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Partial Spatial Equilibria with Fuzzy Constraints

Claude PONSARD

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PARTIAL SPATIAL EQUILIBRIA WITH FUZZY CONSTRAINTS*

0 - Introduction

0.1. It is implicitly accepted by spatial economic analysis that the economic behaviour of agents located in given spaces (market areas, regions, etc.) is precise, that is to say, their behaviour is such that a possible action (consumption, production) is, or is not, preferable to another. In other words, economic agents are assumed to make accurate economic calculations and optimise the objective functions under strict constraints of resource limitation. These objective functions have clearly defined arguments and well-controlled parameters.

0.2. Generally, however, the economic spaces in which agents live are not transparent and the information available to agents is incomplete, imperfect and more or less accurate. This information has varying degrees of credibility, thus its interpretation calls for caution. The more complex a space (for instance an urban space) the less well it is known.

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Moreover, even though an economic agent is relatively well-informed about his spatial environment, he is nevertheless a human being, not a robot. He pursues objectives which are not always rigorously formulated and which sometimes prove to be incompatible or contradictory. Similarly, he appraises imperfectly the constraints which limit his resources and he does not always saturate them strictly.

0.3. While admitting that the behaviour of economic agents living in more or less opaque spaces is imprecise (fuzzy) the analyst must still answer the following questions. Can the description of fuzzy spatial behaviours rely on a suitable and coherent model of economic calculation? Has this description at its disposal specific, novel and sufficiently appropriate mathematical instruments? Existing studies using the theory of fuzzy subsets (1) have to some extent answered these questions. Having formulated a theory of imprecise preferences and constructed a fuzzy utility function [7] first the consumer's spatial equilibrium [9] then the producer's [10] were described in an imprecise context. At the same time these theories of fuzzy spatial behaviour of the consumer and producer were shown to be particular specificationsof a general behaviour model in which the objective and the constraint are both fuzzy [8] .

These approaches have used a F.M.P. (Fuzzy Mathematical Programming), whose the theory was first elaborated by H.Tanaka, T.Okuda and K.Asai [14] .

---

(1) A fuzzy subset A of a referential $\underline{E} = \{x\}$ is formed by the images of the elements x which take their values in a preordered set $\underline{M}$, with Card $\underline{M} \geqslant 2$, by an application μ defined on $\underline{E}$ and with values in $\underline{M}$.
The application $\mu_A(x)$ expresses the degree of membership of the element x to the fuzzy subset A of $\underline{E}$. It is often assumed that M = [0,1] .
The theory of fuzzy subsets was first presented by L.A.Zadeh [16] .
Any reader wishing to go into the subject further might consult the following works given in the bibliography [2] [3] [4] [6] [11] [17] as well as the international journal under reference number [15] .

0.4. In certain cases, only the constraint is fuzzy, whereas the objective is precise. The consumer maximises a traditional utility function under a fuzzy budget constraint . The producer maximises a profit function under an elastic technological constraint.

The aim of the present paper is to provide a solution to this particular type of fuzzy economic calculation by applying a specific method to it. Indeed, D.Ralescu has established that a F.M.P. could be reduced to the solution of a fuzzy integral, called a Sugeno's integral, on the assumption that the objective function is not fuzzy [12] . This proof supplies an interesting result not only for pure mathematics but also for numerical calculus. It follows that this approach provides spatial analysis with an efficient instrument for the modelization of human behaviour and the analysis of partial spatial equilibria in a context where only constraints are fuzzy.

0.5. Remark. To avoid any ambiguity in the notation of mathematical terms, ordinary (non fuzzy) concepts are underlined, whereas fuzzy concepts are not. For example, $A \subset \underline{E}$ is read: A is a fuzzy subset of the referential $\underline{E}$.

1. A General Model of Spatial Behaviour With a Fuzzy Constraint

1.1. Let us assume an economic agent, a consumer or a producer, who pursues his own particular objectives in the space where he is located.

A set of actions (or alternatives) in this space is denoted by $\underline{E} = \left\{x_i^j\right\}$ where $i \in \underline{I}$, $\underline{I}$ finite or not, designates the nature of the actions and where $j \in \underline{J}$, $\underline{J}$ finite or not, stands for the places where these actions can be carried out.

In the description of the general model that follows, the notation can be simplified without causing any ambiguity by assuming: $\underline{E} = \left\{x\right\}$.

The economic agent formulates a precise objective on the elements of $\underline{E}$ and must take into account a fuzzy constraint, ie, a constraint which is "more or less" strict, on the elements of the same set $\underline{E}$ (1).

An objective is a subset of $\underline{E}$, denoted by $\underline{F}$, defined by a function f on $\underline{E}$ such that:

$$f: \underline{E} \longmapsto [0, +\infty[$$
$$\forall x \in \underline{E},\ x \longmapsto f(x) \in [0, +\infty[$$

where f(x) measures the objective under consideration.

A fuzzy constraint is a fuzzy subset of $\underline{E}$, denoted by C, defined by its membership function μ_C such that:

$$\mu_C: \underline{E} \longmapsto [0,1]$$
$$\forall x \in \underline{E},\ x \longmapsto \mu_C(x) \in [0,1]$$

where $\mu_C(x)$ expresses the degree of satisfaction with the fuzzy constraint.

1.2. The problem consists in maximising the function f on the fuzzy subset C, that is to determine:

$$\sup_{x \in C} f(x) = \sup_{x \in \underline{E}} [f(x) \wedge \mu_C(x)]$$

This relation means that the best possible alternative is the maximal element of the intersection of the subset $\underline{F}$ and the fuzzy subset C (2).

