## Partial-Norm of Entanglement: Entanglement Monotones That are not Monogamous

Yu Guo

School of Mathematical Sciences, Inner Mongolia University, Hohhot, Inner Mongolia 010021, People's Republic of China

E-mail: guoyu3@aliyun.com

Abstract. Quantum entanglement is known to be monogamous, i.e., it obeys strong constraints on how the entanglement can be distributed among multipartite systems. Almost all the entanglement monotones so far are shown to be monogamous. We explore here a family of entanglement monotones with the reduced functions are concave but not strictly concave and show that they are not monogamous. They are defined by four kinds of the "partial-norm" of the reduced state, which we call them *partial-norm of entanglement*, minimal partial-norm of entanglement, reinforced minimal partial-norm of entanglement, and *partial negativity*, respectively. This indicates that, the previous axiomatic definition of the entanglement monotone needs supplemental agreement that the reduced function should be strictly concave since such a strict concavity can make sure that the corresponding convex-roof extended entanglement monotone is monogamous. Here, the reduced function of an entanglement monotone refers to the corresponding function on the reduced state for the measure on bipartite pure states.

*Keywords*: Entanglement monotone, Partial-norm of entanglement, Partial negativity, Monogamy

Entanglement, as a quintessential manifestation of quantum mechanics [1, 2, 3], has shown to be a crucial resource in various quantum information processing tasks [1, 6, 4, 7, 5]. The most striking property of entanglement is its distributability, that is, the impossibility of sharing entanglement unconditionally across many subsystems of a composite quantum system [9, 8]. Understanding how entanglement can be quantified and distributed over many parties reveals fundamental insights into the nature of quantum correlations [10] and has profound applications in both quantum communication [11, 12, 13] and other area of physics [14, 15, 16, 17, 18, 11, 19]. Particularly, monogamy law of quantum correlation is the predominant feature that guarantees the quantum key distribution secure [8, 20].

Quantitatively, the monogamy of entanglement is described by an inequality, involving a bipartite entanglement monotone. The term "monotone" refers to the fact that a proper measure of entanglement cannot increase on average under local operations and classical communication (LOCC) [21, 23, 22]. Recall that the traditional monogamy relation of entanglement measure E is quantitatively displayed as an inequality of the following form:

$$E(A|BC) \ge E(AB) + E(AC), \tag{1}$$

where the vertical bar indicates the bipartite split across which the (bipartite) entanglement is measured. However, Eq. (1) is not valid for many entanglement measures but  $E^{\alpha}$  satisfies the relation for some  $\alpha > 0$  [24, 9, 25]. Intense research has been undertaken in this direction. It has been proved that the squashed entanglement and the one-way distillable entanglement are monogamous [26], and almost all the bipartite entanglement measures so far are monogamous for the multiqubit system or monogamous on pure states [24, 9, 14, 25, 27, 28, 29]. However, for the higher dimensional system, it is difficult to check the monogamy of entanglement measure according to Eq. (1) in general. Consequently, the definition of the monogamy is then improved as [30]: a measure of entanglement E is monogamous if for any  $\rho^{ABC} \in S^{ABC}$ that satisfies the disentangling condition, i.e.,

$$E(A|BC) = E(AB), \qquad (2)$$

we have that E(AC) = 0, where  $\mathcal{S}^X$  denotes the set of all density matrices acting on the state space  $\mathcal{H}^X$ . It is equivalent to the traditional monogamy relation in Eq. (1) for any continuous measure E [30]: a continuous measure E is monogamous according to this definition if and only if there exists  $0 < \alpha < \infty$  such that  $E^{\alpha}(A|BC) \geq E^{\alpha}(AB) + E^{\alpha}(AC)$ , for all  $\rho^{ABC} \in \mathcal{S}^{ABC}$  with fixed dim  $\mathcal{H}^{ABC} = d < \infty$ . Such a definition simplifies the justification of the monogamy of entanglement measure greatly [30, 31].

Recall that, a function  $E : \mathcal{S}^{AB} \to \mathbb{R}_+$  is called a measure of entanglement if (1)  $E(\sigma^{AB}) = 0$  for any separable density matrix  $\sigma^{AB} \in \mathcal{S}^{AB}$ , and (2) Ebehaves monotonically under LOCC. Moreover, convex measures of entanglement that do not increase *on average* under LOCC are called entanglement monotones [21]. Let E be a measure of entanglement on bipartite states. We define  $E_F(\rho^{AB}) \equiv$   $\min \sum_{j=1}^{n} p_j E(|\psi_j\rangle \langle \psi_j|^{AB})$ , where the minimum is taken over all pure state decompositions of  $\rho^{AB} = \sum_{j=1}^{n} p_j |\psi_j\rangle \langle \psi_j|^{AB}$ . That is,  $E_F$  is the convex roof extension of E. Vidal [21, Theorem 2] showed that for any entanglement measure E,  $E_F$  above is an entanglement monotone if

$$h\left(\rho^{A}\right) = E\left(|\psi\rangle\langle\psi|^{AB}\right) \tag{3}$$

is concave, i.e.  $h[\lambda\rho_1 + (1-\lambda)\rho_2] \ge \lambda h(\rho_1) + (1-\lambda)h(\rho_2)$  for any states  $\rho_1$ ,  $\rho_2$ , and any  $0 \le \lambda \le 1$ . Hereafter, we call h the reduced function of E and  $\mathcal{H}^A$  the reduced subsystem for convenience.

