Partial Domination - the Isolation Number of a Graph

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Abstract. We prove the following result: If G be a connected graph on $n \geq 6$ vertices, then there exists a set of vertices D with $|D| \leq \frac{n}{3}$ and such that $V(G) \setminus N[D]$ is an independent set, where $N[D]$ is the closed neighborhood of D. Furthermore, the bound is sharp. This seems to be the first result in the direction of partial domination with constrained structure on the graph induced by the non-dominated vertices, which we further elaborate in this paper.

1. Introduction

Many variants of the basic topic of domination can be formulated as follows. Let G be a graph and D a set of vertices of G such that $D \cup N(D) = V(G)$, where $N(D)$ is the set of neighbors of vertices in D that do not belong to D. Given two graph properties $\mathcal{P}$ and $\mathcal{Q}$, we say that D is a $(\mathcal{P}, \mathcal{Q})$-dominating set if the graph induced by D has property $\mathcal{P}$ and the graph induced by $N(D)$ has property $\mathcal{Q}$.

Among the best known properties $\mathcal{P}$ are: *connected domination*, where the graph induced by D is connected (see [8, 11, 21]), *k-connected domination*, where the graph induced by D is k-connected (see [22, 29]), *total domination* where the graph induced by D has no isolates (see [19, 20]), *paired domination*, where the graph induced by D has a perfect matching (see [12]), and many more.

Among the best known properties $\mathcal{Q}$ are: *k-domination*, where $|N(u) \cap D| \geq k$ for every $u \in N(D)$ (see [14, 18]), *locating domination*, where $N(u) \cap D \neq N(v) \cap D$ for any $u, v \in N(D)$ (see [28]), *fair domination*, introduced in [7], where $|N(u) \cap D| = |N(v) \cap D|$ for any $u, v \in N(D)$, and many more.

There are also variants of domination which set a condition on both D and $N(D)$, like the *k-tuple domination*, where $|N[u] \cap D| \geq k$ for every $u \in V(G)$ (see [10, 16, 27]), or the *identifying codes*, where $\emptyset \neq N(u) \cap D \neq N(v) \cap D \neq \emptyset$ for any $u, v \in V(G)$ (see [15]). See also the survey [9] for more information on multiple domination parameters.

Here we are going to present, for the first time to our knowledge, the following extension of this model to the case that $D \cup N(D) \neq V(G)$, that is, we do not necessarily assume that D is a dominating set.

The reason for doing so is because there may be situations in which we do not necessarily need to dominate all the vertices in a graph. Let us illustrate this by the following example. Suppose G represents a communication network, and a security agency wants to detect every conversation between two members,

i.e. two adjacent vertices, in the network. We can imagine some centers forming the vertices of D which can listen to all nodes in $N[D]$ and, as long as $V \setminus N[D]$ is an independent set, they can still listen to all conversations having located microphones in all vertices of D.

Let G be a graph, $D \subseteq V(G)$ a set, $N(D)$ the neighborhood of D and $R(D) = V(G) \setminus N[D]$ the remainder with respect to D. When $R(D) = \emptyset$, D is a dominating set.

So, in addition to the properties $\mathcal{P}$ and $\mathcal{Q}$ imposed on D and $N(D)$, we would like to impose a property $\mathcal{R}$ on $R(D)$, where $\mathcal{R}$ can be the property of $R(D)$ being a k-independent set or inducing a forest, a planar graph, a K_k-free graph, or a k-colorable graph, to mention some examples.

Problems involving partitions of this type $D \cup N(D) \cup R(D) = V(G)$ with D having property $\mathcal{P}$, $N(D)$ having property $\mathcal{Q}$ and $R(D)$ having property $\mathcal{R}$ are called constrained partial domination problems.

In this paper, we shall consider simple constrained partial domination problems with no restriction on properties $\mathcal{P}$ or $\mathcal{Q}$ but with $R(D)$ having certain property $\mathcal{R}$ as mentioned above: being a k-independent set, or inducing a forest, a planar graph, a K_k-free graph or a k-colorable graph, etc.

1.1. Notation

For notation and Graph Theory terminology we in general follow [31]. Specifically, let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ of order $n(G) = |V|$ and edge set $E = E(G)$ of size $m(G) = |E|$. For a subset $S \subseteq V$, the *open neighborhood* of S is the set $N_G(S) = \{u \in V \setminus S \mid uv \in E, v \in S\}$, while the *closed neighborhood* of S is the set $N_G[S] = N_G(S) \cup S$. When $S = \{v\}$, we write $N_G(v)$ and $N_G[v]$ for the open and closed neighborhoods of v, respectively. The *degree* of a vertex v is $\deg_G(v) = |N_G(v)|$. When the graph is clear from the context, we may write $N(S), N[S], N(v), N[v], \deg(v)$. The *minimum degree* $\delta(G)$ of a graph G is the minimum among all vertex degrees of G. Likewise, the *maximum degree* $\Delta(G)$ of G is the maximum among all vertex degrees of G. We call a vertex v *isolate* or say it is *isolated* in G if $\deg_G(v) = 0$. Moreover, a vertex v is called *leaf* if $\deg_G(v) = 1$.

The *complement* $\overline{G}$ of a graph $G = (V, E)$ is the graph consisting of the vertex set V and all edges between vertices of V that do not belong to E. Given a subset $S \subseteq V$, the graph $G[S]$ denotes the subgraph of G induced by S and, for an integer $t \geq 1$, tG denotes the graph consisting of t vertex disjoint copies of G. Given a graph H, we say that G is *H-free* if G does not contain H as a subgraph (observe that H has not to be necessarily induced).

The *complete graph* on n vertices is denoted by K_n and we write $K_{p,q}$ for the *complete bipartite graph* with partition sets of p and q vertices. Further, the *cycle* and the *path* on n vertices are written as C_n and P_n, respectively. We define the *cartesian product* of two graphs G_1 and G_2 as the graph $G_1 \square \ast G_2$ with vertex set $V(G_1) \times V(G_2)$ and such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1 v_1 \in E(G_1)$. For $t, s \geq 1$, the graph $G = P_s \square P_t$ is called *grid graph*.

We call a subset $S \subseteq V$ *dominating* in $G = (V, E)$ if $|N(v) \cap S| \geq 1$ for all $v \in V \setminus S$. The minimum cardinality of a dominating set of G is denoted by $\gamma(G)$. On the other hand, S is called *independent* if $G[S]$ is the empty graph and with $\alpha(G)$ we denote the maximum cardinality of an independent set of G. For $k \geq 0$, S is a k-*independent set* if $\Delta(G[S]) \leq k$ and $\alpha_k(G)$ denotes the maximum cardinality of a k-independent set of G. Observe that 0-independent is the same as independent.

1.2. The Isolation Number of a Graph

We will now introduce a more formal definition and language for the main protagonist of this paper: the isolation number of a graph.

Let $G = (V, E)$ be a graph and $\mathcal{F}$ a family of graphs. We call a set of vertices $S \subseteq V$ $\mathcal{F}$-*isolating* if the graph induced by the set $R(S) = V \setminus N[S]$ contains no member of $\mathcal{F}$ as a subgraph.

Example 1.1. *Consider the following families $\mathcal{F}$ of graphs.*

(1) If $\mathcal{F} = \{K_1\}$, then an $\mathcal{F}$-isolating set coincides with the usual definition of a dominating set.
(2) If $\mathcal{F} = \{K_2\}$, then the vertices not dominated by the $\mathcal{F}$-isolating set form an independent set.

(3) *If $\mathcal{F} = \{K_{1,k+1}\}$ for an integer $k \geq 0$, then the set of vertices not dominated by the $\mathcal{F}$-isolating set induces a graph of maximum degree at most k or, in other words, these vertices form a k-independent set (see [6, 9] for more information and recent results).*

(4) *If $\mathcal{F} = \{C_k \mid k \geq 3\}$, then the vertices not dominated by the $\mathcal{F}$-isolating set induce a forest.*

(5) *If $\mathcal{F}$ is the family of all trees on k vertices, then the non-dominated vertices form a subgraph whose components have all order at most $k - 1$.*

The minimum cardinality of an $\mathcal{F}$-isolating set of a graph G will be denoted $\iota(G, \mathcal{F})$ and called the $\mathcal{F}$-*isolation number* of G. When $\mathcal{F} = \{H\}$, we will set $\iota(G, \mathcal{F}) = \iota_H(G)$. In case $\mathcal{F} = \{K_{1,k+1}\}$, we shall use the notation $\iota_k(G)$, and when $k = 0$ we will write for short $\iota(G)$ instead of $\iota_0(G)$. Finally, $\iota_k(G)$ will be called the k-*isolation number* and $\iota(G)$ just *isolation number*.

1.3. Structure of the Paper

The paper is organized as follows:

In Section 2 we give some basic properties and examples concerning $\iota(G, \mathcal{F})$, relating $\mathcal{F}$-isolating sets to dominating sets, as well as some concrete constructions that will be useful in later sections. Later on we will mainly deal with $\iota(G)$ and sometimes $\iota_k(G)$.

In Section 3, we consider upper bounds on $\iota(G)$ and $\iota_k(G)$ in terms of order, maximum degree and minimum degree with special emphasis on G being a connected graph, and prove some sharpness results as well.

In Section 4, we consider lower bounds on $\iota(G)$ and $\iota_k(G)$ in terms of average degree, maximum degree and minimum degree and prove some sharpness results.

In Section 5, we consider some classes of graphs such as trees, maximal outerplanar graphs, claw-free graphs and grid graphs, and compute some sharp results for $\iota(G)$ for graphs in these families.

In Section 6, we deal with Nordhaus-Gaddum type results for $\iota(G) + \iota(\overline{G})$ and prove sharp upper bounds.

In the closing section 7, we introduce some open problems for further research.

2. Basic Examples and Facts

Example 2.1. *Consider the following examples.*

(1) *For $k \leq n - 1$, $\iota_k(\overline{K_n}) = 0$ and $\iota_k(K_n) = 1$.*

(2) *$\iota(C_n) = \lceil n/4 \rceil$, $\iota_1(C_n) = \lceil \frac{n}{5} \rceil$, $\iota(P_n) = \lceil (n-1)/4 \rceil$ and $\iota_1(P_n) = \lceil \frac{n-2}{5} \rceil$.*

(3) *If P is the Petersen-graph, $\iota(P) = 3$, $\iota_1(P) = 2$ and $\iota_2(P) = 1$.*

(4) *ι is a non-monotone parameter with respect to edge-deletion:*

$$\iota(K_5) = 1 < \iota(C_5) = 2 > \iota(P_5) = 1.$$

Lemma 2.2. *Let G be a graph on n vertices and $\mathcal{F}$ and $\mathcal{F}'$ be two families of graphs. The following assertions hold.*

(i) *If $\mathcal{F}' \subseteq \mathcal{F}$, then $\iota(G, \mathcal{F}') \leq \iota(G, \mathcal{F})$.*

(ii) *If $F_1, F_2 \in \mathcal{F}$ such that $F_1 \subseteq F_2$ and $\mathcal{F}' = \mathcal{F} \setminus \{F_2\}$, then $\iota(G, \mathcal{F}) = \iota(G, \mathcal{F}')$.*

(iii) *If for all $F' \in \mathcal{F}'$ there is an $F \in \mathcal{F}$ such that $F \subseteq F'$, then $\iota(G, \mathcal{F}') \leq \iota(G, \mathcal{F})$.*

Proof. (i) Let $\mathcal{F}' \subseteq \mathcal{F}$ and let S be a minimum $\mathcal{F}$-isolating set of G. Then, clearly, S is an $\mathcal{F}'$-isolating set of G and thus $\iota(G, \mathcal{F}') \leq \iota(G, \mathcal{F})$.

(ii) By item (i), the inequality $\iota(G, \mathcal{F}') \leq \iota(G, \mathcal{F})$ is clear. So let S be a minimum $\mathcal{F}'$-isolating set of G. Then $G - N[S]$ is $\mathcal{F}'$-free and thus, in particular, F_1-free. Since F_1 is a subgraph of F_2, $G - N[S]$ has to be F_2-free, too. Hence, $G - N[S]$ is $\mathcal{F}$-free and we obtain $\iota(G, \mathcal{F}) \leq |S| = \iota(G, \mathcal{F}')$. Now the both inequalities imply the result.

