Arch. Rational Mech. Anal. 116 (1991) 101–113.

Partial Regularity for Stationary Harmonic Maps into Spheres

LAWRENCE C. EVANS

Communicated by D. KINDERLEHRER

1. Introduction

In an interesting recent paper [12], F. HÉLEIN has shown that any weakly harmonic mapping from a two-dimensional surface into a sphere is smooth.

I present here a kind of generalization to higher dimensions, asserting in effect that a stationary harmonic mapping from an open subset of $\mathbb{R}^n (n \geqq 3)$ into a sphere is smooth, except possibly for a closed singular set of $(n-2)$-dimensional Hausdorff measure zero. My proof crucially depends upon several of HÉLEIN's observations (as streamlined by P.-L. LIONS).

To state the result precisely let us suppose that m, $n \geqq 2$, U is a smooth open subset of $\mathbb{R}^n$, and S^{m-1} denotes the unit sphere in $\mathbb{R}^m$. A function u in the Sobolev space $H^1(U; \mathbb{R}^m)$, $u = (u^1, \ldots, u^m)$, belongs to $H^1(U; S^{m-1})$ provided $|u| = 1$ a.e. in U.

Definition. A function $u \in H^1(U; S^{m-1})$ is a *weakly harmonic* mapping of U into S^{m-1} provided

$$-\Delta u = |Du|^2 u \quad \text{in } U. \tag{1.1}$$

This system of partial differential equations is to hold in the weak sense, that is,

$$\int_U Du : Dw \, dx = \int_U |Du|^2 u \cdot w \, dx \tag{1.2}$$

for each test function $w \in H^1(U; \mathbb{R}^m) \cap L^\infty(U; \mathbb{R}^m)$ having compact support, $w = (w^1, \ldots, w^m)$. We employ the notation

$$Du = \left(\left(\frac{\partial u^i}{\partial x_k} \right) \right)_{\substack{1 \leq k \leq n \\ 1 \leq i \leq m}},$$

$$Du : Dw = \frac{\partial u^i}{\partial x_k} \frac{\partial w^i}{\partial x_k}, \quad |Du|^2 = Du : Du.$$

Now let $g: \partial U \to S^{m-1}$ be a given smooth function. It is easy to check that (1.1) is the Euler-Lagrange equation for the variational problem of minimizing the Dirichlet energy

$$I[w] = \int_U |Dw|^2 \, dx$$

among functions w lying in the admissible class

$$\mathscr{A} \equiv \{w \in H^1(U; S^{m-1}) \mid w = g \text{ on } \partial U \text{ in the trace sense}\}.$$

If u is a minimizer of $I[\cdot]$ within $\mathscr{A}$, then u in addition to (1.2) satisfies the integral identity

$$\int_U |Du|^2 (\operatorname{div} \zeta) - 2u^i_{x_k} u^i_{x_l} \zeta^k_{x_l} \, dx = 0 \tag{1.3}$$

for each vector field $\zeta \in C^1(U; \mathbb{R}^n)$, $\zeta = (\zeta^1, \ldots, \zeta^n)$, having compact support within U. To deduce (1.3) set $u^t(x) \equiv u(x + t\zeta(x)) \in \mathscr{A}$ and compute $\frac{d}{dt} I[u^t]|_{t=0} = 0$. Observe also that (1.3) follows directly from (1.2) if u is smooth: take $w = (Du)\,\zeta$.

Definition. A function $u \in H^1(U; S^{m-1})$ is a *weakly stationary harmonic map from* U *into the sphere* S^{m-1} if u satisfies the identities (1.2), (1.3) for all test functions w, ζ as above.

The idea is that (1.2) says u is stationary with respect to variations of the target S^{m-1}, whereas (1.3) says u is stationary with respect to variations of the domain U. See SCHOEN's article [18] for more information. In particular we recall that a stationary mapping u satisfies the *monotonicity inequalities*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} |Du|^2 \, dy \leqq \frac{1}{R^{n-2}} \int_{B(x,R)} |Du|^2 \, dy \quad \text{for all concentric balls } B(x,r) \subset B(x,R) \subset U. \tag{1.4}$$

This was apparently first proved by PRICE [17]. A quick derivation of (1.4) follows. Given $B(0,r) \subset U$, set

$$\zeta = \phi(|y|)\, y$$

in (1.3), where

$$\phi(s) = \begin{cases} 1 & \text{for } s \leqq r, \\ 1 + \dfrac{r-s}{h} & \text{for } r \leqq s \leqq r+h, \\ 0 & \text{for } s \geqq r+h. \end{cases}$$

