Partial Differential Equations with Numerical Methods

Stig Larsson and Vidar Thomée, Springer 2003, 2005

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p. 44, l. 17: $a_j \pm \frac{1}{2}hb_j \geq 0$ **should be** $a_j \pm \frac{1}{2}hb_j > 0$

p. 44, l. 6-: nonnegative **should be** positive

List of corrections, October 10, 2006. Page numbers refer to the second corrected printing 2005.

p. 94, l. Problem 6.6: Problem A.15 **should be** Problem A.14

p. 83, l. (6.16): $\Big\| v - \sum_{j=1}^{N} (v, \varphi_j)\varphi_j \Big\| \leq C\lambda_{N+1}^{-1/2}$ **should be** $\Big\| v - \sum_{j=1}^{N} (v, \varphi_j)\varphi_j \Big\| \leq \lambda_{N+1}^{-1/2} \|\nabla v\|$

p. 40, l. 1-: $\int_\Omega f \, \mathrm{d}x$ **should be** $\dfrac{1}{|\Omega|} \int_\Omega f \, \mathrm{d}x$

p. 31, l. 2: $L_1(\mathbf{R}^d)$ **should be** $L_1(B)$ in view of (3.14)

p. 31, l. 5-: $\int_{|x|=\epsilon} \varphi \dfrac{\partial U}{\partial n} \, \mathrm{d}s$ **should be** $-\int_{|x|=\epsilon} \varphi \dfrac{\partial U}{\partial n} \, \mathrm{d}s$

List of corrections, February 13, 2006. Page numbers refer to the second corrected printing 2005.

p. 88, l. 4: $N_\rho \approx \rho^2 b^2/\pi$ **should be** $N_\rho \approx \rho^2 b^2/(4\pi)$

p. 88, l. 5: $\lambda_n = \lambda_{ml} \approx \rho^2 \approx \pi N_\rho/b^2 \approx \pi n/b^2$ **should be** $\lambda_n = \lambda_{ml} \approx \rho^2 \approx 4\pi N_\rho/b^2 \approx 4\pi n/b^2$

p. 158, l. 4: $n \geq 1$ **should be** $n \geq 0$

p. 236, l. 2: if it **should be** if it is

List of corrections, August 24, 2005.

Most of the following errors have been corrected in the second corrected printing 2005.

p. 3, l. 12-: $\to \infty$ **should be** $t \to \infty$

p. 6, l. 1-: $\Big(\int_\Omega vw \, \mathrm{d}x \Big)^{1/2}$ **should be** $\int_\Omega vw \, \mathrm{d}x$

p. 7, l. 15: we we **should be** we

p. 9, l. 1-: definition of b should be $b = \dfrac{v_\mathrm{f}\sigma_\mathrm{f}L}{\lambda_\mathrm{f}} \dfrac{\sigma}{\sigma_\mathrm{f}} \dfrac{v}{v_\mathrm{f}}$

p. 10, l. 1.21: $b - \nabla \cdot a$ **should be** $b - \nabla a$

p. 16, l. 2-: for ϵ **should be** for $\epsilon > 0$

p. 23, l. Problem 2.2: where c is a positive constant

p. 27, l. 1: $\leq$ **should be** $=$ (in two places)

p. 27, l. 3: $\min_{\bar\Omega} u \leq \min \big\{ \min_{\Gamma} u, 0 \big\}$ **should be** $\min_{\bar\Omega} u \geq \min \big\{ \min_{\Gamma} u, 0 \big\}$

p. 30, l. 5: by parts **should be** by parts twice

p. 30, l. 6: . **should be** ,

p. 31, l. 3-: $\Big| \int_{|x|=\epsilon} \dfrac{\partial \varphi}{\partial n} U \, \mathrm{d}s \Big| = \Big| \dfrac{1}{2\pi} \log(\epsilon) \int_{|x|=\epsilon} \dfrac{\partial \varphi}{\partial n} \, \mathrm{d}s \Big| \leq \epsilon |\log(\epsilon)| \|\nabla \varphi\|_{\mathcal{C}} \to 0$