In Ref. [30], according to definition (2), we showed that  $E_F$  is monogamous whenever  $E_F$  is defined via Eq. (3) with h is strictly concave additionally. Except for the Rényi  $\alpha$ -entropy of entanglement with  $\alpha > 1$ , all other measures of entanglement, that were studied intensively in literature, correspond on pure bipartite state to strict concave functions of the reduced density matrix. Theses include the original entanglement of formation [32], tangle [33], concurrence [34, 33], G-concurrence [35], Tsallis entropy of entanglement [36], and the entanglement measures induced by the fidelity distances [37]. Nevertheless, we are not sure yet whether the entanglement monotone is monogamous if the reduced function is concave but not strictly concave. The purpose of this paper is to address such a issue. We explore the entanglement monotone suggested in Ref. [38], from which we also obtain another two entanglement monotones. We also investigate the partial negativity which is defined as the norm of the negative part of the state after partial transposition. The reduced functions of these quantities are not strictly concave, and they are not equivalent to each other. We then show that they are not monogamous. This is the first time to prove that there exist entanglement monotones that are not monogamous in the light of the disentangling condition. Our results establish a more closer relation between the monogamy of an entanglement monotone and the strict concavity of the reduced function and suggest that we should require the strict concavity of the reduced function for any "fine" entanglement monotone. Moreover, comparing with other reduced functions for which the corresponding entanglement measures are shown to be monogamous, we find that if the reduced function is defined on all of the eigenvalues of the reduced state it is strictly concave and vice versa in general.

Let  $|\psi\rangle = \sum_{j=1}^{r} \lambda_j |e_j\rangle^A |e_j\rangle^B$  be the Schmidt decomposition of  $|\psi\rangle \in \mathcal{H}^{AB}$ , where  $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ , and r is the Schmidt rank of  $|\psi\rangle$ . In 1999, Vidal proposed an entanglement monotone in Ref. [38], i.e.,

$$E_k\left(|\psi\rangle\right) = \sum_{i=k}^r \lambda_i^2, \quad k \ge 2.$$
(4)

In particular,

$$E_2(|\psi\rangle) = \sum_{i=2}^r \lambda_i^2 = 1 - \lambda_1^2 = 1 - \|\rho^A\|,$$
(5)

where  $\rho^A = \operatorname{tr}_B |\psi\rangle\langle\psi|$ ,  $\|\cdot\|$  is the operator norm, i.e.,  $\|X\| = \sup_{|\psi\rangle} \|X|\psi\rangle\|$ . Hereafter, we call  $E_2$  the partial-norm of entanglement in the sense that  $1 - \|\rho^A\|$  counts for only a portion of the norm  $\|\rho^A\|$  for the qubit case. Obviously,  $E_2 \ge 0$  for any  $|\psi\rangle \in \mathcal{H}^{AB}$ and  $E_2(|\psi\rangle) = 0$  if and only if  $|\psi\rangle$  is separable. For mixed state,  $E_2(\rho)$  is defined by the convex-roof extension. Generally,  $\|A + B\| = \|A\| + \|B\|$  does not guarantee  $A = \alpha B$ for hermitian operators A and B, so this reduced function  $h(\rho) = 1 - \|\rho\|$  is not strictly concave. We next illustrate with counter-examples that  $E_2$  is not monogamous.

**Theorem 1.**  $E_2$  is not monogamous.

Let

$$\begin{aligned} |\psi_0\rangle^{AB} &= a_0 |0\rangle^A |0\rangle^B + a_1 |1\rangle^A |1\rangle^B + a_2 |2\rangle^A |2\rangle^B, \\ |\psi_1\rangle^{AB} &= a_0' |0\rangle^A |3\rangle^B + a_1' |1\rangle^A |2\rangle^B + a_2' |2\rangle^A |1\rangle^B \end{aligned}$$

with 
$$a_0^2 = a'_0^2 \ge \frac{1}{2}$$
,  $a'_1 a_2 \ne a_1 a'_2$ ,  $\sum_i a_i^2 = \sum_i a'_i^2 = 1$ ,  $a_0 > a_1 \ge a_2$ ,  $a'_0 > a'_1 \ge a'_2$ , and

$$|\Phi\rangle = \frac{1}{\sqrt{2}} \left( |\psi_0\rangle^{AB} |0\rangle^C + |\psi_1\rangle^{AB} |1\rangle^C \right).$$
(6)

