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Partial regularity of free discontinuity sets, II

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Partial Regularity of Free Discontinuity Sets, II

LUIGI AMBROSIO – NICOLA FUSCO – DIEGO PALLARA

1. – Introduction

In this paper we continue the study, started in [5], of the regularity of (quasi) minimizers of a class of free discontinuity problems including the minimization of the Mumford-Shah functional

$$G(u,K)=\int_{\Omega\setminus K}\left[|\nabla u|^2+\alpha(u-g)^2\right]dx+\beta\mathcal{H}^{n-1}(K)\,, \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is an open set, $g\in L^\infty(\Omega)$, $\alpha,\ \beta>0$ and $\mathcal{H}^{n-1}(K)$ is the $(n-1)$-dimensional Hausdorff measure of K. In (1.1), K varies in the class of relatively closed subsets of Ω and $u\in C^1(\Omega\setminus K)$ (see [20], [6], [19], [16], [11]). Notice that, given K, the optimal function u is the solution of the Neumann problem

$$\Delta u=\alpha(u-g),\qquad \frac{\partial u}{\partial n}=0 \tag{1.2}$$

in $\Omega\setminus K$. Namely, we prove in Theorem 3.1 that any optimal free discontinuity set K is a $C^{1,\alpha}$ hypersurface except for a closed singular set S satisfying $\mathcal{H}^{n-1}(S)=0$. Moreover, we give a characterization of singular points which could be exploited to get further information on the dimension and the structure of S.

The starting point of this paper is a criterion, proved in [5], for the $C^{1,\alpha}$ local regularity of optimal sets K. The criterion involves the rate of decay of the quantity

$$\mathcal{F}(x,\rho)=\rho^{-n-1}\min_{A\in\mathcal{A}}\int_{B_\rho(x)\cap K}\operatorname{dist}^2(y,A)\,d\mathcal{H}^{n-1}(y)$$

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measuring the flatness of K (here $\mathcal{A}$ denotes the set of affine hyperplanes in $\mathbb{R}^n$) and of the scaled Dirichlet energy

$$\mathcal{D}(x,\rho) = \rho^{1-n} \int_{B_\rho(x)} |\nabla u|^2 \, dy.$$

The application of such a criterion to our partial regularity result is based on two decay estimates. The first one, concerning the flatness improvement has been already proved in [5] (see Theorem 3.4 below). Heuristically, (see [5]) a first variation argument and the rectifiability of K show that the Dirichlet energy controls the mean curvature of K. Hence, the proof of flatness improvement has a more geometric flavour and involves some typical arguments of minimal surfaces theory.

The second decay estimate is concerned with the Dirichlet energy. Assuming the decay estimate to be false, we find a sequence of quasi minimizers (z_h, C_h) and a sequence of balls B_{ρ_h}. Scaling and normalizing, we obtain a sequence of quasi minimizers (v_h, K_h) in a fixed ball B_R whose gradients weakly converge to the gradient of a function v which is harmonic in the upper and lower half balls, with zero normal derivative on $\{x_n = 0\}$. An easy reflection argument shows that the scaled Dirichlet energy of v on B_ρ decays like ρ. To transfer this property to the sequence v_h (thus getting a contradiction) we need to prove *strong* convergence of the gradients. To obtain this convergence at least in a smaller ball, we introduce functions obtained by taking the composition of v_h with deformation maps pushing a large part of K_h on suitable minimal surfaces. An energy comparison argument leads to the strong convergence of gradients and to the desired contradiction.

A feature of our problem is that the flatness improvement theorem involves also the Dirichlet energy, and viceversa. For this reason an iteration argument leading to the improvement of *both* quantities is needed. The idea is to use the decay of $\mathcal{F}(x,\rho)$ if $\mathcal{F}(x,\rho)$ controls $\mathcal{D}(x,\rho)$ and to use the decay of $\mathcal{D}(x,\rho)$ if $\mathcal{F}(x,\rho)$ is much smaller than $\mathcal{D}(x,\rho)$. This iteration argument seems to be a natural tool for treating the partial regularity of free discontinuity problems. Partial regularity results for minimizers of the Mumford-Shah functional have also been obtained in the two dimensional case by Bonnet and David (see [7] and [13]).

2. – Quasi minimizers

In the following a natural number $n \geq 2$ will be fixed and we omit the dependence of several constants on n. Since we will often deal with $(n-1)$-dimensional sets, we will use the notation m for $n-1$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. For any function $u \in L^1(\Omega)$ the set S_u denotes the the complement of the Lebesgue set of u, i.e.,

$$x \notin S_u \quad \Longleftrightarrow \quad \exists\, z \in \mathbb{R} \text{ s.t. } \lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho(x)} |u(y) - z|\, dy = 0 .$$

The space $SBV(\Omega)$, introduced by De Giorgi and Ambrosio in [15], consists of all functions u with finite total variation in Ω such that the singular part (with respect to Lebesgue measure) $D^s u$ of the distributional derivative Du is supported in S_u, i.e.,

$$|D^s u|(\Omega \setminus S_u) = 0 .$$

In the following, ∇u stands for the density (with respect to Lebesgue measure) of Du. Notice that

$$u \in SBV(\Omega) \quad \Longrightarrow \quad u \in W^{1,1}(\Omega \setminus \overline{S}_u) \tag{2.1}$$

because the singular part of the distributional derivative of u is supported in S_u. In $SBV(\Omega)$ many variational problems involving volume and surface energies can be formulated and have a solution (see [1], [2], [3], [4]). In this paper very little of the SBV theory will be used, because a density lower bound for minimizers (see Proposition 2.4 below and (2.6)) allows a reduction to the study of pairs $(u, K(u))$, with $u \in W^{1,1}(\Omega \setminus K(u))$ and $K(u) = \overline{S}_u$.

Now we give the definitions of local minimizers and quasi minimizers assuming $\beta = 1$. This assumption is not really restrictive, by a scaling argument.

DEFINITION 2.1 (local minimizers). *We say that $u \in SBV_{\mathrm{loc}}(\Omega)$ is a local minimizer in Ω if*

$$\int_A |\nabla u|^2\, dx + \mathcal{H}^m(S_u \cap A) < +\infty \qquad \forall A \subset\subset \Omega \tag{2.2}$$

and

$$\int_A |\nabla u|^2\, dx + \mathcal{H}^m(S_u \cap A) \le \int_A |\nabla v|^2\, dx + \mathcal{H}^m(S_v \cap A)$$

whenever $v \in SBV_{\mathrm{loc}}(\Omega)$ and $\{v \ne u\} \subset\subset A \subset\subset \Omega$.

DEFINITION 2.2 (quasi minimizers). *We will call deviation from minimality* $\mathrm{Dev}(u, \Omega)$ *of a function $u \in SBV_{\mathrm{loc}}(\Omega)$ satisfying* (2.2) *the smallest $\lambda \in [0, +\infty]$ such that*

$$\int_A |\nabla u|^2\, dx + \mathcal{H}^m(S_u \cap A) \le \int_A |\nabla v|^2\, dx + \mathcal{H}^m(S_v \cap A) + \lambda$$

for any $v \in SBV_{\mathrm{loc}}(\Omega)$ such that $\{v \ne u\} \subset\subset A \subset\subset \Omega$. Clearly, $\mathrm{Dev}(u, \Omega) = 0$ *if and only if u is a local minimizer in Ω. Moreover, we say that u is a quasi minimizer in Ω if there exists a nondecreasing function $\omega(t) : (0, +\infty) \to [0, +\infty)$ such that $\omega(t) \downarrow 0$ as $t \downarrow 0$ and*

$$\mathrm{Dev}(u, B_\rho(x)) \le \rho^m \omega(\rho) \tag{2.3}$$

for any ball $B_\rho(x) \subset \Omega$. We denote by $\mathcal{M}_\omega(\Omega)$ the class of functions satisfying (2.3).

