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Partial Ray Expansion Required to Suitably Approximate the Exact Wave Solution

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Summary

Whenever synthetic seismograms are computed for layered media with the aid of ray theories only a partial ray expansion is possible. We present criteria that may be incorporated in a computer algorithm to obtain an automatic selection of rays which will minimize the error in the partial ray expansion below a preselected level. The program, results of which are presented in this paper, can also be used for the study of multiple reflected and converted phases or for media with dipping interfaces.

1. Introduction

Various ray theories which decompose the total wave field in a layered medium into contributions attributed to individual rays have been used to advantage in the generation and analysis of synthetic seismograms. Among these are generalized ray theory (Spencer 1960) in which the displacement field is decomposed into contributions from an infinite set of rays, each ray being evaluated by a numerical solution of the impulse response (Bortfield 1967; Muller 1968, 1970) or by the Cagniard–de Hoop method (Helmberger 1968; Gilbert & Helmberger 1972). The solution for each ray is exact but the computations are very complex so that only a limited number of rays are used in the solution. An alternative method is asymptotic ray theory (Hron & Kanasewich 1971; Hron 1972) in which the field quantities are expressed as an infinite power series of reciprocal frequency combined with a space-dependent vector which is independent of frequency. The displacement field also becomes decomposed into an infinite set of rays but a greater number may be taken into the approximation by limiting the power series of each to the first non-zero term in the expansion. Although different techniques are used to evaluate the ray amplitudes, the final result is always a synthetic seismogram in which the kinematic and dynamic characteristics of particular rays may be studied.

Ray theories are limited invariably by computational difficulties associated with the selection of the phases in the partial ray expansion. Very often the selection is based on the personal intuition or experience of the seismologist. If the selection is poor then the seismograms are misleading for rays with significant amplitudes will be omitted. An initial attempt at producing a computer algorithm that would generate systematically a set of rays, determine their kinematic and dynamic analogues and test for significant amplitude levels was reported by Hron & Kanasewich (1971). The algorithm for ray generation proved to be very efficient so that seismograms based on the evaluation of more than 150 000 individual rays were not exceptional (Hron 1972). However, no estimate of the accuracy of this partial ray expansion was possible because the convergence of the complete expansion had not been established. This difficulty has been removed by Cisternas, Betancourt & Leiva (1973) who showed that the complete ray expansion can be obtained for a solid layered medium by expanding the Rayleigh matrix rather than the Rayleigh determinant into a power series in which each term can be interpreted as a physical ray.

The subject of this paper is to show that a partial ray expansion consisting of rays generated by a program described by Hron (1973c) represents a very good approximation to the exact solution with a predetermined accuracy specified in the input. Comparison is made with synthetic seismograms generated for a plane wave incident from below on a layered crust which may be computed using the Fourier transform of the crustal transfer function (Hanon 1964) obtained from a Haskell–Thomson matrix formulation (Haskell 1962; Thomson 1950). The wave solution obtained using a matrix formulation is exact and may be used as a basis for determining the accuracy of a partial ray expansion.

2. Basic concepts

The algorithm (Hron 1971, 1972, 1973c), designed to minimize the number of ray computations, is based on an amplitude criteria. The media may consist of a series of homogeneous layers, with arbitrary curvilinear or dipping interfaces, overlying a halfspace. In this paper the source of compressional waves will be a plane wave impinging on the lower interface and the interfaces will be restricted to be non-dipping so that we can compare the results with a wave solution using a matrix formulation. The receiver will be located at the surface. The model is a typical continental section and is given in Table 1.

In general, ray amplitudes decrease rapidly with increasing number of reflections as well as with increasing number of conversions. Therefore ray classification is based on the number of segments, NSEG, and on the number of conversions, JCONV experienced by the ray between the source and receiver. For 4 layers over a halfspace the minimum number of segments is 5. The maximum number of conversions in each set is JCONV = NSEG − 1. The minimum amplitude for an acceptable ray is specified as a percentage, Δ, of the maximum ray amplitude encountered in the first set of rays. Table 2 summarizes the number of rays created, NCREA, and the per cent of these used, NRUSED, as Δ was varied between 0·33 and 15 per cent. There is no geometric spreading of the ray tube for a source at infinity and therefore the ray amplitudes are evaluated with the zero order approximation of asymptotic ray theory. This consists of the product of reflection and transmission coefficients of plane waves.

One additional, but optional, parameter is available to limit the computation. If NRUSED (%) falls below a preset value for any value of JCONV no further rays are calculated in this set and computation proceeds to the next set of rays in which the number of segments is incremented by 2. In Table 2 the limit on NRUSED was 7 per cent. When the source and receiver are at different depths there is no great advantage in classifying rays into kinematic and dynamic analogues and this was not done in this case.

Table 1

Parameters of the Alberta Model

Layer	VP ($km\,s^{-1}$)	VS ($km\,s^{-1}$)	Dens ($g\,cm^{-3}$)	Thickness (km)
1	2·31	1·33	2·04	1·10
2	3·06	1·77	2·21	0·92
3	6·50	3·75	2·73	31·38
4	7·15	4·13	3·20	9·80
5	8·08	4·66	3·45	—

Table 2

Numbers of created rays NRCREA with JCONV conversions and NSEG segments as compared with the numbers of rays NRUSED (in per cent) used for the construction of seismograms because their amplitudes were greater than $\Delta_1 = 0{\cdot}33$ *per cent (1st column),* $\Delta_2 = 1$ *per cent (2nd column),* $\Delta_2 = 5$ *per cent (3rd column) and* $\Delta = 15$ *per cent (4th column) of the strongest amplitude*

SET	NSEG	JCONV	NRCREA	$NRUSED_1$ (%)	$NRUSED_2$ (%)	$NRUSED_3$ (%)	$NRUSED_4$ (%)
1	5	0	1	100	100	100	100
		1	4	100	100	100	50
		2	6	100	83	16	0
		3	4	100	0	0	0
		4	1	100	0	0	0
2	7	0	4	100	50	50	0
		1	24	83	41	12	0
		2	60	68	11	0	0
		3	80	52	3	0	0
		4	60	18	0	0	0
		5	24	0	0	0	0
3	9	0	13	76	23	7	7
		1	104	46	14	2	0
		2	364	30	4	0	0
		3	728	20	0	0	0
		4	910	6	0	0	0
4	11	0	40	40	15	0	0
		1	400	21	3	0	0
		2	1800	14	0	0	0
		3	4800	7	0	0	0
5	13	0	121	25	1	1	0
		1	1452	8	0	0	0
Total no. of rays			11000	1380	99	21	5
Percentage of total used			—	12·55	0·90	0·19	0·04

An important advantage of ray theory is that individual arrivals are identified with their ray paths. A numerical code is used to identify the rays. Table 3 lists 179 unconverted rays identified by an ordinal number, JRAY, and found during computation. The converted phases are then computed from this file of basic unconverted rays. The total number of converted rays with NSEG segments and JCONV conversions is given by

$$C^{\text{NSEG}-1}_{\text{JCONV}} = \frac{(\text{NSEG}-1)!}{\text{JCONV}!\,(\text{NSEG}-1-\text{JCONV})!}. \tag{1}$$

It is assumed that the incident wave in the first segment is fixed as either a compressional (P) or shear (S) phase.