(iii) Let S be a minimum $\mathcal{F}$-isolating set of G. Then $G - N[S]$ is $\mathcal{F}$-free. Since for all $F' \in \mathcal{F}'$ there is an $F \in \mathcal{F}$ such that $F \subseteq F'$, $G - N[S]$ is also $\mathcal{F}'$-free. Hence, S is an $\mathcal{F}'$-isolating set of G and thus $\iota(G, \mathcal{F}') \leq \iota(G, \mathcal{F})$. $\quad\square$

In the sequel of this paper and in view of Lemma 2.2 (ii), we will consider only families of graphs without inclusions among its members.

Lemma 2.3. *Let G be a graph on n vertices and $\mathcal{F}$ a family of graphs. The following assertions hold.*

(i) $\iota(G, \mathcal{F}) \le \gamma(G)$.
(ii) $\iota(G, \mathcal{F}) = \min\{\iota(G[A], \mathcal{F}) + \gamma(G[B]) \mid A \cup B \text{ is a partition of } V\}$.
(iii) $\iota(G, \mathcal{F}) = \min\{\gamma(G \setminus H) \mid H \text{ is an induced } \mathcal{F}\text{-free subgraph of } G\}$.

Proof. (i) Since every dominating set is also an $\mathcal{F}$-isolating set, clearly $\iota(G, \mathcal{F}) \le \gamma(G)$.
(ii) Let $V = A \cup B$ be a partition of V. Let I be a minimum $\mathcal{F}$-isolating set of $G[A]$ and D a minimum dominating set of $G[B]$. Then $I \cup D$ is an $\mathcal{F}$-isolating set of G and hence $\iota(G, \mathcal{F}) \le |I| + |D| = \iota(G[A]) + \gamma(G[B])$. Conversely, let S be a minimum $\mathcal{F}$-isolating set of G and let $A = V \setminus N[S]$ and $B = N[S]$. Then $G[A]$ is $\mathcal{F}$-free and hence $\iota(G[A], \mathcal{F}) = 0$. Moreover, $|S| \ge \gamma(G[B])$. Thus, $\iota(G, \mathcal{F}) = |S| \ge \iota(G[A], \mathcal{F}) + \gamma(G[B])$.
(iii) Let H be an induced $\mathcal{F}$-free graph of G. Then, by item (ii), $\iota(G, \mathcal{F}) \le \iota(H, \mathcal{F}) + \gamma(G \setminus H) = \gamma(G \setminus H)$. On the other side, let S be a minimum $\mathcal{F}$-isolating set of G and let $H = G - N[S]$. Then H is $\mathcal{F}$-free. Again by item(ii), it follows that $|S| = \iota(G, \mathcal{F}) \le \iota(H), \mathcal{F}) + \gamma(G \setminus H)$. □

Lemma 2.4. *Let G be a graph on n vertices. The following assertions hold.*

(i) *For a graph H,* $\iota_H(G) \le \gamma(H) \left\lfloor \frac{n}{n(H)} \right\rfloor$ *and this is sharp if* $\gamma(H) = 1$.

(ii) *For a family of graphs $\mathcal{F}$,* $\iota(G, \mathcal{F}) \le \sup_{F \in \mathcal{F}} \frac{\gamma(F)}{n(F)} n$.

Proof. (i) We will prove by induction on n that $\iota_H(G) \le \gamma(H) \lfloor \frac{n}{n(H)} \rfloor$. If $1 \le n \le n(H)$, then G is either H-free or H is a spanning subgraph of G, i.e. $n = n(H)$. In the first case we have $\iota_H(G) = 0$ and the statement follows trivially. In the second, using Lemma 2.3 (i), we have $\iota_H(G) \le \gamma(G) \le \gamma(H) = \gamma(H) \lfloor \frac{n}{n(H)} \rfloor$ and we are done. Hence, assume that G has order $n \ge n(H) + 1$ and suppose the statement holds for any graph with less than n vertices. If G is H-free, then clearly $\iota_H(G) = 0$ and we are done. Otherwise let us consider a copy H^* of H in G and let $G' = G \setminus H^*$. Then, by the induction hypothesis, $\iota_H(G') \le \gamma(H) \lfloor \frac{n(G')}{n(H)} \rfloor$. Let S be a minimum H-isolating set of G' and let D be a minimum dominating set of H^*. Since $G - (N_G[S \cup D])$ is a subgraph of $G' - N_{G'}[S]$ and the latter is H-free, $S \cup D$ is an H-isolating set of G. Hence,

$$\iota_H(G) \le |S \cup D| = |S| + \gamma(H^*) = |S| + \gamma(H) \le \gamma(H) \left\lfloor \frac{n(G')}{n(H)} \right\rfloor + \gamma(H) = \gamma(H) \left\lfloor \frac{n}{n(H)} \right\rfloor.$$

For the sharpness, consider the graph $G = tH$, where H a nontrivial graph with $\gamma(H) = 1$. Then $n = n(G) = t \cdot n(H)$ and, clearly, $\iota_H(G) = t = \lfloor \frac{n}{n(H)} \rfloor = \gamma(H) \lfloor \frac{n}{n(H)} \rfloor$.
(ii) Let $q(\mathcal{F}) = \sup_{F \in \mathcal{F}} \frac{\gamma(F)}{n(F)}$. We will prove the statement by induction on n. If G is $\mathcal{F}$-free, then we have $\iota(G, \mathcal{F}) = 0 \le q(\mathcal{F})n$. If $F \subseteq G$ and $n(F) = n$ for some $F \in \mathcal{F}$, then let D be a minimum dominating set of F. Clearly, D is a dominating set of G and thus, using Lemma 2.3 (i), $\iota(G, \mathcal{F}) \le \gamma(G) \le \gamma(F) \le q(\mathcal{F})n(F) = q(\mathcal{F})n$. This covers the cases where $n \le \min\{n(F) \mid F \in \mathcal{F}\}$. Suppose that $n > \min\{n(F) \mid F \in \mathcal{F}\}$ and that we have proved the statement for all graphs of order less than n. We can assume that G is not $\mathcal{F}$-free for otherwise it is done. Also the case $F \subseteq G$ and $n(F) = n(G)$ for an $F \in \mathcal{F}$ works as above. Hence, we can assume that there is a graph $F \in \mathcal{F}$ contained in G such that the graph G' obtained after deleting the vertices of F in G is not empty. Let S be a minimum $\mathcal{F}$-isolating set of G' and D a minimum dominating set of F. By the induction hypothesis, $|S| = \iota(G', \mathcal{F}) \le q(\mathcal{F})n(G')$. Moreover, $|D| = \gamma(F) \le q(\mathcal{F})n(F)$. Since $S \cup D$ is an $\mathcal{F}$-isolating set of G, it follows that

$$\iota(G, \mathcal{F}) \le |S| + |D| \le q(\mathcal{F})n(G') + q(\mathcal{F})n(F) = q(\mathcal{F})n.$$

□

Corollary 2.5. *Let G be a graph on n vertices. Then the following statements hold.*

  (i) Let $k \geq 0$ be an integer such that $\Delta(F) \geq k + 1$ for all $F \in \mathcal{F}$. Then $\iota(G, \mathcal{F}) \leq \iota_k(G)$.

  (ii) If $\mathcal{F}$ is a family of non-empty graphs, then $\iota(G, \mathcal{F}) \leq \iota(G)$.

  (iii) $\iota_k(G) \leq \frac{1}{k+2}n$ and this is sharp;

Proof. (i) Since $\Delta(F) \geq k + 1$, the graph $K_{1,k+1} \subseteq F$ for all $F \in \mathcal{F}$ and thus $\iota(G, \mathcal{F}) \leq \iota_k(G)$ follows from 2.2 (iii).
(ii) This is item (i) for $k = 0$.
(iii) This follows directly by Lemma 2.4 (i) using $H = K_{1,k+1}$. $\square$

Given a graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, we define a bipartite graph $B(G)$ the following way. Let $V_1 = \{v_1^1, v_2^1, \ldots, v_n^1\}$ and $V_2 = \{v_1^2, v_2^2, \ldots, v_n^2\}$ be the two partite sets of $B(G)$ and, for $i, j \in \{1, 2, \ldots, n\}$, let v_i^1 be adjacent to v_j^2 if and only if either $i = j$ or v_i is adjacent to v_j in G. Note that $|B(G)| = 2n$ and $\delta(B(G)) = \delta(G) + 1$.

The following (technical) theorem will be used several times in the sequel.

Theorem 2.6. *Let G be a graph of order n. Let $\mathcal{F}$ be a family of graphs with $r = \min\{n(F) \mid F \in \mathcal{F}\}$. Then*

  (i) $\gamma(G) \leq \iota(G \square K_r, \mathcal{F}) \leq \min\{n, (r-1)\gamma(G) + \iota(G, \mathcal{F}), r\iota(G, \mathcal{F}) + \alpha(G, \mathcal{F})\} \leq \min\{n, r\gamma(G)\}$

  (ii) $\gamma(G) \leq \iota(B(G)) \leq \gamma(B(G)) \leq 2\gamma(G)$.

Proof. (i) Let $V(G) = \{x_1, x_2, \ldots, x_n\}$ and let $H = G \square K_r$ be given by the vertex set $V(H) = V_1 \cup V_2 \cup \ldots \cup V_r$ with $V_i = \{x_1^i, x_2^i, \ldots, x_n^i\}$ where $H[V_i] \cong G$ for $i = 1, 2, \ldots, r$ and such that x_k^i is adjacent to x_k^j for any $1 \leq i < j \leq r$ and $1 \leq k \leq n$. For the first inequality, let S be a minimum $\mathcal{F}$-isolating set of H. Let Q be the set of vertices which are not dominated by S in H and let $Q_i = V_i \cap Q$ and $S_i = V_i \cap S$, $1 \leq i \leq r$. We will show that the set $D = \bigcup_{i=1}^{r}\{x_k^1 \mid x_k^i \in S_i\}$ is a dominating set of $H[V_1] \cong G$. Fix an integer $k \in \{1, 2, \ldots, r\}$. If $x_k^1 \in V_1 \setminus Q_1$ then either $x_k^1 \in S_1$ or x_k^1 is dominated by a vertex in S_1 or x_k^1 is dominated by a vertex $x_k^j \in S_j$ for some $j \in \{2, 3, \ldots, r\}$. The first and third cases imply that $x_k^1 \in D$ while from the second case follows that x_k^1 is dominated by a vertex in D. Hence suppose that $x_k^1 \in Q_1$. Note that, for any $\ell \in \{1, 2, \ldots, n\}$, $H[\{x_\ell^i \mid 1 \leq i \leq r\}] \cong K_r$ is not $\mathcal{F}$-free. Thus, since $H[Q]$ is $\mathcal{F}$-free, there has to be a $j \in \{2, 3, \ldots, r\}$ such that $x_k^j \notin Q_j$. Hence, either $x_k^j \in S_j$, or there is a vertex $x_{k'}^j \in S_j$ that dominates $x_{k'}^j$, or x_k^j is dominated by a vertex $x_k^{j'} \in S_{j'}$ for some $j' \in \{1, 2, \ldots, r\}$, $j \neq j'$. This implies that either $x_k^1 \in D$ or $x_{k'}^1$ dominates x_k^1. Hence, D is a dominating set of $H[V_1] \cong G$ and thus $\gamma(G) \leq |D| \leq \sum_{i=1}^{r}|S_i| = |S| = \iota(H, \mathcal{F})$.
For the second inequality, note first that V_1 is a dominating set of H and hence also an $\mathcal{F}$-isolating set of H. Thus $\iota(H, \mathcal{F}) \leq n$. Now let D and S be, respectively, a minimum dominating set and a minimum $\mathcal{F}$-isolating set of G and define $D_i = \{x_k^i \mid x_k \in D\}$ and $S_i = \{x_k^i \mid x_k \in S\}$, for $1 \leq i \leq r$. Then both $\bigcup_{i=1}^{r-1}D_i \cup S_r$ and $\bigcup_{i=1}^{r-1}S_i \cup (V_r \setminus S_r)$ are $\mathcal{F}$-isolating sets of H. Now the properties $|D_i| = \gamma(G)$, $|S_i| = \iota(G, \mathcal{F})$ and $|V_r \setminus S_r| \leq \alpha(G, \mathcal{F})$ yield the desired inequality.
The last inequality is due to Lemma 2.3 (i).
(ii) For the first inequality, let S be a minimum isolating set of $B(G)$. Let $D = \{v_k \mid v_k^i \in S \text{ for some } i\}$. Let $Q = V(B(G)) \setminus N_{B(G)}[S]$ be the set of isolated vertices in $B(G)$. Let v_l be a vertex in $V(G) \setminus D$. Then $v_l^i \notin S$ for $i = 1, 2$. Since v_l^1 is adjacent to v_l^2, one of both vertices, say v_l^1, is not in Q. This implies that $v_l^1 \in N_{B(G)}(S)$. Hence, there is a neighbor of v_l^1 in S and thus there is a vertex in D which is adjacent to v_l in G. It follows that D is a dominating set of G, which gives us $\gamma(G) \leq |D| \leq |S| = \iota(B(G))$.
The second inequality follows from Lemma 2.3 (i). For the last inequality, let D be a minimum dominating set of G. Then $D' = \{v_l^1, v_l^2 \mid v_l \in D\}$ is a dominating set of $B(G)$ and thus $2\gamma(G) = 2|D| = |D'| \geq \gamma(B(G))$. $\square$

3. Upper Bounds

In this section, we are going to present some upper bounds on $\iota(G)$ and $\iota_k(G)$ in terms of the order and the maximum and minimum degree of G. We will make emphasis on weather the graph is connected or not.