REMARK 2.3. The canonical example of quasi minimizer is given by the minimum u of the Mumford-Shah functional

$$F(u) = \int_\Omega \left[|\nabla u|^2 + \alpha(u-g)^2\right] dx + \mathcal{H}^m(S_u)\,, \tag{2.4}$$

with $g \in L^\infty(\Omega)$ and $\alpha \geq 0$. Indeed, it is easy to check that

$$F(v) \geq F(M \wedge v \vee -M) \qquad \forall v \in SBV_{\mathrm{loc}}(\Omega)$$

with $M = \|g\|_\infty$. This shows that $\|u\|_\infty \leq M$ and, for any competing function v, the inequality $F(u) \leq F(M \wedge v \vee -M)$ implies that the deviation from minimality of u in $B_\rho(x)$ does not exceed $4\alpha\omega_n\|g\|_\infty^2\rho^n$. Hence $u \in \mathcal{M}_\omega(\Omega)$ with

$$\omega(t) = 4\alpha\omega_n\|g\|_\infty^2 t\,.$$

Existence of minimizers follows by the compactness and lower semicontinuity results of [1] and [4]. Using Proposition 2.4 below it can be seen (see also [16]) that the minimum of F on $SBV_{\mathrm{loc}}(\Omega)$ is equal to the minimum of the functional originally proposed by Mumford-Shah and Blake-Zissermann:

$$G(u,K) = \int_{\Omega\setminus K} \left[|\nabla u|^2 + \alpha(u-g)^2\right] dx + \mathcal{H}^m(K)\,. \tag{2.5}$$

In (2.5), K varies in the relatively closed subsets of Ω and u varies in $C^1(\Omega\setminus K)$.

PROPOSITION 2.4. *Let $u \in \mathcal{M}_\omega(\Omega)$. Then, for any ball $B_\rho(x) \subset \Omega$ centered at $x \in \overline{S}_u$ we have*

$$\rho \leq \rho_\omega \qquad \Longrightarrow \qquad \mathcal{H}^m\big(S_u \cap B_\rho(x)\big) \geq \theta_\omega \rho^m$$

for some positive constants ρ_ω and θ_ω depending only on ω. Moreover,

$$\mathcal{H}^m(\overline{S}_u \cap \Omega \setminus S_u) = 0\,.$$

PROOF. See Theorem 2.7 and Proposition 2.8 of [5]. The density lower bound for minimizers of the Mumford-Shah functional has been proved also in [11] and [19]. □

We will denote the set $\overline{S}_u$ by $K(u)$. If $u \in \mathcal{M}_\omega(\Omega)$, Proposition 2.4 and the quasi minimality condition imply

$$\begin{aligned} &\int_{B_\rho(x)} |\nabla u|^2\,dy + \mathcal{H}^m\big(K(u)\cap B_\rho(x)\big) \\ &\leq \int_{B_\rho(x)} |\nabla v|^2\,dy + \mathcal{H}^m\big(K(v)\cap B_\rho(x)\big) + \rho^m\omega(\rho) \end{aligned} \tag{2.6}$$

for any $v \in SBV_{\mathrm{loc}}(\Omega)$ such that $\{v \neq u\} \subset\subset B_\rho(x) \subset \Omega$.

3. – Statement of the main result

In the following, $\mathcal{A}$ stands for the set of affine hyperplanes in $\mathbb{R}^n$. The main result of our paper is the following:

THEOREM 3.1. *Let $\mathcal{M}_\omega(\Omega)$ be defined in Definition 2.1, $u \in \mathcal{M}_\omega(\Omega)$, and assume that*

$$\omega(\rho) \le c_0 \rho^{2\gamma} \tag{3.1}$$

for some constants $c_0 \ge 0$, $\gamma > 0$ and let $\alpha = \min\{1/4, \gamma\}/(m+2)$. Then, there is a positive constant $\varepsilon_0(c_0, \gamma)$ such that for any $x \in K(u)$ and any ball $B_\rho(x) \subset \Omega$ with $\rho < 1$, the condition

$$\rho^2 \int_{B_\rho(x)} |\nabla u|^2 \, dy + \min_{A \in \mathcal{A}} \int_{B_\rho(x) \cap K(u)} \operatorname{dist}^2(y, A) \, d\mathcal{H}^m(y) < \varepsilon_0 \rho^{m+2} \tag{3.2}$$

implies that $B_{\rho/2}(x) \cap K(u)$ is a $C^{1,\alpha}$ hypersurface.

REMARK 3.2. Let R be the set of regular points of $K(u)$, i.e., the set of those points $x \in K(u)$ such that (3.2) holds for a sufficiently small $\rho \in (0, 1)$. Then, the complement $S = K(u) \setminus R$ is relatively closed in Ω and it is made of all points satisfying either

$$\limsup_{\rho \to 0^+} \rho^{-m-2} \min_{A \in \mathcal{A}} \int_{B_\rho(x) \cap K(u)} \overset{2}{\operatorname{dist}}(y, A) \, d\mathcal{H}^m(y) > 0 \tag{3.3}$$

or

$$\liminf_{\rho \to 0^+} \rho^{-m} \int_{B_\rho(x)} |\nabla u|^2 \, dx \ge \varepsilon_0 \, . \tag{3.4}$$

It is easy to see that $\mathcal{H}^m(S) = 0$. Indeed, (3.3) is not satisfied at any point $x \in K(u)$ where $K(u)$ has an approximate tangent space P (see [21, Th. 11.6]) because, to estimate the integral in (3.3) we can choose $A = x + P$, thus obtaining that the limit is 0. Moreover, a differentiation theorem (see [21, 3.2(1)]) shows that

$$\int_C |\nabla u|^2 \, dx \ge \frac{\varepsilon_0}{2} \mathcal{H}^m(C)$$

for any Borel subset C of the set E of points $x \in \Omega$ satisfying (3.4). The inequality above implies that E has locally finite $\mathcal{H}^m$-measure, hence $\mathcal{L}^n(E) = 0$. Using the inequality again we obtain that $\mathcal{H}^m(E) = 0$.

The proof of Theorem 3.1 is based, among other things, on a $C^{1,\alpha}$ regularity criterion proved in [5]. Let us introduce a notation for the *Dirichlet integral* of u

$$\mathbb{D}(x,\rho)=\int_{B_\rho(x)}|\nabla u|^2\,dy \tag{3.5}$$

and the *flatness* of $K(u)$

$$\mathbb{A}(x,\rho)=\min_{A\in\mathcal{A}}\int_{B_\rho(x)\cap K(u)}\operatorname{dist}^2(y,A)\,d\mathcal{H}^m(y)\,. \tag{3.6}$$

We can state the following result:

THEOREM 3.3. *Let $u\in\mathcal{M}_\omega(\Omega)$ and assume that there exist $C>0,\sigma\in(0,1)$ such that $\omega(\rho)\le C\rho^{2\sigma}$ and*

$$\mathbb{D}(x,\rho)+\rho^{-2}\mathbb{A}(x,\rho)\le C\rho^{m+\sigma}$$

for any ball $B_\rho(x)\subset\Omega$ centered at $x\in K(u)$. Then, $\Omega\cap K(u)$ is a $C^{1,\alpha}$ hypersurface with $\alpha=\sigma/(m+2)$.

PROOF. See [5], Theorem 5.3 and Remark 5.4. □

To verify the assumption of Theorem 3.3 in $B_{\rho/2}(x)$ (with x satisfying (3.2)) we need to know that $\mathbb{A}$ and $\mathbb{D}$ satisfy good decay properties. The following typical decay property of $\mathbb{A}$ has been proved in Theorem 6.2 of [5]: it says that if $K(u)$ is sufficiently flat near x, it is flatter on a smaller scale, provided $\rho^2\mathbb{D}(x,\rho)$ and $\rho^{m+2}\sqrt{\omega(\rho)}$ are comparable with $\mathbb{A}(x,\rho)$ (remember that the Dirichlet integral is related to the curvature of $K(u)$).

THEOREM 3.4 (flatness improvement). *For any choice of $\beta\in(0,1/112)$ and $\eta>0$ there exists $\varepsilon_1(\beta,\eta,\omega)>0$ such that, for any $u\in\mathcal{M}_\omega(B_\rho(x))$, the conditions*

$$\mathbb{A}(x,\rho)\le\varepsilon_1\rho^{m+2},\qquad \mathbb{A}(x,\beta\rho)\ge\eta\rho^2\left[\mathbb{D}(x,\rho)+\rho^m\sqrt{\omega(\rho)}\right]$$

and $x\in K(u)$ imply

$$\mathbb{A}(x,\beta\rho)\le C\beta^{m+4}\mathbb{A}(x,\rho)$$

with C depending only on ω.

The following theorem is the natural counterpart of the flatness improvement theorem: it says that in the regions where $\mathbb{A}$ is much smaller than $\mathbb{D}$ there is improvement of $\mathbb{D}$.