The coding number, NOFRAY, for any ray is specified by the equation

$$\text{NOFRAY} = 1000 \cdot \text{JRAY} + 100 \cdot \text{JCONV} + \text{JC}. \tag{2}$$

JC is an integer that codifies the points of incidence at which phase conversions occur for any particular ray with ordinal number, JRAY, and having JCONV conversions. An index 0_i, $i = 1, 2, \ldots, \text{NSEG}-1$, specifies, in sequence along the ray, the points of incidence where a transmission or reflection coefficient is required. Table 4 reproduces the output from the algorithm generating JC and it is seen that a hierarchal structure is followed in which a ray with the least segments and conversions

Table 3

Family of unconverted rays prepared for the use in the program ' layered crust '

MAXSFG= 13, LAYERS= 4

SET 1	5
1	5 4 3 2 1
SET 2	7
	12000
2	5 4 3 2 1 1 1
3	5 4 3 2 2 2 1
4	5 4 3 3 3 2 1
5	5 4 4 4 3 2 1
SET 3	9
6	5 4 3 2 1 1 1 1 1
7	5 4 3 2 1 1 2 2 1
8	5 4 3 2 2 2 1 1 1
9	5 4 3 3 3 2 1 1 1
10	5 4 4 4 3 2 1 1 1
11	5 4 3 2 2 2 2 2 1
12	5 4 3 2 2 3 3 2 1
13	5 4 3 3 3 2 2 2 1
14	5 4 4 4 3 2 2 2 1
15	5 4 3 3 3 3 3 2 1
16	5 4 3 3 4 4 3 2 1
17	5 4 4 4 3 3 3 2 1
18	5 4 4 4 4 4 3 2 1
SET 4	11
19	5 4 3 2 1 1 1 1 1 1 1
20	5 4 3 2 1 1 1 1 2 2 1
21	5 4 3 2 1 1 2 2 1 1 1
22	5 4 3 2 2 2 1 1 1 1 1
23	5 4 3 3 3 2 1 1 1 1 1
24	5 4 4 4 3 2 1 1 1 1 1
25	5 4 3 2 1 1 2 2 2 2 1
26	5 4 3 2 2 2 1 1 2 2 1
27	5 4 3 2 2 2 2 2 1 1 1
28	5 4 3 2 1 1 2 3 3 2 1
29	5 4 3 3 3 2 1 1 2 2 1
30	5 4 3 2 2 3 3 2 1 1 1
31	5 4 3 3 3 2 2 2 1 1 1
32	5 4 4 4 3 2 1 1 2 2 1
33	5 4 4 4 3 2 2 2 1 1 1
34	5 4 3 3 3 3 3 2 1 1 1
35	5 4 3 3 4 4 3 2 1 1 1
36	5 4 4 4 3 3 3 2 1 1 1
37	5 4 4 4 4 4 3 2 1 1 1
38	5 4 3 2 2 2 2 2 2 2 1
39	5 4 3 2 2 2 2 3 3 2 1
40	5 4 3 2 2 3 3 2 2 2 1
61	5 4 3 2 1 1 1 1 2 2 1 1 1
62	5 4 3 2 1 1 2 2 1 1 1 1 1
63	5 4 3 2 2 2 1 1 1 1 1 1 1
64	5 4 3 3 3 2 1 1 1 1 1 1 1
65	5 4 4 4 3 2 1 1 1 1 1 1 1
66	5 4 3 2 1 1 2 2 1 1 2 2 1
67	5 4 3 2 1 1 1 1 2 2 2 2 1
68	5 4 3 2 2 2 1 1 1 1 2 2 1
69	5 4 3 2 1 1 2 2 2 2 1 1 1
70	5 4 3 2 2 2 1 1 2 2 1 1 1
71	5 4 3 2 2 2 2 2 1 1 1 1 1
72	5 4 3 2 1 1 1 1 2 3 3 2 1
73	5 4 3 2 1 1 2 3 3 2 1 1 1
74	5 4 3 3 3 2 1 1 1 1 2 2 1
75	5 4 3 3 3 2 1 1 2 2 1 1 1
76	5 4 3 2 2 3 3 2 1 1 1 1 1
77	5 4 3 3 3 2 2 2 1 1 1 1 1
78	5 4 4 4 3 2 1 1 1 1 2 2 1
79	5 4 4 4 3 2 1 1 2 2 1 1 1
80	5 4 4 4 3 2 2 2 1 1 1 1 1
81	5 4 3 3 3 3 3 2 1 1 1 1 1
82	5 4 3 3 4 4 3 2 1 1 1 1 1
83	5 4 4 4 3 3 3 2 1 1 1 1 1
84	5 4 4 4 4 4 3 2 1 1 1 1 1
85	5 4 3 2 1 1 2 2 2 2 2 2 1
86	5 4 3 2 2 2 1 1 2 2 2 2 1
87	5 4 3 2 2 2 2 2 1 1 2 2 1
88	5 4 3 2 2 2 2 2 2 2 1 1 1
89	5 4 3 2 1 1 2 2 2 3 3 2 1
90	5 4 3 2 1 1 2 3 3 2 2 2 1
91	5 4 3 2 2 2 1 1 2 3 3 2 1
92	5 4 3 2 2 3 3 2 1 1 2 2 1
93	5 4 3 3 3 2 1 1 2 2 2 2 1
94	5 4 3 3 3 2 2 2 1 1 2 2 1
95	5 4 3 2 2 2 2 3 3 2 1 1 1
96	5 4 3 2 2 3 3 2 2 2 1 1 1
97	5 4 3 3 3 2 2 2 2 2 1 1 1
98	5 4 4 4 3 2 1 1 2 2 2 2 1
99	5 4 4 4 3 2 2 2 1 1 2 2 1
100	5 4 4 4 3 2 2 2 2 2 1 1 1
101	5 4 3 2 1 1 2 3 3 3 3 2 1
102	5 4 3 3 3 2 1 1 2 3 3 2 1
103	5 4 3 3 3 3 3 2 1 1 2 2 1
104	5 4 3 2 2 3 3 3 3 2 1 1 1
105	5 4 3 3 3 2 2 3 3 2 1 1 1
121	5 4 3 3 4 4 4 4 3 2 1 1 1
122	5 4 4 4 3 3 4 4 3 2 1 1 1
123	5 4 4 4 4 4 3 3 3 2 1 1 1
124	5 4 4 4 4 4 4 4 3 2 1 1 1
125	5 4 3 2 2 2 2 2 2 2 2 2 1
126	5 4 3 2 2 2 2 2 2 3 3 2 1
127	5 4 3 2 2 2 2 3 3 2 2 2 1
128	5 4 3 2 2 3 3 2 2 2 2 2 1
129	5 4 3 3 3 2 2 2 2 2 2 2 1
130	5 4 4 4 3 2 2 2 2 2 2 2 1
131	5 4 3 2 2 3 3 2 2 3 3 2 1
132	5 4 3 2 2 2 2 3 3 3 3 2 1
133	5 4 3 3 3 2 2 2 2 3 3 2 1
134	5 4 3 2 2 3 3 3 3 2 2 2 1
135	5 4 3 3 3 2 2 3 3 2 2 2 1
136	5 4 3 3 3 3 3 2 2 2 2 2 1
137	5 4 3 2 2 2 2 3 4 4 3 2 1
138	5 4 3 2 2 3 4 4 3 2 2 2 1
139	5 4 4 4 3 2 2 2 2 3 3 2 1
140	5 4 4 4 3 2 2 3 3 2 2 2 1
141	5 4 3 3 4 4 3 2 2 2 2 2 1
142	5 4 4 4 3 3 3 2 2 2 2 2 1
143	5 4 4 4 4 4 3 2 2 2 2 2 1
144	5 4 3 2 2 3 3 3 3 3 3 2 1
145	5 4 3 3 3 2 2 3 3 3 3 2 1
146	5 4 3 3 3 3 3 2 2 3 3 2 1
147	5 4 3 3 3 3 3 3 3 2 2 2 1
148	5 4 3 2 2 3 3 3 4 4 3 2 1
149	5 4 3 2 2 3 4 4 3 3 3 2 1
150	5 4 3 3 3 2 2 3 4 4 3 2 1
151	5 4 3 3 4 4 3 2 2 3 3 2 1
152	5 4 4 4 3 2 2 3 3 3 3 2 1
153	5 4 4 4 3 3 3 2 2 3 3 2 1
154	5 4 3 3 3 3 4 4 3 2 2 2 1
155	5 4 3 3 4 4 3 3 3 2 2 2 1
156	5 4 4 4 3 3 3 3 3 2 2 2 1
157	5 4 3 2 2 3 4 4 4 4 3 2 1
158	5 4 4 4 3 2 2 3 4 4 3 2 1
159	5 4 4 4 4 4 3 2 2 3 3 2 1
160	5 4 3 3 4 4 4 4 3 2 2 2 1
161	5 4 4 4 3 3 4 4 3 2 2 2 1
162	5 4 4 4 4 4 3 3 3 2 2 2 1
163	5 4 4 4 4 4 4 4 3 2 2 2 1
164	5 4 3 3 3 3 3 3 3 3 3 2 1
165	5 4 3 3 3 3 3 3 4 4 3 2 1