3.1. Bounds in Terms of Order: Connected Graphs

The following theorem, which is mentioned in the abstract, is one of the main results of this paper and a prototype of problems we pose in Section 7.

Theorem 3.1. *Let G be a connected graph on $n \geq 3$ vertices and different from C_5. Then $\iota(G) \leq \frac{n}{3}$ and this bound is sharp.*

Proof. We will prove the statement by induction on n. Let G be a connected graph on $n \geq 3$ vertices and different from the cycle C_5. If $n = 3$, evidently $\iota(G) = 1 = \frac{n}{3}$. So suppose $n > 3$ and assume the statement is valid for all graphs different from C_5 and with less than n vertices. Select a vertex $r \in V(G)$ and let T be a BFS-tree of G rooted in r. Denote by L the set of leaves of T. Let $V_0 = \{r\}$ and let V_i denote the set of vertices of the i-th generation after r. Let ℓ be the last generation of vertices in T. So, $V = V_0 \cup V_1 \cup V_2 \cup \ldots \cup V_\ell$ and, since $n > 3$, $\ell \geq 1$.

If $\ell = 1$, then $\{r\}$ is an isolating set of G and the statement holds trivially. Hence, we may assume that $\ell \geq 2$.

Suppose now that, for some $k \leq \ell - 1$, there is a vertex $u \in V_k$ such that $N_T(u) \cap V_{k+1} \subseteq L$ and $|N_T(u) \cap V_{k+1}| \geq 2$. Let $U = \{u\} \cup (N_T(u) \cap V_{k+1})$ and $G^* = G - U$. If $U = V(G)$, then clearly u is an isolating set of G and $\iota(G) = 1 \leq \frac{n}{3}$. Hence suppose $V(G) \setminus U \neq \emptyset$. Then G^* is connected. If $n(G^*) \leq 2$, then $\{u\}$ is an isolating set of G and $\iota(G) = 1 \leq \frac{n}{3}$ holds. If $G^* \cong C_5$, let $G^* = x_1x_2x_3x_4x_5x_1$ being x_1 the father of u in T. Then $\{u, x_3\}$ is an isolating set of G and, as $n \geq 8$, $\iota(G) \leq 2 \leq \frac{n}{3}$ is fulfilled. If G^* is different from C_5 and $n(G^*) \geq 3$, then, by the induction hypothesis, there is an isolating set S^* of G^* with $|S^*| \leq \frac{n-|U|}{3}$. Note that $S^* \cup \{u\}$ is an isolating set of G. Hence, since $|U| \geq 3$, we obtain $\iota(G) \leq \frac{n-|U|}{3} + 1 \leq \frac{n}{3}$ and we are done. So we can assume in the following that, for any $k \leq \ell - 1$, all vertices $u \in V_k \setminus L$ fulfill either
(i) u has only one child v and $v \in L$ or
(ii) u has grandchildren.

Since $\ell \geq 2$, there exists a vertex $u \in V_{\ell-2}$ such that it has grandchildren. Consider now the following cases.

Case 1: u has only one child v in T. Since $v \in V_{\ell-1}$, v has no grandchildren. Hence, by the assumption above, v has only one child, say w, which is a leaf in T. Let $G^* = G - \{u, v, w\}$. Then G^* is nonempty and connected. We distinguish the following subcases.

Subcase 1.1: $n(G^*) \leq 2$. Then $n \leq 5$ and it is straightforward to check that, with exception of $G \cong C_5$, $\iota(G) \leq \frac{n}{3}$.

Subcase 1.2: $G^* \cong C_5$. Then $n = 8$. Let $G^* = x_1x_2x_3x_4x_5x_1$ being x_1 the father of u in T. If $x_5 \notin N_G(w)$, then $\{u, x_3\}$ is an isolating set of G. Similarly, if $x_2 \notin N_G(w)$, then $\{u, x_4\}$ is an isolating set of G. If, on the other side, x_2 and x_5 are both neighbors of w in G, then, depending if $x_4 \notin N_G(u)$ or $x_4 \in N_G(u)$, either $\{w, x_2\}$ or $\{u, x_2\}$ is an isolating set of G. Hence, in all cases we obtain an isolating set with 2 vertices and so $\iota(G) \leq 2 \leq \frac{n}{3}$.

Subcase 1.3: $n(G^*) \geq 3$ and $G^* \neq C_5$. Then, by induction, there is an isolating set S^* of G^* with $|S^*| \leq \frac{n-3}{3}$. Since $S^* \cup \{v\}$ is an isolating set of G, we obtain easily $\iota(G) \leq \frac{n-3}{3} + 1 = \frac{n}{3}$.

Case 2: u has at least two children in T. Let A the set of children of u and B the set of grandchildren of u in T and define $U = \{u\} \cup A \cup B$. Let $B_1 = \{x \in B \mid N_G(x) \cap (V \setminus U) = \emptyset\}$ and $B_2 = B \setminus B_1$. Finally, set $U' = U \setminus B_2$ and let I be the set of isolated vertices in $G[B_1]$. Now we have two subcases.

Subcase 2.1: $B_1 \setminus I = \emptyset$. Then $\{u\}$ is an isolating set of $G[U']$. Now let $G^* = G - U'$. Clearly, G^* is either empty or connected. Namely, if $B_2 = \emptyset$, then G^* consists of the vertices from $T - U'$ which is a tree with a leaf in the father of u. On the other side, if $B_2 \neq \emptyset$, then by definition all its vertices are adjacent to some vertex of the tree $T - U$, which is also a tree with a leaf in the father of u. Again, we consider here three different subcases.

(i) $n(G^*) \leq 2$. If $n(G^*) = 0$, then $B_2 = \emptyset$ and $\{u\}$ is a separating set of G. If $n(G^*) = 1$, then, as all the vertices of B_2 have a neighbor in $V(G) - U$, it forces again $B_2 = \emptyset$. Hence, again, $\{u\}$ is an isolating set of G. Thus, in both cases we have $\iota(G) = 1 \leq \frac{n}{3}$. If $n(G^*) = 2$ let $V(G^*) = \{x, y\}$, being x the father of u in T. If $N_G(y) \cap B_1 = \emptyset$, then $B_1 \cup \{y\}$ is an independent set and, since u dominates $A \cup \{x\}$, $\{u\}$ is an isolating set of G and thus

$\iota(G) = 1 \leq \frac{n}{3}$. Finally, suppose that $N_G(y) \cap B_1 \neq \emptyset$. Then, $|B_1| \geq 1$ and, with $|A| \geq 2$ and $n(G^*) \geq 2$, we have $n \geq 6$. As clearly $\{u, x\}$ is an isolating set of G, it follows again $\iota(G) \leq 2 \leq \frac{n}{3}$.

(ii) $G^* \cong C_5$. Again, let $G^* = x_1 x_2 x_3 x_4 x_5 x_1$ being x_1 the father of u in T. Since the vertices in $B_1 = I$ form an independent set and have no neighbors in G^*, it is not difficult to see that $\{u, x_3\}$ is an isolating set of G (the set $A \cup \{x_1, x_2, x_4\}$ is dominated and te rest $B_1 \cup \{x_5\}$ is independent). Hence, as $n \geq 8$, $\iota(G) \leq 2 \leq \frac{n}{3}$.

(iii) $n(G^*) \geq 3$ and $G^* \neq C_5$. Let S^* be a minimum isolating set of G^*. By the induction hypothesis, $|S^*| \leq \frac{n-|U'|}{3}$. Moreover, since $N_G(B_1) \cap V(G^*) = \emptyset$ and B_1 is an independent set in G, $S^* \cup \{u\}$ is an isolating set of G. Hence, $\iota(G) \leq |S^* \cup \{u\}| \leq \frac{n-|U'|}{3} + 1 \leq \frac{n}{3}$ and we are done.

Subcase 2.2: $B_1 \setminus I \neq \emptyset$. Then $\delta(G[B_1 \setminus I]) \geq 1$ and $|B_1 \setminus I| \geq 2$. Let $y, z \in B_1$ be two adjacent vertices in G and let $x \in A$ be the father of y in T. Define $G^* = G - \{x, y, z\}$. Note that, by assumption and since $x \in V_{\ell-1}$, x can have only one child, which is y. This implies that G is connected and $n(G^*) \geq 2$. For the last, we distinguish the following three subcases.

(i) $n(G^*) = 2$. Since u has at least two children in T and $n(G^*) = 2$, $|A| = 2$, say, $A = \{x, v\}$. Then, clearly, $G^* = uv$. As z has to have a father in A different from x, it follows that G is either the 5-cycle $C = uxyzvu$ or the graph $C + vx$. Since by hypothesis $G \neq C$, it follows that $G = C + vx$. Thus, $\{x\}$ is an isolating set and $\iota(G) = 1 \leq \frac{n}{3}$.

(ii) $G^* \cong C_5$. Then $n = 8$. Let $G^* = u u_1 u_2 u_3 u_4 u$ and assume, without loss of generality that $u_1 \in A$ is the father of z in T. Recall that, in G, none of the vertices of $V(G) \setminus U$ is adjacent to the vertices in B_1. In particular, u_3 and y are not adjacent. Hence, it follows that $\{u, u_1\}$ is an isolating set of G, implying that $\iota(G) \leq 2 \leq \frac{n}{3}$.

(iii) $n(G^*) \geq 3$ and $G \neq C_5$. Then, by the induction hypothesis, there is an isolating set S^* of G^* with $|S^*| \leq \frac{n-3}{3}$. Since $S^* \cup \{y\}$ is an isolating set of G, we obtain again $\iota(G) \leq |S^*| + 1 \leq \frac{n-3}{3} + 1 = \frac{n}{3}$ and we are done.

For the sharpness, consider the following graphs.

(i) Let G_1 be the graph consisting of an arbitrary connected graph H_1 on n_1 vertices, each of which is attached to a K_2 by means of an edge. Then $\delta(G_1) = 1$ and $\iota(G_1) = n_1 = \frac{n(G_1)}{3}$.

(ii) Let G_2 be the graph consisting of an arbitrary connected graph H_2 on n_2 vertices, each of which is attached to a K_2 by means of two edges (these edges going each to a different vertex of the K_2). Then $\delta(G_2) = 2$ and $\iota(G_2) = n_2 = \frac{n(G_2)}{3}$. $\square$

Corollary 3.2. *Let G be a graph on n vertices with no component on less than 3 vertices. Then the following holds.*

(i) $\iota(G) \leq \frac{2n}{5}$ and equality holds if and only if G is the union of vertex disjoint copies of C_5.

(ii) Let $G_1, G_2, \ldots, G_k$ be the components of G. If $G_i \neq C_5$ for all $1 \leq i \leq k$, then $\iota(G) \leq \displaystyle\sum_{1 \leq i \leq k} \left\lfloor \frac{n(G_i)}{3} \right\rfloor \leq \left\lfloor \frac{n}{3} \right\rfloor$.

3.2. Bounds in Terms of Order and Maximum Degree

Lemma 3.3. *Let G be a graph of order n and with vertex set V. Consider a subset $S \subseteq V$. Then the following statements hold.*

(i) $\iota(G) \leq \frac{n-|N(S)|+|S|}{2}$.

(ii) If $\delta(G - N[S]) \geq 2$, then $\iota(G) \leq \frac{2n-2|N(S)|+3|S|}{5}$.

(iii) If every component of $G - N[S]$ has at least 3 vertices and no C_5-component, then $\iota(G) \leq \frac{n-|N(S)|+2|S|}{3}$.

Proof. (i) This follows because of $\iota(G) \leq \iota(G - N[S]) + |S| \leq \frac{n-|N[S]|}{2} + |S| = \frac{n-|N(S)|+|S|}{2}$.