After some calculations we send $h \to 0^+$ and deduce for almost every r that

$$(n-2) \int_{B(0,r)} |Du|^2 \, dy = -\frac{2}{r} \int_{\partial B(0,r)} |(Du)\, y|^2 \, dH^{n-1} + r \int_{\partial B(0,r)} |Du|^2 \, dH^{n-1}.$$

Discarding the first term on the right we compute

$$\frac{d}{dr}\left(\frac{1}{r^{n-2}} \int_{B(0,r)} |Du|^2 \, dy\right) \geqq 0$$

for almost every r satisfying $0 < r < \operatorname{dist}(0, \partial U)$. Inequality (1.4) for $x = 0$ follows.

We now state our main result, the following regularity assertion:

Theorem 1. *Assume that* $u \in H^1(U; S^{m-1})$ *is a weakly harmonic mapping which satisfies the monotonicity inequalities* (1.4). *Then there exists an open subset* $V \subset U$ *such that*

$$u \in C^\infty(V; S^{m-1}), \tag{1.5}$$

$$H^{n-2}(U - V) = 0. \tag{1.6}$$

Here H^{n-2} denotes $(n-2)$-dimensional Hausdorff measure.

Remarks. (i) If $n = 2$, the monotonicity inequalities (1.4) are automatic and $H^{n-2} = H^0$ is counting measure. Thus a *weakly harmonic mapping from* $U \subset \mathbb{R}^2$ *into* S^{m-1} *is everywhere smooth.* We consequently recover Hélein's Theorem [12].

(ii) If $n \geqq 3$, the monotonicity (1.4) is a consequence of domain stationarity (1.3). Hence a *stationary harmonic mapping from* $U \subset \mathbb{R}^n$ *into* S^{m-1} *is smooth, except possibly for a closed set of* H^{n-2}*-measure zero.*

Our proof of Theorem 1 depends upon using the constraint $|u| = 1$ to rewrite the right-hand side of the partial differential equation (1.1) to reveal that this term belongs to the Hardy space $\mathscr{H}^1$. This is Hélein's key observation, which for $n = 2$ immediately implies continuity (and therefore smoothness) of u in everywhere in U (cf. Wente [22]). For $n \geqq 3$, we note additionally that monotonicity inequalities (1.4) provide bounds for u in BMO: this turns out to be useful in light of C. Fefferman's identification [5] of $(\mathscr{H}^1)^*$ with BMO.

In §2 we recall the relevant facts about BMO and $\mathscr{H}^1$, and in particular reproduce the observation of Coifman, Lions, Meyer & Semmes [4] that $Du \cdot v \in \mathscr{H}^1$ if $u \in H^1$, $v \in L^2$ and $\operatorname{div} v = 0$. (Their proof was inspired by earlier calculations of Müller [16]).

Sections 3 and 4 establish the proof of Theorem 1. The main idea, as usual for partial regularity, is to show that u is continuous in any region in which the scaled energy is sufficiently small. The new point is that whereas u does not obviously have small oscillation in such a region, it is small in BMO. This turns out to be good enough since $(\mathscr{H}^1)^* = \text{BMO}$. We implement these ideas within a routine blow-up argument, although a direct proof is possible as well.

There is a vast literature on partial regularity for energy *minimizers* in the calculus of variations. Giaquinta's book [7] is the best source for this theory in the unconstrained case, for many further references, etc. See also Schoen &

Uhlenbeck [19] for the basic partial regularity theory for minimizing harmonic maps. Grüter [9] proved that a conformal harmonic map from a surface into a manifold is everywhere smooth, and Schoen [18] extended this assertion to stationary harmonic maps from surfaces. Morrey (cf. [15]) had much earlier shown that a harmonic minimizer from a surface into a manifold is smooth. Other interesting related papers include Bethuel & Brezis [1], Giaquinta, Modica & Souček [8], Hardt & Lin [11], Hardt, Kinderlehrer & Lin [10], Luckhaus [14], etc.

The following proof makes explicit and exact use of the structure of the target sphere S^{m-1}. Chen [3] and Shatah [20] seem to be have been the first to deduce useful analytic consequences from symmetries of the target manifolds for the general weakly harmonic maps. They showed in particular that a weak limit in H^1 of solutions of system (1.1) is still a solution; this is interesting since the term $|Du|^2 u$ is not weakly continuous. Very recently Hélein [13] has extended his regularity theory to cover weakly harmonic maps from surfaces into homogeneous spaces; Schoen and Uhlenbeck have made similar observations. I conjecture the partial regularity theory set forth in this paper will extend as well.