We prove that this problem can be put in the following form (3):

$$\sup_{x \in C} f(x) = \sup_{\alpha \in [0,1]} [\alpha \wedge \sup_{x \in \underline{C}_\alpha} f(x)]$$

where $\underline{C}_\alpha$, with $\alpha \in [0,1]$, designates the α-cut of the fuzzy subset C, that is to say the (non fuzzy) set such that:

(1) We prove that there is no loss of generality by handling this simple case in which there is only one constraint. When there are several constraints, all one has to do is to consider their intersections [6].
(2) In fuzzy algebra the intersection operator (∩) is the min (denoted by Λ) and that of the union (U) is the max (denoted by V).
(3) Cf.Appendix 5.1. and 5.2.

$$\underline{C}_\alpha = \left\{ x;\ x \in \underline{E} :\ \mu_C(x) \geqslant \alpha \right\} .$$

The solution to this problem can be reduced to the solution of a Sugeno's fuzzy integral [1] [12] [13] .

For that purpose, we first need a definition of a fuzzy measure (or valuation).

Let $\underline{\mathfrak{F}}$ be a monotone family of (non-fuzzy) subsets of $\underline{E}$ with the following properties:

(P.1.) $\emptyset \in \underline{\mathfrak{F}}$; $\underline{E} \in \underline{\mathfrak{F}}$

(P.2.) If $\underline{F}_i \in \underline{\mathfrak{F}}$ and $\{\underline{F}_i\}$ is monotone, then

$$\lim_{i \to \infty} \underline{F}_i \in \underline{\mathfrak{F}}$$

A fuzzy measure in Sugeno meaning, denoted by v, is a function defined on $\underline{\mathfrak{F}}$ and with values in the interval $[0, +\infty[$

$$v : \underline{\mathfrak{F}} \subset \underline{\mathcal{P}}(\underline{E}) \longmapsto [0, +\infty[\qquad \text{such that (1):}$$

(P.3.) $v(\emptyset) = 0$; $v(\underline{E}) = 1$

(P.4.) $(\underline{F}_j \in \underline{\mathfrak{F}}, \underline{F}_k \in \underline{\mathfrak{F}}, \underline{F}_j \subseteq \underline{F}_k) \Rightarrow v(\underline{F}_j) \leqslant v(\underline{F}_k)$

(P.5.) $(\text{If } \underline{F}_i \in \underline{\mathfrak{F}}, \underline{F}_i \subseteq \underline{F}_{i+1}) \Rightarrow v(\bigcup_{i=0}^{\infty} \underline{F}_i) = \lim_{i \to \infty} v(\underline{F}_i)$

We now define the function μ_f on the monotone family $\underline{\mathfrak{F}}$ such that:

$$\mu_f : \underline{\mathfrak{F}} \subset \underline{\mathcal{P}}(\underline{E}) \longmapsto [0, +\infty[$$

$$\forall\, \underline{F} \in \underline{\mathfrak{F}},\ \underline{F} \longmapsto \mu_f(\underline{F}) = \sup_{x \in \underline{F}} f(x)$$

with $\sup_{\emptyset} f(x) = 0$ and $\sup_{x \in \underline{E}} f(x) = 1$

It can be verified that the function μ_f is a fuzzy measure in Sugeno's sense of the word (2).

---

(1) We note that $\underline{\mathfrak{F}}$ is not a σ-algebra on $\underline{E}$. Indeed $\underline{\mathfrak{F}}$ does not have the property of complementation, but only that of monotonicity (P.2.). Moreover, v is not a measure in the traditional sense of the word. The valuation v does not possess the property of additivity, only that of monotonicity in the inclusion (P.4). A measure is a particular case of a fuzzy measure.

(2) Cf. Appendix 5.3.

1.3. μ_f is a Sugeno's fuzzy measure, the fuzzy integral of a measurable function f can be defined:

$$f : \underline{E} \longmapsto [0, +\infty[$$

$$\int_{\underline{E}} f \, d\mu_f = \sup_{\alpha \in [0,1]} [\alpha \wedge \mu_f \{f \geqslant \alpha\}]$$

with $\{f \geqslant \alpha\} = \{x;\ x \in \underline{E}\ /\ f(x) \geqslant \alpha\}$.

This is Sugeno's definition of a fuzzy integral [5] [13] .

1.3.1. We therefore establish the following proposition:

Theorem: If the function $F(\alpha) = \mu_f \{f \geqslant \alpha\}$ is continuous, μ_f being a fuzzy measure, then $\bar{\alpha} \geqslant 0$ exists such that:

$$\int_{\underline{E}} f \, d\mu_f = \mu_f \{f \geqslant \bar{\alpha}\}$$

Indeed let F be a function such that:

$$F : [0, +\infty[\longmapsto [0, +\infty[$$

First we show that we cannot have $F(\alpha) \geqslant \alpha$, $\forall \alpha \in [0,1]$.
Indeed, let $\alpha_1 > \alpha_2 \geqslant 0$. Then:

$$\{x;\ x \in \underline{E}\ /\ f(x) \geqslant \alpha_2\} \supset \{x;\ x \in \underline{E}\ /\ f(x) \geqslant \alpha_1\}$$

and

$\mu_f \{f \geqslant \alpha_2\} > \mu_F \{f \geqslant \alpha_1\}$, because the function μ is increasing (1). Let $F(\alpha_2) > F(\alpha_1)$.

It is therefore not possible to have at once, $\forall \alpha \in [0,1]$:

$$\begin{cases} F(\alpha_2) \geqslant \alpha_2 \\ F(\alpha_1) \geqslant \alpha_1 \\ \alpha_1 > \alpha_2 \\ F(\alpha_2) > F(\alpha_1) \end{cases}$$

---

(1) The function μ is increasing since it is a fuzzy measure [Property (P.4.) of a fuzzy measure].

which proves that $F(\alpha) \geqslant \alpha$, $\forall \alpha \in [0,1]$ is impossible. Consequently, $\alpha_0 \in [0,1] \,/\, F(\alpha_0) \leqslant \alpha_0$ exists.

Let us now assume that $\varphi(\alpha) = F(\alpha) - \alpha$. This function is continuous since F is by assumption continuous. We have:

$$\varphi(0) = F(0) \geqslant 0$$

$$\varphi(\alpha_0) = F(\alpha_0) - \alpha_0 \leqslant 0$$

From the theorem of intermediate values applied to φ , there exists $\bar{\alpha} \geqslant 0$ such that $\varphi(\bar{\alpha}) = 0$, let $F(\bar{\alpha}) = \bar{\alpha}$. Thus $\bar{\alpha}$ is a fixed point.