(ii) If $\delta(G - N[S]) \geq 2$, then there is no component of order smaller than 3 and, hence, by Corollary 3.2(i), we infer that $\iota(G) \leq \iota(G - N[S]) + |S| \leq \frac{2(n-|N[S]|)}{5} + |S| = \frac{2n-2|N(S)|+3|S|}{5}$.

(iii) Since every component of $G - N[S]$ has at least 3 vertices and no C_5-component, it follows by Theorem 3.1 $\iota(G) \leq \iota(G - N[S]) + |S| \leq \frac{n-|N[S]|}{3} + |S| = \frac{n-|N(S)|+2|S|}{3}$. $\square$

Theorem 3.4. *Let G be a graph on n vertices and maximum degree Δ.*

(i) Then $\iota(G) \leq \frac{n-\Delta+1}{2}$ and this is sharp for all values of Δ with $\frac{n}{2} \leq \Delta \leq n - 3$.

(ii) *If there is a vertex v of maximum degree Δ and all components of $G - N[v]$ contain at least 3 vertices and no C_5-component, then $\iota(G) \leq \frac{n-\Delta+2}{3}$.*

(iii) *If G is a graph of maximum degree Δ, then $\iota_1(G) \leq \frac{n-\Delta+2}{3}$ and this is sharp.*

Proof. (i) Let x be a vertex of maximum degree Δ. Let $G' = G - N_G[x]$ and let I the set of isolated vertices in G'. Then there is a dominating set D of $G' - I$ with $|D| \leq \frac{n(G')-|I|}{2} \leq \frac{n(G')}{2} = \frac{n-\Delta-1}{2}$. Since $D \cup \{x\}$ is an isolating set of G, we obtain $\iota(G) \leq |D| + 1 \leq \frac{n-\Delta+1}{2}$. The sharpness can be seen with the following construction. For two positive integers p and q let x be the center of a star $K_{1,2p+q}$, from which exactly $2p$ edges are subdivided. Now join by pairs the leaves of the subdivided edges with an edge. The graph G obtained this way has $n(G) = n$, $\iota(G) = p + 1 = \frac{n-(2p+q)+1}{2}$ and $\Delta = \frac{n+q-1}{2}$, where $1 \leq q \leq n - 5$.

(ii) This follows directly from Lemma 3.3 (iii) with $S = \{v\}$.

(iii) Let v be a vertex of maximum degree Δ. Consider the graph $G^* = G - N[v]$. Let $A \subseteq V(G^*)$ be the set of vertices of all components of at most 2 vertices in G^*. Let B be the set of vertices of all C_5-components of G^*. Let I be a minimum $\{K_2\}$-isolating set of $G[B]$. Then clearly $|I| = \frac{|B|}{5}$. Finally, let J be a minimum isolating set of $G^* - (A \cup B)$ (if $V(G^*) \setminus (A \cup B) = \emptyset$, set $J = \emptyset$). Note that $\{v\} \cup I \cup J$ is a $\{K_2\}$-isolating set of G and, by Theorem 3.1, $|J| \leq \frac{n(G^*)-|A|-|B|}{3}$. Hence, we obtain

$$
\begin{aligned}
\iota_1(G) &\leq 1 + |I| + |J| \leq 1 + \frac{|B|}{5} + \frac{n(G^*)-|A|-|B|}{3} \\
&\leq 1 + \frac{|B|}{3} + \frac{n(G^*)-|B|}{3} = 1 + \frac{|V(G)-N[v]|}{3} = \frac{n-\Delta+2}{3}.
\end{aligned}
$$

Let r, s, t be non-negative integers with $r + s + t \geq 1$. Then the graph $F_{r,s,t} = rK_3 \cup sP_3 \cup tC_6$ has $\iota_1(F_{r,s,t}) = r + s + 2t = \frac{n(F_{r,s,t})}{3}$ and thus the bound is sharp. $\square$

3.3. Bounds in Terms of Order and Minimum Degree

Denote by $f(\delta, k) = \inf\{\alpha \mid \iota_k(G) \leq \alpha|G| \text{ for every graph } G \text{ with minimum degree } \delta\}$. In case $k = 0$ we shall use the notation $f(\delta)$.

Theorem 3.5. *The following statements hold.*

(i) *For $\delta \geq 1$, $\frac{2}{\delta+3} \leq f(\delta) \leq \frac{\ln(\delta+1)+\frac{1}{2}}{\delta+1}$.*

(ii) *$f(1) = \frac{1}{2}$, $f(2) = \frac{2}{5}$ and $f(3) = \frac{1}{3}$ and this is sharp.*

(iii) *For $\delta \geq k + 1$,*

$$
(1 - o(1))\frac{\ln(\delta + 1)}{(k + 2)(\delta + 1)} \leq f(\delta, k) \leq \frac{\ln(\delta + \frac{1}{2}) + 1}{\delta + 1}.
$$

Proof. (i) We first prove the lower bound. Here for, consider the following graph. Let n be divisible by an integer $r \geq 4$ and let H be a complete graph K_r of order r to which the edges of a Hamiltonian cycle are deleted. Let G be the graph consisting of $\frac{n}{r}$ copies of H. Then $\delta = \delta(G) = r - 3$ and each vertex dominates all vertices on the copy of H it belongs with exception of two adjacent vertices, showing that $\iota(G) = 2\frac{n}{r} = \frac{2}{\delta+3}n$. For the upper bound, we follow the proof for the probabilistic upper bound for the domination number due to Alon (see [2]) but, instead of including all non-dominated vertices, we only need to take at most the half of those which are not isolated. So let G be a graph with minimum degree $\delta \geq 1$. Let $p \in [0, 1]$. Select a set of vertices A independently at random such that $P(v \in A) = p$. Let I be the set of isolated vertices in $V \setminus A$ and let $B = V \setminus (N[A] \cup I)$. Since there are no isolated vertices in B, we know by Ore [25], that there is a dominating set D of $G[B]$ such that $|D| \leq \frac{|B|}{2}$. Then, clearly, $A \cup D$ is an isolating set of G. Note that $E[|D|] \leq E[\frac{|B|}{2}] = \frac{1}{2}E[|B|]$. Hence, since

$$
P(v \in B) = P(v \in V \setminus N[A]) = (1 - p)^{\deg(v)+1} \leq (1 - p)^{\delta+1},
$$

we obtain, using $1 - x \le e^{-x}$ for $x \ge 0$,

$$E[|A \cup D|] \le E[|A|] + \frac{1}{2}E[|B|] = pn + \frac{1}{2}(1 - p)^{\delta+1}n \le \left(p + \frac{1}{2}e^{-p(\delta+1)}\right)n.$$

Since the function $f(x) = e^{-x(\delta+1)}$ attains its minimum when $x = \frac{\ln(\delta+1)}{\delta+1}$, we can take $p = \frac{\ln(\delta+1)}{\delta+1}$ in order to obtain the minimum value in the above inequality chain. Hence, the expected cardinality for an isolating set is at most

$$\frac{\ln(\delta+1) + \frac{1}{2}}{\delta+1} \, n,$$

which gives the desired upper bound for $\iota(G)$.

(ii) The inequalities $f(1) \ge \frac{1}{2}$, $f(2) \ge \frac{2}{5}$ and $f(3) \ge \frac{1}{3}$ follow from item (i). Let now G be a graph of order n. If G has no isolated vertices, then $\gamma(G) \le \frac{n}{2}$ [25], and hence by Lemma 2.3 (i) $\iota(G) \le \frac{n}{2}$. If $\delta(G) \ge 2$, it is well known that $\gamma(G) \le \frac{2}{5}n$ unless G belongs to a family of 7 exceptional graphs (P_4 and six graphs of order 7) [24]. For P_4 we have $\iota(P_4) = 1$ and for the other 6 exceptional graphs on 7 vertices we checked that there is an isolating set on $2 = \frac{2}{7}n$ vertices. Hence, using Lemma 2.3 (i) for all other graphs G with minimum degree $\delta \ge 2$, we obtain in all cases $\iota(G) \le \frac{2}{5}n$. Finally, if G has minimum degree $\delta \ge 3$, then all components of G have at least 4 vertices and none of them is a C_5 and hence Theorem 3.1 yields $\iota(G) \le \frac{1}{3}n$.

(iii) For the lower bound, consider a graph G of minimum degree $\delta(G) = \delta - 1 \ge 0$ such that $\gamma(G) = (1 - o(1))\frac{\ln\delta}{\delta}n(G)$, whose existence was given by Alon and Wormald in [3]. Now let $H = G \square K_{k+2}$. Then $n(H) = (k+2)n(G)$ and $\delta(H) = \delta(G) + k + 1 \ge k + 1$. By Theorem 2.6 (i), it follows

$$\iota_k(H) \ge \gamma(G) = (1 - o(1))\frac{\ln\delta}{\delta}n(G)$$

$$= (1 - o(1))\frac{(\delta+1)\ln\delta}{\delta\ln(\delta+1)} \cdot \frac{\ln(\delta+1)}{(k+2)(\delta+1)} \cdot (k+2)n(G)$$

$$\ge (1 - o(1))\frac{\ln(\delta+1)}{(k+2)(\delta+1)}n(H),$$

and we are done. For the upper bound, let G be any graph on n vertices and with minimum degree $\delta \ge k+1$. By Lemma 2.3 (i), we have $\iota_k(G) \le \gamma(G)$. Hence, by the bound on domination due to Arnautov, Lovász and Payan [4, 23, 26] (for which Alon gave the probabilistic proof cited above in the proof of item (iii)), $\iota_k(G) \le \gamma(G) \le \frac{\ln(\delta+1)+1}{\delta+1}n$. $\quad\square$

Note that the lower bounds in items (i) and (iii) of Theorem 3.5 are both in order as the first gives a better (and explicit) lower bound for small values of δ.

Theorem 3.6. *Let G be a bipartite graph on n vertices and minimum degree δ. Then $\iota(G) \le \frac{\ln\delta+1}{2\delta}n$ and this is nearly sharp.*

Proof. Let G be bipartite with bipartition $V_1 \cup V_2$. Let $|V_i| = n_1$ and $|V_2| = n_2$ and assume that $n_1 \le n_2$. Choose a subset $A \subseteq V_1$ each vertex of it independently and at random with probability $P(v \in A) = p$. Let $B \subseteq V_2$ be the set of vertices in V_2 having no neighbor in A. Then $A \cup B$ is an isolating set of G and thus $\iota(G)$ is at most as large as the expected size of $|A \cup B|$. Note that

$$E[|A \cup B|] = E[|A|] + E[|B|] = n_1 p + \sum_{v \in V_2}(1 - p)^{\deg(v)} \le n_1 p + n_2(1 - p)^{\delta}.$$

Considering the function $f(x) = n_1 x + n_2(1 - x)^{\delta}$ and its derivative $f'(x) = n_1 - n_2\delta(1 - x)^{\delta-1}$, we can see that $f'(x) = 0$ when $(1 - x)^{\delta-1} = \frac{n_1}{n_2\delta}$. Since $n_1 \le n_2$, we obtain $(1 - x)^{\delta-1} \le \frac{1}{\delta}$ and thus $x \ge 1 - (\frac{1}{\delta})^{\frac{1}{\delta-1}}$. Hence, we can choose $p = 1 - (\frac{1}{\delta})^{\frac{1}{\delta-1}}$. It follows that

$$\iota(G) \le E[|A \cup B|] \le n_1 p + n_2(1 - p)^{\delta} = n_1\left(1 - \left(\frac{1}{\delta}\right)^{\frac{1}{\delta-1}}\right) + n_2\frac{1}{\delta} \le n_1\frac{\ln\delta}{\delta} + n_2\frac{1}{\delta}.$$

Since $n_1 \leq n_2$ and the worst case is when $n_1 = n_2 = \frac{n}{2}$, we infer that $\iota(G) \leq \frac{\ln \delta + 1}{2\delta} n$.

For the sharpness, consider an Alon-Wormald [3] $(\delta - 1)$-regular graph G on n vertices having $\gamma(G) = (1 - o(1))\frac{\ln \delta}{\delta} n$. Further, take the bipartite graph $B(G)$ described just before Theorem 2.6 and note that $|B(G)| = 2n$, $\delta(B(G)) = \delta$. Now, with Theorem 2.6 (ii), we obtain

$$\iota(B(G)) \geq \gamma(G) \geq (1 - o(1))\frac{\ln \delta}{\delta} n = (1 - o(1))\frac{\ln \delta}{2\delta} n(B(G)),$$

showing that the upper bound given above is nearly sharp. $\square$

In the following theorem, we will show that $f(\delta, k)$ is monotonically decreasing as δ grows. This implies in particular that, for any graph G with minimum degree $\delta(G) \geq \delta$, $\iota_k(G) \leq f(\delta, k)n(G)$.