Finally, let me note that S. Chanillo [2] has recently found a completely elementary derivation of the main inequality used in this work, namely,

$$\left| \int_{\mathbb{R}^n} fg \cdot Dh \, dx \right| \leqq C \|Df\|_{L^2} \|g\|_{L^2}, \tag{1.7}$$

where $f,\ h \in H^1(\mathbb{R}^n)$, $g \in L^2(\mathbb{R}^n; \mathbb{R}^n)$, div $g = 0$, and

$$\sup \frac{1}{r^{n-2}} \int_{B(x,r)} |Dh|^2 \, dy \equiv C' < \infty.$$

The constant C in (1.7) depends only on n and C'. The interested reader can verify that we could invoke (1.7) in place of the $\mathscr{H}^1$/BMO inequality (2.4).

I am very grateful to H. Brezis for explaining F. Hélein's proof to me.

§ 2. BMO and $\mathscr{H}^1$

This section reviews definitions and properties of the spaces BMO and $\mathscr{H}^1$. See Fefferman & Stein [6] or Torchinsky [21] for more information.

If $f: \mathbb{R}^n \to \mathbb{R}$ is locally summable, we set

$$\|f\|_* \equiv \sup \left\{ \fint_{B(x,r)} |f - (f)_{x,r}| \, dy \,\middle|\, x \in \mathbb{R}^n,\ r > 0 \right\}, \tag{2.1}$$

where

$$(f)_{x,r} \equiv \fint_{B(x,r)} f \, dy = \frac{1}{|B(x,r)|} \int_{B(x,r)} f \, dy.$$

We say that f has *bounded mean oscillation* provided $\|f\|_* < \infty$. Note that $\|f\|_* = 0$ if and only if f is constant a.e.

Assume now that $g \in L^1(\mathbb{R}^n)$. Let ϕ be any smooth function with support in the unit ball, $\int_{\mathbb{R}^n} \phi \, dx = 1$. We set

$$g^*(x) \equiv \sup_{r>0} \left| \frac{1}{r^n} \int g(y)\, \phi\left(\frac{x-y}{r}\right) dy \right| \tag{2.2}$$

and say that g belongs to the *Hardy space* $\mathscr{H}^1$ if $g^* \in L^1(\mathbb{R}^n)$. We write

$$\|g\|_{\mathscr{H}^1(\mathbb{R}^n)} = \|g^*\|_{L^1(\mathbb{R}^n)}. \tag{2.3}$$

See [4] for equivalent definitions. Observe that $g \in \mathscr{H}^1$ implies $\int_{\mathbb{R}^n} g\, dx = 0$.

A fundamental theorem of C. Fefferman [5], [6] asserts that $(\mathscr{H}^1)^* =$ BMO and in particular provides the inequality

$$\left| \int_{\mathbb{R}^n} fg \, dx \right| \leqq C \|f\|_* \, \|g\|_{\mathscr{H}^1(\mathbb{R}^n)} \tag{2.4}$$

for $f \in L^\infty(\mathbb{R}^n)$, $g \in \mathscr{H}^1(\mathbb{R}^n)$. The constant C depends only on n.

Finally we reproduce for the reader's convenience a result of Coifman, Lions, Meyer & Semmes [4], based upon important contributions due to Müller [16].

Proposition 2.1. *Assume* $u \in H^1(\mathbb{R}^n)$, $v \in L^2(\mathbb{R}^n; \mathbb{R}^n)$, *and*

$$\operatorname{div} v = 0 \quad \textit{in the distribution sense.} \tag{2.5}$$

Then $Du \cdot v \in \mathscr{H}^1(\mathbb{R}^n)$, *with the bound*

$$\|Du \cdot v\|_{\mathscr{H}^1(\mathbb{R}^n)} \leqq C\left(\|u\|^2_{H^1(\mathbb{R}^n)} + \|v\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^n)}\right). \tag{2.6}$$

Proof. Clearly $Du \cdot v \in L^1(\mathbb{R}^n)$. Now fix ϕ as above, choose $x \in \mathbb{R}^n, r > 0$, and set $\phi_r(y) = \phi\left(\dfrac{x-y}{r}\right)$. Then

$$\frac{1}{r^n} \int_{\mathbb{R}^n} Du \cdot v\, \phi_r \, dy = \frac{-1}{r^n} \int_{B(x,r)} \left(u - (u)_{x,r}\right) v \cdot D\phi_r \, dy$$

by (2.5). Thus

$$\left| \frac{1}{r^n} \int_{\mathbb{R}^n} Du \cdot v\, \phi_r \, dy \right| \leqq \frac{C}{r^{n+1}} \int_{B(x,r)} |u - (u)_{x,r}| \, |v| \, dy.$$