p. 33, l. 14-: formulation (3.23) **should be** formulation (3.20)

p. 35, l. 5-: $\dfrac{\partial u}{\partial n}$ **should be** $a\dfrac{\partial u}{\partial n}$

p. 38, l. 14: $m, k = 1$ **should be** $j, k = 1$

p. 39, l. 11-: Hint: $v(x) = v(y) + \int_{y_1}^{x_1} D_1 v(s, x_2)\, \mathrm{d}s + \int_{y_2}^{x_2} D_2 v(y_1, s)\, \mathrm{d}s$.

p. 44, l. 13: of the **should be** of the absolute values of the

p. 44, l. 12-: $\min_j U_j \le \min\{U_0, U_M, 0\}$ **should be** $\min_j U_j \ge \min\{U_0, U_M, 0\}$

p. 45, l. 2-: **delete** $+b_j(u'(x_j) - \hat{\partial} u(x_j))$

p. 46, l. 12-: inter **should be** interior

p. 49, l. 17: dominant **should be** dominant, i.e., $\sum_{j \ne i} |a_{ij}| \le a_{ii}$

p. 49, l. 17: Hint: assume $a_j \pm \frac{1}{2} h b_j \ge 0$.

p. 54, l. 5: with $\|v\|_{K_j} = \|v\|_{L_2(K_j)}$ and $|v|_{2,K_j} = |v|_{H^2(K_j)}$

p. 54, l. 10: $)^{1/2}$ **should be** $\Big)^{1/2}$

p. 55, l. 9: v **should be** u

p. 56, l. 21: $\le s$ **should be** $\le k$

p. 61, l. 11-: . **should be** ,

p. 65, l. 12: We then find **should be** We then find, for $2 \le s \le r$,

p. 65, l. 13: r **should be** s

p. 65, l. 14: These ... **should be** These estimates thus show a reduced convergence rate $O(h^s)$ if $v \in H^s$ with $s < r$.

p. 73, l. 20-: $\|I_h v - v\|_{\mathcal{C}(K_j)}$ **should be** $\|I_h v - v\|_{\mathcal{C}(K_j)} = \|I_h(v - Q_1 v) + (Q_1 v - v)\|_{\mathcal{C}(K_j)}$

p. 81, l. 11: dimension n **should be** dimension m

p. 87, l. Example 6.2: $\int_0^1$ **should be** $\int_0^b$

p. 88, l. 1: a_0 **should be** $a_0 > 0$

p. 88, l. 9: $a_{j+1/2}U_{j+1} + (a_{j+1/2} + a_{j-1/2})U_j - a_{j-1/2}U_{j-1}$ **should be** $a_{j+1/2}U_{j+1} - (a_{j+1/2} + a_{j-1/2})U_j + a_{j-1/2}U_{j-1}$

p. 93, l. Problem 6.3: Assume that Ω is such that (3.36) holds.

p. 96, l. 3: he **should be** the

p. 97, l. 7-: $g = P^{-1}u$ **should be** $g = P^{-1}f$

p. 112, l. 11: Bu **should be** By

p. 112, l. 18: has **should be** have

p. 115, l. 11: $\hat{v}_j^k \mathrm{e}^{-\lambda_j t}$ **should be** $\hat{v}_i \mathrm{e}^{-\lambda_i t}$

p. 115, l. 3-: C_1 **should be** $\frac{1}{2}C_1$

p. 117, l. 8: t^{-k} **should be** $t^{-m-s/2}$

p. 117, l. 3-: $D_t^m E(t) v(\cdot, t)$ **should be** $D_t^m E(t) v$

p. 117, l. 15: (6.4) **should be** Theorem 6.4

p. 119, l. 4: $D_t e$ **should be** $D_t E$

p. 119, l. (8.27): $=$ **should be** $\le$

p. 123, l. 3: $(\bar{x}, \bar{t})$ **should be** $(\tilde{x}, \tilde{t})$

p. 124, l. 15: $|u(x,t)| \le \mathrm{e}^{c|x|^2}$ **should be** $|u(x,t)| \le M\mathrm{e}^{c|x|^2}$

p. 133, l. 3: $\sum_p a_p \mathrm{e}^{\mathrm{i}(j-p)\xi_0}$ **should be** $\epsilon \sum_p a_p \mathrm{e}^{\mathrm{i}(j-p)\xi_0}$

p. 150, l. 1-: **should be** Since $u_h(t) \in S_h$ we may choose $\chi = u_h(t)$ ...