After tracing over subsystems we are left with

$$\begin{split} \rho^{AB} &= \frac{1}{2} \left( |\psi_0\rangle \langle \psi_0|^{AB} + |\psi_1\rangle \langle \psi_1|^{AB} \right), \\ \rho^{AC} &= \frac{1}{2} \left[ \left( a_0^2 |0\rangle \langle 0|^A + a_1^2 |1\rangle \langle 1|^A + a_2^2 |2\rangle \langle 2|^A \right) \otimes |0\rangle \langle 0|^C \right. \\ &\quad + \left( a'_0^2 |0\rangle \langle 0|^A + a'_1^2 |1\rangle \langle 1|^A + a'_2^2 |2\rangle \langle 2|^A \right) \otimes |1\rangle \langle 1|^C \\ &\quad + \left( a_1 a'_2 |1\rangle \langle 2|^A + a_2 a'_1 |2\rangle \langle 1|^A \right) \otimes |0\rangle \langle 1|^C \\ &\quad + \left( a_1 a'_2 |2\rangle \langle 1|^A + a_2 a'_1 |1\rangle \langle 2|^A \right) \otimes |1\rangle \langle 0|^C \right], \\ \rho_0^A &= a_0^2 |0\rangle \langle 0|^A + a_1^2 |1\rangle \langle 1|^A + a'_2^2 |2\rangle \langle 2|^A , \\ \rho_1^A &= a_0^2 |0\rangle \langle 0|^A + a'_1^2 |1\rangle \langle 1|^A + a'_2^2 |2\rangle \langle 2|^A \end{split}$$

and

$$\rho^{A} = a_{0}^{2}|0\rangle\langle0|^{A} + \frac{1}{2}(a_{1}^{2} + {a'}_{1}^{2})|1\rangle\langle1|^{A} + \frac{1}{2}(a_{2}^{2} + {a'}_{2}^{2})|2\rangle\langle2|^{A},$$

where  $\rho_{0,1}^A = \operatorname{tr}_b |\psi_{0,1}\rangle \langle \psi_{0,1}|^{AB}$ . From here it follows that  $E_2(|\Phi\rangle^{A|BC}) = 1 - a_0^2$ . We next show that  $E_2(\rho^{AB}) = E_2(|\Phi\rangle^{A|BC})$  but  $E_2(\rho^{AC}) > 0$ , namely,  $E_2$  is not monogamous. For any pure state ensemble of  $\rho^{AB} = \sum_i p_i |\phi_i\rangle \langle \phi_i|^{AB}$ , we have

$$p_i |\phi_i\rangle^{AB} = \frac{1}{\sqrt{2}} \left( u_{i0} |\psi_0\rangle^{AB} + u_{i1} |\psi_1\rangle^{AB} \right)$$

for any *i*, where  $|u_{i0}|^2 + |u_{i1}|^2 \leq 1$ , which yields the largest eigenvalue of  $\sigma_i^A = \text{tr}_B |\phi_i\rangle\langle\phi_i|^{AB}$  is always  $a_0$ . Thus

$$E_2(\rho^{AB}) = E_2(|\Phi\rangle^{A|BC}) = 1 - a_0^2$$

as desired. On the other hand, we let  $|x\rangle^{AC} = a_1|1\rangle^A|0\rangle^C + a'_2|2\rangle^A|1\rangle^C$  and  $|y\rangle^{AC} = a_2|2\rangle^A|0\rangle^C + a'_1|1\rangle^A|1\rangle^C$ , then

$$\rho^{AC} = a_0^2 |0\rangle \langle 0|^A \otimes (|0\rangle \langle 0|^C + |1\rangle \langle 1|^C) + \frac{1}{2} |x\rangle \langle x|^{AC} + \frac{1}{2} |y\rangle \langle y|^{AC}.$$

It is easy to see that  $\rho_{AC}^{T_A}$  is not positive whenever  $a'_1 a_2 \neq a_1 a'_2$ , and thus  $E_2(\rho^{AC}) > 0$ , here  $T_X$  denotes the partial transpose transformation with respect to the subsystem X.

If the reduced subsystem is two-dimensional, we consider the three-qubit case with no loss of generality. Any pure state  $|\psi\rangle$  in  $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$  can be expressed as [39]

$$|\psi\rangle^{ABC} = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle$$

up to local unitary transformation, where  $\lambda_i \geq 0$ ,  $0 \leq \varphi \leq \pi$ ,  $\sum_i \lambda_i^2 = 1$ . The reduced states  $\rho^{AB} = p|x_1\rangle\langle x_1| + (1-p)|x_2\rangle\langle x_2|$  with  $\sqrt{p}|x_1\rangle = \lambda_2|10\rangle + \lambda_4|11\rangle$  and  $\sqrt{(1-p)}|x_2\rangle = \lambda_0|00\rangle + \lambda_1 e^{i\varphi}|10\rangle + \lambda_3|11\rangle$ , and

$$\rho^{A} = \left(\begin{array}{cc} \lambda_{0}^{2} & \lambda_{0}\lambda_{1}e^{-i\varphi} \\ \lambda_{0}\lambda_{1}e^{i\varphi} & \lambda_{1}^{2} + \lambda_{2}^{2} + \lambda_{3}^{2} + \lambda_{4}^{2} \end{array}\right).$$