THEOREM 3.5 (energy improvement). *For any choice of $\beta \in (0, 6/7)$ there exist $\varepsilon_2(\beta, \omega) > 0$ and $\eta(\beta, \omega) \in (0, 1)$ such that, for any $u \in \mathcal{M}_\omega(B_\rho(x))$, the conditions*

$$\mathbb{D}(x, \rho) \leq \varepsilon_2 \rho^m, \quad \mathbb{A}(x, \rho) \leq \varepsilon_2 \rho^{m+2}, \quad \rho^{m+2}\sqrt{\omega(\rho)} + \mathbb{A}(x, \beta\rho) \leq \eta\rho^2 \mathbb{D}(x, \rho)$$

and $x \in K(u)$ imply

$$\mathbb{D}\Big(x, \frac{\beta}{25}\rho\Big) \leq \beta^n \mathbb{D}(x, \rho).$$

A large part of this paper will be devoted to the proof of the energy improvement theorem. Using a careful choice of the constants in both decay theorems, and using in an essential way the independence of the constant C of Theorem 3.4 on β and η, we will be able to perform in Lemma 6.1 and Corollary 6.2 a joint iteration leading to the desired estimates on $\mathbb{A}$ and $\mathbb{D}$.

4. – Preliminary results

The following theorem is concerned with the asymptotic behaviour of sequences of quasi minimizers assuming that Dirichlet energy tends to zero and the jump sets become flatter and flatter.

THEOREM 4.1. *Let $(u_h) \subset \mathcal{M}_\omega(B_R)$, $(S_h) \subset \mathcal{A}$ be satisfying the conditions*

$$\lim_{h\to+\infty} \int_{B_\rho} |\nabla u_h|^2\, dx + \mathrm{Dev}(u_h, B_\rho) + \int_{B_\rho\cap K(u_h)} \mathrm{dist}^2(x, S_h)\, d\mathcal{H}^m = 0$$

for any $\rho \in (0, R)$ and

$$\lim_{h\to+\infty} \mathrm{dist}\big(0, K(u_h)\big) = 0, \qquad \lim_{h\to+\infty} S_h = S.$$

Then S contains the origin and the following properties hold:
(i) *for any $\rho \in (0, R)$ the height*

$$\max_{x\in\overline{B}_\rho\cap K(u_h)} \mathrm{dist}(x, S)$$

is infinitesimal as $h \to +\infty$;
(ii) *denoting by $S^\pm_{R/4}$ the two connected components of*

$$\{x \in B_{R/2} : \mathrm{dist}(x, S) \geq R/4\},$$

we have

$$\liminf_{h\to+\infty} \Big| \int_{S^+_{R/4}} u_h\, dx - \int_{S^-_{R/4}} u_h\, dx \Big| > 0;$$

(iii) *the measures $\mathcal{H}^m \llcorner K(u_h)$ weakly converge to $\mathcal{H}^m \llcorner S$ in B_R.*

PROOF. Let $x_h \in K(u_h)$ be converging to 0 and

$$\rho_h = \min\left\{\frac{\operatorname{dist}(0, S_h)}{4}, \rho_\omega, \frac{R}{4}\right\}.$$

Using Proposition 2.4, if (a subsequence of) ρ_h is not infinitesimal, for h large enough we get

$$\begin{aligned}\int_{B_{R/2}\cap K(u_h)} \operatorname{dist}^2(x, S_h)\, d\mathcal{H}^m &\geq \int_{B_{\rho_h}(x_h)\cap K(u_h)} \operatorname{dist}^2(x, S_h)\, d\mathcal{H}^m \\ &\geq \theta_\omega \rho_h^m \min_{B_{\rho_h}(x_h)\cap K(u_h)} \operatorname{dist}^2(x, S_h) \\ &\geq \theta_\omega \rho_h^m \min_{B_{2\rho_h}} \operatorname{dist}^2(x, S_h) \geq \frac{\theta_\omega}{4}\rho_h^{m+2}\end{aligned}$$

a contradiction. Hence, the distances of S_h from 0 are infinitesimal, S contains the origin and

$$\lim_{h\to+\infty} \int_{B_\rho} |\nabla u_h|^2\, dx + \operatorname{Dev}(u_h, B_\rho) + \int_{B_\rho\cap K(u_h)} \operatorname{dist}^2(x, S)\, d\mathcal{H}^m = 0$$

for any $\rho \in (0, R)$. Then, (i) follows again by the density lower bound (see [5, Proposition 5.1]). The properties (ii), (iii) are proved in Step 4 and Step 5 of Theorem 4.3 [5] (see also Lemma 6.1 of the same paper). □

The next lemma is based on a well known reflection argument.

LEMMA 4.2. *Let B^+ be the upper unit half ball in $\mathbb{R}^n$ and $v : B^+ \to \mathbb{R}$. For $a \in (0, 1)$, let $S_a^+ = \{(z, y) \in B_1 : y > a\}$ and let us assume that $v \in H^1(S_a^+)$ for any $a > 0$. Then*

(i) *v belongs to $H^1(B^+)$ if and only if $\nabla v \in L^2(B^+)$;*

(ii) *if $v \in H^1(B^+)$ and*

$$\int_{B^+} \langle \nabla v, \nabla \phi\rangle\, dx = 0 \qquad \forall \phi \in C_0^1(B_1) \tag{4.1}$$

the function $\tilde{v}(z, y) = v(z, -y)$ is the harmonic extension of v to B_1.

PROOF. (i) One implication is trivial. Assuming $\nabla v \in L^2(B^+)$, we need only to show that $v \in L^2(B^+)$. Clearly, the functions $v_k = k \wedge v \vee -k$ belong to $H^1(B^+)$ and $\|\nabla v_k\|_2 \leq \|\nabla v\|_2$ for any $k > 0$; the Poincaré inequality (see [22, Th. 4.4.2]) implies

$$\int_{B^+} |v_k - \bar{v}_k|^2\, dx \leq c \int_{B^+} |\nabla v_k|^2\, dx$$

for some dimensional constant c, with $\bar{v}_k$ equal to the average of v_k on $S^+_{1/2}$. Letting $k \to +\infty$ we get

$$\int_{B^+} |v - \bar{v}|^2 \, dx \leq c \int_{B^+} |\nabla v|^2 \, dx$$

hence $v \in L^2(B^+)$.
(ii) It is well known that $\tilde{v} \in H^1(B_1)$. For any test function $\phi \in C^1_0(B_1)$, using two times (4.1) and a change of variables we get

$$\begin{aligned}\int_{B_1} \langle \nabla\tilde{v}, \nabla\phi\rangle \, dx &= \int_{B^+} \langle \nabla v, \nabla\phi\rangle \, dx + \int_{B_1 \setminus B^+} \langle \nabla\tilde{v}, \nabla\phi\rangle \, dx \\ &= \int_{B_1 \setminus B^+} \langle \nabla\tilde{v}, \nabla\phi\rangle \, dx = \int_{B^+} \langle \nabla v, \nabla\tilde{\phi}\rangle \, dx = 0\end{aligned}$$

with $\tilde{\phi}(z, y) = \phi(z, -y)$. Hence, $\tilde{v}$ is harmonic. □

For any 1-Lipschitz function v on ∂B_R, let us denote by $\tilde{v} : B_R \to \mathbb{R}$ the solution of the minimum problem

$$\min\left\{\int_{B_R} \sqrt{1 + |\nabla u|^2} \, dx : u \text{ 1-Lipschitz}, \ u = v \text{ on } \partial B_R\right\} \tag{4.2}$$

and let

$$\mathcal{E}(v) = \int_{B_R} \sqrt{1 + |\nabla\tilde{v}|^2} \, dx .$$

We are interested to estimate the convergence to 0 of $\mathcal{E}(v) - \omega_n R^n$ as $\|v\|_{H^1(\partial B_R)}$ tends to 0.

LEMMA 4.3 (area excess estimate). *Let $v : \partial B_R \to \mathbb{R}$ be a 1/2-Lipschitz function. For any $\alpha \geq 1$ we have*

$$\mathcal{E}(v) - \omega_n R^n \leq \left[\frac{\alpha^2}{R}\|v\|^2_{L^2(\partial B_R)} + \frac{R\|\nabla v\|^2_{L^2(\partial B_R)}}{n + 2\alpha - 2}\right] \tag{4.3}$$

provided $\|v\|_\infty < R/(2\alpha)$.

PROOF. The transformation

$$u(x) \mapsto \hat{u}(y) = \frac{1}{R}u(Ry)$$

preserves the Lipschitz constant and maps functions on B_R to functions on B_1. Hence, we can assume with no loss of generality that $R = 1$.