p. 150, l. 1-: $U^n \in S_h$ **should be** $u_h \in S_h$

p. 150, l. 1-: $\chi = u$ **should be** $\chi = u_h$

p. 154, l. 7: 10.1 **should be** 10.3

p. 155, l. 1: $\Big(\int_0^t \|\rho_t\|_2 \, \mathrm{d}s\Big)^{1/2}$ **should be** $\Big(\int_0^t \|\rho_t\|^2 \, \mathrm{d}s\Big)^{1/2}$

p. 155, l. 9-: v **should be** w (four times)

p. 155, l. 3-: v **should be** w

p. 156, l. 12: Φ **should be** Φ_j

p. 158, l. 4-: method **should be** a method

p. 160, l. 2-: and (8.18). **should be** (8.18), and Problem 8.10.

p. 165, l. 9: **delete** which we may assume to be symmetric,

p. 169, l. 1: 11.2 **should be** 11.3

p. 169, l. 10-: bounded **should be** bounded or unbounded

p. 179, l. 16: $+\|f\|\|u\|$ **should be** $+2\|f\|\|u\|$ and $C_1 = 1$

p. 204, l. 5: 13.3 **should be** 13.1

p. 226, l. 6: $w = \lambda v$ **should be** $w = \lambda v$ or $v = \lambda w$

p. 227, l. (A.4): w **should be** u

p. 233, l. 14: for $1 \le p < \infty$. **should be** for $1 \le p < \infty$, if Γ is sufficiently smooth.

p. 232, l. 3: The latter **should be** If Ω is bounded, then the latter

p. 233, l. 8: $1 \le p \le \infty$, and **should be** $1 \le p \le \infty$ if Ω is bounded, and

p. 234, l. 9: C^1 **should be** $\mathcal{C}^1$

p. 235, l. 10: for any l. **should be** for any $l \ge k$, if Γ is sufficiently smooth.

p. 237, l. 14-: $\mathcal{C}(\bar\Omega) \subset H^k(\Omega)$ **should be** $H^k(\Omega) \subset \mathcal{C}(\bar\Omega)$

p. 237, l. 4-: $\mathcal{C}^\ell(\bar\Omega) \subset H^k(\Omega)$ **should be** $H^k(\Omega) \subset \mathcal{C}^\ell(\bar\Omega)$

p. 239, l. 4: $L_2(\mathbf{R})$ **should be** $L_2(\mathbf{R}^d)$

p. 240, l. 3: $\mathrm{e}^{-\mathrm{i}x\cdot\xi}$ **should be** $\mathrm{e}^{-\mathrm{i}z\cdot\xi}$

p. 242, l. 5: $\|v\|_{W_1^2} \le |\Omega|^{1/2}\|v\|_{H^2}$ **should be** $\|v\|_{W_1^2} \le C\|v\|_{H^2}$

p. 242, l. 12: $\nabla \hat v$ **should be** $\hat\nabla \hat v$

Here is an improved version of Theorem 6.4.