It is straightforward that (1)  $|\psi\rangle$  is genuinely entangled if and only if  $\lambda_0 > 0$ ,  $\lambda_2^2 + \lambda_4^2 > 0$ and  $\lambda_3^2 + \lambda_4^2 > 0$ , (2)  $\rho^{AB}$  is separable iff  $\lambda_3 = 0$ , and (3)  $\rho^{AC}$  is separable iff  $\lambda_2 = 0$ . If  $E_2(|\psi\rangle^{A|BC}) = E_2(\rho^{AB})$ , then

$$E_2(\rho^{AB}) = \sum_k p_k E_2(|\phi_k\rangle)$$

for any  $\rho^{AB} = \sum_k p_k |\phi_k\rangle \langle \phi_k |$  according to Corollary 5 in Ref. [30]. This leads to the minimal eigenvalue of

$$(1-p)\operatorname{tr}_{B}|x_{2}\rangle\langle x_{2}| = \begin{pmatrix} \lambda_{0}^{2} & \lambda_{0}\lambda_{1}e^{-\mathrm{i}\varphi} \\ \lambda_{0}\lambda_{1}e^{\mathrm{i}\varphi} & \lambda_{1}^{2}+\lambda_{3}^{2} \end{pmatrix}$$

coincides with that of  $\rho^A$ , which yields either  $\lambda_2 = \lambda_4 = 0$ , or  $\lambda_1 = 0$  and  $\lambda_0 \leq \lambda_3$ . That is,  $\rho^{AC}$  could be entangled. Therefore  $E_2$  is still not monogamous whenever the reduced subsystem is two dimensional.

Let  $\lambda_{\min}$  be the minimal positive Schmidt coefficient of  $|\psi\rangle$ . We define

$$E_{\min}(|\psi\rangle) = \begin{cases} \lambda_{\min}^2, & \lambda_{\min} < 1, \\ 0, & \lambda_{\min} = 1 \end{cases}$$
(7)

for pure state and then define by means of the convex-roof extension for mixed state. Denoting by

$$\|\rho\|_{\min} = \begin{cases} \lambda_{\min}^2, & \lambda_{\min} < 1, \\ 0, & \lambda_{\min} = 1. \end{cases}$$

$$\tag{8}$$

it turns out that

$$E_{\min}(|\psi\rangle) = h(\rho^A) = \|\rho^A\|_{\min}.$$

We call  $E_{\min}$  the minimal partial-norm of entanglement, which reflects as the minimal case of the partial-norm. It is clear that  $E_{\min}(\rho) = 0$  iff  $\rho$  is separable. Let  $\delta(\rho) = (\delta_1, \delta_2, \ldots, \delta_d)$  for any state  $\rho \in \mathcal{S}$  with dim  $\mathcal{H} = d$ , where  $\delta_i$ s are the eigenvalues,  $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_d$ . The concavity of h is clear since

$$\delta[t\rho + (1-t)\sigma] \prec t\delta(\rho) + (1-t)\delta(\sigma),$$

which implies

$$||t\rho + (1-t)\sigma||_{\min} \ge t||\rho||_{\min} + (1-t)||\sigma||_{\min},$$

where " $\prec$ " is the majorization relation between probability distributions. Thus  $E_{\min}$  is an entanglement monotone.

By now, except for the convex-roof extension of the negativity N [41], denoted by  $N_F$ , all the reduced functions of convex-roof extended entanglement monotones in previous literature are shown to be strictly concave. Here N is defined as [40]  $N(\rho) = \sum_i \mu_i$  with  $\mu_i$ s are the eigenvalues of the negative part of  $\rho^{T_A}$ . In order to show that the reduced function of  $N_F$ , denoted by  $h_N$ , is strictly concave. We give the following statement at first, which is a complementary of Vidal [21, Theorem 2].

**Proposition 2.** Let E be an entanglement measure with the reduced function h defined as Eq. (3). If E is an entanglement monotone, then h is concave.

*Proof.* Let  $\rho$  and  $\sigma$  be any given two states in  $\mathcal{S}^A$ ,  $0 \leq t \leq 1$ . Taking  $|\psi\rangle^{AB}$  and  $|\phi\rangle^{AB}$  in  $\mathcal{H}^{AB}$  such that  $\rho = \operatorname{tr}_B |\psi\rangle \langle \psi|^{AB}$  and  $\sigma = \operatorname{tr}_B |\phi\rangle \langle \phi|^{AB}$ , we let

$$|\Psi\rangle^{ABC} = \sqrt{t} |\psi\rangle^{AB} |0\rangle^{C} + \sqrt{1-t} |\phi\rangle^{AB} |1\rangle^{C}$$

be a pure state in  $\mathcal{H}^{ABC}$ . Consider a LOCC  $\{I^A \otimes I^B \otimes |0\rangle \langle 0|^C, I^A \otimes I^B \otimes |1\rangle \langle 1|^C\}$  acting on  $|\Psi\rangle^{ABC}$ , we obtain the output

$$\left\{t|\psi\rangle\langle\psi|^{AB}\otimes|0\rangle\langle0|^{C},(1-t)|\phi\rangle\langle\phi|^{AB}\otimes|1\rangle\langle1|^{C}\right\},$$

where  $I^{A,B}$  is the identity operator acting on  $\mathcal{H}^{A,B}$ . This leads to

$$E(|\Psi\rangle^{A|BC}) \ge tE(|\psi\rangle^{AB}|0\rangle^{C}) + (1-t)E(|\phi\rangle^{AB}|1\rangle^{C})$$

since E is an entanglement monotone, which is equivalent to

$$h(t\rho + (1-t)\sigma) \ge th(\rho) + (1-t)h(\sigma),$$

that is, h is concave.