It remains to prove that:

$$\int_{\underline{E}} f \, d\,\mu_f = \bar{\alpha}$$

Two cases may occur:

- Let $\alpha < \bar{\alpha}$ and $\alpha \geqslant 0$.

Then: $F(\alpha) \geqslant F(\bar{\alpha})$. But, $F(\bar{\alpha}) = \bar{\alpha}$ and $\bar{\alpha} > \alpha$

Hence: $F(\alpha) > \alpha$ and $\alpha \wedge F(\alpha) = \alpha$

$\alpha \wedge F(\alpha) < \bar{\alpha}$

- Let $\alpha \geqslant \bar{\alpha}$

Then: $F(\alpha) \leqslant F(\bar{\alpha})$. But, $F(\bar{\alpha}) = \bar{\alpha}$ and $\bar{\alpha} \leqslant \alpha$

Hence: $F(\alpha) \leqslant \alpha$ and $\alpha \wedge F(\alpha) = F(\alpha)$

$\alpha \wedge F(\alpha) \leqslant \bar{\alpha}$

Thus, in both cases: $\alpha \wedge F(\alpha) \leqslant \bar{\alpha}$, $\forall \alpha \in [0,1]$

Consequently:

$$\sup_{\alpha \in [0,1]} [\alpha \wedge \mu_f \{f \geqslant \alpha\}] \leqslant \bar{\alpha}$$

Since we have:

$$\bar{\alpha} = \bar{\alpha} \wedge F(\bar{\alpha}) \quad \text{and} \quad \bar{\alpha} \wedge F(\bar{\alpha}) \leqslant \sup_{\alpha \in [0,1]} [\alpha \wedge \mu_f \{f \geqslant \alpha\}]$$

it can be verified that the only possible case is:

$$\sup_{\alpha \in [0,1]} [\alpha \wedge \mu_f \{f \geqslant \alpha\}] = \bar{\alpha}$$

[Q.E.D.]

1.3.2. It only remains to apply this fundamental theorem to the function f under consideration, i.e.:

$$f : \underline{E} \longmapsto [0,+\infty[$$

and to the function μ_f.

We derive the following proposition:

Corrolary: If the function $F(\alpha) = \sup_{x \in \underline{C}_\alpha} f(x)$ is continuous, then there exists $\bar{\alpha} \in [0,1]$ such that $\sup_{x \in C} f(x) = \sup_{x \in \underline{C}_{\bar{\alpha}}} f(x)$

Indeed, from the previous theorem, $\bar{\alpha} \in [0,1]$ exists such that:

$$\int_{\underline{E}} f \, d\mu_f = \mu_f \left\{ f \geq \bar{\alpha} \right\}$$

or by definition of a Sugeno's integral:

$$\sup_{\alpha \in [0,1]} [\alpha \wedge \mu_f \left\{ f \geq \alpha \right\}] = \mu_f \left\{ f \geq \bar{\alpha} \right\} = \sup_{x \in \underline{C}_{\bar{\alpha}}} f(x)$$

Now, we have:

$$\sup_{\alpha \in [0,1]} [\alpha \wedge \mu_f \left\{ f \geq \alpha \right\}] = \sup_{x \in \underline{E}} [f(x) \wedge \mu_C(x)] = \sup_{x \in \underline{C}_{\bar{\alpha}}} f(x)$$

Hence the solution:

$$\sup_{x \in C} f(x) = \sup_{x \in \underline{C}_{\bar{\alpha}}} f(x)$$

[Q.E.D.]

1.3.3. The question of which conditions are necessary for a function F to be continuous has been examined thoroughly in the framework of the models using F.M.P. [9] [10] [14] . The same conditions are, of course, valid in the present study.

1.4. Illustration

Let us take a simple example. Let us adopt:

- a set of alternatives : $\underline{E} = [0,3]$
- an objective function : $f(x) = e^{-x}$, $0 \leqslant x \leqslant 3$,

which is monotone and decreasing.

- a fuzzy constraint : $\mu_C(x) = \frac{x^2}{10}$ which is monotone and increasing.

We want to find the value of Sugeno's fuzzy integral:

$$\int_{\underline{E}} f \, d \, \mu_f = \sup_{\alpha \in [0,1]} [\alpha \wedge \mu_f \{f \geqslant \alpha\}]$$

First we need to know the sets $\{f \geqslant \alpha\}$ for $\alpha = 0,\ldots,1$.
We have:

$$\{x \,/\, f(x) \geqslant 0\} = [0, 3.0]$$
$$\{x \,/\, f(x) \geqslant 0.1\} = [0, -\ell n\, 0.1] = [0, 2.30259]$$
$$\{x \,/\, f(x) \geqslant 0.2\} = [0, -\ell n\, 0.2] = [0, 1.60943]$$
$$\{x \,/\, f(x) \geqslant 0.3\} = [0, -\ell n\, 0.3] = [0, 1.20397]$$
$$\{x \,/\, f(x) \geqslant 0.4\} = [0, -\ell n\, 0.4] = [0, 0.91629]$$
$$\{x \,/\, f(x) \geqslant 0.5\} = [0, -\ell n\, 0.5] = [0, 0.69314]$$
$$\{x \,/\, f(x) \geqslant 0.6\} = [0, -\ell n\, 0.6] = [0, 0.51082]$$
$$\{x \,/\, f(x) \geqslant 0.7\} = [0, -\ell n\, 0.7] = [0, 0.35667]$$
$$\{x \,/\, f(x) \geqslant 0.8\} = [0, -\ell n\, 0.8] = [0, 0.22314]$$
$$\{x \,/\, f(x) \geqslant 0.9\} = [0, -\ell n\, 0.9] = [0, 0.10536]$$
$$\{x \,/\, f(x) \geqslant 1.0\} = \emptyset$$

Next we calculate: $\alpha \wedge \mu_f \{f \geqslant \alpha\}$

It follows:

$$0.0 \wedge \frac{(3)^2}{10} = 0.0 \wedge 0.9 = 0.0$$

$$0.1 \wedge \frac{(2.30259)^2}{10} = 0.1 \wedge 0.530192 = 0.1$$

$$0.2 \wedge \frac{(1.60943)^2}{10} = 0.2 \wedge 0.2590264 = 0.2$$

$$0.3 \wedge \frac{(1.20397)^2}{10} = 0.3 \wedge 0.1449543 = 0.1449543$$

There is no need to go on to $\alpha = 0.4$, since the value required lies between 0.2 and 0.1449543.