Choose any $2 < p < 2^* = \dfrac{2n}{n-2} \leqq \infty$ and let $1 < q \equiv \dfrac{p}{p-1} < 2$. Then

$$\left|\frac{1}{r^n}\int Du\cdot v\phi_r\,dy\right| \leqq \frac{C}{r^{n+1}}\left(\int_{B(x,r)}|u-(u)_{x,r}|^p\,dx\right)^{1/p}\left(\int_{B(x,r)}|v|^q\,dx\right)^{1/q}$$

$$\leqq \frac{C}{r^{1+n/p}}\left(\int_{B(x,r)}|u-(u)_{x,r}|^p\,dx\right)^{1/p}\left(\fint_{B(x,r)}|v|^q\right)^{1/q}$$

$$\leqq C\left(\fint_{B(x,r)}|Du|^r\,dx\right)^{1/r}\left(\fint_{B(x,r)}|v|^q\right)^{1/q},$$

where $p = r^*$, that is, $r = \dfrac{pn}{p+n} < 2$. Consequently,

$$\left|\frac{1}{r^n}\int_{\mathbb{R}^n} Du\cdot v\phi_r\,dy\right| \leqq CM(|Du|^r)^{1/r}M(|v|^q)^{1/q}$$

$$\leqq C[M(|Du|^r)^{2/r} + M(|v|^q)^{2/q}],$$

$M(\cdot)$ denoting the Hardy-Littlewood maximal function. Now $|Du|^r \in L^{2/r}$, $2/r > 1$. Thus

$$\|M(|Du|^r)\|_{L^{2/r}} \leqq C\||Du|^r\|_{L^{2/r}},$$

and so

$$\int_{\mathbb{R}^n} M(|Du|^r)^{2/r}\,dx \leqq C\int_{\mathbb{R}^n}|Du|^2\,dx.$$

Similarly,

$$\int_{\mathbb{R}^n} M(|v|^q)^{2/q}\,dx \leqq C\int_{\mathbb{R}^n}|v|^2\,dx.$$

Consequently we deduce

$$(Du\cdot v)^* \equiv \sup_{r>0}\left|\frac{1}{r^n}\int_{\mathbb{R}^n} Du\cdot v\phi_r\,dy\right| \in L^1,$$

$$\|(Du\cdot v)^*\|_{L^1} \leqq C(\|u\|^2_{H^1} + \|v\|^2_{L^2}). \qquad \square$$

§ 3. Energy decay and blow-up

This section and the next provide the proof of Theorem 1. Assume henceforth that $u \in H^1(U; S^{m-1})$ satisfies the hypothesis of Theorem 1. If $B(y, r) \subset U$, we define the scaled energy

$$E(x,r) \equiv \frac{1}{r^{n-2}}\int_{B(x,r)}|Du|^2\,dy. \tag{3.1}$$

The key to the proof of Theorem 1 is this assertion about energy:

Proposition 3.1. *There exist constants* $0 < \varepsilon_0, \tau < 1$ *such that*

$$E(x, r) \leqq \varepsilon_0 \tag{3.2}$$

implies

$$E(x, \tau r) \leqq \tfrac{1}{2} E(x, r) \tag{3.3}$$

for all $x \in U$, $0 < r < \operatorname{dist}(x, \partial U)$.

Proof. We argue by contradiction, for $\tau > 0$ selected as below. Were (3.2), (3.3) false, there would exist balls $B(x_k, r_k) \subset U$ such that

$$E(x_k, r_k) \equiv \lambda_k^2 \to 0, \tag{3.4}$$

whereas

$$E(x_k, \tau r_k) > \tfrac{1}{2} \lambda_k^2. \tag{3.5}$$

We rescale our variables to the unit ball $B(0, 1) \subset \mathbb{R}^n$, as follows. If $z \in B(0, 1)$, write

$$v_k(z) \equiv \frac{u(x_k + r_k z) - a_k}{\lambda_k}, \tag{3.6}$$

where

$$a_k \equiv \fint_{B(x_k, r_k)} u \, dy = (u)_{x_k, r_k}$$

denotes the average of u over $B(x_k, r_k)$, $k = 1, \ldots$.