By Proposition 2,  $N_F$  is an entanglement monotone since N is an entanglement monotone and thus the reduced function  $h_N$  is concave. Note here that, in Ref. [41], there is a gap in the proof of the concavity of  $h_N$ : the second inequality of the last part in page 2 is wrong since  $|\phi_k\rangle$  is not necessarily a basis (i.e., it is just an orthogonal set but not complete) in general. We show that  $h_N$  is strictly concave as well. We assume to obtain a contradiction that  $h_N$  is not strictly concave. Then there exists  $\rho^A =$  $p\rho_1^A + (1-p)\rho_2^A \in S^A$  with  $\operatorname{spec}(\rho_1^A) \neq \operatorname{spec}(\rho_2^A)$ , but  $h_N(\rho^A) = ph_N(\rho_1^A) + (1-p)h_N(\rho_2^A)$ , here  $\operatorname{spec}(X)$  denotes the spectrum of X. Let

$$\rho^{AB} = p|\psi_1\rangle\langle\psi_1|^{AB} + (1-p)|\psi_2\rangle\langle\psi_2|^{AB}$$

with  $|\psi_i\rangle^{AB} = \sum_j \lambda_{ij} |e_{ij}\rangle^A |e_{ij}\rangle^B$  is the Schmidt decomposition of  $|\psi_i\rangle^{AB}$ , i = 1, 2, where  $\operatorname{tr}_B |\psi_i\rangle\langle\psi_i|^{AB} = \rho_i^A$ , and

$$\langle e_{ij}|e_{kl}\rangle^B = \delta_{ik}\delta_{jl}$$

We take

$$|\widetilde{\Psi}\rangle^{ABC} = \sqrt{p}|\psi_1\rangle^{AB}|0\rangle^C + \sqrt{1-p}|\psi_2\rangle^{AB}|1\rangle^C,$$

then for any ensemble of  $\rho^{AB} = \sum_k q_k |\phi_k\rangle \langle \phi_k|^{AB}$ ,

$$\sum_{k} q_{k} N(|\phi_{k}\rangle^{AB}) = \sum_{k} q_{k} h_{N}(\rho_{k}^{A})$$

$$\geq \sum_{k} \left[ p|u_{k1}|^{2} h_{N}(\rho_{1}^{A}) + (1-p)|u_{k2}|^{2} h_{N}(\rho_{2}^{A}) \right]$$

$$= ph_{N}(\rho_{1}^{A}) + (1-p)h_{N}(\rho_{2}^{A}),$$

where

$$\sqrt{q_k}|\phi_k\rangle^{AB} = u_{k1}\sqrt{p}|\psi_1\rangle^{AB} + u_{k2}\sqrt{1-p}|\psi_2\rangle^{AB}$$

It turns out that  $N(\rho^{AB}) = N(|\tilde{\Psi}\rangle^{A|BC})$ . But  $|\tilde{\Psi}\rangle^{ABC}$  does not admit the form  $|\psi\rangle^{AB_1}|\psi\rangle^{B_2C}$  up to some local unitary operation, where  $B_1B_2$  means  $\mathcal{H}^B$  has a subspace isomorphic to  $\mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2}$  and up to local unitary on system  $B_1B_2$ , which contradicts with Theorem 3 in [29]. Thus  $h_N$  is strictly concave. That is, all the reduced functions of the monogamous entanglement monotones so far are strictly concave.

We now go back to discuss the monogamy of  $E_{\min}$ . Clearly, if the reduced system is two-dimensional, then  $E_{\min} = E_2$ , which is not monogamous. For higher dimensional case, we consider a pure state as in Eq. (6) just by replacing  $a_0^2 = a'_0^2 \ge \frac{1}{2}$ ,  $a_0 > a_1 \ge a_2$ ,  $a'_0 > a'_1 \ge a'_2$ , with  $a_0 = a'_0$ ,  $a_1 \ge a_2 > a_0$ ,  $a'_1 \ge a'_2 > a'_0$ , from which one can conclude that  $E_{\min}$  is not monogamous.

However,  $E_{\min}$  does not achieve the maximal value for the maximally entangled state. For making up the disadvantages, we can define

$$E'_{\min}(|\psi\rangle) = \begin{cases} \lambda_{\min}^2 S_r(|\psi\rangle), & \lambda_{\min} < 1, \\ 0, & \lambda_{\min} = 1, \end{cases}$$
(9)

for pure state and then define by means of the convex-roof extension for mixed state, where  $S_r(|\psi\rangle)$  denotes the Schmidt rank of  $|\psi\rangle$ . We call it the *reinforced minimal partial*norm of entanglement.  $E'_{\min}$  is equal to  $2E_{\min}$  for any  $2 \otimes n$  state. In such a case,  $E'_{\min}$ reaches the maximal quantity for the maximally entangled state but not only for these states. In addition, it is easy to follow that  $E'_{\min}$  is also an entanglement monotone and is not monogamous.