41	5	4	3	3	3	2	2	2	2	2	1		
42	5	4	4	4	3	2	2	2	2	2	1		
43	5	4	3	2	2	3	3	3	3	2	1		
44	5	4	3	3	3	2	2	3	3	2	1		
45	5	4	3	3	3	3	3	2	2	2	1		
46	5	4	3	2	2	3	4	4	3	2	1		
47	5	4	4	4	3	2	2	3	3	2	1		
48	5	4	3	3	4	4	3	2	2	2	1		
49	5	4	4	4	3	3	3	2	2	2	1		
50	5	4	4	4	4	4	3	2	2	2	1		
51	5	4	3	3	3	3	3	3	3	2	1		
52	5	4	3	3	3	3	4	4	3	2	1		
53	5	4	3	3	4	4	3	3	3	2	1		
54	5	4	4	4	3	3	3	3	3	2	1		
55	5	4	3	3	4	4	4	4	3	2	1		
56	5	4	4	4	3	3	4	4	3	2	1		
57	5	4	4	4	4	4	3	3	3	2	1		
58	5	4	4	4	4	4	4	4	3	2	1		
SFT 5 13													
59	5	4	3	2	1	1	1	1	1	1	1	1	1
60	5	4	3	2	1	1	1	1	1	1	2	2	1

106	5	4	3	3	3	3	3	2	2	2	1	1	1
107	5	4	3	2	1	1	2	3	4	4	3	2	1
108	5	4	4	4	3	2	1	1	2	3	3	2	1
109	5	4	3	3	4	4	3	2	1	1	2	2	1
110	5	4	4	4	3	3	3	2	1	1	2	2	1
111	5	4	3	2	2	3	4	4	3	2	1	1	1
112	5	4	4	4	3	2	2	3	3	2	1	1	1
113	5	4	3	3	4	4	3	2	2	2	1	1	1
114	5	4	4	4	3	3	3	2	2	2	1	1	1
115	5	4	4	4	4	4	3	2	1	1	2	2	1
116	5	4	4	4	4	4	3	2	2	2	1	1	1
117	5	4	3	3	3	3	3	3	3	2	1	1	1
118	5	4	3	3	3	3	4	4	3	2	1	1	1
119	5	4	3	3	4	4	3	3	3	2	1	1	1
120	5	4	4	4	3	3	3	3	3	2	1	1	1

166	5	4	3	3	3	3	4	4	3	3	3	2	1
167	5	4	3	3	4	4	3	3	3	3	3	2	1
168	5	4	4	4	3	3	3	3	3	3	3	2	1
169	5	4	3	3	4	4	3	3	4	4	3	2	1
170	5	4	3	3	3	3	4	4	4	4	3	2	1
171	5	4	4	4	3	3	3	3	4	4	3	2	1
172	5	4	3	3	4	4	4	4	3	3	3	2	1
173	5	4	4	4	3	3	4	4	3	3	3	2	1
174	5	4	4	4	4	4	3	3	3	3	3	2	1
175	5	4	3	3	4	4	4	4	4	4	3	2	1
176	5	4	4	4	3	3	4	4	4	4	3	2	1
177	5	4	4	4	4	4	3	3	4	4	3	2	1
178	5	4	4	4	4	4	4	4	3	3	3	2	1
179	5	4	4	4	4	4	4	4	4	4	3	2	1

Table 4

List of combinations used for creation of converted rays from individual sets of unconverted rays

SET	NSEG	JCONV	JC	i				
1	5	1	1)	1				
			2)	2				
			3)	3				
			4)	4				
1	5	2	1)	1	2			
			2)	1	3			
			3)	1	4			
			4)	2	3			
			5)	2	4			
			6)	3	4			
1	5	3	1)	1	2	3		
			2)	1	2	4		
			3)	1	3	4		
			4)	2	3	4		
1	5	4	1)	1	2	3	4	
2	7	1	1)	1				
			2)	2				
			3)	3				
			4)	4				
			5)	5				
			6)	6				
2	7	2	1)	1	2			
			2)	1	3			
			3)	1	4			
			4)	1	5			
			5)	1	6			
			6)	2	3			
			7)	2	4			
			8)	2	5			
			9)	2	6			
			10)	3	4			
			11)	3	5			
			12)	3	6			
			13)	4	5			
			14)	4	6			
			15)	5	6			
2	7	4	1)	1	2	3	4	
			2)	1	2	3	5	
			3)	1	2	3	6	
			4)	1	2	4	5	
			5)	1	2	4	6	
			6)	1	2	5	6	
			7)	1	3	4	5	
			8)	1	3	4	6	
			9)	1	3	5	6	
			10)	1	4	5	6	
			11)	2	3	4	5	
			12)	2	3	4	6	
			13)	2	3	5	6	
			14)	2	4	5	6	
			15)	3	4	5	6	
2	7	5	1)	1	2	3	4	5
			2)	1	2	3	4	6
			3)	1	2	3	5	6
			4)	1	2	4	5	6
			5)	1	3	4	5	6
			6)	2	3	4	5	6
3	9	1	1)	1				
			2)	2				
			3)	3				
			4)	4				
			5)	5				
			6)	6				
			7)	7				
			8)	8				
3	9	2	1)	1	2			
			2)	1	3			
			3)	1	4			
			4)	1	5			
			5)	1	6			
			6)	1	7			
			7)	1	8			
3	9	3	1)	1	2	3		
			2)	1	2	4		
			3)	1	2	5		
			4)	1	2	6		
			5)	1	2	7		
			6)	1	2	8		
			7)	1	3	4		
			8)	1	3	5		
			9)	1	3	6		
			10)	1	3	7		
			11)	1	3	8		
			12)	1	4	5		
			13)	1	4	6		
			14)	1	4	7		
			15)	1	4	8		
			16)	1	5	6		
			17)	1	5	7		
			18)	1	5	8		
			19)	1	6	7		
			20)	1	6	8		
			21)	1	7	8		
			22)	2	3	4		
			23)	2	3	5		
			24)	2	3	6		
			25)	2	3	7		
			26)	2	3	8		
			27)	2	4	5		
			28)	2	4	6		
			29)	2	4	7		
			30)	2	4	8		
			31)	2	5	6		
			32)	2	5	7		
			33)	2	5	8		
			34)	2	6	7		
			35)	2	6	8		
			36)	2	7	8		
			37)	3	4	5		
			38)	3	4	6		
3	9	4	1)	1	2	3	4	
			2)	1	2	3	5	
			3)	1	2	3	6	
			4)	1	2	3	7	
			5)	1	2	3	8	
			6)	1	2	4	5	
			7)	1	2	4	6	
			8)	1	2	4	7	
			9)	1	2	4	8	
			10)	1	2	5	6	
			11)	1	2	5	7	
			12)	1	2	5	8	
			13)	1	2	6	7	
			14)	1	2	6	8	
			15)	1	2	7	8	
			16)	1	3	4	5	
			17)	1	3	4	6	
			18)	1	3	4	7	
			19)	1	3	4	8	
			20)	1	3	5	6	
			21)	1	3	5	7	
			22)	1	3	5	8	
			23)	1	3	6	7	
			24)	1	3	6	8	
			25)	1	3	7	8	
			26)	1	4	5	6	
			27)	1	4	5	7	
			28)	1	4	5	8	
			29)	1	4	6	7	
			30)	1	4	6	8	
			31)	1	4	7	8	
			32)	1	5	6	7	
			33)	1	5	6	8	
			34)	1	5	7	8	
			35)	1	6	7	8	
			36)	2	3	4	5	
			37)	2	3	4	6	
			38)	2	3	4	7	
			39)	2	3	4	8	
			40)	2	3	5	6	
			41)	2	3	5	7	