Utilizing (3.1), (3.4), (3.5) and (3.6) we verify that

$$\sup_k \int_{B(0,1)} |v_k|^2 \, dz < \infty, \qquad \int_{B(0,1)} |Dv_k|^2 \, dz = 1, \tag{3.7}$$

but

$$\frac{1}{\tau^{n-2}} \int_{B(0,\tau)} |Dv_k|^2 \, dz > \tfrac{1}{2} \quad (k = 1, 2, \ldots). \tag{3.8}$$

The sequence $\{v_k\}_{k=1}^{\infty}$ is thus bounded in $H^1(B(0,1); \mathbb{R}^m)$, whence there exists a subsequence (which we reindex as necessary and denote also by $\{v_k\}_{k=1}^{\infty}$) such that

$$v_k \to v \qquad \text{strongly in } L^2(B(0, 1); \mathbb{R}^m) \tag{3.9}$$

$$Dv_k \rightharpoonup Dv \quad \textit{weakly} \quad \text{in } L^2(B(0, 1); M^{m \times n}), \tag{3.10}$$

$M^{m \times n}$ being the space of real $m \times n$ matrices.

Next select any smooth function $w: B(0, 1) \to \mathbb{R}^m$ with compact support. Define

$$w_k(y) \equiv w\left(\frac{y - x_k}{r_k}\right) \qquad (y \in B(x_k, r_k)).$$

Since u is a weakly harmonic, (1.2) gives

$$\int_{B(x_k, r_k)} Du : Dw_k \, dy = \int_{B(x_k, r_k)} |Du|^2 \, u \cdot w_k \, dy.$$

We rescale this identity to the unit ball, obtaining thereby the equality

$$\int_{B(0,1)} Dv_k : Dw \, dz = \lambda_k \int_{B(0,1)} |Dv_k|^2 \, (a_k + \lambda_k v_k) \cdot w \, dz. \tag{3.11}$$

Observe also that (3.6) implies

$$|a_k + \lambda_k v_k|^2 = 1 \quad \text{a.e. in } B(0, 1). \tag{3.12}$$

We send k to infinity in (3.11), invoking (3.7) and (3.12) to deduce

$$\int_{B(0,1)} Dv : Dw \, dz = 0. \tag{3.13}$$

This equality obtains for all w as above. Consequently

$$\Delta v = 0 \quad \text{in } B(0, 1)$$

in the weak, and therefore classical, sense. Hence v is smooth, and we have the bound

$$\|Dv\|^2_{L^\infty(B(0,\frac{1}{2}),M^{m\times n})} \leqq C \int_{B(0,1)} v^2 \, dz < \infty.$$

In particular, therefore,

$$\frac{1}{\tau^{n-2}} \int_{B(0,\tau)} |Dv|^2 \, dz \leqq C\tau^2 < \tfrac{1}{2}, \tag{3.14}$$

provided we adjust $0 < \tau < \frac{1}{2}$ to be small enough.

Next we utilize Proposition 4.1, to be proved in §4 following. This asserts

$$Dv_k \to Dv \quad \textit{strongly} \text{ in } L^2(B(0, \tfrac{1}{2}); M^{m\times n}). \tag{3.15}$$

But then (3.8) forces

$$\frac{1}{\tau^{n-2}} \int_{B(0,\tau)} |Dv|^2 \, dz \geqq \tfrac{1}{2},$$

a contradiction to (3.14). □

Proof of Theorem 1. Set

$$V \equiv \{x \in U \mid E(x, r) < \varepsilon_0 \text{ for some } 0 < r < \operatorname{dist}(x, \partial U)\}.$$

Then V is open, and standard covering arguments imply $H^{n-2}(U - V) = 0$. Furthermore if $x \in V$, we have

$$E(y, r) \leqq Cr^\gamma \tag{3.16}$$

for some $\gamma > 0$, $C > 0$, all y near x, and all sufficiently small radii $r > 0$. See GIAQUINTA [7] for a proof of (3.16) from Proposition 3.1. It follows that u is uniformly Hölder continuous with exponent $\gamma/2$ on each compact subset of V: again see GIAQUINTA [7]. Hence $u \in C^{0,\gamma/2}(V; S^{m-1})$, and routine elliptic regularity theory then proves that $u \in C^\infty(V; S^{m-1})$; see SCHOEN & UHLENBECK [19] and references therein. □

§ 4. Compactness

All of the calculations and assertions in § 3 are routine, except for the compactness assertion (3.15), to which we now turn our attention. First, select a smooth cutoff function $\zeta: \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\begin{aligned} &0 \leqq \zeta \leqq 1, \\ &\zeta \equiv 1 \quad \text{on } B(0, \tfrac{1}{2}), \\ &\zeta \equiv 0 \quad \text{on } \mathbb{R}^n - B(0, \tfrac{5}{8}). \end{aligned} \tag{4.1}$$

Lemma 4.1. *The sequence* $\{\zeta v_k\}_{k=1}^\infty$ *is bounded in* $\mathrm{BMO}(\mathbb{R}^n; \mathbb{R}^m)$.