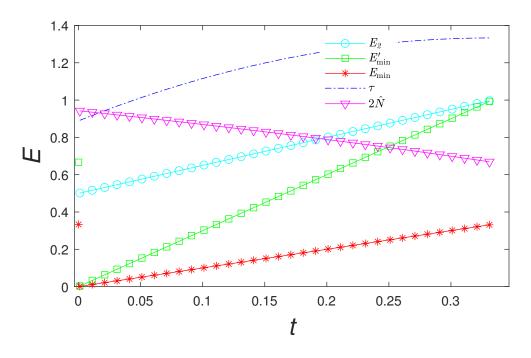
Let  $|\psi\rangle$ ,  $|\phi\rangle$ ,  $|\varphi\rangle$ ,  $|\xi\rangle$ , and  $|\zeta\rangle$  be pure states with the reduced states, respectively, are diag(2/3, 1/6, 1/6), diag(1/3, 1/3, 1/3), diag(3/5, 2/5, 0), diag(2/5, 2/5, 15), and diag(4/5, 1/5, 0). Then we arrive at

$$E_{2}(|\varphi\rangle) < E_{2}(|\phi\rangle) \text{ but } E_{\min}(|\phi\rangle) < E_{\min}(|\varphi\rangle),$$
  

$$E_{2}(|\varphi\rangle) < E_{2}(|\xi\rangle) \text{ but } E'_{\min}(|\xi\rangle) < E'_{\min}(|\varphi\rangle),$$
  

$$E_{\min}(|\psi\rangle) < E_{\min}(|\zeta\rangle) \text{ but } E'_{\min}(|\zeta\rangle) < E'_{\min}(|\psi\rangle)$$

That is, these three measures are not equivalent to each other.



**Figure 1.** (color online). Comparing  $E_2$ ,  $E'_{\min}$  with the tangle  $\tau$  for  $|\psi\rangle$  with Schmidt numbers  $(\sqrt{2/3-t}, \sqrt{1/3}, \sqrt{t}), 0 \le t \le 1/3$ .

The maximal value of  $E_2$  is (d-1)/d. We thus, in order to get a normalized measure, replace  $E_2$  by  $dE_2/(d-1)$ . Hereafter the notation  $E_2$  refers to the normalized one. For the  $2 \otimes n$  system,  $E_2$  coincides with  $E'_{\min}$  but not for  $m \otimes n$  system with  $2 < m \leq n$ . For any pure state  $|\psi\rangle$  with Schmidt numbers p and 1-p in  $2 \otimes n$  system,  $p \leq 1/2$ , it is immediate that

$$E_2(|\psi\rangle) = 2E_{\min}(|\psi\rangle) = E'_{\min}(|\psi\rangle) = 2p,$$

and

$$\tau(|\psi\rangle) = 2p(1-p).$$

Here,  $\tau$  is the tangle, which is defined as the square of concurrence, i.e.,  $\tau(|\psi\rangle) = 2(1 - \operatorname{tr} \rho_A^2)$ ,  $\rho_A = \operatorname{tr}_B |\psi\rangle\langle\psi|$ . That is  $E_2 \geq \tau$  and both of them are monotonically increasing with  $0 \leq p \leq 1/2$ .

We now compute these three entanglement monotones for the qutrit-qutrit pure state and then compare them with tangle. It can be easily calculated since they are homogeneous. We consider  $|\psi\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3$  with Schmidt numbers  $(\sqrt{2/3} - t, \sqrt{1/3}, \sqrt{t}),$  $0 \leq t \leq 1/3$ , and  $|\phi\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3$  with Schmidt numbers  $(\sqrt{p}, \sqrt{q}, \sqrt{1-p-q})$ for illustration purposes,  $p \geq q$ . The behaviours of theses quantities for these two states are depicted in Fig. 1 and Fig. 2, respectively. In the case of t = 0,  $E'_{\min}(|\psi\rangle) = 2/3 = 2E_{\min}(|\psi\rangle)$  and  $\tau(|\psi\rangle) = 8/9$ . For the case of p + q = 1,  $E_2(|\psi\rangle) = 3q/2 < E'_{\min}(|\psi\rangle) = 2q$ . That is,  $E_{\min}$  and  $E'_{\min}$  are not continuous and are not equivalent. The upper bounds of these quantities can be easily derived. Let  $\rho$  be a state in  $\mathcal{S}^{AB}$ , and  $E_2(\rho) = \sum_i p_i E_2(|\psi_i\rangle)$ , then

$$E_2(\rho) = \sum_i p_i E_2(|\psi_i\rangle)$$
  
=  $\frac{d}{d-1} \sum_i p_i \left(1 - \|\rho_i^A\|\right)$   
=  $\frac{d}{d-1} \left[1 - \left(\sum_i p_i \|\rho_i^A\|\right)\right]$   
 $\leq \frac{d}{d-1} \left(1 - \|\sum_i p_i \rho_i^A\|\right)$   
=  $\frac{d}{d-1} \left(1 - \|\rho^A\|\right).$ 

That is

$$E_2(\rho) \le \frac{d}{d-1} \min\{1 - \|\rho^A\|, 1 - \|\rho^B\|\}.$$
(10)

Analogously,

$$E_{\min}(\rho) \le \min\{\|\rho^A\|_{\min}, \|\rho^B\|_{\min}\}$$
(11)

and

$$E'_{\min}(\rho) \le \min\{r_A \| \rho^A \|_{\min}, r_B \| \rho^B \|_{\min}\},$$
(12)

where  $r_{A,B}$  is the rank of  $\rho^{A,B}$ .

