathfrak{N}''_{w\sigma})$. Further, it is easily shown that $\widehat{(\mathfrak{N}''_{w\sigma})}''_{w\sigma} = \widehat{\mathfrak{N}''_{w\sigma}}$. Thus, $\widehat{\mathfrak{N}''_{w\sigma}}$ is a partial GW*-algebra on $\hat{\mathscr{D}}(\mathfrak{N}''_{w\sigma})$. This completes the proof.

Similarly we have the following

Corollary 4.8. *Let $\mathfrak{N}$ be a †-invariant subset of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$. Then the following statements are equivalent:*

(1) *$\hat{\mathfrak{N}}''_{w\sigma}$ is a partial GW*-algebra on $\hat{\mathscr{D}}(\mathfrak{N})$.*

(2) *$\hat{\mathfrak{N}}''_{w\sigma}$ is *-symmetric.*

(3) $\hat{\mathfrak{N}}''_{w\sigma} = [(\mathfrak{N}'_w)']^{s^*} = \mathscr{R}^p((\mathfrak{N}'_w)', \hat{\mathscr{D}}(\mathfrak{N}))$

(4) $\mathfrak{N}'_w \hat{\mathscr{D}}(\mathfrak{N}) = \hat{\mathscr{D}}(\mathfrak{N})$.

The statement (4) in Theorem 4.7 implies the statement (4) in Corollary 4.8, and hence if $\widehat{\mathfrak{N}''_{w\sigma}}$ is a partial GW*-algebra on $\hat{\mathscr{D}}(\mathfrak{N}''_{w\sigma})$, then $\hat{\mathfrak{N}}''_{w\sigma}$ is a partial GW*-algebra on $\hat{\mathscr{D}}(\mathfrak{N})$.

Theorem 4.7 shows that the two definitions of partial GW*-algebra are indeed equivalent. If $\mathfrak{M}$ is a partial GW*-algebra in the sense of Definition 4.6, we observe [19] that its bounded part $\bar{\mathfrak{M}}_b \equiv (\mathfrak{M}'_w)'$ is a von Neumann algebra. Then it follows from Theorem 4.7 that, for $\mathfrak{M}_0 \equiv (\mathfrak{M}'_w)' = \bar{\mathfrak{M}}_b$, one has $\mathfrak{M} = [\mathfrak{M}_0 \upharpoonright \mathscr{D}]^{s^*} = \mathfrak{M}''_{w\sigma} = \mathscr{R}^p(\mathfrak{M}_0, \mathscr{D})$. Conversely, if $\mathfrak{M}$ satisfies the conditions of Definition 3.4, we get $(\mathfrak{M}'_w)' = \mathfrak{M}_0$ and therefore $\mathfrak{M}'_w\mathscr{D} = \mathscr{D}$, which implies, by Theorem 4.7 (4), that $\mathfrak{M}$ is a partial GW*-algebra in the sense of Definition 4.6.

These definitions seem to be a good choice, for the resulting partial GW*-algebras have all the expected properties, in particular they appear as a natural generalization of von Neumann algebras. Indeed, in the bounded case, a $*$-invariant subset $\mathfrak{N}$ of $\mathscr{B}(\mathscr{H})$, containing the identity, is a von Neumann algebra iff $\mathfrak{N}'' = \mathfrak{N}$ and then it equals $[\mathfrak{N}]^{s*}$. Furthermore, if a partial GW*-algebra $\mathfrak{M}$ on $\mathscr{D}$ leaves $\mathscr{D}$ invariant, i.e. $\mathfrak{M} \subset \mathscr{L}^\dagger(\mathscr{D})$, then $\mathfrak{M}$ is an EW†-algebra [8], that is, a symmetric Op*-algebra $\mathfrak{M}$ such that $\mathfrak{M}_b$ is a von Neumann algebra.

We may try also to generalize to partial Op*-algebras, and in particular to Op*-algebras, the other familiar statement: if $\mathfrak{N}$ is a $*$-invariant subset of $\mathscr{B}(\mathscr{H})$, then $\mathfrak{A} \equiv \mathfrak{N}''$ is the minimal von Neumann algebra containing $\mathfrak{N}$ and it verifies the relation $\mathfrak{A}' = \mathfrak{N}'$. The first part of the sentence, namely existence of a canonical partial GW*-algebra containing $\mathfrak{N}$, goes over easily, provided $\mathfrak{N}'_w$ is an algebra. In the sequel, we will write for short $\mathfrak{N}^\varepsilon \equiv \varepsilon_w(\mathfrak{N})''_{w\sigma}$. We will discuss the question of minimality later on.

Proposition 4.9. *Suppose $\mathfrak{N}$ is a †-invariant subset of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ such that $\mathfrak{N}'_w$ is an algebra. Then $\widehat{(\varepsilon_w(\mathfrak{N}))''_{w\sigma}}$ and $\widehat{\mathfrak{N}^\varepsilon} \equiv \widehat{\varepsilon_w(\mathfrak{N})''_{w\sigma}}$ are partial GW*-algebras on $\hat{\mathscr{D}}(\varepsilon_w(\mathfrak{N}))$ and $\hat{\mathscr{D}}(\mathfrak{N}^\varepsilon)$, respectively.*

Proof. If follows from Theorem 3.9 of I that $\varepsilon_w(\mathfrak{N})$ is a †-invariant subset of $\mathscr{L}^\dagger(\mathscr{D}_w(\mathfrak{N}), \mathscr{H})$ such that $\varepsilon_w(\mathfrak{N}) \succ \mathfrak{N}$, $\varepsilon_w(\mathfrak{N})'_w = \mathfrak{N}'_w$ and $\varepsilon_w(\mathfrak{N})'_w \mathscr{D}_w(\mathfrak{N}) = \mathscr{D}_w(\mathfrak{N})$, which implies that $\varepsilon_w(\mathfrak{N})'_w \hat{\mathscr{D}}(\varepsilon_w(\mathfrak{N})) = \hat{\mathscr{D}}(\varepsilon_w(\mathfrak{N}))$ and $\varepsilon_w(\mathfrak{N})'_w \hat{\mathscr{D}}(\mathfrak{N}^\varepsilon) = \hat{\mathscr{D}}(\mathfrak{N}^\varepsilon)$. Then the statement of the proposition follows from Theorem 4.7 and Corollary 4.8.