Theorem 1. *The eigenfunctions* $\{\varphi_j\}_{j=1}^{\infty}$ *of (6.5) form an orthonormal basis for* L_2*. The series* $\sum_{j=1}^{\infty} \lambda_j (v, \varphi_j)^2$ *is convergent if and only if* $v \in H_0^1$*. Moreover,*

$$\|\nabla v\|^2 = a(v, v) = \sum_{j=1}^{\infty} \lambda_j (v, \varphi_j)^2, \quad \textit{for all } v \in H_0^1. \tag{1}$$

Proof. By our above discussion it follows that for the first statement it suffices to show (6.13) for all v in H_0^1, which is a dense subspace of L_2. We shall demonstrate that

$$\Big\| v - \sum_{j=1}^{N} (v, \varphi_j) \varphi_j \Big\| \le \lambda_{N+1}^{-1/2} \|\nabla v\|, \quad \text{for all } v \in H_0^1, \tag{2}$$

which then implies (6.13) in view of Theorem 6.3.

To prove (2), set $v_N = \sum_{j=1}^{N} (v, \varphi_j) \varphi_j$ and $r_N = v - v_N$. Then $(r_N, \varphi_j) = 0$ for $j = 1, \dots, N$, so that

$$\frac{\|\nabla r_N\|^2}{\|r_N\|^2} \ge \inf \Big\{ \|\nabla v\|^2 : v \in H_0^1, \|v\| = 1, (v, \varphi_j) = 0, j = 1, \dots, N \Big\} = \lambda_{N+1},$$

and hence

$$\|r_N\| \le \lambda_{N+1}^{-1/2} \|\nabla r_N\|.$$

It now suffices to show that the sequence $\|\nabla r_N\|$ is bounded. We first recall from Theorem 6.1 that $a(\varphi_i, \varphi_j) = 0$ for $i \ne j$, so that $a(r_N, v_N) = 0$. Hence $a(v, v) = a(v_N, v_N) + 2a(v_N, r_N) + a(r_N, r_N) = a(v_N, v_N) + a(r_N, r_N)$ and

$$\|\nabla r_N\|^2 = a(r_N, r_N) = a(v, v) - a(v_N, v_N) \le a(v, v) = \|\nabla v\|^2,$$

which completes the proof of (2).

For the proof of the second statement, we first note that, for $v \in H_0^1$,

$$\sum_{j=1}^{N} \lambda_j (v, \varphi_j)^2 = a(v_N, v_N) = a(v, v) - a(r_N, r_N) \le a(v, v),$$

and we conclude that $\sum_{j=1}^{\infty} \lambda_j (v, \varphi_j)^2 < \infty$. Conversely, we assume that $v \in L_2$ and $\sum_{j=1}^{\infty} \lambda_j (v, \varphi_j)^2 < \infty$. We already know that $v_N \to v$ in L_2 as $N \to \infty$. To obtain convergence in H^1 we note that, with $M > N$,

$$\alpha \|v_N - v_M\|_1^2 \le \|\nabla (v_N - v_M)\|^2 = \sum_{j=N+1}^{M} \lambda_j (v, \varphi_j)^2 \to 0 \text{ as } N \to \infty.$$

Hence, v_N is a Cauchy sequence in H^1 and converges to a limit in H^1. Clearly, this limit is the same as v. By the trace theorem (Theorem A.4) v_N is also a Cauchy sequence in $L_2(\Gamma)$, and since $v_N = 0$ on Γ we conclude that $v = 0$ on Γ. Hence, $v \in H_0^1$. Finally, (1) is obtained by letting $N \to \infty$ in $a(v_N, v_N) = \sum_{j=1}^{N} \lambda_j (v, \varphi_j)^2$. □

Here is an improved version of Theorem 13.1.

Theorem 2. *Let u_h and u be the solutions of (13.2) and (13.1). Then we have, for $t \ge 0$,*

$$
\begin{aligned}
\|u_{h,t}(t) - u_t(t)\| &\le C\Big(|v_h - R_h v|_1 + \|w_h - R_h w\|\Big) \\
&\quad + Ch^2\Big(\|u_t(t)\|_2 + \int_0^t \|u_{tt}\|_2 \, ds\Big), \\
\|u_h(t) - u(t)\| &\le C\Big(|v_h - R_h v|_1 + \|w_h - R_h w\|\Big) \\
&\quad + Ch^2\Big(\|u(t)\|_2 + \int_0^t \|u_{tt}\|_2 \, ds\Big), \\
|u_h(t) - u(t)|_1 &\le C\Big(|v_h - R_h v|_1 + \|w_h - R_h w\|\Big) \\
&\quad + Ch\Big(\|u(t)\|_2 + \int_0^t \|u_{tt}\|_1 \, ds\Big).
\end{aligned}
$$