When  $k \geq 3$ ,  $E_k$  is not a faithful entanglement monotone, and it is not monogamous either. Another entanglement measure that lack of investigating the monogamy is the Schmidt number, which is regarded as a universal entanglement measure [42], defined by [43]

$$S_r(\rho) = \min_{p_i, |\psi_i\rangle} \max_{|\psi_i\rangle} S_r(|\psi_i\rangle), \tag{13}$$

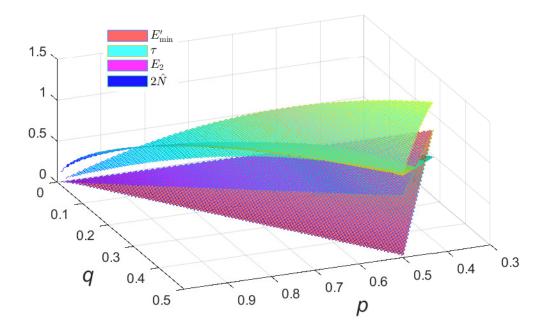
where the minimum is taken over all decomposition  $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$ . It is also not monogamous since both the Schmidt number of  $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$  and that of its two reduced states are 2.

In addition, let  $\rho^{T_A}$  be the partial transpose of  $\rho$ , one may consider the partial-norm of the negative part of  $\rho^{T_A}$ ,  $N\rho^-$ . For example, we take

$$\hat{N}(\rho) = \|N\rho^{-}\|.$$
(14)

We call it *partial negativity* hereafter. Take  $\rho = |\psi\rangle\langle\psi|$  with  $|\psi\rangle = \sum_{j} \lambda_{j} |e_{j}\rangle^{A} |e_{j}\rangle^{B}$  as the Schmidt decomposition of  $|\psi\rangle$ . Then  $\hat{N}(|\psi\rangle) = \lambda_{1}\lambda_{2}$ , and the corresponding reduced function is

$$\hat{h}(\rho^A) = \sqrt{\delta_1 \delta_2},\tag{15}$$



**Figure 2.** (color online). Comparing  $E_2$ ,  $E'_{\min}$  with the tangle  $\tau$  for  $|\phi\rangle$  with Schmidt numbers  $(\sqrt{p}, \sqrt{q}, \sqrt{1-p-q}), p+q < 1.$ 

where  $\delta_1 = \lambda_1^2$ ,  $\delta_2 = \lambda_2^2$ .  $\hat{N}$  can still be regarded as a kind of partial norm as  $\sqrt{\delta_1 \delta_2} \leq \delta_1 = \|\rho^A\|$ , in other words,  $\hat{N}$  is also a kind of partial norm of entanglement. By definition,  $\hat{N}(|\psi\rangle^{ab}) = 0$  if and only if it is separable, and

 $0 < \hat{N}(\rho) \le N(\rho)$ 

for any non-positive partial transpose state  $\rho$ . A simple comparison between  $\hat{N}$  and  $E_2$ ,  $E_{\min}$ ,  $E'_{\min}$  are given in Fig. 1 and Fig. 2, which indicate that they are not equivalent to each other. For the two-qubit case,  $2\hat{N}_F$  coincides with the *G*-concurrence [35]. We conjecture that  $\hat{h}$  is concave [44].  $\hat{h}$  is strictly concave on  $\mathcal{S}(\mathcal{H})$  with dim  $\mathcal{H} = 2$  since it reduced to an elementary symmetric function [45, p. 116], but it is not true for the higher dimensional case. In order to see this, we take

$$\rho = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which yields  $\hat{h}(\frac{1}{2}\rho + \frac{1}{2}\sigma) = \frac{1}{2}\hat{h}(\rho) + \frac{1}{2}\hat{h}(\sigma)$ . We now assume that  $\hat{N}$  is an entanglement monotone, then we can conclude the following.

**Theorem 3.**  $\hat{N}$  and  $\hat{N}_F$  are not monogamous whenever the reduced subsystem has dimension greater than 2.

We show this statement by a counter-example. Let

$$|\widetilde{\Omega}\rangle^{ABC} = \lambda_0 |0\rangle^A |00\rangle^{BC} + \lambda_1 |1\rangle^A |10\rangle^{BC} + \lambda_2 |2\rangle^A |11\rangle^{BC}$$
(16)

with  $\lambda_0 \geq \lambda_1 \geq \lambda_2 > 0$ , it turns out that

$$\hat{N}(|\hat{\Omega}\rangle^{A|BC}) = \hat{N}(\rho^{AB}) = \lambda_0 \lambda_1$$

but

$$\hat{N}(\rho^{AC}) = \lambda_1 \lambda_2 > 0.$$

That is,  $\hat{N}$  is not monogamous. Moreover, from this example, we can also get  $\hat{N}_F$  is not monogamous either in light of  $\hat{N} \leq \hat{N}_F$ .