Theorem 3.7. *For $\delta \geq 1$, $f(\delta, k) \geq f(\delta + 1, k)$.*

Proof. Since $f(\delta + 1, k) = \lim_{n \to \infty} f(\delta + 1, k, n)$, there is a sequence of graphs $(H_i)_{i \geq 1}$ with $\delta(H_i) = \delta + 1$ such that $n_i = n(H_i)$ tends to infinity as i grows and

$$\lim_{i \to \infty} \frac{\iota_k(H_i)}{n_i} = f(\delta + 1, k).$$

Note that the infimum is either a minimum but then it is obtained by arbitrarily many copies of the graph that realizes the minimum or there is a sequence as above with $\lim_{i \to \infty} \frac{\iota_k(H_i)}{n_i} = f(\delta + 1, k)$. So, in any case, we can use such a sequence. Consider now the graphs $G_i = H_i \cup K_{\delta+1}$. Then $\delta(G_i) = \delta$, $n_\iota(G_i) = n_i + \delta + 1$ and $\iota_k(G_i) \geq \iota_k(H_i)$ for $i \geq 1$ and we obtain

$$f(\delta, k) \geq \lim_{i \to \infty} \frac{\iota_k(G_i)}{n(G_i)} \geq \lim_{i \to \infty} \frac{\iota_k(H_i)}{n_i + \delta + 1} = \lim_{i \to \infty} \frac{\iota_k(H_i)n_i}{n_i(n_i + \delta + 1)} = f(\delta + 1, k)$$

and we are done. $\square$

3.4. Bounds in Terms of Order and Minimum Degree: Connected Graphs

Theorem 3.8. *Let $\mathcal{F}$ be a family of graphs and let $\delta \geq 1$. Let H be a connected graph with $\iota(H, \mathcal{F}) = q$. Then, for arbitrarily large n, if $\delta = 1$ and $\delta(H) = 1$ or if $\delta \geq 2$ and $\delta(H) \geq \delta$, there is a connected graph G on n vertices such that $\delta(G) = \delta$ and*

$$\iota(G, \mathcal{F}) \geq \frac{q}{n(H) + 1} n.$$

Proof. Let H be a connected graph with $\delta(H) \geq \delta \geq 2$ or $\delta(H) = \delta = 1$ and let $\iota(H, \mathcal{F}) = q$. Define a graph G according to the following cases.

Case 1: $\delta = 1$. Take $t \geq \delta + 1$ copies $H_1, H_2, \ldots, H_t$ of H and a connected graph G^* on t vertices $v_1, v_2, \ldots, v_t$. For $1 \leq i \leq t$, select one vertex u_i from $V(H_i)$ such that in $V(H_i) \setminus \{u_i\}$ there is still one vertex of degree 1 in H_i and connect u_i and v_i by an edge. Then $n(G) = n = t(n(H) + 1)$ and, clearly, $\delta(G) = 1$.

Case 2: $\delta \geq 2$. Take $t \geq \delta + 1$ copies $H_1, H_2, \ldots, H_t$ of H and a connected graph G^* on t vertices $v_1, v_2, \ldots, v_t$ and minimum degree $\delta(G^*) = \delta - 1 \geq 1$. For $1 \leq i \leq t$, select one vertex u_i from $V(H_i) = V_i$ and connect u_i and v_i by an edge. Then $n(G) = n = t(n(H) + 1)$ and, clearly, $\delta(G) = \delta$.

Now, in both cases, let S be a minimum $\mathcal{F}$-isolating set of G and let $S_i = S \cap V_i$, for $1 \leq i \leq t$. Fix one $i \in \{1, 2, \ldots, t\}$. If $v_i \notin S$, then $H_i - N_{H_i}[S_i] \subseteq G - N_G[S]$ and thus $H_i - N_{H_i}[S_i]$ is $\mathcal{F}$-free. Hence, S_i is an $\mathcal{F}$-isolating set of H_i, which yields $|S_i| \geq q$. On the other side, if $v_i \in S$, then we have $H_i - N_{H_i}[S_i \cup \{u_i\}] \subseteq G - N_G[S]$ and thus $H_i - N_{H_i}[S_i \cup \{u_i\}]$ is $\mathcal{F}$-free. This implies that $S_i \cup \{u_i\}$ is an $\mathcal{F}$-isolating set of H_i and thus $|S_i \cup \{u_i\}| \geq r$, from which we deduce $|S_i| \geq q - 1$. For both constructions, it follows that $|(V_i \cup \{v_i\}) \cap S| \geq k$ for $1 \leq i \leq t$, yielding $\iota(G, \mathcal{F}) = |S| \geq tq = \frac{q}{n(H)+1} n$. $\square$

The following two corollaries follow partially from Theorem 3.8. We shall need the following notation. We define a parameter $f_c(\delta, k)$ the following way. Let

$$f_c(\delta, k, n) = \inf\{\alpha \in \mathbb{R} : \iota(G, k) \le \alpha\, n(G), G \text{ connected graph } n(G) \ge n, \delta(G) = \delta\}.$$

Observe that $f_c(\delta, k, n) \le f_c(\delta, k, n+1) \le 1$. Hence, for fixed δ and k, $f_c(\delta, k, n)$ is a monotone non-decreasing sequence, which is bounded from above. So, we may define

$$f_c(\delta, k) = \lim_{n \to \infty} f_c(\delta, k, n).$$

Further, in case $k = 0$ we shall use the notation $f_c(\delta)$ for $f_c(\delta, 0)$.

Corollary 3.9. *The following statements hold.*

(i) $f_c(1) = f_c(2) = \frac{1}{3}$.

(ii) For $\delta \ge 3$, $\frac{2}{\delta+4} \le f_c(\delta) \le f(\delta) \le \frac{\ln(\delta+1)+\frac{1}{2}}{\delta+1}$.

(iii) For $\delta \ge k + 1$, $(1 - o(1))\frac{\ln(\delta+1)}{(k+2)(\delta+1)} \le f_c(\delta, k) \le f(\delta, k) \le \frac{\ln(\delta+\frac{1}{2})+1}{\delta+1}$.

Proof. (i) Since the graphs yielding the sharpness in the bound of Theorem 3.1 are connected and arbitrarily large, this is clear.

(ii) Consider the graph $H = K_r - C_r$, a complete graph on $r \ge 4$ vertices without a Hamiltonian cycle. Then $n(H) = r = \delta(H) + 3$ and $\iota(G) = 2$ and, thus, Theorem 3.8 gives $\frac{2}{\delta+4} \le f_c(\delta)$. Since, clearly, $f_c(G) \le f(G)$, the last inequality follows from Theorem 3.5 (i).

(iii) This follows from the construction of the graph yielding the lower bound of Theorem 3.5 (iii), which is already a connected arbitrarily large graph. $\square$

Corollary 3.10. *The following statements hold.*

(i) For $\mathcal{F} = \{C_k \mid k \ge 3\}$, there are arbitrarily large connected graphs G for which $\iota(G, \mathcal{F}) \ge \frac{1}{4}n(G)$.

(ii) There are arbitrarily large connected graphs G for which $\iota_{K_k}(G) \ge \frac{1}{k+1}n(G)$.

(iii) For $\mathcal{F}_k$, the family of all trees of order $k \ge 2$, there are arbitrarily large connected graphs G for which $\iota(G, \mathcal{F}_k) \ge \frac{1}{k+1}n(G)$.

Proof. For (i), (ii) and (iii), apply Theorem 3.8 taking, respectively, $H = C_3, K_k$ and $K_{1,k-1}$. $\square$

Now we are going to show that $f_c(\delta, k)$ is monotonically decreasing as δ grows (where $\delta \ge 2$). This implies in particular that, for any connected graph G with minimum degree $\delta(G) \ge \delta$, $\iota_k(G) \le f_c(\delta, k)n(G)$.

Theorem 3.11. *For $\delta \ge 2$, $f_c(\delta, k) \ge f_c(\delta + 1, k)$.*

Proof. Since $f_c(\delta+1, k) = \lim_{n \to \infty} f_c(\delta+1, k, n)$, there is a sequence $(H_i)_{i \ge 1}$ of connected graphs H_i with $\delta(H_i) = \delta+1$, $n_i = n(H_i)$, $q_i = \iota_k(H_i)$ and $\lim_{i \to \infty} n_i$ tending to infinity such that

$$\lim_{i \to \infty} \frac{q_i}{n_i} = f_c(\delta + 1, k).$$

By Theorem 3.8, there are connected graphs G_i with $\delta(G_i) = \delta \ge 2$ with $\frac{\iota_k(G_i)}{n(G_i)} \ge \frac{q_i}{n_i+1}$. Then we have

$$f_c(\delta, k) \ge \lim_{i \to \infty} \frac{\iota_k(G_i)}{n(G_i)} \ge \lim_{i \to \infty} \frac{q_i}{n_i + 1} = \lim_{i \to \infty} \frac{q_i n_i}{n_i(n_i + 1)} = \lim_{i \to \infty} \frac{q_i}{n_i} = f_c(\delta, k)$$

and we are done. $\square$

4. Lower Bounds

Let G be a graph. With G^2 we denote the *power-2 graph* of G, that is, the graph that arises from G by adding all edges between vertices within distance 2. Recall also that $\alpha_k(G)$ is the k-independence number of G.

Theorem 4.1. *Let G be a graph with minimum degree δ and maximum degree Δ.*

  (i) If $\delta \geq k + 1$, then $\iota_k(G) \geq \gamma(G^2)$.

  (ii) If $\Delta \geq k + 1$, $\iota_k(G) \geq \frac{n+1-\alpha_k(G)}{\Delta+1}$ and this is sharp.

Proof. (i) Let S be a minimum k-isolating set of G. Then every vertex in $N(S)$ has a neighbor in S. Also, since $\delta \geq k + 1$ and $G - N[S]$ is $K_{1,k+1}$-free, every vertex in $V(G) \setminus N[S]$ has a neighbor in $N(S)$. Hence, every vertex in $V(G) \setminus N[S]$ is within distance 2 from a vertex of S and thus S is a dominating set of G^2, yielding $\gamma(G^2) \leq \iota_k(G)$.

(ii) Let S be a minimum k-isolating set of G. Since $\Delta \geq k + 1$, $S \neq \emptyset$. Let $x \in S$. Then, Since $G - N[S]$ is $K_{1,k+1}$-free and x has no neighbors in $V(G) \setminus N[S]$, $(V(G) \setminus N[S]) \cup \{x\}$ is a k-independent set of G and, therefore, $\alpha_k(G) \geq n - |N[S]| + 1$. Since every vertex in S has at most Δ neighbors in $N(S)$, we obtain

$$\alpha_k(G) \geq n - |N[S]| + 1 = n - |S| - |N(S)| + 1 \geq n - |S| - \Delta|S| + 1 = n - (\Delta + 1)|S| + 1.$$

This implies $\iota_k(G) = |S| \geq \frac{n+1-\alpha_k(G)}{\Delta+1}$. For the sharpness, consider a graph G consisting of Δ copies $G_1, G_2, \ldots, G_\Delta$ of K_{k+1} and a vertex x. Now select a vertex $x_i \in V(G_i)$ and include the edges xx_i, $1 \leq i \leq \Delta$. Then $\{x\}$ is a minimum k-isolating set and $V(G) \setminus \{x_1, x_2, \ldots, x_\Delta\}$ is a maximum k-independent set. Hence, $\iota_k(G) = 1 = \frac{\Delta(k+1)+2-(\Delta k+1)}{\Delta+1} = \frac{n(G)+1-\alpha_k(G)}{\Delta+1}$. $\square$

In the following theorem, we give a lower bound for the isolation number $\iota(G)$ of a graph G in terms of its maximum degree and average degree. For subsets $A, B \subseteq V(G)$, we will use the notation $m(A, B)$ for the number of edges with one vertex in A and one in B.