2 7 3

1)	1	2	3
2)	1	2	4
3)	1	2	5
4)	1	2	6
5)	1	3	4
6)	1	3	5
7)	1	3	6
8)	1	4	5
9)	1	4	6
10)	1	5	6
11)	2	3	4
12)	2	3	5
13)	2	3	6
14)	2	4	5
15)	2	4	6
16)	2	5	6
17)	3	4	5
18)	3	4	6
19)	3	5	6
20)	4	5	6

8)	2	3
9)	2	4
10)	2	5
11)	2	6
12)	2	7
13)	2	8
14)	3	4
15)	3	5
16)	3	6
17)	3	7
18)	3	8
19)	4	5
20)	4	6
21)	4	7
22)	4	8
23)	5	6
24)	5	7
25)	5	8
26)	6	7
27)	6	8
28)	7	8

39)	3	4	7
40)	3	4	8
41)	3	5	6
42)	3	5	7
43)	3	5	8
44)	3	6	7
45)	3	6	8
46)	3	7	8
47)	4	5	6
48)	4	5	7
49)	4	5	8
50)	4	6	7
51)	4	6	8
52)	4	7	8
53)	5	6	7
54)	5	6	8
55)	5	7	8
56)	6	7	8

42)	2	3	5	8
43)	2	3	6	7
44)	2	3	6	8
45)	2	3	7	8
46)	2	4	5	6
47)	2	4	5	7
48)	2	4	5	8
49)	2	4	6	7
50)	2	4	6	8
51)	2	4	7	8
52)	2	5	6	7
53)	2	5	6	8
54)	2	5	7	8
55)	2	6	7	8
56)	3	4	5	6
57)	3	4	5	7
58)	3	4	5	8
59)	3	4	6	7
60)	3	4	6	8
61)	3	4	7	8
62)	3	5	6	7
63)	3	5	6	8
64)	3	5	7	8
65)	3	6	7	8
66)	4	5	6	7
67)	4	5	6	8
68)	4	5	7	8
69)	4	6	7	8
70)	5	6	7	8

SET	NSEG	JCONV	JC	i
4	11	1	1)	1
			2)	2
			3)	3
			4)	4
			5)	5
			6)	6
			7)	7
			8)	8
			9)	9
			10)	10
4	11	2	1)	1 2
			2)	1 3
			3)	1 4
			4)	1 5
			5)	1 6
			6)	1 7
			7)	1 8
			8)	1 9
			9)	1 10
			10)	2 3
			11)	2 4
			12)	2 5
			13)	2 6
			14)	2 7
			15)	2 8
			16)	2 9
			17)	2 10
			18)	3 4
			19)	3 5
			20)	3 6
			21)	3 7
			22)	3 8
			23)	3 9
			24)	3 10
			25)	4 5
			26)	4 6
			27)	4 7
			28)	4 8
			29)	4 9
			30)	4 10

SET	NSEG	JCONV	JC	i
			39)	6 10
			40)	7 8
			41)	7 9
			42)	7 10
			43)	8 9
			44)	8 10
			45)	9 10
4	11	3	1)	1 2 3
			2)	1 2 4
			3)	1 2 5
			4)	1 2 6
			5)	1 2 7
			6)	1 2 8
			7)	1 2 9
			8)	1 2 10
			9)	1 3 4
			10)	1 3 5
			11)	1 3 6
			12)	1 3 7
			13)	1 3 8
			14)	1 3 9
			15)	1 3 10
			16)	1 4 5
			17)	1 4 6
			18)	1 4 7
			19)	1 4 8
			20)	1 4 9
			21)	1 4 10
			22)	1 5 6
			23)	1 5 7
			24)	1 5 8
			25)	1 5 9
			26)	1 5 10
			27)	1 6 7
			28)	1 6 8
			29)	1 6 9
			30)	1 6 10
			31)	1 7 8
			32)	1 7 9

SET	NSEG	JCONV	JC	i
			41)	2 3 8
			42)	2 3 9
			43)	2 3 10
			44)	2 4 5
			45)	2 4 6
			46)	2 4 7
			47)	2 4 8
			48)	2 4 9
			49)	2 4 10
			50)	2 5 6
			51)	2 5 7
			52)	2 5 8
			53)	2 5 9
			54)	2 5 10
			55)	2 6 7
			56)	2 6 8
			57)	2 6 9
			58)	2 6 10
			59)	2 7 8
			60)	2 7 9
			61)	2 7 10
			62)	2 8 9
			63)	2 8 10
			64)	2 9 10
			65)	3 4 5
			66)	3 4 6
			67)	3 4 7
			68)	3 4 8
			69)	3 4 9
			70)	3 4 10
			71)	3 5 6
			72)	3 5 7
			73)	3 5 8
			74)	3 5 9
			75)	3 5 10
			76)	3 6 7
			77)	3 6 8
			78)	3 6 9
			79)	3 6 10
			80)	3 7 8

SET	NSEG	JCONV	JC	i
			89)	4 5 9
			90)	4 5 10
			91)	4 6 7
			92)	4 6 8
			93)	4 6 9
			94)	4 6 10
			95)	4 7 8
			96)	4 7 9
			97)	4 7 10
			98)	4 8 9
			99)	4 8 10
			100)	4 9 10
			101)	5 6 7
			102)	5 6 8
			103)	5 6 9
			104)	5 6 10
			105)	5 7 8
			106)	5 7 9
			107)	5 7 10
			108)	5 8 9
			109)	5 8 10
			110)	5 9 10
			111)	6 7 8
			112)	6 7 9
			113)	6 7 10
			114)	6 8 9
			115)	6 8 10
			116)	6 9 10
			117)	7 8 9
			118)	7 8 10
			119)	7 9 10
			120)	8 9 10
5	13	1	1)	1
			2)	2
			3)	3
			4)	4
			5)	5
			6)	6
			7)	7

31)	5	6
32)	5	7
33)	5	8
34)	5	9
35)	5	10
36)	6	7
37)	6	8
38)	6	9

33)	1	7	10
34)	1	8	9
35)	1	8	10
36)	1	9	10
37)	2	3	4
38)	2	3	5
39)	2	3	6
40)	2	3	7

81)	3	7	9
82)	3	7	10
83)	3	8	9
84)	3	8	10
85)	3	9	10
86)	4	5	6
87)	4	5	7
88)	4	5	8

8)	8
9)	9
10)	10
11)	11
12)	12

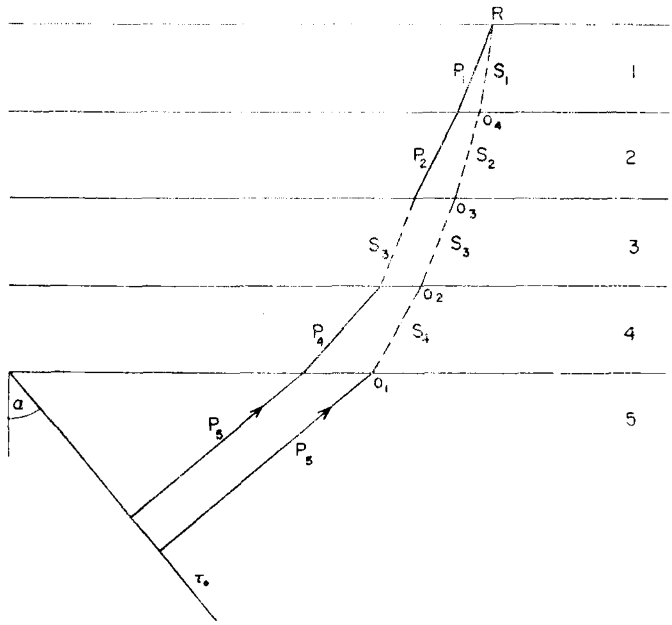


FIG. 1. A schematic picture of two different rays originating on the same plane wavefront, τ_0, impinging on the bottom of the layered crust with the angle of incidence $90° - \alpha$. Points of incidence O_1 are the only points where the conversion from P to S mode propagation or vice versa can occur.

and the earliest points of incidence, i, with a conversion, is given the lowest unassigned integer. Table 4 lists only those values of i where a phase conversion occurs.