Let us make an interpolation between $\alpha = 0.2$ and $\alpha = 0.3$. This operation is justified since f(x) and $\mu_C(x)$ are monotone and have variations in opposite directions.
We obtain:

$$\{x \,/\, f(x) \geqslant 0.21\} = [\,0,\ -\ell n\ 0.21\,] = [\,0,\ 1.56065\,]$$
$$\{x \,/\, f(x) \geqslant 0.22\} = [\,0,\ -\ell n\ 0.22\,] = [\,0,\ 1.51413\,]$$
$$\{x \,/\, f(x) \geqslant 0.23\} = [\,0,\ -\ell n\ 0.23\,] = [\,0,\ 1.46968\,]$$

And so on.
We obtain:

$$0.21 \wedge \frac{(1.56065)^2}{10} = 0.21 \wedge 0.243562 = 0.21$$

$$0.22 \wedge \frac{(1.51413)^2}{10} = 0.22 \wedge 0.229258 = 0.22$$

$$0.23 \wedge \frac{(1.46968)^2}{10} = 0.23 \wedge 0.215995 = 0.215995$$

We make another interpolation between $\alpha = 0.22$ and $\alpha = 0.23$.
We obtain:

$$\{x \,/\, f(x) \geqslant 0.221\} = [\,0,\ -\ell n\ 0.221\,] = [\,0,\ 1.50959\,]$$
$$\{x \,/\, f(x) \geqslant 0.222\} = [\,0,\ -\ell n\ 0.222\,] = [\,0,\ 1.50508\,]$$
$$\{x \,/\, f(x) \geqslant 0.223\} = [\,0,\ -\ell n\ 0.223\,] = [\,0,\ 1.50058\,]$$
$$\{x \,/\, f(x) \geqslant 0.224\} = [\,0,\ -\ell n\ 0.224\,] = [\,0,\ 1.49610\,]$$

And son on.
We obtain:

$$0.221 \wedge \frac{(1.50959)^2}{10} = 0.221 \wedge 0.227887 = 0.221$$

$$0.222 \wedge \frac{(1.50508)^2}{10} = 0.222 \wedge 0.226526 = 0.222$$

$$0.223 \wedge \frac{(1.50058)^2}{10} = 0.223 \wedge 0.225175 = 0.223$$

$$0.224 \wedge \frac{(1.49610)^2}{10} = 0.224 \wedge 0.223834 = 0.223834$$

We reiterate for α included between 0.223 and 0.224.

We have:

$$\{x \;/\; f(x) \geqslant 0.2231\} = [\ 0,\ -\ell n\ 0.2231\ ] = [\ 0,\ 1.50014\ ]$$

...

$$\{x \;/\; f(x) \geqslant 0.2239\} = [\ 0,\ -\ell n\ 0.2239\ ] = [\ 0,\ 1.49666\ ]$$

We obtain

$$0.2231 \wedge \frac{(1.50014)^2}{10} = 0.2231 \wedge 0.225041 = 0.2231$$

...

$$0.2239 \wedge \frac{(1.49666)^2}{10} = 0.2239 \wedge 0.223967 = 0.2239$$

Thus the value of the fuzzy integral is equal to 0.22395 with an error of 10^{-4}.

2. Application to the analysis of the consumer's fuzzy spatial equilibrium

2.1. The analysis of the consumer's spatial equilibrium with a fuzzy budget constraint is a particular specification of the general model developed in Section 1.

2.1.1. The consumer's space is characterized by a finite set of localities where consumer goods are supplied. One of these localities where the consumer is implanted is called the demand spot. If the consumer is mobile or if the goods are transportable he can consume, according to his preferences, either where he lives, or at a supply point, or at any locality of his choice. In the latter case we consider that at least one product is supplied at this locality, at a price which might be zero, and enters into consumption (a tourist site, for instance, which supplies one or more market or non-market goods, such as a nature reserve, a picturesque landscape,etc.). If a product is not transportable, the locality of consumption is necessarily determined at its supply point.

The set of all the located consumer goods is denoted by $\underline{X} = \left\{\underline{X}_i^j\right\}$ where the index i stands for the nature of the good, with i = 1,...,n, and where the index j designates its supply point, with j = 1,...,m. The quantities of goods $\underline{X}_i^j$ are expressed by positive real numbers, or zero, denoted by x_i^j. We have $\underline{X} = \mathbb{R}^{mn}$ and a consumption, at a point in the consumer's space, is an element of $\mathbb{R}^{mn}$.

<u>2.1.2.</u> From the set $\underline{X}$, we define the set of all the possible located consumptions, denoted by $\underline{K}$, and assumed to be countable. A vector of $\underline{K}$ describes a complex of located goods. It is denoted by ${}^h k$, with h = 1,...,g,...,ℓ,....

Thus ${}^h k = [\ {}^h x_i^j\ ]$ and $\underline{K} \subset \mathbb{R}^{mn}$.

Using these notations, it is possible to check that $\underline{K}$ is a specification of the set of all possible actions described in paragraph 1.1.