Theorem 4.2. *Let G be a graph on n vertices with average degree d and maximum degree Δ. Then*

$$\iota(G) \geq \frac{\mathrm{d}n}{2\Delta^2}.$$

Moreover, equality holds if and only if, for an integer $t \geq 1$, G is a bipartite graph with partition sets A and B, $|A| \leq |B|$, where $A = \{a_{i,j} \mid 1 \leq i \leq t, \ 1 \leq j \leq \Delta\}$ and $S = \{s_i \mid 1 \leq i \leq t\} \subseteq B$, such that the following holds: $N(s_i) = \{a_{i,j} \mid 1 \leq j \leq \Delta\}$, $|N(a) \cap B| = \Delta$ for all $a \in A$ and $|N(b) \cap A| \leq \Delta$ for all $b \in B$.

Proof. Let $V = V(G)$ and let S be a minimum isolating set of G. We will bound the number of edges from above. Clearly, $m(S, N[S]) \leq \Delta|S|$. Moreover, since the vertices of $N(S)$ have all at least one neighbor in S, we have $m(N(S), V \setminus S) \leq (\Delta - 1)|N(S)| \leq (\Delta - 1)\Delta|S|$. Hence,

$$m(G) \leq \Delta|S| + (\Delta - 1)\Delta|S| = \Delta^2|S| = \Delta^2\iota(G).$$

Thus it follows $\iota(G) \geq \frac{m(G)}{\Delta^2} = \frac{\mathrm{d}n}{2\Delta^2}$.

Now suppose that we have a graph G with average degree d and maximum degree Δ such that $\iota(G) = \frac{\mathrm{d}n}{2\Delta^2}$. Then, for any minimum isolating set S of G, we have equalities in all the above inequalities and thus $m(S, N[S]) = \Delta|S|$ and $m(N(S), V \setminus S) = (\Delta - 1)|N(S)| = (\Delta - 1)\Delta|S|$. Hence, both $N(S)$ and $S \cup (V \setminus N[S])$ are independent sets. Setting $A = N(S)$ and $B = S \cup (V \setminus N[S])$, $S = \{s_i \mid 1 \leq i \leq t\}$, $N(s_i) = \{a_{i,j} \mid 1 \leq j \leq \Delta\}$, and $A = \{a_{i,j} \mid 1 \leq i \leq t, \ 1 \leq j \leq \Delta\}$ it is clear that G is of the form described in the statement of the theorem. For the converse, consider a bipartite graph G with partite sets A and B, $|A| \leq |B|$, where $A = \{a_{i,j} \mid 1 \leq i \leq t, \ 1 \leq j \leq \Delta\}$ and $S = \{s_i \mid 1 \leq i \leq t\} \subseteq B$, such that the following holds: $N(s_i) = \{a_{i,j} \mid 1 \leq j \leq \Delta\}$, $|N(a) \cap B| = \Delta$ for all $a \in A$ and $|N(b) \cap A| \leq \Delta$ for all $b \in B$. Clearly, S is an isolating set of G. Moreover, $m(G) = \Delta|A| = \Delta^2 t$. Hence, by the above inequality, $\Delta^2 t = m(G) \leq \Delta^2\iota(G) \leq \Delta^2|S| = \Delta^2 t$. Thus, $\iota(G) = \frac{m(G)}{\Delta^2} = \frac{\mathrm{d}n}{2\Delta^2}$. $\square$

Observe that, for an r-regular graph G of order n and attaining the bound of Theorem 4.2, we have $\gamma(G) = \frac{n}{r+1}$, while $\iota(G) = \frac{n}{2r}$. Hence, we have here another example where the parameters $\iota(G)$ and $\gamma(G)$ differ considerably, namely here on a factor of $\frac{1}{2}$.

5. Bounds for Certain Families of Graphs

In the following two theorems, we will deal with the k-isolation number of trees. Given a tree T, we will call a vertex $x \in V(T)$ a *support vertex* of T if x is neighbor of a leaf, i.e. a vertex of degree one. Moreover, an *inner vertex* of T is a vertex that is not a leaf.

Theorem 5.1. *Let T be a tree on n vertices and different from $K_{1,k+1}$. Then $\iota_k(T) \leq \frac{n}{k+3}$ and this is sharp.*

Proof. Let T be a tree different from $K_{1,k+1}$. We will prove the statement by induction on the number of vertices of T. If $n \leq k + 2$, then clearly T has no $K_{1,k+1}$ as a subgraph and $\iota_k(G) = 0 \leq \frac{n}{k+3}$. Suppose now that T is a tree on $n \geq k + 3$ vertices and assume that, for any tree on less than n vertices and different from $K_{1,k+1}$, the above inequality holds. We now distinguish two cases.

Case 1. Suppose that T has a support vertex v of degree $\deg(v) \neq k + 1$.
Let u be a leaf adjacent to v and define $T' = T - u$. Then T' is a tree on $n - 1$ vertices. If $T' = K_{1,k+1}$, then $T = K_{1,k+2}$ or T is isomorphic to a $K_{1,k+1}$ with a subdivided edge. Since in both cases $\{v\}$ is a k-isolating set of T and $n = k + 3$, the inequality $\iota_k(T) \leq \frac{n(T)}{k+3}$ holds trivially. Hence, we may assume that $T' \neq K_{1,k+1}$ and, by the induction hypothesis, $\iota_k(T') \leq \frac{n(T')}{k+3}$. Now, due to the degree condition on v, observe that u cannot belong to any subtree isomorphic to $K_{1,k+1}$ and, thus, any k-isolating set of T' is also a k-isolating set of T. Hence, $\iota_k(T) \leq \iota_k(T') \leq \frac{n(T')}{k+3} < \frac{n}{k+3}$ and we are done.

Case 2. Suppose that all support vertices of T have degree $k + 1$.
Observe first that the diameter of T cannot be less than 3: otherwise, T would be a star $K_{1,r}$ and since the support vertices have all degree $k + 1$, $r = k + 1$ and thus $T = K_{1,k+1}$, which is a contradiction to the assumptions. Hence, $\operatorname{diam}(T) \geq 3$. Let $P = x_0 x_1 \ldots x_d$ be a diametral path of T. If $d = \operatorname{diam}(T) = 3$, it is easy to see that $\{x_1\}$ is a k-isolating set of T and so $\iota_k(G) \leq 1 \leq \frac{n}{k+3}$. Hence, we may assume that $d = \operatorname{diam}(T) \geq 4$. Let T_1 and T_2 be the trees resulting after removing the edge $x_3 x_4$ from T, where T_1 is the tree containing x_3 and T_2 is the tree containing x_4. Since all support vertices of T have degree $k + 1$, T_1 has at least $k + 3$ vertices and $\{x_3\}$ is a k-isolating set of T_1. If, further, $\operatorname{diam}(T_2) \leq 2$, then $\{x_3\}$ is a k-isolating set of T itself and clearly $\iota_k(T) \leq 1 \leq \frac{n}{k+3}$. On the other side, if $\operatorname{diam}(T_2) \geq 3$, then $T_2 \neq K_{1,k+1}$ and, by the induction hypothesis, we have $\iota_k(T_2) \leq \frac{n(T_2)}{k+3}$. Now let I be a minimum k-isolating set of T_2. Then $I \cup \{x_3\}$ is a k-isolating set of T, which implies

$$\iota_k(T) \leq |I| + 1 = \iota_k(T_2) + 1 \leq \frac{n(T_2)}{k+3} + 1 \leq \frac{n(T_2)}{k+3} + \frac{n(T_1)}{k+3} = \frac{n}{k+3},$$

and we are done.

For the sharpness, consider a path P on t vertices $v_1, v_2, \ldots, v_t$ and t copies of $K_{1,k+1}$ such that each vertex v_i of P is joined by an edge to one of the leaves of the i-th copy of $K_{1,k+1}$. Define the in this way constructed tree by T. Clearly, the vertices of P are a k-isolating set of T and we cannot come out with less since for each copy of $K_{1,k+1}$ we need at least one vertex in the k-isolating set. $\square$

Theorem 5.2. *Let T be a tree on n vertices in which all non-leaves have equal degree $r \geq k + 3$. Then $\iota_k(T) \leq \frac{n-2}{2(r-1)}$ and this is sharp.*

Proof. Let I be the set of inner vertices and L the set of leaves of T. Since all inner vertices of T have degree $r \geq k + 3$, we have the following equality chain:

$$2(n - 1) = 2m(T) = r|I| + |L| = r|I| + n - |I| = n + (r - 1)|I|.$$

This implies that $|I| = \frac{n-2}{r-1}$. Let now D be minimum dominating set of the tree $T - L$, resulting from removing all leaves of T. Then D is a k-isolating set of T and we have, with Lemma 2.3 (iii) and Ore's inequality,

$$\iota_k(T) \leq \gamma(T - L) \leq \frac{|I|}{2} = \frac{n-2}{2(r-1)}.$$

For the sharpness, construct a tree T the following way. Take a path $P = x_1 x_2 \dots x_t$ on $t \geq 3$ vertices and attach to each of the inner vertices x_i of P a leaf v_i, where $2 \leq i \leq t - 1$. Attach now, to each vertex v_i, $r - 1$ leaves and, to each vertex x_i, $r - 2$ leaves, for $2 \leq i \leq t - 1$. Then, any k-isolating set of the resulting tree T contains either x_i or v_i for each $2 \leq i \leq t - 1$. Moreover, $\{x_2, x_3, \dots, x_{t-1}\}$ is a k-isolating set of T. Hence, $\iota_k(T) = t - 1 = \frac{n(T)-2}{2(r-1)}$. $\square$

For the theorem coming next, we shall need the following result from Campos and Wakabayashi [5].

Theorem 5.3 ([5]). *Let G be a maximal outerplanar graph on $n \geq 4$ vertices and t vertices of degree 2. Then, $\gamma(G) \leq \frac{n+t}{4}$.*

Next theorem shows that, for a maximal outerplanar graph, at most $\frac{1}{4}$ of the vertices are needed for an isolating set.

Theorem 5.4. *Let G be a maximal outerplanar graph on $n \geq 4$ vertices. Then $\iota(G) \leq \frac{n}{4}$ and this is sharp.*

Proof. If $n = 4$, then G is the complete graph on 4 vertices minus an edge, which has one vertex dominating all others and thus $\iota(G) = 1$. If $n = 5$, then again, since G is a triangulation of the pentagon, there has to be a vertex dominating all others. When $n = 6, 7$, we use the fact that a maximal outerplanar graph has $m = 2n - 3$ edges. It is also well known that a maximal outerplanar graph has at least two vertices of degree 2. If $n = 6$, these two facts imply that there has to be a vertex of degree at least 4, otherwise we would have $18 = 2m = \sum_{v \in V(G)} \deg(v) \leq 2 \cdot 2 + 4 \cdot 3 = 16$, a contradiction. Since a vertex of degree 4 in a 6-vertex graph forms an isolating set, we have $\iota(G) = 1$. Let now $n = 7$. Note that the only possible degree sequences of 7 vertices satisfying $\sum_{v \in V(G)} \deg(v) = 2m = 2(2n - 3) = 22$ and having at least 2 vertices of degree 2 are $5, 4, 4, 3, 2, 2, 2$ and $4, 4, 4, 3, 3, 2, 2$. Let $C = v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_1$ be the cycle surrounding the outerface of G and suppose that v_1 has degree 4. If v_1 is not-adjacent to two non-consecutive vertices of the cycle different from v_2 and v_7, then $\{v\}$ is an isolating set. Hence suppose that v_1 is not-adjacent to two consecutive vertices on the cycle different from v_2 and v_7. Without loss of generality, due to symmetry reasons, we can assume that either $N(v) = \{v_2, v_3, v_4, v_7\}$ or $N(v) = \{v_2, v_3, v_6, v_7\}$. In the first case, since G is a triangulation of C, v_4 has to be adjacent to v_7. Then v_4 is a vertex of degree at least 4 which dominates all but at most the two non-adjacent vertices v_2 and v_6, and hence $\iota(G) = 1$. In the second case, i.e. when v_1 is adjacent to v_3 and v_6, then $v_3 v_6 \in E(G)$. Then either $v_3 v_5$ or $v_4 v_6 \in E(G)$, implying that either v_3 or v_6 has degree 5, which leads to $\iota(G) = 1$.

Let now G be a maximal outerplanar graph on $n \geq 8$ vertices. Let N_2 be the set of vertices of degree 2 in G and let $n_2 = |N_2|$. Note that (for $n \geq 4$) N_2 is an independent set. Hence, $n_2 \leq \frac{n}{2}$. Let $G^* = G - N_2$. Note that (for $n \geq 5$) the deletion of any vertex from N_2 creates at most one new vertex of degree 2. Hence, G^* has at most $n_2^* \leq n_2$ vertices of degree 2. Further, G^* is a maximal outerplanar graph on $n^* \geq \frac{n}{2} \geq 4$ vertices. By Theorem 5.3, it follows that $\gamma(G^*) \leq \frac{n^*+n_2^*}{4}$. Hence, by Lemma 2.3 (iii), $\iota(G) \leq \gamma(G \setminus G[N_2]) = \gamma(G^*) \leq \frac{n^*+n_2^*}{4} \leq \frac{n^*+n_2}{4} = \frac{n}{4}$ and we are done.