Proof. Fix any point $z_0 \in B(0, \frac{7}{8})$ and any radius $0 < r \leqq \frac{1}{8}$. Write

$$y_k \equiv x_k + r_k z_0 \in B(x_k, \tfrac{7}{8} r_k).$$

From the monotonicity inequalities (1.4) we have

$$\begin{aligned} \frac{1}{(r r_k)^{n-2}} \int_{B(y_k, r r_k)} |Du|^2 \, dy &\leqq \frac{8^{n-2}}{r_k^{n-2}} \int_{b(y_k, \frac{1}{8} r_k)} |Du|^2 \, dy \\ &\leqq \frac{8^{n-2}}{r_k^{n-2}} \int_{B(x_k, r_k)} |Du|^2 \, dy \\ &= 8^{n-2} \lambda_k^2. \end{aligned}$$

Rescaling this estimate we obtain the bound

$$\frac{1}{r^{n-2}} \int_{B(z_0, r)} |Dv_k|^2 \, dz \leqq 8^{n-2}$$

for $k = 1, \ldots$ and all $0 < r \leqq \frac{1}{8}$, $z_0 \in B(0, \frac{7}{8})$. Consequently

$$\fint_{B(z_0, r)} |v_k - (v_k)_{z_0, r}| \, dz \leqq C < \infty \tag{4.2}$$

for k, r and z_0 as above. Since $\{v_k\}_{k=1}^\infty$ is bounded in $L^2(B(0, 1); \mathbb{R}^m)$, the John-Nirenberg inequality implies

$$\{v_k\}_{k=1}^\infty \text{ is bounded in } L^p(B(0, \tfrac{7}{8}); \mathbb{R}^m) \quad (1 \leqq p < \infty). \tag{4.3}$$

As ζ is smooth,

$$|(\zeta v_k)_{z_0, r} - \zeta (v)_{z_0, r}| \leqq C r \fint_{B(z_0, r)} |v_k| \, dz \quad \text{on } B(z_0, r)$$

for any ball $B(z^0, r)$. Thus if $z_0 \in B(0, \frac{3}{4})$, $0 < r \leqq \frac{1}{8}$, we have

$$\begin{aligned}\fint_{B(z_0,r)} |\zeta v_k - (\zeta v_k)_{z_0,r}|\, dz &\leqq \fint_{B(z_0,r)} |v_k - (v_k)_{z_0,r}|\, dz + Cr \fint_{B(z_0,r)} |v_k|\, dz \\ &\leqq C + \frac{C}{r^{n-1}} \int_{B(z_0,r)} |v_k|\, dz \quad \text{by (4.2)} \\ &\leqq C + \frac{C}{r^{n-1}} \left(\int_{B(0,\frac{7}{8})} |v_k|^n\, dz \right)^{1/n} r^{n(1-\frac{1}{n})} \\ &\leqq C < \infty, \quad \text{by (4.3)}.\end{aligned}$$

This same inequality obtains for $z_0 \in \mathbb{R}^n - B(0, \frac{3}{4})$, $0 < r \leqq \frac{1}{8}$, since $\zeta \equiv 0$ on $\mathbb{R}^n - B(0, \frac{5}{8})$. We recall that

$$\sup_k \|\zeta v_k\|_{L^1} < \infty$$

to conclude the proof. □

Next define

$$b^{ij}_{k,l} \equiv v^j_{k,x_l}(a^i_k + \lambda_k v^i_k) - v^i_{k,x_l}(a^j_k + \lambda_k v^j_k)$$

for $1 \leqq i, j \leqq m$, $1 \leqq l \leqq n$, $k = 1, \ldots$.

Lemma 4.2. *For each function $\phi \in H^1(B(0, 1)) \cap L^\infty(B(0, 1))$ with compact support,*

$$\int_{B(0,1)} \phi_{x_l} b^{ij}_{k,l}\, dz = 0 \tag{4.4}$$

for $1 \leqq i, j \leqq m, k = 1, \ldots$.