Let $\underline{\mathcal{K}}$ be a monotone family of subsets of $\underline{K}$. Let us assume that it has the properties (P.1.) and (P.2.) described in paragraph 1.2. Indeed $\underline{K}$ is the set on which the consumer must make his choice. Property (P.1.) implies that a monotone family of subsets of $\underline{K}$ necessarily has as elements the empty set (the consumer intends not to consume at a given moment) and the full set, that is, $\underline{K}$ itself (the consumer effects a trade-off between all the possible located consumptions). Property (P.2.) implies that if a sequence of subsets of possible located consumption goods which is monotone is an element of $\underline{\mathcal{K}}$, then the reunion and intersection of its components are the elements of $\underline{\mathcal{K}}$ and the limit of this sequence belongs to $\underline{\mathcal{K}}$.

2.1.3. The consumer's space is bestowed with a given price system which is peculiar to the consumer since it is a C.I.F. price system.

The consumer's location (location of his residence) serves as a point of reference, wherever the consumption is actually physically carried out. If the consumer , located at a point denoted by $j^\star$, consumes at that point, the transport costs equal the costs of bringing the goods from their respective supply points to point $j^\star$. If the consumer prefers to go to a supply point to consume a complex of goods, the transport costs equal the sum of the costs of moving the agent and the costs of bringing the goods to that place. Some of these costs may be zero.

For any goods i, with i=1,...,n, and for any place j, with j=1,...,m, we denote by t_i^j the transport costs associated with the consumption of a unit of the good i made in place j which is effected either at the place $j^\star$ or at the place j, with $j \neq j^\star$.

Let ${}^f p_i^j$ be the F.O.B. price of a unit of the good $\underline{X}_i^j$.

The C.I.F. unit price of this good at the consumption place, denoted by ${}^c p_i^j$, is equal to:

$${}^c p_i^j = {}^f p_i^j + t_i^j \text{ , with } i=1,\ldots,n, \text{ and } j=1,\ldots,m.$$

A C.I.F. price system in the consumer's space, denoted by ${}^c p$, is therefore a point of $\mathbb{R}^{mn}$ such that, for any good i and any place j, the real number ${}^c p_i^j$ is the C.I.F. price of the good at its place of consumption. For a complex of goods ${}^h k \in \underline{\bar{K}}$ and for the spatial C.I.F. price system ${}^c p$, the value of this consumption is by definition equal to:

$${}^c p \,.\, {}^h k = \sum_{i=1}^{n} \sum_{j=1}^{m} {}^c p_i^j \,.\, {}^h x_i^j \,.$$

2.2. The consumer's aim is the best possible satisfaction of his needs.

2.2.1. The consumer compares the complexes of located goods and evaluates them with respect to his preference system. His preferences are assumed to be expressed numerically. More precisely, it is assumed that the conditions of existence and of continuity of a utility function are satisfied.

We have an objective function denoted by f, such that:

$$f : \underline{K} \longmapsto [0, +\infty[$$
$$\forall\, {}^{h}k \in \underline{K},\ {}^{h}k \longmapsto f({}^{h}k) \in [0, +\infty[$$

with the usual properties:

$$\forall ({}^{m}k, {}^{n}k) \in \underline{K}^2,\ {}^{m}k \succ {}^{n}k \iff f({}^{m}k) > f({}^{n}k)$$
$${}^{m}k \sim {}^{n}k \iff f({}^{m}k) = f({}^{n}k)\ .$$

2.2.2. In a static analysis, a consumer has a given budget which corresponds to the value of all the goods in his possession, whatever their respective locations. This wealth is represented by the real number $w \in \mathbb{R}$.

For a set of possible located consumptions $\underline{K} \subset \mathbb{R}^{mn}$, for a C.I.F. price system ${}^{c}p \in \mathbb{R}^{mn}$ and a wealth $w \in \mathbb{R}$, we define a budget set, denoted by $\underline{B}$, $\underline{B} \subset \underline{K}$, by:

$$\underline{B} = \left\{ {}^{h}k;\ \forall\, {}^{h}k \in \underline{K} : {}^{c}p\ .\ {}^{h}k \leqslant w \right\}$$

where ${}^{c}p.{}^{h}k \leqslant w$ designates the budget constraint and implies that the value of a consumption cannot exceed the consumer's wealth.

In classical theory, the optimal consumption maximises the consumer's utility on this budget set. The set of all the possible consumptions is partitioned into two classes: that of the consumptions said to be efficient and that of the non-efficient consumptions. For a given budget constraint, the class of the efficient consumptions is the set of the consumer's technical optima (or efficiency boundary) and the consumer's choice is directed towards the elementsof this set.

However this theory only holds true in the particular case of a precise behaviour. In general, the budget constraint is "more or less" limiting.

Indeed, the consumer does not necessarily have accurate information about his wealth. He cannot draw up an inventory of his wealth every time he chooses a consumption. He does not have all the data necessary for such a calculation and what is more, he doesn't even effect such a precise estimation. Similarly, his knowledge about his C.I.F. price system is incomplete and imperfect. The economic space in which he lives lacks transparence.

Moreover, a consumer who barely earns a living wage is forced to match his consumption with his wealth. However, over and above this living wage, he can effect a trade-off between saving and consumption. The greater his wealth, the more elastic the constraint.

If we admit that there exists an imprecise frontier between "a little more" consumption and "a little less" saving, or the contrary, according to the consumer's preferences and his associated utility function, then we admit that the set of efficient located consumptions is not reduced to the classical efficiency frontier, but that it is a fuzzy subset of the set of all the possible located consumptions. Indeed, any complex of located goods belongs "more or less" to the set of consumptions compatible with the budget constraint.