The two rays, $P_5\, S_4\, S_3\, S_2\, S_1$ and $P_5\, P_4\, S_3\, P_2\, P_1$ with NOFRAY 1101 and 1204, respectively, in Fig. 1, may be used to demonstrate the coding system. From the number of ray segments and the layer sequence we see that JRAY = 1 from Table 3. There is one conversion (JCONV = 1) in the first ray and two in the second ray (JCONV = 2). The first ray was converted at O_1, the first encounter with an interface, therefore JC = 1 according to Table 4. In the second ray conversions take place during the second and third encounters with an interface, O_2 and O_3, so JC = 4 according to Table 4. The coding number, NOFRAY, and the amplitude to one significant figure is presented beneath the synthetic seismogram at the time of the arrival of the ray. For example, ray $P_5\, P_4\, P_3\, S_2\, S_1$, which is the largest arrival with amplitude $1. \times 10^0$ on the horizontal component in Fig. 2(a), is listed as NOFRAY = 1103 with amplitude 1EO at a time of 0·48s after the initial arrival.

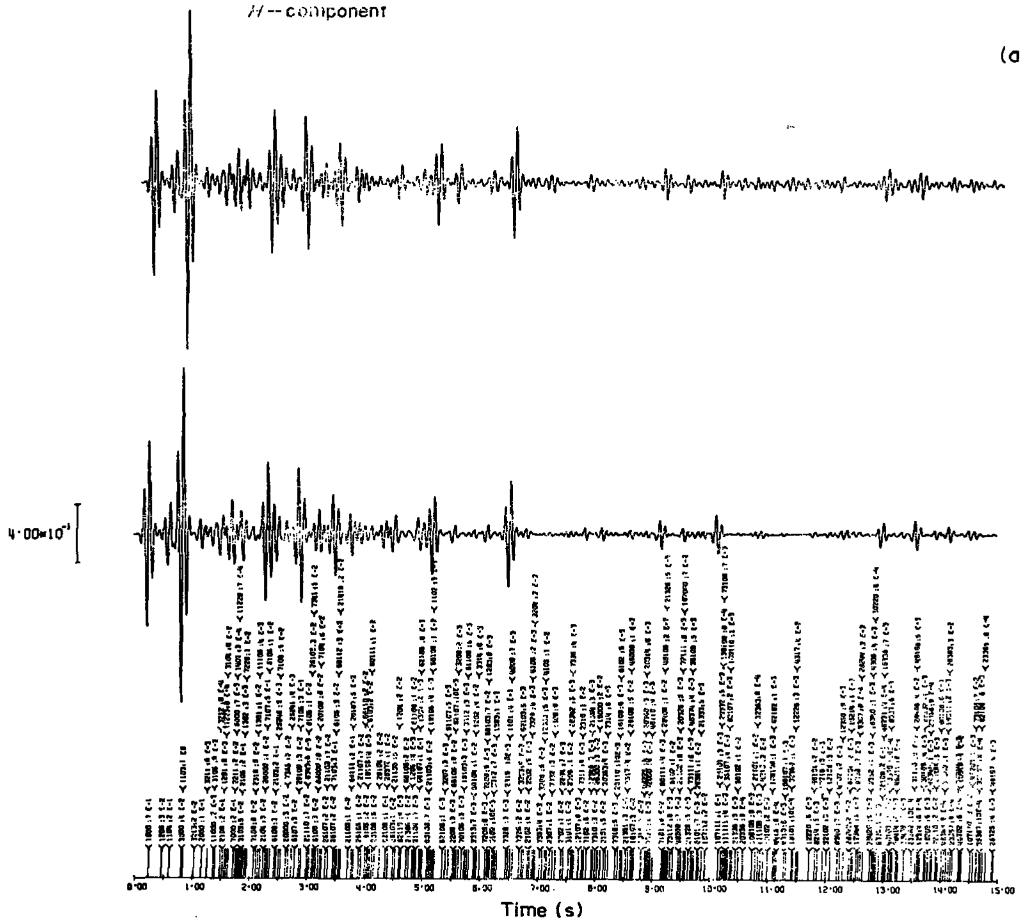
It is necessary to establish that the exact solution can be decomposed into an infinite number of contributions attributed to physical rays. Cisternas *et al.* (1973) showed that all $4n+2$ amplitudes of elastic waves existing in a n layered medium overlying a halfspace can be considered as the components of amplitude at the interface, arranged in the form of a vector $\mathbf{x}$:

$$\mathbf{x} = \bar{\Omega}\mathbf{x} + \mathbf{x}_s$$

or (3)

$$\mathbf{x}_s = (\bar{\mathbf{I}} - \bar{\Omega})\,\mathbf{x}$$

where $\bar{\Omega}$ is a $4n+2$ by $4n+2$ matrix of modified reflection and transmission coefficients, vector $\mathbf{x}_s$ represents the source function and $\bar{\mathbf{I}}$ is the identity matrix. The equation can



be rewritten as

$$\mathbf{x} = (\bar{\mathbf{I}} - \bar{\mathbf{\Omega}})^{-1}\,\mathbf{x}_s$$

or as a power series

$$x = \sum_{j=0}^{\infty} \bar{\mathbf{\Omega}}^j\,\mathbf{x}_s \tag{4}$$

where all terms in the power series can be interpreted as the contributions of real physical rays. This establishes the equivalence of any wave solution that can be obtained with a ray expansion.

A time history of the surface motion of a layered system responding to a pulse of plane waves is obtained by convolving the crustal impulse function, $h(t)$, with the impulse response of the source, $S(t)$. In the frequency domain the convolution is transformed into a multiplication of a complex crustal transfer function, $H(\omega)$, with the complex response of the source, $S(\omega)$. The synthetic seismogram is obtained by an inverse Fourier transform using a fast Fourier algorithm.