Proof. We compute

$$\begin{aligned}\int_{B(0,1)} \phi_{x_l} b^{ij}_{k,l}\, dz &= \int_{B(0,1)} \phi_{x_l}[v^j_{k,x_l}(a^i_k + \lambda_k v^i_k) - v^i_{k,x_l}(a^j_k + \lambda_k v^j_k)]\, dz \\ &= \int_{B(0,1)} v^j_{k,x_l}((a^i_k + \lambda_k v^i_k)\, \phi)_{x_l} - v^i_{k,x_l}((a^j_k + \lambda_k v^j_k)\, \phi)_{x_l}\, dz \\ &= \lambda_k \int_{B(0,1)} |Dv_k|^2 [(a^j_k + \lambda_k v^j_k)(a^i_k + \lambda_k v^i_k) - (a^i_k + \lambda_k v^i_k)(a^j_k + \lambda_k v^j_k)]\, \phi\, dz \\ &= 0, \quad \text{according to (3.11)}. \quad \square\end{aligned}$$

Lemma 4.3. *For each $1 \leqq i, j \leqq m$, the sequence $\{(\zeta v^i_k)_{x_l} b^{ij}_{k,l}\}_{k=1}^\infty$ is bounded in $\mathscr{H}^1(\mathbb{R}^n)$.*

Proof. This is an immediate consequence of Lemma 4.2 and Proposition 2.1. □

Proposition 4.1. *The rescaled functions* $\{Dv_k\}_{k=1}^{\infty}$ *converge strongly in* $L^2(B(0,\frac{1}{2});$ $M^{m\times n})$.

Proof. Subtracting (3.13) from (3.11) we find

$$\int_{B(0,1)} (Dv_k - Dv):Dw\,dz = \lambda_k \int_{B(0,1)} |Dv_k|^2(a_k+\lambda_k v_k)\cdot w\,dz \tag{4.5}$$

for smooth $w: B(0,1)\to\mathbb{R}^m$ with compact support. By approximation the same identity obtains for $w\in H_0^1(B(0,1);\mathbb{R}^m)\cap L^\infty(B(0,1);\mathbb{R}^m)$. We now insert

$$w\equiv \zeta^2(v_k - v)$$

into (4.5). The left-hand side of (4.5) is

$$\begin{aligned} L_k &\equiv \int_{B(0,1)} \zeta^2|Dv_k - Dv|^2\,dz + 2\int_{B(0,1)} \zeta(v_k - v)\cdot(Dv_k - Dv)\cdot D\zeta\,dz \\ &\geqq \int_{B(0,\frac{1}{2})} |Dv_k - Dv|^2\,dz + o(1) \end{aligned} \tag{4.6}$$

as $k\to\infty$, in view of (3.9) and (3.10). The right-hand side of (4.5) reads

$$\begin{aligned} R_k &\equiv \lambda_k \int_{B(0,1)} \zeta^2|Dv_k|^2(a_k+\lambda_k v_k)\cdot(v_k - v)\,dz \\ &= \lambda_k \int_{B(0,1)} \zeta^2 v^j_{k,x_l}\left(v^j_{k,x_l}(a^i_k+\lambda_k v^i_k)(v^i_k - v^i)\right)dz \\ &= \lambda_k \int_{B(0,1)} \zeta^2 v^j_{k,x_l}\left(v^j_{k,x_l}(a^i_k+\lambda_k v^i_k) - v^i_{k,x_l}(a^j_k+\lambda_k v^j_k)\right)(v^i_k - v^i)\,dz, \end{aligned}$$

the last equality holding in light of (3.12). (Here is HÉLEIN's trick.) Thus

$$\begin{aligned} R_k &= \lambda_k \int_{B(0,1)} \zeta^2 v^j_{k,x_l} b^{ij}_{k,l}(v^i_k - v^i)\,dz \\ &= \lambda_k \int_{\mathbb{R}^n} (\zeta v^j_k)_{x_l} b^{ij}_{k,l}(\zeta(v^i_k - v^i))\,dz - \lambda_k \int_{\mathbb{R}^n} v^j_k \zeta_{x_l} b^{ij}_{k,l}\zeta(v^i_k - v^i)\,dz \\ &\equiv \lambda_k(R^1_k + R^2_k). \end{aligned} \tag{4.7}$$

Now

$$\sup_k |R^2_k| < \infty,$$

since $\{v_k\}_{k=1}^{\infty}$ is bounded in $L^4(B(0,\frac{7}{8});\mathbb{R}^m)$ and $\{b^{ij}_{k,l}\}_{k=1}^{\infty}$ is bounded in $L^2(B(0,\frac{7}{8}))$.