Let $v_\alpha(x) = |x|^\alpha v(x/|x|)$. By the equations (in polar coordinates)

$$\nabla_r v_\alpha = \alpha r^{\alpha-1} v(\omega), \qquad \nabla_\omega v_\alpha = r^\alpha \nabla_\omega v(\omega)$$

we infer that $|\nabla_r v_\alpha| \leq 1/2$ and $|\nabla_\omega v_\alpha / r| \leq 1/2$, hence v_α is a 1-Lipschitz function. In particular

$$\begin{aligned}
\mathcal{E}(v) - \omega_n \leq \mathcal{E}(v_\alpha) - \omega_n &\leq \frac{1}{2} \int_{B_1} |\nabla v_\alpha|^2 \, dx \\
&\leq \left[\alpha^2 \|v\|^2_{L^2(\partial B_1)} \int_0^1 r^{n+2\alpha-3} \, dr + \|\nabla_\omega v\|^2_{L^2(\partial B_1)} \int_0^1 r^{n+2\alpha-3} \, dr \right] \\
&\leq \left[\alpha^2 \|v\|^2_{L^2(\partial B_1)} + \frac{\|\nabla v\|^2_{L^2(\partial B_1)}}{n+2\alpha-2} \right].
\end{aligned}$$

□

LEMMA 4.4 (deformation lemma). *Let $g^+, g^-, g : \overline{B_1^m} \to \mathbb{R}$ be Lipschitz functions such that $g^+ = g^- = g$ on ∂B_1^m and*

$$-2 < g^-(z) \leq g(z) \leq g^+(z) < 2 \qquad \forall z \in B_1^m .$$

Let C be the cylinder $B_1^m \times (-3, 3)$, $\Gamma(g) \subset C$ be the graph of g, $W \subset C$ be the open set above the graph of g^+ and below the graph of g^-.

Then, for any $v \in SBV(C)$ there exists $w \in SBV(C)$ such that the traces of v and w on ∂C are equal and

$$\int_C |\nabla w|^2 \, dx \leq M \int_W |\nabla v|^2 \, dx, \qquad \mathcal{H}^m\big(S_w \setminus \Gamma(g)\big) \leq M \mathcal{H}^m\big(S_v \cap W\big)$$

with M depending only on the Lipschitz constants of g^+, g^-, g.

PROOF. Let $x = (z, y)$, $\Phi : C \to C$ be the map defined by

$$\Phi(z, y) = \begin{cases} \big(z, 3 + (3 - y)(-1 - g^+(z) + g(z))\big) & \text{if } \ 2 < y < 3 ; \\ \big(z, y - g^+(z) + g(z)\big) & \text{if } \ g^+(z) < y \leq 2 ; \\ \big(z, g(z)\big) & \text{if } \ g^-(z) < y \leq g^+(z) ; \\ \big(z, y - g^-(z) + g(z)\big) & \text{if } \ -2 < y \leq g^-(z) ; \\ \big(z, -3 + (y + 3)(1 - g^-(z) + g(z))\big) & \text{if } \ -3 < y \leq -2 . \end{cases}$$

It is easy to check that the restriction of Φ to ∂C is the identity map, $\Phi : W \to C \setminus \Gamma(g)$ is invertible and the norm of the Jacobian matrices of Φ and Φ^{-1} can be uniformly estimated with the Lipschitz constants of g^+, g^-, g. The function

$w := v \circ \Phi^{-1}$ (arbitrarily defined on $\Gamma(g)$) has the same trace of v on ∂C and $S_w \setminus \Gamma(g) \subset \Phi(S_v \cap W)$, hence

$$\mathcal{H}^m(S_w \setminus \Gamma(g)) \leq M\mathcal{H}^m(S_v \cap W).$$

Moreover

$$\int_C |\nabla w(x)|^2\,dx = \int_C \left|\nabla v(\Phi^{-1}(x))\nabla\Phi^{-1}(x)\right|^2 dx$$
$$= \int_W \left|\nabla v(x)\nabla\Phi^{-1}(\Phi(x))\right|^2 |\det J\Phi(x)|\,dx \leq M\int_W |\nabla v|^2\,dx. \;\square$$

THEOREM 4.5 (Lipschitz approximation). *There exist constants $R(\omega) > 0$ and $P(\omega) > 0$ with the following property: let $u \in \mathcal{M}_\omega(B_{7r})$ for some $r \leq R$ and let*

$$A = \int_{B_{6r}\cap K(u)\cap S_u} |T^\perp x|^2\,d\mathcal{H}^m, \qquad D = \int_{B_{6r}} |\nabla u|^2\,dx, \qquad L = r^m\sqrt{\frac{\mathrm{Dev}(u, B_{6r})}{(6r)^m}}.$$

Then, if $PA < r^{m+2}$ and $K(u) \cap B_{r/16} \neq \emptyset$ there exists a Lipschitz function $f : T \to T^\perp$ with Lipschitz constant less than $1/2$ such that

$$\sup|f|^{m+2} \leq PA, \qquad \int_{B^m_{r/4}} |f|^2\,dz \leq A + P\left(\frac{A}{r^2} + D + L\right)(PA)^{2/(m+2)} \tag{4.4}$$

$$\int_{B^m_{r/4}} |\nabla f|^2\,dz \leq P\left[\frac{A}{r^2} + D + L\right] \tag{4.5}$$

and, denoting by $\Gamma(f)$ the graph of f, we have

$$\mathcal{H}^m(B_r \cap K(u) \setminus \Gamma(f)) \leq P\left[\frac{A}{r^2} + D + L\right]. \tag{4.6}$$

PROOF. Let $R = R(\omega, 1/2)$ be the costant defined before Theorem 5.2 of [5]. We will indicate by c the generic constants depending only on n and ω appearing in the estimates of Theorem 5.2 of [5]. Then, this theorem states that $A/r^{m+2} < c$ and $K(u) \cap B_{r/16} \neq \emptyset$ imply the existence of a Lipschitz continuous function $f : T \to T^\perp$ with Lipschitz constant less than $1/2$ such that $\sup|f|^{m+2} \leq cA$ and

$$\begin{aligned}&\mathcal{H}^m(B_r \cap K(u) \setminus \Gamma(f)) + \mathcal{H}^m(X)\\ &\leq c\left[\int_{B_{5r}} |\nabla u|^2\,dx + \int_{B_{5r}\cap K(u)} \|S_x - T\|^2\,d\mathcal{H}^m\right]\end{aligned} \tag{4.7}$$

where X is the projection on $B^m_{r/4}$ of $\Gamma(f) \setminus K(u)$ and $\|S_x - T\|$ represents the distance of the approximate tangent space S_x to $K(u)$ at x from T (see [5] for details).

Using the tilt lemma (see Lemma 6.1 of [5]) we obtain (4.6) from (4.7) and the estimate $\mathcal{H}^m(X) \le c(A/r^2 + D + L)$. The second inequality in (4.4) can be easily deduced from the first one by evaluating separately the integral on X and the integral on $B^m_{r/4} \setminus X$.

Finally, the inequality (4.5) can be proved with the same method using the tilt lemma and noticing that $|\nabla f| \le 1/2$ and

$$\|S_x - T\|^2 = 2(1 - \nu_n) \ge \frac{|\nabla f(z)|^2}{1 + |\nabla f(z)|^2} \ge \frac{1}{2} |\nabla f(z)|^2$$

where ν is the normal to the graph of f at $x = (z, f(z))$, with $z \in X$ differentiability point of f (see Step 2 in Theorem 6.2 of [5]). □

5. – Proof of the energy improvement theorem

We assume without loss of generality that $\omega(t) > 0$ for any $t > 0$. As usual, to prove Theorem 3.5 we argue by contradiction, getting sequences $(z_h) \subset \mathcal{M}_\omega(B_{\rho_h}(y_h))$, (S_h), $(\tilde{S}_h) \subset \mathcal{A}$ satisfying

$$\int_{B_{\rho_h}(y_h)} |\nabla z_h|^2 \, dx < \gamma_h \rho_h^m, \qquad \int_{B_{\rho_h}(y_h) \cap K(z_h)} \operatorname{dist}^2(x, S_h) \, d\mathcal{H}^m < \gamma_h \rho_h^{m+2} \tag{5.1}$$

$$\rho_h^{m+2} \sqrt{\omega(\rho_h)} + \int_{B_{\beta\rho_h}(y_h) \cap K(z_h)} \operatorname{dist}^2(x, \tilde{S}_h) \, d\mathcal{H}^m \le \gamma_h \rho_h^2 \int_{B_{\rho_h}(y_h)} |\nabla z_h|^2 \, dx \tag{5.2}$$

for a suitable infinitesimal sequence γ_h and

$$\int_{B_{\beta\rho_h/25}(y_h)} |\nabla z_h|^2 \, dx > \beta^n \int_{B_{\rho_h}(y_h)} |\nabla z_h|^2 . \tag{5.3}$$

To rescale all the functions to a fixed ball we will use the following remark.