The next proposition summarizes the situation for partial Op*-algebras.

Proposition 4.10. *Let $\mathfrak{M}$ be a t_{s*}-closed partial Op*-algebra. Then:*

(i) *$\bar{\mathfrak{M}}_b$ is a von Neumann algebra.*

(ii) *If $[\mathfrak{M}_b]^{s*} = \mathfrak{M}$, then $\mathfrak{M} = \mathfrak{M}''_{w\sigma}$ and it is a partial SV*-algebra.*

(iii) *If $[\mathfrak{M}_b]^{s*} = \mathfrak{M}$, then $\widehat{(\varepsilon_w(\mathfrak{M}))''_{w\sigma}}$ and $\widehat{\mathfrak{M}^\varepsilon} = \widehat{\varepsilon_w(\mathfrak{M})''_{w\sigma}}$ are partial GW*-algebras on $\hat{\mathscr{D}}(\varepsilon_w(\mathfrak{M}))$ and $\hat{\mathscr{D}}(\mathfrak{M}^\varepsilon)$ respectively.*

(iv) *If $[\mathfrak{M}_b]^{s*} = \mathfrak{M}$ and $\mathfrak{M}_b\mathscr{D} \subset \mathscr{D}$, then $(\mathfrak{M}, \mathfrak{M}_b)[t_{s*}]$ is a topological quasi $*$-algebra.*

Proof. (i) is obvious.

(ii) follows from Proposition 3.1.1 of [16].

(iii) If $[\mathfrak{M}_b]^{s*} = \mathfrak{M}$, then $\mathfrak{M}'_w$ is an algebra; so the statement follows from Proposition 4.9.

(iv) follows from Proposition 4.2.

It is clear that if $\mathfrak{M}$ is a fully closed and t_{s*}-closed partial Op*-algebra with $\mathfrak{M}'_b\mathscr{D} \subset \mathscr{D}$ and we require that $(\mathfrak{M}, \mathfrak{M}_b)[t_{s*}]$ is a topological quasi $*$-algebra, then $\mathfrak{M}$ is automatically a partial GW*-algebra.

If we require $\mathfrak{M}$ to be self-adjoint, the situation simplifies

Proposition 4.11. *Let $\mathfrak{M}$ be a t_{s*}-closed and self-adjoint partial Op*-algebra. If $(\mathfrak{M}, \mathfrak{M}_b)[t_{s*}]$ is a topological quasi $*$-algebra, then $\mathfrak{M}$ is a partial*

GW-algebra.*

Proof. $\mathfrak{M}_b$ is clearly a von Neumann algebra and $(\mathfrak{M}_b)'_w = \mathfrak{M}'_w$; thus $\mathfrak{M}'_b$ maps $\mathscr{D}$ into $\mathscr{D}^*(\mathfrak{M}) = \mathscr{D}$.

We turn now to the question of the minimality of the partial GW*-algebra generated by $*$-invariant set of operators of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$. In the bounded case, if $\mathfrak{N} = \mathfrak{N}^* \subset \mathscr{B}(\mathscr{H})$, then $\mathfrak{A} = \mathfrak{N}''$ is the *only* von Neumann algebra that verifies the relation $\mathfrak{A}' = \mathfrak{N}'$, and it is the smallest one that contains $\mathfrak{N}$. In the general case, $\mathfrak{N} = \mathfrak{N}^\dagger \subset \mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$, there might be many partial Op*-algebras $\mathfrak{M}$ such that $\mathfrak{M}'_w = \mathfrak{N}'_w$, living on *different* domains. As the following theorem shows, there is a distinguished one among them, namely $\widehat{\mathfrak{N}}^\varepsilon \equiv \widehat{\varepsilon_w(\mathfrak{N})''_{w\sigma}}$: it is a partial GW*-algebra over $(\mathfrak{N}'_w)'$, and moreover it is *minimal*, in the sense that it lives on the smallest possible domain. To show this, however, we need a new notion, that of *embedding*.

Let $\mathfrak{N}$ be a †-invariant subset of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$. In I, Section 3 we have considered the partial Op*-algebra $\mathfrak{M}_w[\mathfrak{N}]$ generated by $\mathfrak{N}$, on the *same* domain $\mathscr{D}$. We have also extended the subset $\mathfrak{N}$ to a set $\varepsilon(\mathfrak{N})$ living on a larger domain, but the two sets are still in one-to-one correspondence. In general we need more: how does one compare two (partial) Op*-algebras living on *different* domains $\mathscr{D}_1$, $\mathscr{D}_2$?

Definition 4.12. Given $\mathscr{D}_1 \subset \mathscr{D}_2$, let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two partial Op*-algebras on $\mathscr{D}_1$ and $\mathscr{D}_2$, respectively. Then $\mathfrak{M}_2$ is said to embedded in $\mathfrak{M}_1$ if $\mathscr{D}_1 \subset \mathscr{D}_2$ and $\mathfrak{M}_2 \restriction \mathscr{D}_1 \subset \mathfrak{M}_1$, which is denoted by $\mathfrak{M}_2 \subsetneqq\!\!\to \mathfrak{M}_1$.