To see the sharpness, consider an arbitrary maximal outerplanar graph on $2p$ vertices such that $v_1 v_2 \dots v_{2p} v_1$ is the outercycle. Insert new vertices w_i and edges $v_i w_i$, for $1 \leq i \leq 2p$, and also the edges $v_{2j-1} w_{2j}$ and $w_{2j-1} w_{2j}$, for $1 \leq j \leq p$. Then we have constructed again a maximal outerplanar graph G on $n = 4p$ vertices, now with the outercycle going along the paths $v_{2j-1} w_{2j-1} w_{2j} v_{2j}$ in consecutive order, for $1 \leq j \leq p$. Note that, any isolating set of G has to contain at least one vertex from each of these paths, otherwise the adjacent vertices w_{2j-1} and w_{2j} would not be dominated. Hence, $\iota(G) \geq \frac{n}{4}$. On the other hand, $\{w_{2j-1} \mid 1 \leq j \leq p\}$ is an isolating set of G with $p = \frac{n}{4}$ vertices. Thus, we have $\iota(G) = \frac{n}{4}$. $\square$

In the following theorem, we consider *claw-free graphs*, i. e. graphs which do not contain a $K_{1,3}$ as an induced subgraph.

Theorem 5.5. *Let G be a claw-free graph on n vertices with average degree d, maximum degree Δ and minimum degree δ. Then the following lower bounds on $\iota(G)$ hold:*

(i) $\iota(G) \geq \frac{\delta(n+1)+2}{(\delta+2)(\Delta+1)}$,

(ii) $\iota(G) \geq \frac{2\,\mathrm{d}\,n}{3\Delta^2+2\Delta}$.

Proof. (i) In [13], the authors prove the bound $\alpha(G) \geq \frac{r-1}{r-1+\delta}n$ for graphs on n vertices and minimum degree δ and without induced $K_{1,r}$. Setting $r = 3$ in this result, and combining it with the bound on Theorem 4.1 (ii), leads to

$$\iota(G) \geq \frac{n+1-\alpha(G)}{\Delta+1} \geq \frac{n+1-\frac{2n}{\delta+2}}{\Delta+1} = \frac{\delta(n+1)+2}{(\delta+2)(\Delta+1)},$$

and we are done.

(ii) Let G be claw-free and let S be a minimum isolating set of G. Similarly as in Theorem 4.2, we will bound the number of edges of G from above. Consider a vertex $x \in S$ and let $N_x = N(x) \setminus S$, $G_x = G[N[N_x] \cup \{x\}]$ and $t_x = |N_x|$. Differently from the proof of Theorem 4.2, since G is claw-free, $N(v)$ has to contain enough edges such that any independent set of three vertices is avoided. This holds in particular also for N_x. The minimum number of edges contained in a graph with independence number at most 2 is equal to the number of edges of the complement of the Turán graph (the triangle-free ones). That is, in the worst case, $G[N_x]$ consists of two cliques of equal or almost equal order, depending on the parity of t_x. Hence, $G[N_x]$ has at most $\binom{\lceil t_x/2 \rceil}{2} + \binom{\lfloor t_x/2 \rfloor}{2}$ edges. Hence, we have the following.

$$
\begin{aligned}
m(G_x) &= t_x + m(G_x - x) \\
&= t_x + \sum_{v \in N_x} \deg_{G_x-x}(v) - m(G[N_x]) \\
&\leq t_x + t_x(\Delta-1) - m(G[N_x]) \\
&= t_x\Delta - m(G[N_x]).
\end{aligned}
$$

If t_x is odd, we have $m(G[N_x]) \geq \binom{(t_x+1)/2}{2} + \binom{(t_x-1)/2}{2} = \frac{(t_x-1)^2}{4}$ and thus

$$m(G_x) \leq t_x\Delta - \frac{(t_x-1)^2}{4} \leq \Delta^2 - \frac{(\Delta-1)^2}{4} = \frac{3\Delta^2+2\Delta-1}{4} < \frac{3\Delta^2+2\Delta}{4}.$$

If t_x is even, we have $m(G[N_x]) \geq 2\binom{t_x/2}{2} = \frac{t_x(t_x-2)}{4}$ and thus

$$m(G_x) \leq t_x\Delta - \frac{t_x(t_x-2)}{4} \leq \Delta^2 - \frac{\Delta(\Delta-2)}{4} = \frac{3\Delta^2+2\Delta}{4}.$$

We obtain now the following upper bound on the number of edges of G:

$$m(G) \leq \sum_{x \in S} m(G_x) \leq |S|\frac{3\Delta^2+2\Delta}{4},$$

which implies $\iota(G) = |S| \geq \frac{4m(G)}{3\Delta^2+2\Delta} = \frac{2d}{3\Delta^2+2\Delta}n$. $\square$

Observe that, for an r-regular claw-free graph G on n vertices, the bound of Theorem 5.5 gives $\iota(G) \geq \frac{2n}{3r+2}$, which is on around a factor of $\frac{4}{3}$ better than the bound $\iota(G) \geq \frac{n}{2r}$ of Theorem 4.2 for r-regular graphs.

In the following theorem, we shall consider grids and other similar graphs.

Theorem 5.6. *Let $t, s \geq 3$ be two integers. Then*

$$\frac{st}{8} \leq \iota(C_s \square C_t) \leq \frac{st}{8} + \frac{3(s+t+3)}{8},$$

$$\frac{st}{8} - \frac{t}{16} \leq \iota(P_s \square C_t) \leq \frac{st}{8} + \frac{3s+t+3}{8},$$

$$\frac{st}{8} - \frac{s+t}{16} \leq \iota(P_s \square P_t) \leq \frac{st}{8} + \frac{s+t+1}{8}.$$

Further, the bound $\frac{st}{8} \leq \iota(C_s \square C_t)$ is sharp for $s, t \equiv 0 \pmod 4$.

Proof. Let $G = C_s \square C_t$. The lower bound follows directly from Theorem 4.2 using the fact that G is 4-regular. For the sharpness, let $s, t \equiv 0 \pmod 4$ and consider the isolating set consisting of all vertices on a row and column congruent to 1 (mod 4) or in a row and column congruent to 3 (mod 4). For the upper bound, we will construct an isolating set the following way. Let $V(G) = \{(i, j) \mid 1 \le i \le s, 1 \le j \le t\}$ be the set of vertices of G where, for $A_i = \{(i, j) \mid 1 \le j \le t\}$ and $B_j = \{(i, j) \mid 1 \le i \le s\}$, $G[A_i] \cong C_t$ and $G[B_j] \cong C_s$. Let $S_1 = \{(i, j) \mid i, j \equiv 1 \pmod 4\}$ and $S_2 = \{(i, j) \mid i, j \equiv 3 \pmod 4\}\}$. Then $S = S_1 \cup S_2$ is an isolating set of G with

$$\iota(G) \le |S| \le 2 \left\lceil \frac{s}{4} \right\rceil \left\lceil \frac{t}{4} \right\rceil \le \frac{(s+3)(t+3)}{8} = \frac{st}{8} + \frac{3(s+t+3)}{8}.$$

Now let $H = P_s \square C_t$. Then H has $2t$ vertices of degree 3 and $t(s-2)$ vertices of degree 4. Hence, $d(G) = \frac{6t + 4t(s-2)}{st} = \frac{2t(2s-1)}{st}$ and Theorem 4.2 yields $\iota(H) \ge \frac{t(2s-1)}{16} = \frac{st}{8} - \frac{t}{16}$. For the upper bound, define the vertices as in the previous case such that $G[A_i] \cong C_t$ and $G[B_j] \cong P_s$ and set $S_1 = \{(i, j) \mid i, j \equiv 1 \pmod 4\}$ and $S_2 = \{(i, j) \mid i, j \equiv 3 \pmod 4\}\}$. Further, define

$$S^* = \begin{cases} \{(s, j) \mid j \equiv 1 \pmod 4\}, & \text{if } s \equiv 2 \pmod 4 \\ \{(s, j) \mid j \equiv 3 \pmod 4\}, & \text{if } s \equiv 0 \pmod 4 \\ \emptyset, & \text{if } s \equiv 1, 3 \pmod 4 \end{cases}$$

and $S = S_1 \cup S_2 \cup S^*$. Then S is an isolating set of H. Observe that S has $\lceil \frac{s}{2} \rceil$ rows of $\lceil \frac{n}{4} \rceil$ vertices each, and thus we obtain

$$\iota(G) \le |S| \le \left\lceil \frac{s}{2} \right\rceil \left\lceil \frac{t}{4} \right\rceil \le \frac{(t+3)(s+1)}{8} = \frac{st}{8} + \frac{3s+t+3}{8}.$$

Finally, let $J = P_s \square P_t$. Observe that J has 2 vertices of degree 2, $2s + 2t - 8$ vertices of degree 3 and $(s-2)(t-2)$ vertices of degree 4. This gives $d(G) = \frac{4st - 2s - 2t}{st}$ and thus we have with Theorem 4.2 $\iota(G) \ge \frac{2st - s - t}{16} = \frac{st}{8} - \frac{s+t}{16}$. For the upper bound, define the vertices as above such that $G[A_i] \cong P_t$ and $G[B_j] \cong P_s$ and set $S_1 = \{(i, j) \mid i, j \equiv 1 \pmod 4\} \cup \{(s, j) \mid j \equiv 1 \pmod 4\}$ and $S_2 = \{(i, j) \mid i, j \equiv 3 \pmod 4\}\} \cup \{(i, t) \mid i \equiv 3 \pmod 4\}$. Further, define

$$S^* = \begin{cases} \{(s, j) \mid j \equiv 1 \pmod 4\}, & \text{if } s \equiv 0 \pmod 4 \\ \{(s, j) \mid j \equiv 3 \pmod 4\}, & \text{if } s \equiv 2 \pmod 4 \\ \emptyset, & \text{if } s \equiv 1, 3 \pmod 4, \end{cases}$$

$$T^* = \begin{cases} \{(i, t) \mid i \equiv 1 \pmod 4\}, & \text{if } t \equiv 0 \pmod 4 \\ \{(i, t) \mid j \equiv 3 \pmod 4\}, & \text{if } t \equiv 2 \pmod 4 \\ \emptyset, & \text{if } t \equiv 1, 3 \pmod 4, \end{cases}$$

and $S = S_1 \cup S_2 \cup S^* \cup T^*$. Then S has $\lceil \frac{s}{4} \rceil$ rows of vertices (i, j) with $i, j \equiv 1 \pmod 4$, each having $\lceil \frac{t}{4} \rceil$ vertices. Similarly, S has $\lceil \frac{s-2}{4} \rceil$ rows of vertices (i, j) with $i, j \equiv 3 \pmod 4$, each having $\lceil \frac{t-2}{4} \rceil$ vertices. Hence,

$$\iota(G) \le |S| \le \left\lceil \frac{s}{4} \right\rceil \left\lceil \frac{t}{4} \right\rceil + \left\lceil \frac{s-2}{4} \right\rceil \left\lceil \frac{t-2}{4} \right\rceil.$$

The last inequality is worst when $s, t \equiv 3 \pmod 4$, and thus

$$\iota(G) \le 2 \frac{(s+1)(t+1)}{16} = \frac{st}{8} + \frac{s+t+1}{8}.$$

$\square$

Observe that, according to [1, 17], for $s \ge t \ge 16$, the domination number of the grid graph $P_s \square P_t$ is equal to $\frac{st}{5} + O(s+t)$, which is around a factor of $\frac{8}{5}$ larger than the bounds for the isolation number of grid graphs from Theorem 5.6.

6. Nordhaus-Gaddum Bounds

In this section, we will deal with Nordhaus-Gaddum bounds for the isolation number. Observe that $\iota(G) = 0$ if and only if $G = \overline{K_n}$. Since, for $n \geq 2$, $\iota(K_n) = 1$ it follows that $\iota(G) + \iota(\overline{G}) \geq 1$ for $n \geq 2$ and this is sharp. The following results give upper bounds on the sum $\iota(G) + \iota(\overline{G})$.