Analogous to that of the logarithmic negativity, we define the *logarithmic partial* negativity by

$$\hat{N}_l(\rho) = \log_2[\hat{N}(\rho) + 1].$$
(17)

It is straightforward that  $\hat{N}_l$  is not convex. For any LOCC acting on  $\rho^{ab}$  that leaves the output states  $\{p_i\sigma_i\}$ , we have

$$\sum_{i} p_{i} \hat{N}_{l}(\sigma_{i}) = \sum_{i} p_{i} \log_{2} x_{i} \le \log_{2} \sum_{i} p_{i} x_{i} \le \log_{2} [\hat{N}(\rho) + 1] = \hat{N}_{l}(\rho)$$

since  $\log_2$  is concave and  $\hat{N}$  is non-increasing on average under LOCC by assumption, where  $x_i = \hat{N}(\sigma_i) + 1$ . Therefore it is also an entanglement monotone and is not monogamous (hereafter, we still call it an entanglement monotone even though it is not convex as in Ref. [22]).

In sum, for the sake of distinguishing these entanglement monotones so far in the sense of monogamy law, we suggest the term informationally complete entanglement *monotone*, which means that its reduced function is related to all its eigenvalues. For example, the entanglement of formation is informationally complete since the von Neumann entropy is defined on all of the eigenvalues which include all the information of the entanglement, but  $E_2$ ,  $E_{\min}$ ,  $E'_{\min}$ ,  $\hat{N}$ ,  $\hat{N}_F$ , and  $\hat{N}_l$  are not the case except for the two-dimensional case since they just capture "partial information" of the entanglement. The worst one is the Schmidt number, which reflects the least information of the entanglement, and of course is not informationally complete. Our discussion supports that, for an entanglement monotone  $E_F$  with reduced function h,  $E_F$  is monogamous if and only if it is informationally complete, and in turn, iff h is strictly concave (the "if" part is proved [31]). So the axiomatic definition of an entanglement monotone should be improved as follows. Let E be a nonnegative function on  $\mathcal{S}^{AB}$  with  $E(|\psi\rangle) = h(\rho^A)$  for pure state. We call E a strict entanglement monotone if (i)  $E(\sigma^{AB}) = 0$  for any separable density matrix  $\sigma^{AB} \in \mathcal{S}^{AB}$ , (ii) E behaves monotonically decreasing under LOCC on average, and (iii) the reduced function h is strictly concave. We use henceforth the term strict entanglement monotone to distinguish it from the previous entanglement monotone.

With such a spirit, except for  $E_2$ ,  $E_{\min}$ ,  $E'_{\min}$ ,  $\hat{N}$ ,  $\hat{N}_F$ ,  $\hat{N}_l$  and the Schmidt number, all the previous entanglement monotones that are shown to be monogamous or monogamous on pure states are strict entanglement monotones, these include the original entanglement of formation, negativity, the squashed entanglement [46], the convex-roof extension of negativity, tangle, concurrence, the relative entropy of entanglement [23], G-concurrence, the Tsallis entropy of entanglement, the conditional entanglement of mutual information [47], and the entanglement measures induced by the fidelity distances, etc. However, it still remains unknown that whether or not the non convex-roof extended strict entanglement monotones in literature are monogamous in addition to the squashed entanglement. We conjecture that all the informationally complete entanglement monotones are monogamous.

As a by-product, we can obtain new coherence measures from the reduced function h of  $E_2$ ,  $E_{\min}$  and  $E'_{\min}$ , respectively. Let

$$C_h(|\psi\rangle) = h(x_0, x_1, \dots, x_{d-1})$$
 (18)

for pure state  $|\psi\rangle = \sum_{i} x_{i} |i\rangle$  under the reference basis  $\{|i\rangle\}_{i=0}^{d-1}$ , and by the convex-roof extension for mixed state, i.e.,

$$C_h(\rho) = \min_{p_j, |\psi_j\rangle} \sum_j p_j C_h(|\psi_j\rangle),$$

where the minimum is taken over all decomposition  $\rho = \sum_{j} p_{j} |\psi_{j}\rangle \langle \psi_{j}|$ . It turns out that (i)  $h(1, 0, \dots, 0) = 0$ , (ii)  $h(\pi(x_{0}, x_{1}, \dots, x_{d-1})) = h(x_{0}, x_{1}, \dots, x_{d-1})$  for any permutation  $\pi$  and any  $(x_{0}, x_{1}, \dots, x_{d-1})$ , and (iii) h is concave. This reveals that  $C_{h}$  is a well-defined coherence measure according to Theorem 1 in Ref. [48]. Also notice here that, the associated function h of all the previous coherence measures defined by means of the convex-roof extension are strictly concave, which are different from  $C_{h}$ .

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