Suppose that $\mathfrak{M}_2 \subsetneqq\!\!\to \mathfrak{M}_1$. Then the map:

$$\imath : X \in \mathfrak{M}_2 \longmapsto X \restriction \mathscr{D}_1 \in \mathfrak{M}_1$$

is a faithful $*$-homomorphism of the partial Op*-algebra $\mathfrak{M}_2$ into the partial Op*-algebra $\mathfrak{M}_1$, but it is not necessarily a $*$-isomorphism; that is, $\mathfrak{M}_2 \restriction \mathscr{D}_1$ is not a partial $*$-subalgebra of $\mathfrak{M}_1$. In particular, if $\varepsilon(\mathfrak{M}) \succ \mathfrak{M}$, then $\varepsilon(\mathfrak{M}) \subsetneqq\!\!\to \mathfrak{M}$ and $\varepsilon^{-1} = \imath$ is a bijection *onto* $\mathfrak{M}$. This suggests a stronger notion of embedding:

Definition 4.13. $\mathfrak{M}_2$ is said to be a *partial $*$-subalgebra* of $\mathfrak{M}_1$ *by restriction* if $\mathfrak{M}_2 \subsetneqq\!\!\to \mathfrak{M}_1$ and $\imath$ is a $*$-isomorphism of $\mathfrak{M}_2$ into $\mathfrak{M}_1$, which is denoted by $\mathfrak{M}_2 \Subset\!\!\to \mathfrak{M}_1$.

In particular, if $\varepsilon(\mathfrak{M}) \succ_w \mathfrak{M}$ or $\varepsilon(\mathfrak{M}) \succ_s \mathfrak{M}$, then $\varepsilon(\mathfrak{M}) \Subset\!\!\to \mathfrak{M}$ and $\imath$ is an isomorphism onto.

Using this notion of embedding, we are now ready to show our point. Notice that the partial GW*-algebra $\widehat{\mathfrak{N}}^\varepsilon \equiv \widehat{\varepsilon_w(\mathfrak{N})''_{w\sigma}}$ verifies the relations $(\widehat{\mathfrak{N}}^\varepsilon)'_w = \mathfrak{N}'_w$ and $(\widehat{\mathfrak{N}}^\varepsilon)'_w \hat{\mathscr{D}}(\mathfrak{N}^\varepsilon) = \hat{\mathscr{D}}(\mathfrak{N}^\varepsilon)$. Denote by $\mathscr{J}$ the set of all weak partial Op*-

algebras $\mathfrak{M}$ which are fully closed and live on domains $\mathscr{E}$ such that $\mathscr{D} \subset \mathscr{E}$, $\mathfrak{M}'_w = \mathfrak{N}'_w$ and $\mathfrak{M}'_w \mathscr{E} = \mathscr{E}$. The embedding relation $\subsetneq$ is a (partial) order on $\mathscr{J}$. Then we state:

Theorem 4.14. *Suppose $\mathfrak{N}$ is a $\dagger$-invariant subset of $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ such that $\mathfrak{N}'_w$ is an algebra. Then the partial GW*-algebra $\widehat{\mathfrak{N}^\varepsilon} \equiv \widehat{\varepsilon_w(\mathfrak{N})''_{w\sigma}}$ is maximal in the ordered set $(\mathscr{T}, \subsetneq)$.*

Warning: According to Definition 4.6, 'maximal' implies 'living on the smallest domain'!

Proof. Let $\mathfrak{M}$ be a fully closed partial Op*-algebra on a domain $\mathscr{E}$ such that $\mathscr{D} \subset \mathscr{E}$, $\mathfrak{M}'_w = \mathfrak{N}'_w$ and $\mathfrak{M}'_w \mathscr{E} = \mathscr{E}$. Since $\mathfrak{N}'_w \mathscr{D} \subset \mathfrak{M}'_w \mathscr{E} = \mathscr{E}$, we have $\mathscr{D}_w(\mathfrak{N}) \subset \mathscr{E}$. Take an arbitrary $Y \in \mathfrak{M}$. Since $\mathfrak{M}'_w = \mathfrak{N}'_w$, $\mathfrak{M}'_w \mathscr{E} = \mathscr{E}$ and $YC\xi = CY\xi$ for each $C \in \mathfrak{M}'_w$ and $\xi \in \mathscr{E}$, it follows that

$$
\begin{aligned}
(Y(\textstyle\sum_{k=1}^n C_k\zeta_k) \mid C(\sum_{j=1}^m C'_j\zeta'_j))
&= \textstyle\sum_k \sum_j (YC_k\zeta_k \mid C C'_j\zeta'_j) \\
&= \textstyle\sum_k \sum_j (C'^*_j C^* C_k Y\zeta_k \mid \zeta'_j) \\
&= \textstyle\sum_k \sum_j (C'^*_j C^* C_k\zeta_k \mid Y^\dagger\zeta'_j) \\
&= (C^*(\textstyle\sum_k C_k\zeta_k) \mid \sum_j C'_j Y^\dagger\zeta'_j) \\
&= (C^*(\textstyle\sum_{k=1}^n C_k\zeta_k) \mid Y^\dagger(\sum_{j=1}^m C'_j\zeta'_j))
\end{aligned}
$$

for each $C \in \varepsilon_w(\mathfrak{N})'_w = \mathfrak{N}'_w$ and $\sum_{k=1}^n C_k\zeta_k$, $\sum_{j=1}^m C'_j\zeta'_j \in \mathscr{D}_w(\mathfrak{N})$, and hence