This idea is translated formally by defining a membership function on the elements of $\underline{K}$ which takes its values in the interval $[0,1]$. This function, denoted by μ_C, is such that:

$$\mu_C : \underline{K} \longmapsto [0,1]$$

$$\forall {}^h k \in \underline{K},\ {}^h k \longmapsto \mu_C({}^h k) \in [0,1]$$

and

$$\mu_C({}^h k) = 1 \quad \text{if} \quad {}^c p.{}^h k = w$$

$$\mu_C({}^h k) = 0 \quad \text{if} \quad {}^c p.{}^h k > w$$

$$\mu_C({}^h k) \in \,]0,1[\quad \text{if} \quad {}^c p.{}^h k < w\ .$$

The function μ_C depends on w. It is monotone and decreasing for the increasing values of w:

$$(w' > w) \Rightarrow [(w' - {}^{c}p \,.\, {}^{h}k) > (w - {}^{c}p \,.\, {}^{h}k)]$$
$$\Rightarrow \mu'_C({}^{h}k) \leqslant \mu_C({}^{h}k)$$

It enables us to construct a fuzzy subset of $\underline{K}$, denoted by C, such that:

$$C = \left\{ {}^{h}k;\ \forall\ {}^{h}k \in \underline{K},\quad \mu_C({}^{h}k) \in [0,1] \right\}.$$

The fuzzy subset C of $\underline{K}$ plays the role of a fuzzy constraint (or elastic constraint) in the consumer's economic calculation.

2.2.3. Let us define the function μ_f on the monotone family $\underline{\mathcal{K}}$ defined in paragraph 2.1.2. such that:

$$\mu_f : \underline{\mathcal{K}} \subset \mathcal{P}(\underline{K}) \longmapsto [0, +\infty[$$
$$\forall \underline{K}_i \in \underline{\mathcal{K}},\quad \underline{K}_i \longmapsto \mu_f(\underline{K}_i) = \sup_{x \in \underline{K}_i} f(x)$$

with $\sup_{\emptyset} f(x) = 0$ and $\sup_{x \in \underline{K}} f(x) = 1$

We have verified that the function μ_f is a Sugeno's fuzzy measure on $\underline{\mathcal{K}}$ (1).

2.2.4. Finally the consumer's choice criterion is to maximise the utility $f({}^{h}k)$ by satisfying as well as possible the fuzzy budget constraint $\mu_C({}^{h}k)$ for any ${}^{h}k$ of $\underline{K}$.

Finding the optimal demand therefore comes down to determining:

$$\sup_{{}^{h}k \in C} f({}^{h}k) = \sup_{{}^{h}k \in \underline{K}} [f({}^{h}k) \wedge \mu_C({}^{h}k)]$$

It suffices to apply the solution developed in paragraphs 1.2. and 1.3.

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(1) Cf. Appendix 5.3.

At equilibrium, the optimal demand is determined and, in the consumer's space, the places to which the consumer directs his particular demands for located goods are known since the optimal complex ${}^{h}k^{\star}$ is equal to the optimal vector $[{}^{h}x_{i}^{j}]^{\star}$ where j is the index of the supply points of the goods indexed i entering the complex indexed h. Finally, the consumer's global volume of saving is determined since he does not saturate his wealth constraint, apart from in particular cases (bare living wage or voluntary zero saving).

3. Application to the analysis of the producer's fuzzy spatial equilibrium

3.1. Similarly, the analysis of the producer's spatial equilibrium with a fuzzy technological constraint is a particular specification of the general model of imprecise behaviour.

3.1.1. The producer's space is characterized by the location of his production unit, by the inputs supply space and the outputs demand space.

The location and technical dimension of the production unit are given in one place indexed $j^{\star}$ which is the point of reference from which the spatial price system is determined.

The production process requires a set of inputs, denoted by $\underline{Y}_{i}^{j}$, where the index i designates their nature, with $i = 1,\ldots,n$, and the index j stands for the places where they are supplied,with $j = 1,\ldots,p$. This process generates a set of outputs, denoted by $\underline{Y}_{i}^{j}$ with $i=n+1,\ldots,m$, which is the index designating the nature of the goods produced, and with $j=p+1,\ldots,q$, which is the index standing for the places where they are demanded.Using these notations, this set $\underline{Y}_{i}^{j}$ has as elements (m-n) outputs demanded at (q-p) places.

The set of located goods peculiar to the producer (inputs and outputs) is denoted by $\underline{Y}$, with $\underline{Y} = \{\underline{Y}_i^j\}$, i=1,...,m and j=1,...,q. Hence the goods space peculiar to the producer has: np + (m-n)(q-p) = n(2p-q) + m(q-p) = k coordinates; by convention let us assume that a single place supplying (respectively demanding) more than one input (resp.more than one output) is represented by the same number of points as distinct goods being supplied (demanded). We have $\underline{Y} = \mathbb{R}^k$.

As is customary, input quantities are represented by negative real numbers and output quantities by positive real numbers. Thus a production, denoted by y, with $y = (y_i^j)$, i = 1,...,m and j = 1,...,q, is represented by a point of $\mathbb{R}^k$.

<u>3.1.2.</u> The set $\underline{Y}$ is partitioned into a subset of all the technically possible productions, denoted by $\underline{Z}$, and its complement in $\underline{Y}$. An element y of $\underline{Z}$ is called a producer's supply.

This set $\underline{Z}$ is a specification of the set of all possible actions described in paragraph 1.1.

Let $\underline{\mathcal{Q}}$ be a monotone family of subsets of $\underline{Z}$. It has properties (P.1.) and (P.2.) described in paragraph 1.2. Property (P.1.) implies firstly the possibility of inaction ($\emptyset \in \underline{\mathcal{Q}}$) and secondly the existence of a trade off between all the possible productions ($\underline{Z} \in \underline{\mathcal{Q}}$). Property (P.2.) implies that if a monotone sequence of the subsets of all the possible productions is an element of $\underline{\mathcal{Q}}$, then the limit of this sequence belongs to $\underline{\mathcal{Q}}$.

<u>3.1.3.</u> The spatial price system peculiar to the producer is given. It depends on the location of the production unit and the distances separating it from the input supply places and the output demand places.

Let ${}^f p_i^j$, with i=1,...,n, and j=1,...,p, be the F.O.B. unit prices of the inputs y_i^j supplied at the places indexed j. The unit transport prices of these inputs to the production

place $j^\star$ are denoted by $t_i^{j \rightarrow j\star}$. The C.I.F. unit prices of inputs sent back to the point $j^\star$, denoted by ${}^c p_i^j$, with $i=1,\ldots,n$, and $j=1,\ldots,p$, are therefore equal to:
${}^c p_i^j = {}^f p_i^j + t_i^{j \rightarrow j\star}$.