REMARK 5.1 (scaling). Let $u \in SBV(\Omega)$ and let $B_\rho(x_0) \subset \Omega$. Then, it is easy to check that

$$u_\rho(y) = \frac{1}{\sqrt{\rho}} u(x_0 + \rho y)$$

belongs to $SBV(\Omega_\rho)$ with $\Omega_\rho = \rho^{-1}(\Omega - x_0)$. Moreover, for any ball $B_\eta(y) \subset \Omega_\rho$ we have

$$\int_{B_\eta(y)} |\nabla u_\rho|^2 \, dx = \rho^{-m} \int_{B_{\eta\rho}(x_0+\rho y)} |\nabla u|^2 \, dx,$$

$$\mathcal{H}^m\big(S_{u_\rho} \cap B_\eta(y)\big) = \rho^{-m}\mathcal{H}^m\big(S_u \cap B_{\eta\rho}(x_0 + \rho y)\big)$$

$$\mathrm{Dev}(u_\rho, B_\eta(y)) = \rho^{-m}\mathrm{Dev}(u, B_{\eta\rho}(x_0 + \rho y)) .$$

In particular, if $\rho \le 1$, the monotonicity of $\omega(\rho)$ shows that

$$u \in \mathcal{M}_\omega(\Omega) \qquad \Longrightarrow \qquad u_\rho \in \mathcal{M}_\omega(\Omega_\rho).$$

Let $R = R(\omega)$ be given by Theorem 4.5. By (5.1), (5.2), (5.3) and our assumption on ω we obtain that $\rho_h \le R$ for h large enough. Hence, scaling the functions by a factor ρ_h/R as in Remark 5.1, we obtain new sequences $(u_h) \subset \mathcal{M}_\omega(B_R(x_h))$, (T_h), $(\tilde{T}_h)$ satisfying the conditions

$$\int_{B_R(x_h)} |\nabla u_h|^2 \, dx < R^m \gamma_h, \qquad \int_{B_R(x_h)\cap K(u_h)} \mathrm{dist}^2(x, T_h) \, d\mathcal{H}^m < \gamma_h R^{m+2} \tag{5.4}$$

$$\sqrt{\omega(\rho_h)} + \int_{B_{\beta R}(x_h)\cap K(u_h)} \mathrm{dist}^2(x, \tilde{T}_h) \, d\mathcal{H}^m \le \gamma_h \int_{B_R(x_h)} |\nabla u_h|^2 \, dx \tag{5.5}$$

and

$$\int_{B_{\beta R/25}(x_h)} |\nabla u_h|^2 \, dx > \beta^n \int_{B_R(x_h)} |\nabla u_h|^2 \, dx . \tag{5.6}$$

The deviation from minimality of u_h in $B_R(x_h)$ satisfies

$$\begin{aligned} \mathrm{Dev}(u_h, B_R(x_h)) &= \frac{\mathrm{Dev}(z_h, B_{\rho_h}(y_h))}{\rho_h^m / R^m} \le R^m \omega(\rho_h) \\ &\le R^m \gamma_h^2 \Big(\int_{B_R} |\nabla u_h|^2 \, dx \Big)^2 . \end{aligned} \tag{5.7}$$

Denoting by T the hyperplane $\{x_n = 0\}$, by a rotation and a translation we can assume with no loss of generality that $\tilde{T}_h = T$, $T(x_h) = 0$ and T_h converges to some hyperplane S. Using (5.5) and the density lower bound it is easy to see, by the same argument used in Theorem 4.1, that $x_h \to 0$ (notice that $x_h \in K(u_h)$). By Theorem 4.1(iii) and (5.4), (5.7) we obtain that $\mathcal{H}^m \llcorner K(u_h)$ weakly converges in B_R to $\mathcal{H}^m \llcorner S$ and S contains the origin. Since

$$0 = \lim_{h \to +\infty} \int_{B_{\beta R}(x_h) \cap K(u_h)} \operatorname{dist}^2(x, \tilde{T}_h)\, d\mathcal{H}^m$$
$$= \lim_{h \to +\infty} \int_{B_{\beta R} \cap K(u_h)} \operatorname{dist}^2(x, T)\, d\mathcal{H}^m = \int_{B_{\beta R} \cap S} \operatorname{dist}^2(x, T)\, d\mathcal{H}^m$$

we infer that $T = S$. Setting now

$$\varepsilon_h = \left(\int_{B_R} |\nabla u_h|^2\, dx \right)^{1/2}$$

and defining $v_h = u_h / \varepsilon_h$ we can rewrite (5.4), (5.5), (5.6) as follows

$$\lim_{h \to +\infty} \varepsilon_h^2 + \int_{B_R(x_h) \cap K(v_h)} \operatorname{dist}^2(x, T)\, d\mathcal{H}^m = 0 \tag{5.8}$$

$$\int_{B_{\beta R}(x_h) \cap K(v_h)} \operatorname{dist}^2(x, T)\, d\mathcal{H}^m = o(\varepsilon_h^2) \tag{5.9}$$

$$\int_{B_R(x_h)} |\nabla v_h|^2\, dx = 1, \qquad \int_{B_{\beta R/25}(x_h)} |\nabla v_h|^2\, dx > \beta^n . \tag{5.10}$$

We point out that, by (2.6), the functions v_h are quasi minimizers in $B_R(x_h)$ of the rescaled functionals

$$\mathcal{F}_h(v) = \int_{B_\rho} |\nabla v|^2\, dx + \frac{1}{\varepsilon_h^2} \mathcal{H}^m(B_\rho \cap K(v)) \tag{5.11}$$

and, denoting by Dev_h the deviation from minimality of $\mathcal{F}_h$, (5.7) yields

$$\lim_{h \to +\infty} \mathrm{Dev}_h(v_h, B_\rho) \le \limsup_{h \to +\infty} \frac{\mathrm{Dev}(u_h, B_R(x_h))}{\varepsilon_h^2} = 0 \tag{5.12}$$

for any $\rho < R$ (notice that $B_\rho \subset B_R(x_h)$ for h large enough).

STEP 1. Let us use $y = x_n$ to denote the "vertical" coordinate. By (5.8) and Theorem 4.1(i) we obtain

$$\lim_{h\to+\infty} \sup_{x\in D\cap K(v_h)} |y| = 0 \qquad \forall D \subset\subset B_R. \tag{5.13}$$

In particular, the maximal height of $K(v_h)$ in $\overline{B}_{R/2}$ is infinitesimal. For any $a > 0$ we denote by S_a^+ (respectively S_a^-) the set of points $x \in B_{R/2}$ such that $y > a$ (respectively $y < -a$), we denote by $c_h^\pm$ the average of v_h on $S_{R/4}^\pm$ and we assume, up to a change of sign, that $c_h^+ \geq c_h^-$. By Theorem 4.1(ii) we obtain that the sequence $\varepsilon_h(c_h^+ - c_h^-)$ has no infinitesimal subsequences, hence $(c_h^+ - c_h^-)$ diverges.

For any $a > 0$ (5.13) implies the existence of $h_a \in \mathbb{N}$ such that $v_h - c_h^\pm$ belongs to $H^1(S_a^\pm)$ for $h \geq h_a$. Moreover, if $a < R/4$, the average of $v_h - c_h^\pm$ is zero on $S_{R/4}^\pm \subset S_a^\pm$ and the Poincaré inequality (see [22, Th. 4.4.2]) implies that the family $\{v_h - c_h^\pm\}_{h\geq h_a}$ is bounded in $H^1(S_a^\pm)$. Possibly extracting a subsequence, by a diagonal argument we can find a function $v : B_{R/2}\setminus T \to \mathbb{R}$ such that $v_h - c_h^\pm$ weakly converges to v in $H^1(S_a^\pm)$, and also pointwise in S_a, for any $a \in (0, R/4)$. By lower semicontinuity of the Dirichlet integral and the first equality in (5.10) we infer

$$\int_{B_R} |\nabla v|^2\,dx \leq 1. \tag{5.14}$$

In particular, Lemma 4.2(i) yields $v \in H^1(S_0^\pm)$.

STEP 2. Now we show that v in harmonic in $B_{R/2}\setminus T$ and its normal derivative on T is zero.

Using (5.12) and comparing $\mathcal{F}_h(v_h)$ with $\mathcal{F}_h(v_h + \phi)$ for $\phi \in C_0^1(B_{R/2}\setminus T)$, it is easy to check that v is harmonic in $B_{R/2}\setminus T$.

Possibly passing to a subsequence we can assume that the measures $|\nabla v_h|^2 \mathcal{L}^n$ are weakly converging in $B_{R/2}$ to some measure μ. Since ∇v_h weakly converges to ∇v in $B_{R/2}$ we have $\mu \geq |\nabla v|^2\mathcal{L}^n$.