$$
Y|\mathscr{D}_w(\mathfrak{N}) \in \mathfrak{N}^\varepsilon \equiv \varepsilon_w(\mathfrak{N})''_{w\sigma}, \quad \text{for all} \quad Y \in \mathfrak{M}, \tag{4.4}
$$

which implies

$$
\hat{\mathscr{D}}(\mathfrak{N}^\varepsilon) = \bigcap_{X \in \mathfrak{N}^\varepsilon} D(\bar{X}) \subset \bigcap_{Y \in \mathfrak{M}} D(\overline{Y \upharpoonright \mathscr{D}_w(\mathfrak{N})})
$$

$$
\subset \bigcap_{Y \in \mathfrak{M}} D(\bar{Y}) = \hat{\mathscr{D}}(\mathfrak{M}) = \mathscr{E}.
$$

Furthermore, we have by (4.4)

$$
\overline{Y \upharpoonright \hat{\mathscr{D}}(\mathfrak{N}^\varepsilon)} = \overline{Y \upharpoonright \mathscr{D}_w(\mathfrak{N})}
$$

for each $Y \in \mathfrak{M}$, and hence

$$
Y \upharpoonright \hat{\mathscr{D}}(\mathfrak{N}^\varepsilon) = \overline{(Y \upharpoonright \mathscr{D}_w(\mathfrak{N}))} \upharpoonright \hat{\mathscr{D}}(\mathfrak{N}^\varepsilon) \in \widehat{\mathfrak{N}^\varepsilon}
$$

for each $Y \in \mathfrak{M}$, which implies $\mathfrak{M} \subsetneq \mathfrak{N}^\varepsilon$. This completes the proof.

If the set $\mathfrak{N}$ of Theorem 4.14 consists of bounded operators only, we get the following result:

Corollary 4.15. *Suppose* $\mathfrak{N}$ *is a* †*-invariant subset of* $\mathscr{B}(\mathscr{D}, \mathscr{H}) \equiv \{X \in \mathscr{L}^\dagger(\mathscr{D}, \mathscr{H});\ X$ *is bounded*$\}$. *Then* $\widehat{(\varepsilon_w(\mathfrak{N}))''_{w\sigma}}$ *equals the von Neumann algebra* $(\mathfrak{N}'_w)'$ *and* $\widehat{\mathfrak{N}^\varepsilon} \equiv \widehat{\varepsilon_w(\mathfrak{N})''_{w\sigma}}$ *is a partial GW*-algebra on* $\hat{\mathscr{D}}(\mathfrak{N}^\varepsilon)$.

Replacing weak commutants by quasi-weak ones in Theorem 4.14, we obtain an entirely similar statement, which is proved in the same way:

Theorem 4.16. *Suppose* $\mathfrak{N}$ *is a* †*-invariant subset of* $\mathscr{L}^\dagger(\mathscr{D}, \mathscr{H})$ *such that* $\mathfrak{N}'_{qw}$ *is an algebra. Then* $\widehat{(\varepsilon_{qw}(\mathfrak{N}))''_{w\sigma}}$ *and* $\widehat{\varepsilon_{qw}(\mathfrak{N})''_{w\sigma}}$ *are partial GW*-algebras on* $\hat{\mathscr{D}}(\varepsilon_{qw}(\mathfrak{N}))$ *and* $\hat{\mathscr{D}}(\varepsilon_{qw}(\mathfrak{N})''_{w\sigma})$, *respectively. In particular,* $\widehat{\varepsilon_{qw}(\mathfrak{N})''_{w\sigma}}$ *is maximal in the ordered set* $(\mathscr{J}_q, \supsetneq)$ *consisting of weak partial Op*-algebras* $(\mathfrak{M}, \mathscr{E})$ *on domains* $\mathscr{E}$ *such that* $\mathfrak{M}'_w = \mathfrak{N}'_{qw}$ and $\mathfrak{M}'_w \mathscr{E} = \mathscr{E}$.

§5. Invariant Positive Sesquilinear Forms on Abelian Partial $*$-Algebras

Powers [2] has shown that a self-adjoint representaion π of an abelian $*$-algebra $\mathfrak{A}$ is standard iff $(\pi(\mathfrak{A})'_w)'$ is abelian. In this section we extend this property to the case of abelian partial Op*-algebras, and apply this result to the study of invariant positive sesquilinear forms on abelian partial $*$-algebras.

Let $(\Omega, \mathscr{B}, m)$ be a σ-finite measure space and $\mathscr{M}(\Omega)$ be the set of all multiplication operators M_f by a.e. finite measurable functions f on Ω. Then $\mathscr{M}(\Omega)$ is the set of all densely defined closed operators in $L^2(\Omega)$ affiliated with the von Neumann algebra $M_{L^\infty(\Omega)}$, which is an abelian EW*-algebra [23]. A fully closed partial Op*-algebra $(\mathfrak{M}, \mathscr{D})$ is said to be *a partial Op*-algebra generated by measurable functions on* Ω if $\bar{\mathfrak{M}} \subset \mathscr{M}(\Omega)$.

We first give some necessary and sufficient conditions for a fully closed partial Op*-algebra to be unitrarily equivalent to a partial Op*-algebra generated by measurable functions.

Theorem 5.1. *Let* $\mathfrak{M}$ *be a fully closed partial Op*-algebra on* $\mathscr{D}$. *Consider the following statements*:

(1) $\mathfrak{M}'_w \mathscr{D} \subset \mathscr{D}$ *and* $(\mathfrak{M}'_w)'$ *is abelian.*

(2) $[(\mathfrak{M}'_w)' \upharpoonright \mathscr{D}]^{s^*}$ *is an abelian, standard partial GW*-algebra on* $\mathscr{D}$.