$$x(T) = \frac{1}{N}\sum_{W=0}^{N-1} H(W)\,S(W)\exp\,(2\pi iWT/N) \tag{5}$$

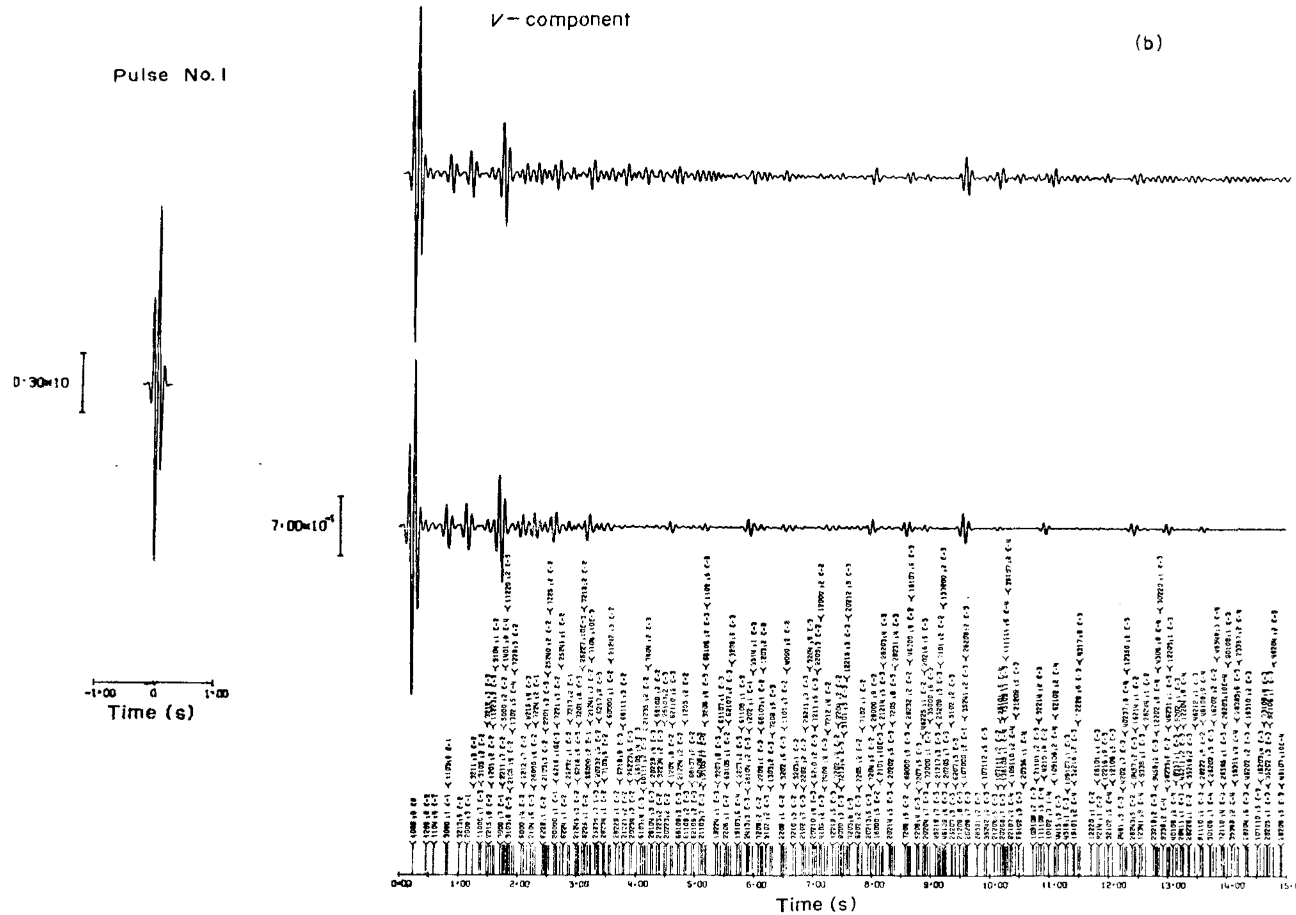


FIG. 2. (a) Horizontal and (b) vertical components comparing the exact solution (upper seismograms) with a partial ray expansion (lower seismograms). The source is a plane compressional wave incident at an angle of 70° on the layered sequence shown in Table 1. The phases are identified by their code number, NOFRAY (equation (2)), which is displayed above the time scale and followed by the relative amplitude written as a floating point number (i.e. 2E0 = 2·10°). The source pulse with predominant frequency of 10 Hs is shown in (b).

where W is the digital frequency index, the time index, T, is limited to $T < \frac{1}{2}N-1$ and

$$N = \frac{1}{\Delta t \cdot \Delta f}. \tag{6}$$

The transfer function for the horizontal and vertical components of ground motion at a given frequency are obtained from matrix elements which are determined by the boundary conditions and the layer parameters (Haskell 1962). The initial position of the wave front is arbitrary and therefore the arrival times of all events displayed on the seismogram were related to the first arrival which was set at zero.

3. Discussion of numerical results

The accuracy of a partial ray expansion as an approximation to the complete ray solution represented by the Thomson–Haskell theory was examined on a four layered medium (Table 1) overlaying a halfspace. Synthetic seismograms were obtained for this model with a plane P wave impinging on the bottom with an angle of incidence of 70°. The striking resemblance of seismograms computed according to Thomson–Haskell theory (Fig. 2(a) and (b)) with the lower seismograms, which represent the sum of individual ray contributions, justifies our partial ray approximation of the exact wave solution. Altogether 1380 rays or 12·55 per cent of 11 000 rays, where amplitude was computed, were used for the construction of the seismograms. The relative error for the first 3 seconds of the seismogram was less than 0·5 per cent as

Table 5

Numbers of rays, NRUSED, used for the construction of seismograms and computational time (CPU in sec) on an IBM 360/67 computer for different values of Δ (in per cent) representing a threshold in terms of strongest amplitude. All weaker arrivals were rejected by the computer

Δ (%)	NRUSED	CPU (s)
0·33	1,380	201
1·0	99	57
5·0	21	48
15·0	5	46

found by comparison with numerical values. The accuracy of the partial ray expansion decreases with later arrivals because of the limitations of the family of basic rays used for computation (Table 3) and the restricted number of conversions. However, all prominent arrivals which appeared in the exact seismograms are present. In addition to this fact, which alone would justify the use of a partial ray expansion, ray theory makes it possible to interpret any event displayed on the seismogram, as it is shown in Fig. 2.

Let us explore the effect of varying Δ, the minimum relative amplitude, for a ray to be included in the partial ray series. Four seismograms are displayed in Fig. 3, each with a different Δ. By observing computational times which were necessary on an IBM 360/67 computer (Table 5) we can see that the seismograms can be produced cheaply for $\Delta > 1$ per cent. The relative error of a partial ray expansion measured with respect to a wave solution cannot be less than Δ. To reduce the relative error of 0·5 per cent in Fig. 2, Δ would have to be set less than 0·33 per cent.

The importance of an unbiased selection of rays is demonstrated in Figs 4 and 5. Four seismograms were obtained for the Alberta model having four different families of rays. The maximum number of conversions was restricted in the seismograms shown in Figs 4(b), (d), 5(b) and (d). The number of sets of unconverted phases were