Now let, ${}^f p_i^j$, with $i=n+1,\ldots,m$, and $j = j^\star$, be the F.O.B. unit prices of the outputs at the place of production. The unit transport prices of taking these outputs from the place indexed $j^\star$ to the demand places indexed j are denoted by $t_i^{j\star \rightarrow j}$. The C.I.F. unit prices of the outputs delivered to points j, denoted by ${}^c p_i^j$, are equal to:

$${}^c p_i^j = {}^f p_i^j + t_i^{j\star \rightarrow j} \quad \text{with } i=n+1,\ldots,m, \text{ and } j=p+1,\ldots,q.$$

A real number representing its price to the producer is associated with each element of the set of located goods peculiar to the producer. ${}^c p$ denotes a C.I.F. price system in the producer's space.

3.2. The producer's aim is to make the maximum profit.

3.2.1. Given a production y in $\underline{Z}$ and a spatial price system ${}^c p$ peculiar to the producer, then his profit, denoted by $\underline{P}$, is by definition the internal product ${}^c py$ such that:

$$\underline{P} = {}^c py = \sum_{i=n+1}^{m} \sum_{j=p+1}^{q} {}^f p_i^j \, y_i^j - \sum_{i=1}^{n} \sum_{j=1}^{p} {}^c p_i^j \, y_i^j$$

The producer must choose a distribution of his located inputs and outputs which maximises $\underline{P}$ under a fuzzy technological constraint. This production, called equilibrium production, is the producer's optimal supply with respect to the spatial price system.

Since ${}^c p$ is given and constant (for given $j^\star$) the profit $\underline{P}$ only depends on y.

We have an objective function, denoted by f, such that:

$$f : \underline{Z} \longmapsto [0, +\infty[$$
$$\forall y \in \underline{Z},\ y \longmapsto f(y) \in [0, +\infty[$$

which is monotone and increasing for the increasing values of y.

<u>3.2.2.</u> The classical theory of the producer partitions the set of all possible productions $\underline{Z}$ into two classes (for a given technological constraint): (a) production said to be efficient and (b) production said to be inefficient. However this theory only holds in the special case where all inputs display maximal technical efficiency and alone determine the quantities of outputs obtained.

Generally however, the efficiency of an input is relative and depends on several factors which are linked to its state and the conditions of its use. These factors are not always measurable. Hence each input's contribution to a product only corresponds to maximal technical norms in very exceptional circumstances.

Moreover, free (non-economic) factors, uncontrollable and fixed factors also influence production so that well-controlled inputs do not alone determine the quantity of outputs.

It follows that any technically possible production is more or less efficient. Instead of partitioning the set of all possible productions, we define the set of "more or less" efficient productions as a fuzzy subset of $\underline{Z}$, denoted by C. Its elements have a membership function, denoted by μ_C, such that:

$$\mu_C : \underline{Z} \longmapsto [0,1]$$
$$\forall y \in \underline{Z},\ y \longmapsto \mu_C(y) \in [0,1]$$

and $\mu_C(y) = 0$ in the assumption of the production of waste

$\mu_C(y) = 1$ if the efficiency of inputs is maximal and if all inputs are well-controlled.

$\mu_C(y) \in \,]0,1[$ in all other cases.

This function enables us to construct a fuzzy subset C of $\underline{Z}$ which is such that:

$$C = \left\{ y;\ \forall\, y \in \underline{Z},\ \mu_C(y) \in [0,1] \right\}$$

This fuzzy subset C of $\underline{Z}$ plays the role of a fuzzy constraint in the producer's economic calculation.

<u>3.2.3.</u> We define the function μ_f on the monotone family $\underline{\mathcal{Z}}$ defined in paragraph 3.1.2. such that:

$$\mu_f : \underline{\mathcal{Z}} \subset \underline{\mathcal{P}}(\underline{Z}) \longmapsto [0, +\infty[$$

$$\forall \underline{Z}_i \in \underline{\mathcal{Z}},\ \underline{Z}_i \longmapsto \mu_f(\underline{Z}_i) = \sup_{y \in \underline{Z}_i} f(y)$$

with $\sup_{\emptyset} f(y) = 0$ and $\sup_{x \in \underline{Z}} f(y) = 1$.

This function is a Sugeno's fuzzy measure on $\underline{\mathcal{Z}}$ (1).

<u>3.2.4.</u> Finally, determining the producer's optimal supply consists in finding:

$$\sup_{y \in C} f(y) = \sup_{y \in \underline{Z}} [f(y) \wedge \mu_C(y)]$$

The solution developed in paragraphs 1.2. and 1.3. holds true.

At equilibrium, the optimal supply is determined and, in the producer's space, the selling places of the outputs and the quantities supplied are known, as well as the places where inputs are purchased and the quantities demanded. Indeed an equilibrium supply is an element $(y_i^j)^*$ of the set $\underline{Y}$, with i=1,...,m, and j=1,...,q, where i=1,...,n, designates the inputs and j=1,...,p, their supply points, and where i=n+1,...,m, stands for the outputs and j=p+1,...,q, their demand points.

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(1) Cf.Appendix <u>5.3.</u>

4. CONCLUSION

4.1. A strict constraint is the limiting case of an elastic constraint; it is not its antithesis. Consequently, the theory of partial spatial equilibria with fuzzy constraints should not be contrasted with the classical theory. The former embodies the latter as a particular case.

4.2. However, interesting though it may be, this aptitude for generality is not the principal quality of this approach. It should be emphasized that this theory contributes new results in that it modifies the description of behaviours and partial spatial equilibria. In particular, the fact that these equilibria result from satisfactory trade-offs between what is desirable and what is "more or less" possible, means that the properties of their optimality are put into perspective.

In addition, the theoretical analysis gains in realism without any loss of rigour. Its operational character is obvious.

4.3. Research should now turn to the elaboration of a theory of general spatial equilibrium in a fuzzy context. Such a task seems difficult. Not only must the convenient assumptions of perfect competition be discarded as soon as the spatial factor is introduced, but also the introduction of fuzziness implies prior creation of new mathematical tools which have proved so necessary to the elaboration of this theory.