(3) $\mathfrak{M}$ *is abelian and standard.*

Then the following implications hold:

$$\begin{array}{ccc} (1) & \Longleftrightarrow & (2) \\ \Downarrow & & \\ (3) & & \end{array}$$

Furthermore, suppose $\mathfrak{M}$ *has a strongly* $\hat{p}$*-cyclic vector* ξ_o, *that is,* $R^w(\mathfrak{M})^{\xi_0}\xi_0$ *is a core for each* $\bar{X}$, $X \in \mathfrak{M}$ [17]. *Then the statements* (1)–(3) *above are equivalent to* *the statement*

(4) *There exists a compact Hausdorff space Z and a bounded Radon measure μ on Z such that $\mathfrak{M}$ is unitarily equivalent to a partial Op*-algebra $(\mathfrak{N}, \mathscr{E})$ generated by measurable functions on Z, such that $(\mathfrak{N}'_w)' = M_{L^\infty(Z,\mu)}$.*

Proof. • (1) ⇒ (2): Since $\mathfrak{M}'_w \mathscr{D} \subset \mathscr{D}$ and $\mathfrak{M}$ is fully closed, it follows from Corollary 4.8 that $[(\mathfrak{M}'_w)' \restriction \mathscr{D}]^{s^*}$ is a partial GW*-algebra on $\mathscr{D}$ and

$$[(\mathfrak{M}'_w)' \restriction \mathscr{D}]^{s^*} = \{X \in \mathscr{L}^\dagger(\mathscr{D}, \mathscr{H}); \bar{X} \text{ is affiliated to } (\mathfrak{M}'_w)'\}. \tag{5.1}$$

Hence it follows from Theorem 5.6.15 of [29] that $X^* = \bar{X}$ for every $X = X^\dagger \in [(\mathfrak{M}'_w)' \restriction \mathscr{D}]^{s^*}$. Hence $[(\mathfrak{M}'_w)' \restriction \mathscr{D}]^{s^*}$ is standard. Since $(\mathfrak{M}'_w)'$ is abelian, we have

$$(X^\dagger \xi | Y\eta) = (Y^\dagger \xi | X\eta)$$

for all $X, Y \in [(\mathfrak{M}'_w)' \restriction \mathscr{D}]^{s^*}$ and $\xi, \eta \in \mathscr{D}$, which implies that $X \in L^w(Y)$ iff $Y \in L^w(X)$ and then $X \square Y = Y \square X$. Hence $[(\mathfrak{M}'_w)' \restriction \mathscr{D}]^{s^*}$ is abelian.

• (2) ⇒ (1): This is trivial.

• (2) ⇒ (3): This follows since $\mathfrak{M}$ is a $*$-subalgebra of the abelin, standard partial GW*-algebra $[(\mathfrak{M}'_w)' \restriction \mathscr{D}]^{s^*}$ on $\mathscr{D}$.

Suppose now there exists a strongly $\hat{p}$-cyclic vector ξ_0 for $\mathfrak{M}$.

• (3) ⇒ (1): Since $\mathfrak{M}$ is abelian and standard, we have

$$R(\mathfrak{M}) \equiv R^w(\mathfrak{M}) = R^s(\mathfrak{M}) = L^w(\mathfrak{M}) = L^s(\mathfrak{M}),$$
$$R(\mathfrak{M})\mathscr{D} \subset \mathscr{D}.$$

Since ξ_0 is strongly $\hat{p}$-cyclic for $\mathfrak{M}$, it follows that $R(\mathfrak{M})\xi_0$ is dense in $\mathscr{H}$, which implies that the map

$$A\xi_0 \in R(\mathfrak{M})\xi_0 \longmapsto A^\dagger \xi_0 \in R(\mathfrak{M})\xi_0$$

extends to an isometry J on $\mathscr{H}$. Then it is easily shown that

$$JX\xi_0 = X^\dagger \xi_0, \; X \in \mathfrak{M}, \tag{5.2}$$

$$JC\xi_0 = C^\dagger \xi_0, \; C \in R(\mathfrak{M})'_w. \tag{5.3}$$

By (5.2), (5.3) and the standardness of $\mathfrak{M}$ we have

$$\begin{aligned}(CXA\xi_0 | B\xi_0) &= (XA\xi_0 | \bar{B}C^*\xi_0)\\ &= ((B^\dagger \square X)A\xi_0 | C^*\xi_0)\\ &= (C\xi_0 | J(B^\dagger \square X)A\xi_0)\\ &= (C\xi_0 | A^\dagger \square (B^\dagger \square X)^\dagger \xi_0)\\ &= (C\xi_0 | \bar{A}^\dagger X^\dagger B\xi_0)\\ &= (CA\xi_0 | X^\dagger B\xi_0)\end{aligned}$$

for all $C \in R(\mathfrak{M})'_w$, $X \in \mathfrak{M}$ and A, $B \in R(\mathfrak{M})$, which implies

$$R(\mathfrak{M})'_w = \mathfrak{M}'_w. \tag{5.4}$$

By the standardness of $\mathfrak{M}$, $\bar{A}$ is self-adjoint for each $A = A^\dagger \in R(\mathfrak{M})$. Let $\bar{A} = \int_{-\infty}^{\infty} \lambda \, dE_A(\lambda)$ be the spectral resolution of $\bar{A}$. In the same way as in Theorem 7.1 of [2], we can prove that $(R(\mathfrak{M})'_w)' = \{E_A(\lambda); A = A^\dagger \in R(\mathfrak{M}), -\infty < \lambda < \infty\}''$ and $E_A(\lambda)E_B(\mu) = E_B(\mu)E_A(\lambda)$ for all $A = A^\dagger$, $B = B^\dagger \in R(\mathfrak{M})$ and $-\infty < \lambda$, $\mu < \infty$. Hence it follows from (5.4) that $(\mathfrak{M}'_w)'$ is abelian.