Finally, the $\mathscr{H}^1$/BMO inequality (2.4) gives

$$\sup_k |R^1_k| \leqq \sum_{i,j=1}^{m} C \sup_k \|\zeta(v^i_k - v^i)\|_* \,\|(\zeta v^j_k)_{x_l} b^{ij}_{k,l}\|_{\mathscr{H}^1} < \infty,$$

according to Lemmas 4.1 and 4.3. Thus $R_k = O(\lambda_k) = o(1)$ as $k \to \infty$. Hence

$$\int_{B(0,\frac{1}{2})} |Dv_k - Dv|^2 \, dz \leqq o(1) \quad \text{as } k \to \infty,$$

and we are done. $\square$

Note added in proof. HÉLEIN has now extended his regularity theory to cover weakly harmonic maps from surfaces into general targets. Also T. RIVIERE has constructed a weakly harmonic (but nonstationary!) map from the unit ball $B(0, 1) \subset \mathbb{R}^3$ into S^2, with singularities forming a line segment.

Acknowledgment. This research was supported in part by the National Science Foundation grant DMS-89-03328.

References

1. F. BETHUEL & H. BREZIS, Regularity of minimizers of relaxed problems for harmonic maps, preprint 1990.
2. S. CHANILLO, Sobolev inequalities involving divergence free maps, *Comm. Part. Diff. Eqs.*, to appear.
3. Y. M. CHEN, Weak solutions to the evolution problem for harmonic maps, preprint.
4. R. COIFMAN, P.-L. LIONS, Y. MEYER & S. SEMMES, Compacité par compensation et espaces de Hardy, *Comptes Rendus Acad. Sci.* Serie I, t. **309** (1981), 945–949.
5. C. FEFFERMAN, Characterizations of bounded mean oscillation, *Bulletin AMS* **77** (1971), 585–587.
6. C. FEFFERMAN & E. STEIN, H^p spaces of several variables, *Acta Math* **129** (1972), 137–193.
7. M. GIAQUINTA, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems,* Princeton Univ. Press, 1989.
8. M. GIAQUINTA, G. MODICA & J. SOUČEK, The Dirichlet energy of mappings with values in a sphere, *Manuscripta Math.* **65** (1989), 489–507.
9. M. GRÜTER, Regularity of weak H-surfaces, *J. Reine Angew. Math.* **329** (1981), 1–15.
10. R. HARDT, D. KINDERLEHRER & F-H. LIN, Stable defects of minimizers of constrained variational principles, *Ann. IHP, Analyse Nonlinéaire* **5** (1988), 297–322.
11. R. HARDT & F-H. LIN, Mappings minimizing the L^p norm of the gradient, *Comm. Pure Appl. Math.* **40** (1987), 556–588.
12. F. HÉLEIN, Regularité des applications faiblement harmoniques entre une surface et une sphere, *Comptes Rendus Acad. Sci.,* to appear.
13. F. HÉLEIN, Regularity of weakly harmonic maps from a surface in a manifold with symmetries, preprint, 1990.
14. S. LUCKHAUS, Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold, *Indiana Univ. Math. Jour.* **37** (1988), 349–367.
15. C. B. Morrey, Jr., *Multiple Integrals in the Calcules of Variations*, Springer-Verlag, 1966.
16. S. MÜLLER, A surprising higher integrability property of mappings with positive determinant, *Bull. Amer. Math. Soc.* **21** (1989), 245–248.
17. P. PRICE, A monotonicity formula for Yang-Mills fields, *Manuscripta Math.* **43** (1983), 131–166.

18. R. SCHOEN, Analytic aspects of the harmonic map problem, in *Seminar on Nonlinear Partial Differential Equations* (edited by S. S. CHERN), MSRI Publications **2**, Springer-Verlag, 1984.
19. R. SCHOEN & K. UHLENBECK, A regularity theory for harmonic maps, *Jour. Diff. Geom.* **17** (1982), 307–335.
20. J. SHATAH, Weak solutions and development of singularities of the SU(2) σ-model, *Comm. Pure Appl. Math.* **41** (1988), 459–469.
21. A. TORCHINSKY, *Real Variable Methods in Harmonic Analysis,* Academic Press, 1986.
22. W. C. WENTE, The Dirichlet problem with a volume constraint, *Manuscripta Math.* **11** (1974), 141–157.

Department of Mathematics
University of California
Berkeley, California 94720

(Received March 14, 1991)