Comparing $\mathcal{F}_h(v_h)$ with $\mathcal{F}_h(\phi(v + c_h^\pm) + (1-\phi)v_h)$ with $\phi \in C_0^1(B_{R/2}\setminus T)$ and using again (5.12) we get the inequality

$$\int_{B_{R/2}} |\nabla v_h|^2\,dx \leq (1+\varepsilon)\int_{B_{R/2}} \phi|\nabla v|^2 + (1-\phi)|\nabla v_h|^2\,dx + \left(1+\frac{1}{\varepsilon}\right)\int_{B_{R/2}} |\nabla\phi|^2(v_h - c_h^\pm - v)^2\,dx + o(1)$$

for any $\varepsilon > 0$. Using the strong convergence of $v_h - c_h^\pm$ to v in $L^2_{\mathrm{loc}}(B_{R/2}\setminus T)$, letting first $h \to +\infty$ then $\varepsilon \to 0$ we find

$$\int_{B_{R/2}} \phi\,d\mu \leq \int_{B_{R/2}} \phi|\nabla v|^2\,dx.$$

Since ϕ is arbitrary, we obtain that $\mu \llcorner (B_{R/2} \setminus T) = |\nabla v|^2 \mathcal{L}^n$, hence the absolutely continuous part of μ is obviously $|\nabla v|^2 \mathcal{L}^n$ and the singular part must be supported in T.

We now claim that v has zero normal derivative on $T \cap B_{R/2}$, i.e.,

$$\int_{S_0^\pm} \langle \nabla v, \nabla \phi \rangle \, dx = 0 \tag{5.15}$$

for any $\phi \in C_0^1(B_{R/2})$. Indeed, let $\phi \in C_0^1(B_{R/2})$, $\varepsilon > 0$ and let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth non decreasing function such that $\psi(-\infty) = 0$, $\psi(+\infty) = 1$ and $\psi'(t) \leq \varepsilon$ for any $t \in \mathbb{R}$. By (5.12) the sequence $z_h = \mathrm{Dev}_h(v_h, B_{R/2})$ is infinitesimal. Hence, denoting by c_h the average of c_h^+ and c_h^-, comparing $\mathcal{F}_h(v_h)$ with $\mathcal{F}_h\big(v_h + \sqrt{z_h}\phi\psi(v_h - c_h)\big)$ and dividing both sides by $\sqrt{z_h}$ we obtain

$$\liminf_{h \to +\infty} \int_{B_{R/2}} \psi(v_h - c_h)\langle \nabla v_h, \nabla \phi \rangle \, dx + \int_{B_{R/2}} \phi\psi'(v_h - c_h)|\nabla v_h|^2 \, dx \geq 0 .$$

Since ∇v_h weakly converges to ∇v in $L^2(B_{R/2}, \mathbb{R}^n)$ and $\psi(v_h - c_h)$ strongly converges to $\chi_{S_0^+}$ (here we use the fact that $c_h^+ - c_h^- \to +\infty$) we obtain

$$\int_{S_0^+} \langle \nabla \phi, \nabla v \rangle \, dx \geq -\varepsilon \|\phi\|_\infty .$$

As ε and ϕ are arbitrary, (5.15) is proved for S_0^+. The argument for S_0^- is similar. By Lemma 4.2(ii) we infer that the restrictions of v to $S_0^\pm$ can be prolonged by reflection along T to harmonic functions on $B_{R/2}$. In particular, v and ∇v are locally bounded in $B_{R/2}$ and

$$\int_{B_\rho} |\nabla v|^2 \, dx \leq \frac{2^n \rho^n}{R^n} \int_{B_{R/2}} |\nabla v|^2 \, dx \leq \frac{2^n \rho^n}{R^n} \qquad \forall \rho \in (0, R/2) . \tag{5.16}$$

STEP 3. Now we prove an estimate from below on the size of $K(v_h)$, namely

$$\mathcal{H}^m\big(C_\rho \cap K(v_h)\big) \geq \omega_m \rho^m - o(\varepsilon_h^2)$$

for any cylinder $C_\rho = B_\rho^m(x_0) \times (-\rho, \rho)$ contained in $B_{R/2}$. Let $Z_h \subset B_\rho^m(x_0)$ be the projection of $C_\rho \cap K(v_h)$ on T; by well know properties of Hausdorff measures it suffices to prove that $\mathcal{L}^m(Z_h) \geq \omega_m \rho^m - o(\varepsilon_h^2)$. Let $G_h = B_\rho^m(x_0) \setminus Z_h$;

given $a \in (0, \rho)$ we have that the functions $v_h(z, \cdot)$ are absolutely continuous in $(-a, a)$ for $\mathcal{L}^m$-almost every $z \in G_h$, hence

$$\begin{aligned}\int_{G_h} |v_h(z,a) - v_h(z,-a)|^2\,dz &\le 2a \int_{G_h}\int_{-a}^{a} \left|\frac{\partial v_h}{\partial y}\right|^2 (z,y)\,dy\,dz \\ &\le 2a \int_{B_{R/2}} |\nabla v_h|^2\,dx \le 2a\,.\end{aligned} \tag{5.17}$$

On the other hand, the integrals

$$\int_{G_h} \left|(v_h(z,a) - c_h^+) - (v_h(z,-a) - c_h^-)\right|^2 dz$$

are bounded because of the convergence of $(v_h - c_h^\pm)$ to v. Since $(c_h^+ - c_h^-)$ tends to $+\infty$ from (5.17) we infer that $\mathcal{L}^m(G_h)$ is infinitesimal, hence

$$\lim_{h\to+\infty} \int_{G_h} \left|(v_h(z,a) - c_h^+) - (v_h(z,-a) - c_h^-)\right|^2 dz = 0 \tag{5.18}$$

for any $a \in (0, \rho)$. Assuming by contradiction that $\mathcal{L}^m(G_{h_k}) \ge \delta\varepsilon_{h_k}^2$ for some $\delta > 0$ and some subsequence (h_k), by (5.17) and (5.18) we obtain

$$\delta\varepsilon_{h_k}^2 (c_{h_k}^+ - c_{h_k}^-)^2 \le \mathcal{L}^m(G_{h_k})(c_{h_k}^+ - c_{h_k}^-)^2 \le 3a$$

for k large enough depending on a. Since a is arbitrary we find that $\varepsilon_{h_k}(c_{h_k}^+ - c_{h_k}^-)$ tends to 0, a contradiction with Theorem 4.1(ii).

STEP 4. Applying Theorem 4.5 to the functions u_h with $r = \beta R/6$ we can find 1/2-Lipschitz functions $f_h : B_{r/4}^m \to \mathbb{R}$ such that (we use (4.4), (4.5), (5.9), (5.7))

$$\lim_{h\to+\infty} \sup |f_h| = 0, \qquad \int_{B_{r/4}^m} |f_h|^2\,dz = o(\varepsilon_h^2) \tag{5.19}$$

$$\int_{B_{r/4}^m} |\nabla f_h|^2\,dz \le Q\varepsilon_h^2 \tag{5.20}$$

$$\mathcal{H}^m\big(B_r \cap K(v_h) \setminus \Gamma(f_h)\big) \le Q\varepsilon_h^2 \tag{5.21}$$

for some constant $Q \ge 0$. Possibly extracting a subsequence, we can assume that the measures

$$\nu_h = \frac{1}{\varepsilon_h^2}\mathcal{H}^m \llcorner \big(K(v_h) \setminus \Gamma(f_h)\big)$$

are weakly converging in B_r to some measure ν. Since the maximal height of $K(v_h)$ is infinitesimal, the measure ν is supported in T.

STEP 5. We will prove that $\mu = \lim |\nabla v_h|^2 \mathcal{L}^n$ is absolutely continuous with respect to $\mathcal{L}^n$. The convergence of the Dirichlet integrals and the final contradiction will follow at once.