- (4) $\Rightarrow$ (3): Since $\bar{\mathfrak{N}} \subset \mathscr{M}(Z)$, it follows from Theorem 5.6.4 of [29] that $\mathfrak{N}$ is standard and $\bar{X}$ is affiliated with $M_{L^\infty(Z,\mu)}$ for each $X \in \mathfrak{N}$, which implies that $\mathfrak{N}$ is abelian. Hence $\mathfrak{M}$ is standard and abelian.
- (2) $\Rightarrow$ (4): Let $\mathfrak{Z}$ be the abelian C^*-algebra on $\mathscr{H}$ generated by $\{E_A(\lambda); A = A^\dagger \in R(\mathfrak{M}), -\infty < \lambda < \infty\}$, Z the spectrum of $\mathfrak{Z}$ and $f \mapsto T_f$ the Gelfand representation of the C^*-algebra $C(Z)$ onto $\mathfrak{Z}$. Then, by the Riesz representation theorem, there exists a bounded Radon measure μ on the compact Hausdorff space Z such that

$$(T_f \xi_0 \mid \xi_0) = \int_Z f \, d\mu, \; f \in C(Z).$$

Since $(\mathfrak{M}'_w)' = \mathfrak{M}'_w = \mathfrak{Z}'$ and ξ_o is cyclic for $\mathfrak{Z}'$, it follows from [30] (Part I, Chapter 7) that the Gelfand representation $f \mapsto T_f$ extends to a *-isomorphism $f \mapsto T_f$ of $L^\infty(Z, \mu)$ onto $(\mathfrak{M}'_w)'$ and

$$(T_f \xi_0 \mid \xi_0) = (M_f 1 \mid 1), \quad f \in L^\infty(Z, \mu),$$

which implies that there exists an isometry U of $L^2(Z, \mu)$ onto $\mathscr{H}$ such that $U L^\infty(Z, \mu) = (\mathfrak{M}'_w)' \xi_0$ and $U^*(\mathfrak{M}'_w)' U = M_{L^\infty(Z,\mu)}$. Hence it follows from (5.1) and Theorem 5.6.4 of [5] that $U^* \mathfrak{M} U$ is a fully closed partial Op*-algebra on $U^*\mathscr{D}$ such that $U^* \mathfrak{M} U \subset \mathscr{M}(Z)$. This completes the proof.

By Theorem 5.1 we have the following

Corollary 5.2. *Let $\mathfrak{M}$ be an abelian partial GW^*-algebra on $\mathscr{D}$ over $\mathfrak{M}_0$. Then $\mathfrak{M}$ is standard. In addition, if $\mathfrak{M}$ has a strongly $\hat{p}$-cyclic vector, then it is unitarily equivalent to a partial GW^*-algebra generated by measurable functions on a compact Hausdorff space.*

Next we apply Theorem 5.1 to the study of invariant positive sesquilinear forms on abelian partial *-algebras.

Let $\mathscr{H}(\mathfrak{A})$ be the Hilbert space obtained by the completion of a maximal Hilbert algebra $\mathfrak{A}$. We denote by π_0 and ρ_0 the left and right regular representations of $\mathfrak{A}$, respectively, and by $J_{\mathscr{H}(\mathfrak{A})}$ (or simply, J) the involution of $\mathscr{H}(\mathfrak{A})$. For each $x \in \mathscr{H}(\mathfrak{A})$ we denote by $\pi_{\mathscr{H}(\mathfrak{A})}(x)$ and $\rho_{\mathscr{H}(\mathfrak{A})}(x)$ the closures of the closable operators defined by

$$\pi_{\mathscr{H}(\mathfrak{A})}(x)\xi = \rho_0(\xi)x \text{ and } \rho_{\mathscr{H}(\mathfrak{A})}(x)\xi = \pi_0(\xi)x, \quad \xi \in \mathfrak{A},$$

respectively. Pallu de la Barrière [31] has shown that

$$\pi_{\mathscr{H}(\mathfrak{A})}(x)^* = \pi_{\mathscr{H}(\mathfrak{A})}(Jx) \text{ and } J\pi_{\mathscr{H}(\mathfrak{A})}(x)J = \rho_{\mathscr{H}(\mathfrak{A})}(Jx), \; x \in \mathscr{H}(\mathfrak{A}). \tag{5.5}$$

Suppose $y \in D(\pi_{\mathscr{H}(\mathfrak{A})}(x))$. Then it follows from (5.5) that

$$Jx \in D(\pi_{\mathscr{H}(\mathfrak{A})}(Jy)) \text{ and } \pi_{\mathscr{H}(\mathfrak{A})}(Jy)\,Jx = J\pi_{\mathscr{H}(\mathfrak{A})}(x)\,y. \tag{5.6}$$

It is easily shown by (5.5) and (5.6) that $\mathscr{H}(\mathfrak{A})$ is an associative partial $*$-algebra equipped with the involution $x \to Jx$ and the following partial multiplication: *$x \in L(y)$ iff $y \in D(\pi_{\mathscr{H}(\mathfrak{A})}(x))$, and then $xy = \pi_{\mathscr{H}(\mathfrak{A})}(x)y$*, so that $R(\mathscr{H}(\mathfrak{A})) = \mathfrak{A}$. The partial $*$-algebra $\mathscr{H}(\mathfrak{A})$ is called the *H-system* of $\mathfrak{A}$.