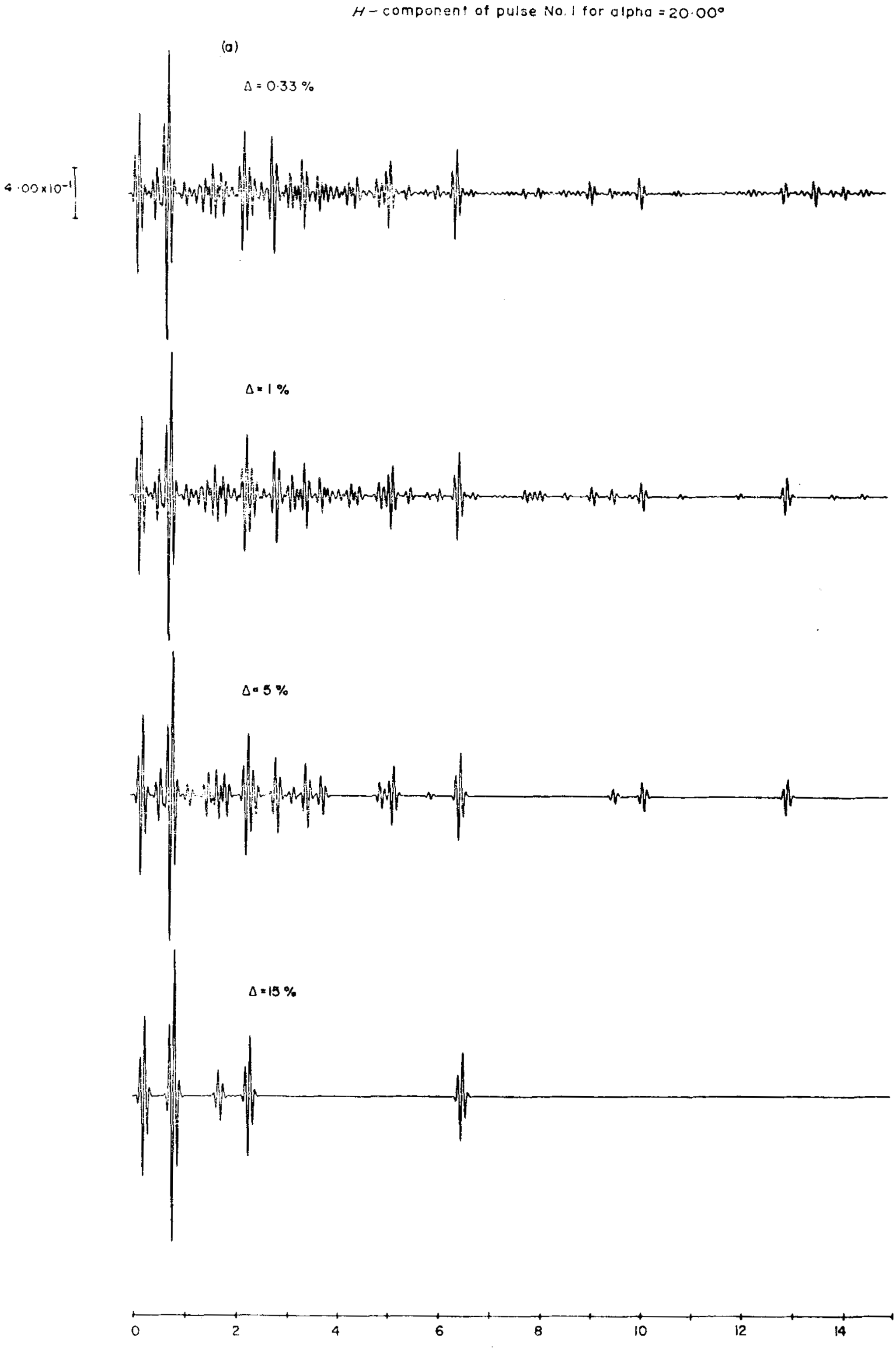

H – component of pulse No. I for alpha = 20·00°
(a)
Δ = 0·33 %
4·00 x 10⁻¹
Δ = 1 %
Δ = 5 %
Δ = 15 %
0
2
4
6
8
10
12
14
S

FIG. 3(a)

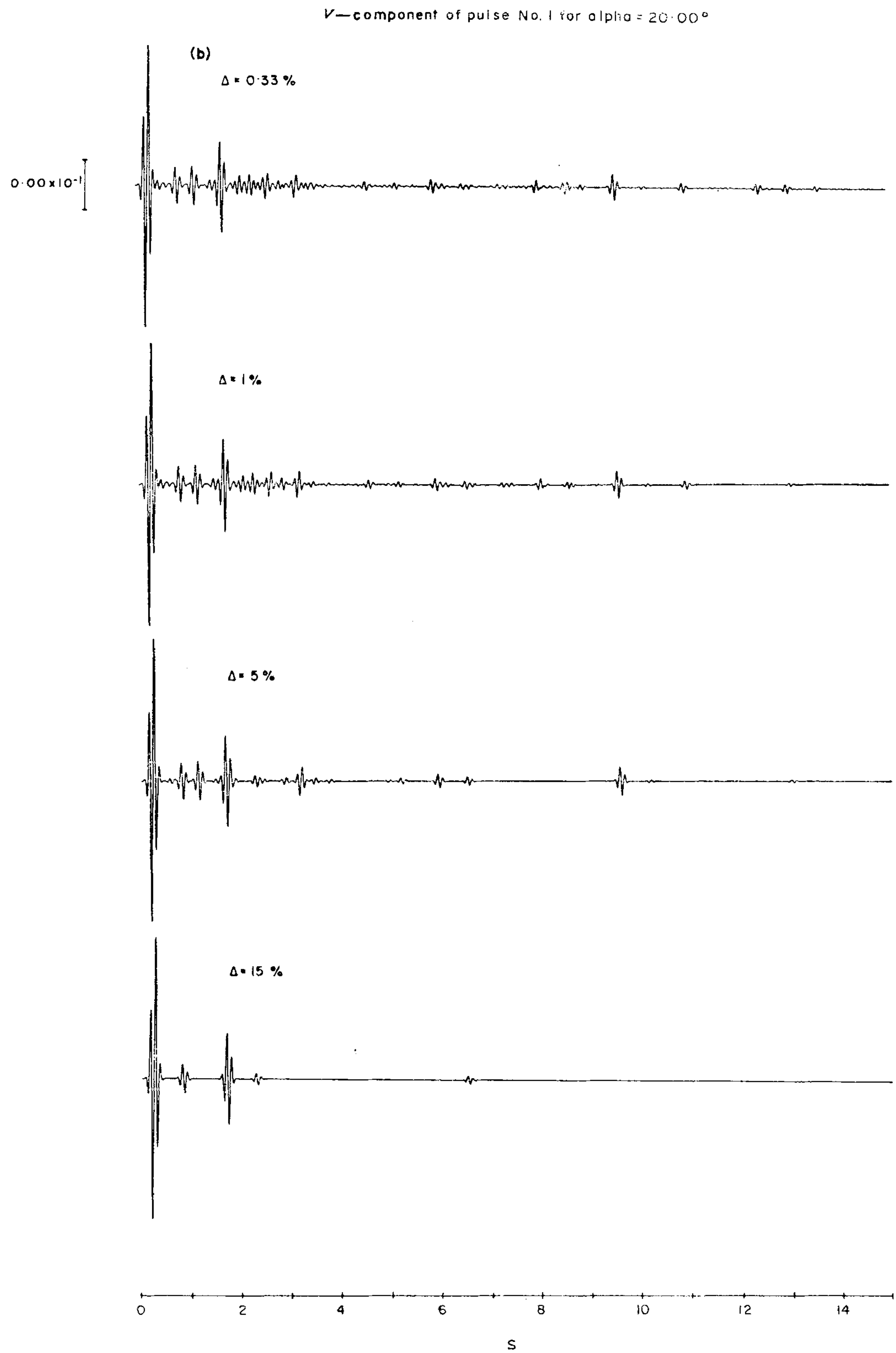


FIG. 3. The seismograms for the (a) horizontal and (b) vertical components computed according to asymptotic ray theory for the model given in Table 1. Four different values of Δ, an amplitude threshold, were used in the construction of seismograms. The uppermost seismograms are identical with those displayed in Fig. 2. Corresponding values of computer (CPU) time and the total numbers of phases are listed in Table 5.