Let us fix a cylinder $C = B^m_\rho(x_0) \times (-3\rho, 3\rho)$ contained in $B_{r/4}$, and assume that ρ and x_0 fulfil the conditions

$$\mu(\partial C) = \nu(\partial C) = 0 \tag{5.22}$$

$$\liminf_{h\to+\infty} \frac{1}{\varepsilon_h^2} \int_{\partial B^m_\rho(x_0)} |\nabla f_h|^2 \, dz \le Q' < +\infty . \tag{5.23}$$

Let $h(k)$ be a subsequence such that

$$\lim_{k\to+\infty} \frac{1}{\varepsilon^2_{h(k)}} \int_{\partial B^m_\rho(x_0)} |\nabla f_{h(k)}|^2 \, dz \le Q'$$

and let $g_k : \overline{B}^m_\rho(x_0) \to \mathbb{R}$ be the 1-Lipschitz solutions of the least area problems (4.2) with $B^m_\rho(x_0)$ in place of B_R and $f_{h(k)}$ in place of v. Let

$$g_k^+(x) = \inf_{y\in\partial B^m_\rho(x_0)} f_{h(k)}(y) + |x-y|, \qquad g_k^-(x) = \sup_{y\in\partial B^m_\rho(x_0)} f_{h(k)}(y) - |x-y| .$$

It is well known (and easy to check) that g_k^+ and g_k^- are respectively the largest and the smallest 1-Lipschitz extensions of the restriction of $f_{h(k)}$ to $\partial B^m_\rho(x_0)$. In particular

$$g_k^-(x) \le g_k(x) \le g_k^+(x) \quad \text{and} \quad g_k^-(x) \le f_{h(k)}(x) \le g_k^+(x) \qquad \forall x \in \overline{B}^m_\rho(x_0) .$$

Since f_h uniformly converges to 0 in $\partial B^m_\rho(x_0)$, the functions $g_k^\pm$ uniformly converge in $\overline{B}^m_\rho(x_0)$ to $\pm\mathrm{dist}(x, \partial B^m_\rho(x_0))$; we set

$$W_\infty = \left\{x = (z, y) \in \overline{C} : |y| \ge \mathrm{dist}\big(z, \partial B^m_\rho(x_0)\big)\right\} .$$

For k large enough we have $\|g_k^\pm\|_\infty < 2\rho$, hence we can apply Lemma 4.4, getting functions $w_k \in SBV(C)$, with the same trace of $v_{h(k)}$ of ∂C, such that

$$\int_C |\nabla w_k|^2 \, dx \le M \int_{W_k} |\nabla v_{h(k)}|^2 \, dx \tag{5.24}$$

and

$$\mathcal{H}^m\big(S_{w_k}\big) \le M\mathcal{H}^m\big(K(v_{h(k)}) \cap W_k\big) + \mathcal{H}^m\big(\Gamma(g_k)\big) \tag{5.25}$$

where W_k denotes the open portion of C above the graph of g_k^+ and below the graph of g_k^-. Using Lemma 4.3 and taking into account (5.19) and (5.20) we have

$$\limsup_{k\to+\infty} \frac{\mathcal{H}^m(\Gamma(g_k)) - \omega_m \rho^m}{\varepsilon_{h(k)}^2} \le \frac{\rho Q'}{m + 2\alpha - 2}$$

for any $\alpha \ge 1$, hence $\mathcal{H}^m(\Gamma(g_k)) \le \omega_m \rho^m + o(\varepsilon_{h(k)}^2)$. Using the fact that W_k does not intersect the graph of $f_{h(k)}$ we obtain the following estimate on $\mathcal{H}^m(S_{w_k})$:

$$\mathcal{H}^m(S_{w_k}) \le \omega_m \rho^m + M\varepsilon_{h(k)}^2 \nu_{h(k)}(W_k) + o(\varepsilon_{h(k)}^2)\,. \tag{5.26}$$

Now, we define

$$\tilde{v}_k(x) = \begin{cases} v_{h(k)}(x) & \text{if } x \notin C\,; \\ w_k(x) & \text{if } x \in C \end{cases}$$

and we compare $\mathcal{F}_{h(k)}(v_{h(k)})$ with $\mathcal{F}_{h(k)}(\tilde{v}_k)$. Taking into account (5.12) and (5.24) we get

$$\begin{aligned} \int_C |\nabla v_{h(k)}|^2\,dx \le M \int_{W_k} |\nabla v_{h(k)}|^2\,dx \\ + \frac{\mathcal{H}^m(S_{w_k} \cap C) - \mathcal{H}^m(S_{v_{h(k)}} \cap C)}{\varepsilon_{h(k)}^2} + o(1)\,. \end{aligned} \tag{5.27}$$

Now, by Step 3, $\mathcal{H}^m(S_{v_{h(k)}} \cap C)$ exceeds $\omega_m \rho^m$ up to an infinitesimal faster than $\varepsilon_{h(k)}^2$. Hence, letting $k \to +\infty$ in (5.27) and using (5.26) we obtain

$$\mu(C) \le M\mu(W_\infty) + M\nu(W_\infty)\,.$$

Finally, our choice of ρ and the fact that μ^s and ν are both supported in T guarantees that $\mu^s(W_\infty) = \nu(W_\infty) = 0$ and

$$\mu(W_\infty) = \int_{W_\infty} |\nabla v|^2\,dx \le 6\omega_m \|\nabla v\|_{L^\infty(B_{R/4})}^2 \rho^n\,.$$

Hence, $\mu(C) \le 6M\omega_m \|\nabla v\|_{L^\infty(B_{R/4})}^2 \rho^n$.

By a standard approximation argument, the inequality remains true even if (5.22) or (5.23) are not fulfilled. Indeed, the set of radii ρ for which (5.22) is false is at most countable, and the set of radii ρ for which (5.23) is false is $\mathcal{L}^1$-negligible, by (5.20), and the Fatou lemma. Since the cylinder $C \subset B_{r/4}$ is arbitrary, it follows that $\mu^s \llcorner B_{r/4} = 0$.

Step 6. End of the proof. From Step 5, it follows that $|\nabla v_h|^2 \mathcal{L}^n$ weakly converges to $|\nabla v|^2 \mathcal{L}^n$ in $B_{\beta R/24}$. In particular,

$$\lim_{h\to+\infty} \int_{B_\rho} |\nabla v_h|^2\,dx = \int_{B_\rho} |\nabla v|^2\,dx \qquad \forall \rho \in (0, \beta R/24)\,.$$

Choosing $\rho = \beta R/25$ and passing to the limit as $h \to +\infty$ in (5.10) we obtain

$$\int_{B_{\beta R/25}} |\nabla v|^2 \, dx \geq \beta^n .$$

This contradicts the energy estimate (5.16) with $\rho = \beta R/25$. □

6. – Proof of partial regularity

In this section we will prove the partial regularity of quasi minimizers, stated in Theorem 3.1. Since ω is a fixed function, we omit in this section the dependence of the constants on ω. Let $c_0 \geq 0$, $\gamma > 0$, be given by (3.1).

If $u \in \mathcal{M}_\omega(B_\rho(x))$ and $x \in K(u)$ we define

$$A(\rho) = \mathbb{A}(x, \rho), \qquad D(\rho) = \mathbb{D}(x, \rho) . \tag{6.1}$$

By Theorem 3.4 we obtain that, for any choice of $\beta \in (0, 1/112)$, A satisfies the following decay property, that for convenience we restate in a slightly different way:

(i) For any $\eta > 0$ there exists $\varepsilon_1(\beta, \eta) > 0$ such that the conditions

$$A(\rho) \leq \varepsilon_1 \rho^{m+2}, \qquad A(\beta\rho) \geq \eta \max\{\rho^2 D(\rho), \sqrt{c_0}\rho^{m+2+\gamma}\}$$

imply $A(\beta\rho) \leq C\beta^{m+4} A(\rho)$, with C depending only on c_0, γ.

Similarly, using Theorem 3.5, we obtain that D satisfies for any $\beta \in (0, 6/7)$ the following decay property:

(ii) there exist $\varepsilon_2(\beta) > 0$ and $\eta(\beta) \in (0, 1)$ such that the conditions

$$D(\rho) \leq \varepsilon_2 \rho^m, \qquad A(\rho) \leq \varepsilon_2 \rho^{m+2}, \qquad \max\{\sqrt{c_0}\rho^{m+2+\gamma}, A(\beta\rho)\} \leq \eta \rho^2 D(\rho)$$

imply $D(\beta\rho/25) \leq \beta^n D(\rho)$.

LEMMA 6.1. *Let $\bar{\rho} \in (0, 1)$, A, $D : (0, \bar{\rho}) \to \mathbb{R}^+$ be nondecreasing functions satisfying* (i) *and* (ii) *above. Then, for $\beta > 0$ sufficiently small* (*see* (6.3) *below*) *there are constants ε, η, $L > 0$ such that, setting*

$$E(\rho) = \left[\eta \frac{D(\rho)}{\rho^m} + \frac{A(\rho)}{\rho^{m+2}}\right] \tag{6.2}$$

the following implication holds

$$E(\beta\rho) > L\rho^\gamma, \ E(\rho) < \varepsilon \implies \left[E(\tau_1\rho) \leq \tau_1^{1/4} E(\rho) \quad or \quad E(\tau_2\rho) \leq \tau_2^{1/4} E(\rho)\right]$$

with $\tau_1 = \beta/25$ and $\tau_2 = \beta^{1/3}$.