For the structure of abelian H-systems we have the following

Proposition 5.3. Let $\mathscr{H}(\mathfrak{A})$ be a H-system of a maximal Hilbert algebra $\mathfrak{A}$ with unit. Then the following statements are equivalent.

(1) $\mathfrak{A}$ is abelian.

(2) The H-system $\mathscr{H}(\mathfrak{A})$ is abelian.

(3) $\mathscr{H}(\mathfrak{A})$ is isometric and isomorphic to the H-system $L^2(Z, \mu) = \mathscr{H}(L^\infty(Z, \mu))$,

where Z is a compact Hausdorff space and μ is a bounded Radon measure on Z.

Proof. The implication (3) ⇒ (2) ⇒ (1) are trivial.

• (1) ⇒ (3): Let ξ_0 be a unit of $\mathfrak{A}$. As shown in the proof of the implication (2) ⇒ (4) in Theorem 5.1, there exists an isometry U of $L^2(Z, \mu)$ onto $\mathscr{H}(\mathfrak{A})$ such that $U1 = \xi_0$ and $U^*\pi_0(\mathfrak{A})U = M_{L^\infty(Z,\mu)}$, where Z is a compact Hausdorff space and μ is a bounded Radon measure on Z. Then it is easily shown that $\pi_{\mathscr{H}(\mathfrak{A})}(Uf)\,Ug = U\pi_{L^2(Z,\mu)}(f)g$ for each $f \in L^2(Z, \mu)$ and $g \in L^\infty(Z, \mu)$, which implies that $f, g \in L^2(Z, \mu)$; $f \in L(g)$ iff $Ug \in D(\pi_{\mathscr{H}(\mathfrak{A})}(Uf))$ iff $Uf \in L(Ug)$, and then $(Uf)(Ug) = Ufg$. It is clear that $J_{\mathscr{H}(\mathfrak{A})}(Uf) = U J_{L^2(Z,\mu)} f$ for each $f \in L^2(Z, \mu)$. Thus $\mathscr{H}(\mathfrak{A})$ and $L^2(Z, \mu)$ are isometric and isomorphic.

Theorem 5.4. *Let $\mathfrak{A}$ be an abelian partial $*$-algebra with unit e and ϕ a $\mathfrak{B}$-invariant positive sesquilinear form on $\mathfrak{A} \times \mathfrak{A}$. Then the following statements are equivalent.*

(1) *$\widehat{\pi_\phi^{\mathfrak{B}}}$ is standard.*

(2) $\pi_\phi^{\mathfrak{B}}(\mathfrak{A})'_w \lambda_\phi(e) \subset \mathscr{D}(\widehat{\pi_\phi^{\mathfrak{B}}})$ and $(\pi_\phi^{\mathfrak{B}}(\mathfrak{A})'_w)' = \pi_\phi^{\mathfrak{B}}(\mathfrak{A})'_w$.

(3) *$\pi_\phi^{\mathfrak{B}}(\mathfrak{A})'_w \lambda_\phi(e) \subset \mathscr{D}(\widehat{\pi_\phi^{\mathfrak{B}}})$ and $\pi_\phi^{\mathfrak{B}}(\mathfrak{A})'_w \lambda_\phi(e)$ is dense in $\mathscr{H}_\phi$.*

(4) *There exists a $*$-homomorphism Φ of $\mathfrak{A}$ into the H-system $L^2(Z, \mu)$ such that*

(4)$_1$ *$\Phi(\mathfrak{B})$ is a core for each $\pi_{L^2(Z,\mu)}(\Phi(x))$, $x \in \mathfrak{A}$;*

$$(4)_2 \quad \phi(x, y) = \int_Z [\Phi(x)](t)\,\overline{[\Phi(y)](t)}\, d\mu(t), \quad x, y \in \mathfrak{A},$$

where Z is a compact Hausdorff space and μ is a bounded Radon measure on Z.
In this case, $\mathfrak{B}$ is maximal in $\mathscr{F}_\phi$.

Proof. Theorem 5.1 implies that (1)$\Leftrightarrow$(2) and (1)$\Rightarrow$(3).
(3)$\Rightarrow$(1): Since $(\pi_\phi^{\mathfrak{B}}(\mathfrak{A})'_w)'\lambda_\phi(e)$ is a maximal Hilbert algebra in $\mathscr{H}_\phi$, it follows that

$$\pi_{\mathscr{H}_\phi}(\lambda_\phi(x))\, C\lambda_\phi(e) = C\lambda_\phi(x) = \overline{\pi_\phi^{\mathfrak{B}}(x)}\, C\lambda_\phi(e)$$

for each $x \in \mathfrak{A}$ and $C \in \pi_\phi^{\mathfrak{B}}(\mathfrak{A})'_w$, where $\pi_{\mathscr{H}_\phi}$ is the left regular representation of the H-system $\mathscr{H}_\phi$. Hence we have

$$\pi_{\mathscr{H}_\phi}(\lambda_\phi(x)) \subset \overline{\pi_\phi^{\mathfrak{B}}(x)}, \quad x \in \mathfrak{A},$$

which implies by (5.5) that

$$\overline{\pi_\phi^{\mathfrak{B}}(x^*)} \subset \pi_\phi^{\mathfrak{B}}(x)^* \subset \pi_{\mathscr{H}_\phi}(\lambda_\phi(x))^* = \pi_{\mathscr{H}_\phi}(\lambda_\phi(x^*)) \subset \overline{\pi_\phi^{\mathfrak{B}}(x^*)},$$