Theorem 6.1. *Let G be a graph with minimum degree δ. Then, if $\iota(\overline{G}) \geq 3$, $\iota(G) + \iota(\overline{G}) \leq \delta + 1$.*

Proof. Observe first that if $\delta(G) \leq 3$, then $\iota(\overline{G}) \leq 2$, since if there is a vertex $y \in V$ with $\deg(y) \leq 3$, then, in $\overline{G}$, $|V \setminus N_{\overline{G}}[y]| \leq 3$ and $V \setminus N_{\overline{G}}[y]$ can be isolated by one more vertex. So we may assume, without loss of generality, that $\delta(G) \geq 4$. The condition $\iota(\overline{G}) \geq 3$ implies that no two vertices form an isolating set in $\overline{G}$, meaning that, for any two vertices $x, y \in V$, there is an edge $uv \in E(\overline{G})$ with both end vertices u, v outside $N_{\overline{G}}[x] \cup N_{\overline{G}}[y]$. In particular, it follows that, in G, $|N_G(x) \cap N_G(y)| \geq 2$ for any two vertices $x, y \in V$ and thus $\mathrm{diam}(G) \leq 2$. Since $\iota(\overline{G}) \geq 3$ we have also that $G \neq K_n$ and, hence, we obtain that $\mathrm{diam}(G) = 2$. Let now u be a vertex of minimum degree δ in G. Then $N_G(u)$ is a dominating and an isolating set in G. Let $Y = V \setminus N_G[u]$ and let X be a minimal isolating set of G such that $X \subseteq N_G(u)$ and $X' = N_G(u) \setminus X$. Note that $V = X \cup X' \cup Y \cup \{u\}$ is a disjoint union. Let $x \in X$. Then, by the minimality condition on X, there are vertices $y, z \in Y \cup X' \cup \{x\}$ such that $yz \in E(G)$ and $(N_G(y) \cup N_G(z)) \cap X = \{x\}$. Hence, in $\overline{G}$, there are vertices $y, z \in Y \cup X' \cup \{x\}$ such that $yz \notin E(\overline{G})$ and $X \setminus \{x\} = (N_{\overline{G}}(y) \cap N_{\overline{G}}(z)) \cap X$. Observe that $X' \neq \emptyset$, since otherwise $\{u, y\}$ would be an isolating set in $\overline{G}$, contradicting $\iota(\overline{G}) \geq 3$. Let $I \subseteq X'$ be the set of all isolated vertices in $\overline{G}[X']$ and let D be a minimum isolating set in $\overline{G}[X' \setminus I]$ (if $X' \setminus I = \emptyset$, set $D = \emptyset$). Then $|D| \leq \frac{|X' \setminus I|}{2}$. We distinguish now the following cases.

Case 1: $I = \emptyset$. Then $|X'| \geq 2$ and thus $|X| \leq \delta - 2$. In this case, $D \cup \{u, y\}$ is an isolating set of $\overline{G}$ and therefore we have

$$\iota(G) + \iota(\overline{G}) \leq |X| + |D \cup \{u, y\}|$$
$$\leq |X| + \frac{|X'|}{2} + 2$$
$$= |X| + \frac{\delta - |X|}{2} + 2$$
$$= \frac{|X| + \delta}{2} + 2$$
$$\leq \frac{2\delta - 2}{2} + 2 = \delta + 1.$$

Case 2: $|I| = 1$, say $I = \{w\}$. Then $|X| = \delta - |X'| \leq \delta - 1$. If $N_G(w) \cap X = \emptyset$, then $X \subseteq N_{\overline{G}}(w)$ and, thus, $D \cup \{u, w\}$ is an isolating set in $\overline{G}$. Hence,

$$\iota(G) + \iota(\overline{G}) \leq |X| + |D \cup \{u, w\}|$$
$$\leq |X| + \frac{|X' \setminus I|}{2} + 2$$
$$= |X| + \frac{\delta - |X| - |I|}{2} + 2$$
$$= \frac{\delta + |X| - |I|}{2} + 2$$
$$\leq \frac{2\delta - 2}{2} + 2 = \delta + 1$$

and we are done. Therefore, we may assume, without loss of generality, that $wx \in E(G)$. In this case, $D \cup \{u, y\}$ is an isolating set in $\overline{G}$. As above, we obtain again $\iota(G) + \iota(\overline{G}) \leq \delta + 1$.

Case 3: $|I| \geq 2$. Then $|X| \leq \delta - 2$ and $D \cup \{u, x, y\}$ is an isolating set of $\overline{G}$. Thus we have

$$
\begin{aligned}
\iota(G) + \iota(\overline{G}) &\leq |X| + |D \cup \{u, x, y\}| \\
&\leq |X| + \frac{|X' \setminus I|}{2} + 3 \\
&\leq |X| + \frac{\delta - |X| - |I|}{2} + 3 \\
&= \frac{\delta + |X| - |I|}{2} + 3 \\
&\leq \frac{2\delta - 4}{2} + 3 = \delta + 1
\end{aligned}
$$

and we are done. $\square$

Theorem 6.2. *Let $f(x)$ be a function defined for $x \in [2, \infty)$ such that $\iota(G) \leq f(\delta)n$ for all graphs G of order n and minimum degree $\delta \geq 2$. Then, for any graph G of order $n \geq \frac{\delta-1}{f(\delta)}$, we have*

$$
\iota(G) + \iota(\overline{G}) \leq nf(\delta) + 2.
$$

Moreover, if the upper bound $f(\delta)n$ is attained for a graph G of order n and minimum degree $\delta \geq 2$, then the above inequality is sharp.

Proof. Let G be a graph of order n and minimum degree $\delta \geq 2$. By Theorem 6.1, if $\iota(\overline{G}) \geq 3$, we have $\iota(G) + \iota(\overline{G}) \leq \delta + 1$. On the other side, if $\iota(\overline{G}) \leq 2$, we obtain $\iota(G) + \iota(\overline{G}) \leq f(\delta)n + 2$. Hence, $\iota(G) + \iota(\overline{G}) \leq \max\{\delta + 1, f(\delta)n + 2\}$. If $n \geq \frac{\delta-1}{f(\delta)}$, then $f(\delta)n + 2 \geq \delta + 1$, implying thus $\iota(G) + \iota(\overline{G}) \leq nf(\delta) + 2$. For the sharpness, assume that $\iota(G) = f(\delta)n(G)$ for a graph G of minimum degree $\delta \geq 2$. Let H be the graph consisting of $\frac{n}{n(G)} \geq 2$ copies of G, where n is an integer divisible by $n(G)$. Then $\iota(H) = \frac{n}{n(G)}f(\delta)n(G) = f(\delta)n$ and $\iota(\overline{G}) = 2$. $\square$

Remark 6.3. *Observe that, if $\delta(G) \leq 1$, then $\iota(\overline{G}) = 1$. Hence, in this case, we have $\iota(G) + \iota(\overline{G}) = \iota(G) + 1$, showing that the bound of Theorem 6.2 cannot be attained.*

Corollary 6.4. *Let G be a graph of oder n and minimum degree δ. Then we have the following bounds.*

 (i) If $\delta = 0$, $\iota(G) + \iota(\overline{G}) \leq \left\lfloor \frac{n+1}{2} \right\rfloor$ and this is sharp.

 (ii) If $\delta = 1$, $\iota(G) + \iota(\overline{G}) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$ and this is sharp.

 (iii) If $\delta = 2$, $\iota(G) + \iota(\overline{G}) \leq \frac{2}{5}n + 2$ and this is sharp.

 (iv) If $\delta = 3$, $\iota(G) + \iota(\overline{G}) \leq \frac{1}{3}n + 2$ and this is sharp.

 (v) If $n \geq \frac{(\delta-1)(\delta+1)}{\ln(\delta+1)+\frac{1}{2}}$, then $\iota(G) + \iota(\overline{G}) \leq \frac{\ln(\delta+1)+\frac{1}{2}}{\delta+1}n + 2$.

Proof. (i) Let G be a graph on n vertices and with minimum degree $\delta = 0$. Let I be the set of isolated vertices in G. Then, by Theorem 3.5(ii), $\iota(G) = \iota(G - I) \leq \left\lfloor \frac{n-|I|}{2} \right\rfloor \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Since in $\overline{G}$ there is a vertex of degree $n - 1$, $\iota(\overline{G}) = 1$ and, hence, $\iota(G) + \iota(\overline{G}) \leq \left\lfloor \frac{n-1}{2} \right\rfloor + 1 = \left\lfloor \frac{n+1}{2} \right\rfloor$. The sharpness follows by considering the graph $G = \frac{n-1}{2}K_2 \cup K_1$, when n is odd, and $G = \frac{n-2}{2}K_2 \cup 2K_1$, when n is even.

(ii) Let G be a graph on n vertices and minimum degree $\delta = 1$. Then, by Theorem 3.5 (ii), we have $\iota(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$ and, since $\overline{G}$ has a vertex of degree $n - 2$, $\iota(\overline{G}) = 1$. Hence, $\iota(G) + \iota(\overline{G}) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$. For the sharpness, consider the graph $G = \frac{n}{2}K_2$, when n is even, and $G = \frac{n-3}{2}K_2 \cup K_{1,2}$, when n is odd.

(iii) - (v) The bounds follow from Theorems 3.5 and 6.2 for $n \geq \frac{\delta-1}{f(\delta)}$, with $f(2) = \frac{2}{5}$, $f(3) = \frac{1}{3}$ and $f(\delta) = \frac{\ln(\delta+1)+\frac{1}{2}}{\delta+1}$. For $\delta \in \{2, 3\}$ and small values of n, that is, $\delta + 1 \leq n < \frac{\delta-1}{f(\delta)}$, we only need to check when $\delta = 3$ and $n = 4$ or 5. In this case, the only possibilities for G are either K_4, $K_5 - e$, or $K_5 - \{e, f\}$, where e and f are the edges of a matching in K_5. Evidently, the bound holds in these cases, too. $\square$

7. Open Problems

This paper, as it is introducing a new subject, offers obviously many possible directions, open problems, conjectures and generalizations. Rather than doing this, we choose to offer some few concrete open problems where a progress to solve them seems doable, and which hopefully will shade more light on which ideas and techniques will be useful in attacking such problems on partial domination with restricted structure imposed on the non-dominated vertices.

Recalling Theorem 3.5, we show $f(\delta) \geq \frac{2}{\delta+3}$ and we prove that this is sharp for $\delta = 1, 2, 3$.

Problem 7.1. *Determine other values of $f(\delta)$. In particular, is $f(4) = \frac{2}{7}$?*

Recall that, by means of Theorem 3.8, we show in Corollary 3.9 that $f_c(1) = f_c(2) = \frac{1}{3}$ and that $f_c(\delta) \geq \frac{2}{\delta+4}$ for $\delta \geq 3$.

Problem 7.2. *Determine other values of $f_c(\delta)$. In particular, is $f_c(3) = \frac{2}{7}$?*

For $\mathcal{F} = \{C_k \mid k \geq 3\}$, we know after Theorem 3.1 and Corollary 3.10 that $\frac{1}{4} \leq \limsup_{n\to\infty}\{\frac{\iota(G,\mathcal{F})}{n(G)} \mid n(G) \geq n\} \leq \frac{1}{3}$. Also for $\mathcal{F} = \{K_{k+1}\}$ and $\mathcal{F} = \mathcal{F}_k$, the family of all trees of order $k \geq 2$, we know, after the same theorem and corollary, that $\frac{1}{k+1} \leq \limsup_{n\to\infty}\{\frac{\iota(G,\mathcal{F})}{n(G)} \mid n(G) \geq n\} \leq \frac{1}{3}$. Hence, we state the following problem.

Problem 7.3. *For $\mathcal{F} = \{C_k \mid k \geq 3\}$, $\mathcal{F} = \{K_{k+1}\}$ and $\mathcal{F} = \mathcal{F}_k$, determine*

$$\limsup_{n\to\infty}\left\{\frac{\iota(G,\mathcal{F})}{n(G)} \mid n(G) \geq n\right\} \text{ and } \sup_{n\to\infty}\left\{\frac{\iota(G,\mathcal{F})}{n(G)} \mid n(G) \geq n\right\}.$$

Problem 7.4. *Determine or give a lower and an upper bound for $\iota(G)$ or $\iota(G,\mathcal{F})$ for further interesting families of graphs.*

Problem 7.5. *Estimate $g(n,\delta) = \max\{\gamma(G) - \iota(G) \mid n(G) = n, \ \delta(G) = \delta\}$. Is it true that $g(n,\delta) \geq cn\frac{ln(\delta+1)}{\delta+1}$ for some constant c with $0 < c < 1$?*

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