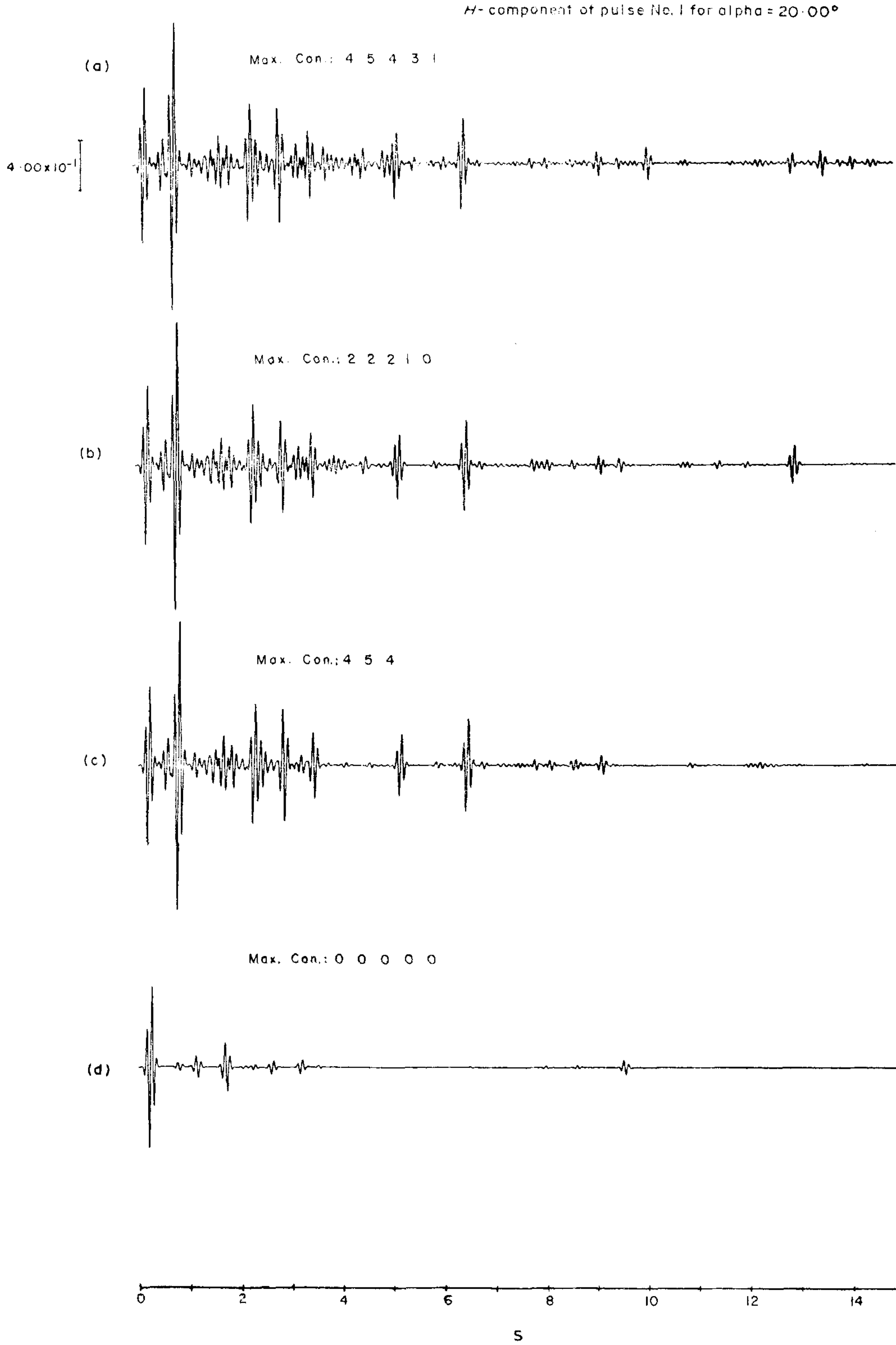


FIG. 4. The horizontal components of elastic wave motion for the model in Table 1 with four different groups of rays included. The symbol ' max con: xxxxx ' denotes the maximum number of conversions (JCONV) in each of the five sets of rays in Table 3. The total number of rays used for the construction of the three lower seismograms were reduced with respect to case ' a ' either by restricting the maximum number of conversions allowed in each of five sets (cases b and d) or by reducing the number of unconverted rays to the first three sets only (case c).

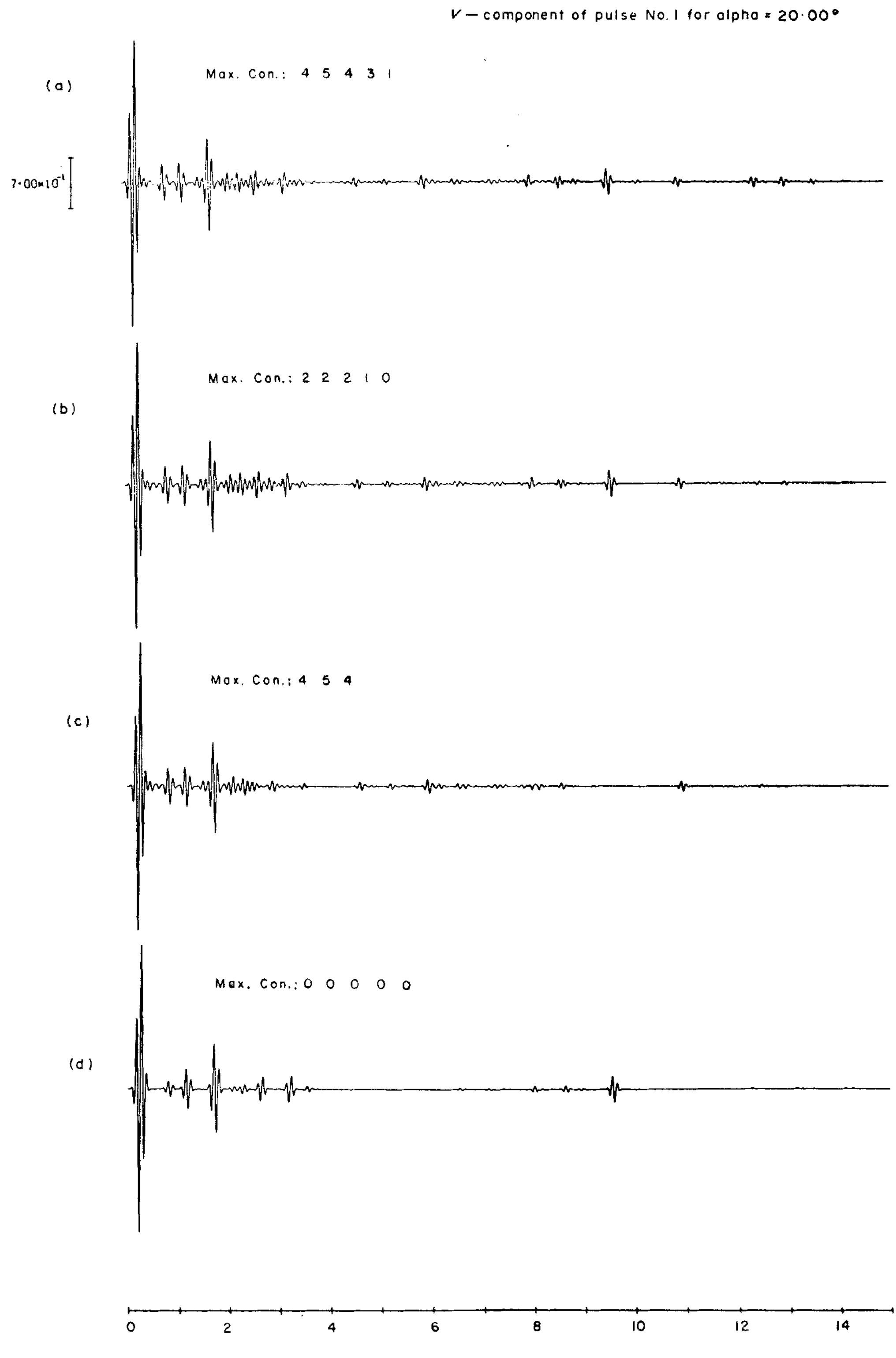


FIG. 5. The vertical component of elastic waves for the model in Table 1 with four different groups of rays included.

restricted in Figs 4(c) and 5(c). We can see very clearly that in all cases where the algorithm for automatic ray generation was impaired by a human interaction the resulting seismograms differed considerably from those which were computed without any interference (Figs 4(a) and 5(a)). We can see that the exclusion of converted phases from computation, a practice used quite frequently by many authors in the past, results in completely false seismograms. This is especially conspicuous on the horizontal component because of the lack of interference among strong converted phases. The importance of converted phases in the interpretation of incident shear waves in teleseismic studies has been discussed recently by Kanasewich, Alpaslan & Hron (1973).

Finally, the agreement between the wave and ray solution (Fig. 2) is reassurance that the reflection and transmission coefficients which are used for evaluation of ray amplitudes with the help of asymptotic ray theory are correct. This is a rather pleasant conclusion considering how many mistakes have been made in this field.

4. Conclusions

The usefulness of approximating the exact wave solution by a partial ray expansion has been demonstrated by comparison of synthetic seismograms evaluated with a Thomson–Haskell matrix formulation and asymptotic ray theory. However, synthetic seismograms evaluated according to any ray theory are subject to considerable error unless an objective computerized system of ray generation is used. A predetermined accuracy of computation can be achieved thereby. Since the incorporation of dip into the individual interfaces does not restrict the generality of our conclusion about a partial ray expansion, the same ray generating algorithm can be used with similar accuracy on body wave propagation in a crust with dipping or curvilinear interfaces.

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University of Alberta,
Alberta, Canada

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