Proof. Writing as usual

$$u_h - u = (u_h - R_h u) + (R_h u - u) = \theta + \rho,$$

we may bound ρ and ρ_t as in the proof of Theorem 10.1 by

$$\|\rho(t)\| + h|\rho(t)|_1 \le Ch^2\|u(t)\|_2, \quad \|\rho_t(t)\| \le Ch^2\|u_t(t)\|_2. \tag{3}$$

For $\theta(t)$ we have, after a calculation analogous to that in (10.14),

$$(\theta_{tt}, \chi) + a(\theta, \chi) = -(\rho_{tt}, \chi), \quad \forall \chi \in S_h, \quad \text{for } t > 0. \tag{4}$$

Imitating the proof of Lemma 13.1, we choose $\chi = \theta_t$:

$$\tfrac{1}{2}\frac{d}{dt}\big(\|\theta_t\|^2 + |\theta|_1^2\big) \le \|\rho_{tt}\|\,\|\theta_t\|.$$

After integration in t we obtain

$$
\begin{aligned}
\|\theta_t(t)\|^2 + |\theta(t)|_1^2 &\le \|\theta_t(0)\|^2 + |\theta(0)|_1^2 + 2\int_0^t \|\rho_{tt}\|\,\|\theta_t\| \,\mathrm{d}s \\
&\le \|\theta_t(0)\|^2 + |\theta(0)|_1^2 + 2\int_0^t \|\rho_{tt}\| \,\mathrm{d}s \max_{s\in[0,t]} \|\theta_t\| \\
&\le \|\theta_t(0)\|^2 + |\theta(0)|_1^2 + 2\Big(\int_0^T \|\rho_{tt}\| \,\mathrm{d}s\Big)^2 + \tfrac{1}{2}\Big(\max_{s\in[0,T]} \|\theta_t\|\Big)^2,
\end{aligned}
$$

for $t \in [0, T]$. This implies

$$\tfrac{1}{2}\Big(\max_{s\in[0,T]} \|\theta_t\|\Big)^2 \le \|\theta_t(0)\|^2 + |\theta(0)|_1^2 + 2\Big(\int_0^T \|\rho_{tt}\| \,\mathrm{d}s\Big)^2$$

and hence

$$\|\theta_t(t)\|^2 + |\theta(t)|_1^2 \le 2\|\theta_t(0)\|^2 + 2|\theta(0)|_1^2 + 4\Big(\int_0^T \|\rho_{tt}\| \,\mathrm{d}s\Big)^2,$$

for $t \in [0, T]$. In particular this holds with $t = T$ where T is arbitrary. Using also bounds for ρ_{tt} similar to (3), we obtain

$$
\begin{aligned}
\|\theta_t(t)\| + \|\theta(t)\| &\le C\Big(\|\theta_t(t)\| + |\theta(t)|_1\Big) \\
&\le C\Big(\|w_h - R_h w\| + |v_h - R_h v|_1\Big) + Ch^2\int_0^t \|u_{tt}\|_2 \,\mathrm{d}s,
\end{aligned}
$$

and

$$|\theta(t)|_1 \leq C\Big(\|w_h - R_h w\| + |v_h - R_h v|_1\Big) + Ch\int_0^t \|u_{tt}\|_1 \,\mathrm{d}s.$$

Together with the bounds in (3) this completes the proof. □

We remark that the choices $v_h = R_h v$ and $w_h = R_h w$ in Theorem 2 give optimal order error estimates for all the three quantities considered, but that other optimal choices of v_h could cause a loss of one power of h, because of the gradient in the first term on the right. This can be avoided by a more refined argument. The regularity requirement on the exact solution can also be reduced.

`/stig`