PROOF. We fix a constant $\beta \in (0, 1/(112)^3)$ so small that

$$25^{m+3}\beta^{1/3} < \left(\frac{\beta}{25}\right)^{1/4}, \qquad 2C\beta^{2/3} < \left(\beta^{1/3}\right)^{1/4} \tag{6.3}$$

and we fix $\eta(\beta) > 0$ according to **(ii)**. Finally, we set

$$\varepsilon = \eta(\beta)\min\left\{\varepsilon_2(\beta), \varepsilon_1\left(\beta, \beta^{(3m+7)/3}\eta(\beta)\right), \varepsilon_1\left(\beta^{1/3}, \beta^{(3m+7)/3}\eta(\beta)\right)\right\}$$

and we choose $L > 0$ satisfying the conditions

$$\frac{\eta(\beta)}{L}\left[\beta^{-m} + \beta^{1/3}\right] < c_0^{-1/2}, \qquad \frac{\beta^{-m-2} + \beta^{-(9m+7)/3}}{L} < c_0^{-1/2}.$$

Let us assume that $E(\rho) < \varepsilon$ and $E(\beta\rho) > L\rho^\gamma$ for some $\rho \in (0, \bar{\rho})$.

CASE 1. Assume that

$$A(\beta\rho) \le \beta^{(3m+7)/3}\eta(\beta)\rho^2 D(\rho). \tag{6.4}$$

Our choice of ε implies $D(\rho) < \rho^m \varepsilon_2(\beta)$; moreover, using the inequality $E(\beta\rho) > L\rho^\gamma$ we get

$$\frac{\eta(\beta)D(\rho)}{\rho^m}\left[\beta^{-m} + \beta^{1/3}\right] > L\rho^\gamma$$

and our choice of L implies $\sqrt{c_0}\rho^{m+\gamma} \le \eta(\beta)D(\rho)$. Hence, we can apply **(ii)**. This, together with (6.4) yields

$$\begin{aligned} E\left(\frac{\beta}{25}\rho\right) &= \frac{A(\frac{\beta}{25}\rho)}{(\frac{\beta}{25}\rho)^{m+2}} + \eta(\beta)\frac{D(\frac{\beta}{25}\rho)}{\left(\frac{\beta}{25}\rho\right)^m} \le \frac{25^{m+2}A(\beta\rho)}{\beta^{m+2}\rho^{m+2}} + 25^m\eta(\beta)\frac{D(\frac{\beta}{25}\rho)}{\beta^m\rho^m} \\ &\le \frac{25^{m+2}\beta^{(3m+7)/3}\eta(\beta)\rho^2 D(\rho)}{\beta^{m+2}\rho^{m+2}} + 25^m\eta(\beta)\frac{\beta^n D(\rho)}{\beta^m\rho^m} \\ &< 25^{m+3}\beta^{1/3}\eta(\beta)\frac{D(\rho)}{\rho^m} \le 25^{m+3}\beta^{1/3}E(\rho). \end{aligned}$$

In this case (6.3) yields $E(\tau_1\rho) \le \tau_1^{1/4}E(\rho)$.

CASE 2. Now we assume that (6.4) does not hold, hence

$$A(\beta^{1/3}\rho) \ge A(\beta\rho) > \beta^{(3m+7)/3}\eta(\beta)\rho^2 D(\rho) \tag{6.5}$$

Our choice of ε implies

$$A(\rho) < \varepsilon_1\left(\beta^{1/3}, \beta^{(3m+7)/3}\eta(\beta)\right)\rho^{m+2}, \qquad A(\rho) < \varepsilon_1\left(\beta, \beta^{(3m+7)/3}\eta(\beta)\right)\rho^{m+2}.$$

Moreover, (6.5) and $E(\beta\rho) > L\rho^\gamma$ yield

$$\frac{A(\beta\rho)}{\rho^{m+2}}\left[\beta^{-m-2}+\beta^{-(9m+7)/3}\right] > L\rho^\gamma$$

and our choice of L yields $\sqrt{c_0}\rho^{m+2+\gamma} \le A(\beta\rho)$. Since $\beta < 1$ we have also $\sqrt{c_0}(\beta^{1/3}\rho)^{m+2+\gamma} \le A(\beta^{1/3}\rho)$.

Hence we can apply (i) with $\beta^{1/3}$ and β. This, together with (6.5), yields

$$\begin{aligned} &E(\beta^{1/3}\rho)\\ &= \frac{A(\beta^{1/3}\rho)}{\beta^{(m+2)/3}\rho^{m+2}} + \eta(\beta)\frac{D(\beta^{1/3}\rho)}{\beta^{m/3}\rho^m} \le C\frac{\beta^{(m+4)/3}A(\rho)}{\beta^{(m+2)/3}\rho^{m+2}} + \eta(\beta)\frac{D(\rho)}{\beta^{m/3}\rho^m}\\ &\le C\beta^{2/3}\frac{A(\rho)}{\rho^{m+2}} + \frac{A(\beta\rho)}{\beta^{(3m+8)/3}\rho^{m+2}} \le C\beta^{2/3}\frac{A(\rho)}{\rho^{m+2}} + C\frac{\beta^{m+4}A(\rho)}{\beta^{(3m+8)/3}\rho^{m+2}}\\ &\le 2C\beta^{2/3}E(\rho)\,. \end{aligned}$$

In this case (6.3) gives $E(\tau_2\rho) \le \tau_2^{1/4}E(\rho)$. □

COROLLARY 6.2. *With the assumptions and the notations of Lemma* 6.1, $E(\rho) < \varepsilon$ *implies*

$$E(r) \le \left(\frac{25}{\beta}\right)^{m+2+\sigma} \max\left\{\left(\frac{r}{\rho}\right)^\sigma E(\rho), \frac{L}{\beta^\sigma}r^\sigma\right\} \qquad \forall r \in (0,\rho]$$

with $\sigma = \min\{1/4, \gamma\}$.

PROOF. Let us assume that $E(\rho) < \varepsilon$ for some $\rho \in (0,\bar\rho)$. We inductively define an infinitesimal sequence of radii (ρ_k) such that $\rho_0 = \rho$, $\rho_{k+1}/\rho_k \in \{\beta, \tau_1, \tau_2\}$ and

$$E(\rho_k) \le \max\left\{\left(\frac{\rho_k}{\rho}\right)^\sigma E(\rho), \frac{L}{\beta^\sigma}\rho_k^\sigma\right\}. \tag{6.6}$$

Clearly, (6.6) holds with $k = 0$. Assume that the inequality is valid for ρ_k; if

$$E(\beta\rho_k) \le L\beta^{-\sigma}(\beta\rho_k)^\sigma$$

we choose $\rho_{k+1} = \beta\rho_k$. Otherwise, $E(\beta\rho_k) \ge L\rho_k^\sigma$ and Lemma 6.1 yields that (6.6) is valid either with $\rho_{k+1} = \tau_1\rho_k$ or with $\rho_{k+1} = \tau_2\rho_k$.

Given $r \in (0,\rho)$ there exists a unique k such that $\rho_{k+1} \le r < \rho_k$; since $\tau_1 = \beta/25 < \beta < \tau_2 = \beta^{1/3}$ we have

$$\begin{aligned} E(r) &\le \left(\frac{\rho_k}{r}\right)^{m+2} E(\rho_k) \le \left(\frac{25}{\beta}\right)^{m+2} E(\rho_k)\\ &\le \left(\frac{25}{\beta}\right)^{m+2} \max\left\{\left(\frac{\rho_k}{\rho}\right)^\sigma E(\rho), L\beta^{-\sigma}\rho_k^\sigma\right\}\\ &\le \left(\frac{25}{\beta}\right)^{m+2+\sigma} \max\left\{\left(\frac{r}{\rho}\right)^\sigma E(\rho), L\beta^{-\sigma}r^\sigma\right\} \end{aligned}$$

and the statement follows. □

CONCLUSION OF THE PROOF. Let ε be given by Corollary 6.2. We need only to choose ε_0 so small that (3.2) implies

$$\left[\eta\frac{\mathbb{D}(y,\rho/2)}{(\rho/2)^m}+\frac{\mathbb{A}(y,\rho/2)}{(\rho/2)^{m+2}}\right]<\varepsilon \qquad \forall y\in B_{\rho/2}(x).$$

By Corollary 6.2 we obtain that the assumptions of Theorem 3.3 are satisfied in $\Omega = B_{\rho/2}(x)$.

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