for each $x \in \mathfrak{A}$. It follows that $\widehat{\pi_\phi^{\mathfrak{B}}}$ is standard. Thus the statements (1)–(3) are equivalent.
(1)$\Rightarrow$(4): Since $(\pi_\phi^{\mathfrak{B}}(\mathfrak{A})'_w)' = \pi_\phi^{\mathfrak{B}}(\mathfrak{A})'_w$, it follows that $(\pi_\phi^{\mathfrak{B}}(\mathfrak{A})'_w)'\lambda_\phi(e)$ is a maximal abelian Hilbert algebra in $\mathscr{H}_\phi$. By Proposition 5.3 there exists an isometric $*$-isomorphism U of the H-system $\mathscr{H}_\phi$ onto the H-system $L^2(Z, \mu)$, where Z is a compact Hausdorff space and μ is a bounded Radon measure on Z. As shown above in the proof of the implication (3)$\Rightarrow$(1), we have

$$\overline{\pi_\phi^{\mathfrak{B}}(x)} = \pi_{\mathscr{H}_\phi}(\lambda_\phi(x)), \quad x \in \mathfrak{A}. \tag{5.7}$$

We introduce on $\lambda_\phi(\mathfrak{A})$ the partial multiplication of the H-system $\mathscr{H}_\phi$; that is, $\lambda_\phi(x) \in L(\lambda_\phi(y))$ iff $\lambda_\phi(y) \in D(\pi_{\mathscr{H}_\phi}(\lambda_\phi(x))) = D(\overline{\pi_\phi^{\mathfrak{B}}(x)})$. We now put $\Phi = U \circ \lambda_\phi$. Take arbitrary $x, y \in \mathfrak{A}$ such that $x \in L(y)$. Since $\pi_\phi^{\mathfrak{B}}$ is a $*$-representation of $\mathfrak{A}$, we have $\pi_\phi^{\mathfrak{B}}(x) \in L^w(\pi_\phi^{\mathfrak{B}}(y))$. Thus it follows from (5.7) that

$$\lambda_\phi(y) = \pi_\phi^{\mathfrak{B}}(y)\lambda_\phi(e) \in D(\pi_\phi^{\mathfrak{B}}(x^*)^*) = D(\overline{\pi_\phi^{\mathfrak{B}}(x)}),$$

which means that $\lambda_\phi(x) \in L(\lambda_\phi(y))$ and $\lambda_\phi(x)\,\lambda_\phi(y) = \lambda_\phi(xy)$. Hence, λ_ϕ is a $*$-homomorphism of the partial $*$-algebra $\mathfrak{A}$ into the H-system $\mathscr{H}_\phi$. Furthermore, since U is a $*$-isomorphism of $\mathscr{H}_\phi$ onto $L^2(Z, \mu)$, it follows that Φ is a $*$-homomorphism of the partial $*$-algebra $\mathfrak{A}$ into the H-system $L^2(Z, \mu)$.

Next we show that $\Phi(\mathfrak{B})$ is a core for each $\pi_{L^2(Z,\mu)}(\Phi(x))$, $x \in \mathfrak{A}$. Indeed, for each $a \in \mathfrak{B}$ there exists a sequence $\{A_n\}$ in $(\pi_\phi^{\mathfrak{B}}(\mathfrak{A})'_w)'$ which converges strongly to $\overline{\pi_\phi^{\mathfrak{B}}(a)}$. Thus we have

$$\lim_{n\to\infty} UA_n\lambda_\phi(e) = U\lambda_\phi(a) = \Phi(a),$$

$$\begin{aligned}\lim_{n\to\infty} \pi_{L^2(Z,\mu)}(\Phi(x))\, U A_n \lambda_\phi(e) &= \lim_{n\to\infty} U(\pi_{\mathscr{H}_\phi}(\lambda_\phi(x))\, A_n\lambda_\phi(e)) \\ &= U(\overline{\pi_\phi^{\mathfrak{B}}(a)}\,\lambda_\phi(x)) \\ &= U\lambda_\phi(xa) \\ &= \Phi(xa),\end{aligned}$$

which means that $\Phi(\mathfrak{B})$ is a core for $\pi_{L^2(Z,\mu)}(\Phi(x))$. For each $x, y \in \mathfrak{A}$ we have

$$\begin{aligned}\phi(x, y) = (\lambda_\phi(x) \mid \lambda_\phi(y)) &= (U\lambda_\phi(x) \mid U\lambda_\phi(y)) \\ &= (\Phi(x) \mid \Phi(y)) \\ &= \int_Z [\Phi(x)](t)\overline{[\Phi(y)](t)}\, d\mu(t).\end{aligned}$$

- (4) ⇒ (1): Since

$$\begin{aligned}(\pi_\phi^{\mathfrak{B}}(x)\lambda_\phi(a) \mid \pi_\phi^{\mathfrak{B}}(y)\lambda_\phi(b)) &= \phi(xa, xb) \\ &= (\pi_{L^2(Z,\mu)}(\Phi(x))\Phi(a) \mid \pi_{L^2(Z,\mu)}(\Phi(x))\Phi(b)),\end{aligned}$$

for all $x, y \in \mathfrak{A}$ and all $a, b \in \mathfrak{B}$, and $\Phi(\mathfrak{B})$ is a core for each $\pi_{L^2(Z,\mu)}(\Phi(x))$, $x \in \mathfrak{A}$, it follows that $\overline{\pi_\phi^{\mathfrak{B}}(x)}$ is unitarily equivalent to $\pi_{L^2(Z,\mu)}(\Phi(x))$, which implies that $\pi_\phi^{\mathfrak{B}}(x)^* = \overline{\pi_\phi^{\mathfrak{B}}(x^*)}$. This completes the proof.

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