5. APPENDIX

5.1. α-cuts of a fuzzy subset of a referential.

5.1.1. Definition. Let a referential $\underline{E} = \{x\}$, and let C be a fuzzy subset of $\underline{E}$. We call α-cut of C, denoted by $\underline{C}_\alpha$ with $\alpha \in [0,1]$ the non fuzzy set such that:

$$\underline{C}_\alpha = \left\{ x;\ \forall x \in \underline{E} : \mu_C(x) \geqslant \alpha \right\}$$

where $\mu_C(x)$ is the characteristic membership function of the element x to the fuzzy subset C which takes its values in the interval $[0,1]$.

5.1.2. Property. The set of α-cuts, denoted by $\left\{\underline{C}_\alpha\right\}_{\alpha\in[0,1]}$ is a decreasing sequence such that:

$$\forall(\alpha_1,\alpha_2) \in [0,1]^2 : \alpha_1 \leqslant \alpha_2 \Rightarrow \underline{C}_{\alpha_1} \supseteq C_{\alpha_2} \quad \text{and } \underline{C}_o = \underline{E}$$

(immediate)

5.1.3. Decomposition Theorem. Let $C \in \mathcal{P}(\underline{E})$, $\mathcal{P}(\underline{E})$ being the fuzzy power set of $\underline{E}$, and $\left\{\underline{C}_\alpha\right\}_{\alpha\in[0,1]}$ its α-cuts.

Hence: $C = \bigcup_{\alpha\in[0,1]} \alpha \cdot \underline{C}_\alpha$

Indeed, we have $\mu_{\underline{C}_{\alpha_i}}(x) = 1 \quad \text{if} \quad \mu_C(x) \geqslant \alpha_i$

$$\mu_{\underline{C}_{\alpha_i}}(x) = 0 \quad \text{if} \quad \mu_C(x) < \alpha_i$$

Hence the membership function of C is written:

$$\mu_{\bigvee_{\alpha_i}[\alpha_i \cdot \underline{C}_{\alpha_i}]}(x) = \bigvee_{\alpha_i} [\alpha_i \cdot \underline{C}_{\alpha_i}] = \bigvee_{\mu_C(x) \geqslant \alpha_i} [\alpha_i] = \mu_C(x)$$

[Q.E.D.]

5.2. Theorem. The following relation holds true:

$$\sup_{x \in C} f(x) = \sup_{\alpha\in[0,1]} [\alpha \wedge \sup_{x \in \underline{C}_\alpha} f(x)]$$

Indeed, from the theorem 5.1.3., we have:

$$\mu_C(x) = \bigvee_{\alpha\in[0,1]} [\alpha \cdot \underline{C}_\alpha] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu_{\underline{C}_\alpha}(x)]$$

By definition, we have:

$$\sup_{x \in C} f(x) = \sup_{x \in \underline{E}} [f(x) \wedge \mu_C(x)]$$

Whence it follows:

$$\sup_{x \in C} f(x) = \sup_{x \in \underline{E}} \Big[f(x) \wedge \big[\bigvee_{\alpha\in[0,1]} (\alpha \wedge \mu_{\underline{C}_\alpha}(x))\big]\Big]$$

$$= \sup_{x \in \underline{E}} [\bigvee_{\alpha\in[0,1]} (\alpha \wedge \mu_{\underline{C}_\alpha}(x) \wedge f(x))]$$

(distributivity of V with respect to Λ)

Therefore we can write:

$$\sup_{x \in C} f(x) = \bigvee_{x \in \underline{E}} \bigvee_{\alpha\in[0,1]} (\alpha \wedge \mu_{\underline{C}_\alpha}(x) \wedge f(x))$$

$$= \bigvee_{\alpha\in[0,1]} \bigvee_{x \in \underline{E}} (\alpha \wedge \mu_{\underline{C}_\alpha}(x) \wedge f(x))$$

(properties of associativity and commutativity)

$$= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \bigvee_{x\in\underline{E}} (\mu_{\underline{C}_\alpha}(x) \wedge f(x))]$$

(since α is independent of x)

By partitioning $\underline{C}_\alpha$, we obtain:

$$\bigvee_{x \in \underline{E}} [\mu_{\underline{C}_\alpha}(x) \wedge f(x)] = \bigvee_{x \in \underline{C}_\alpha} [\mu_{\underline{C}_\alpha}(x) \wedge f(x)] \vee \bigvee_{x \notin \underline{C}_\alpha} [\mu_{\underline{C}_\alpha}(x) \wedge f(x)]$$

$$= \bigvee_{x \in \underline{C}_\alpha} [1 \wedge f(x)] \vee \bigvee_{x \notin \underline{C}_\alpha} [0 \wedge f(x)]$$

$$= \bigvee_{x \in \underline{C}_\alpha} f(x) = \sup_{x \in \underline{C}_\alpha} f(x)$$

We can therefore write:

$$\sup_{x \in C} f(x) = \sup_{\alpha \in [0,1]} [\alpha \wedge \sup_{x \in \underline{C}_\alpha} f(x)]$$

[Q.E.D.]

5.3. Theorem. Let μ_f denote the function defined in paragraph 1.2. This function is a Sugeno's fuzzy measure.
Indeed:

(1) $\mu_f(\emptyset) = 0$; $\mu_f(\underline{E}) = 1$ by construction

(2) Let $\underline{F}_j \subseteq \underline{F}_k$; $\forall x \in \underline{F}_j$, therefore $x \in \underline{F}_k$

Therefore: $\sup_{x \in \underline{F}_j} f(x)$ must be such that the element x belongs to $\underline{F}_k$.
From whence it follows: $\sup_{x \in \underline{F}_k} f(x) \geqslant \sup_{x \in \underline{F}_j} f(x)$

let : $\mu_f(\underline{F}_k) \geqslant \mu_f(\underline{F}_j)$.

(3) Let us assume that $\underline{F} = \bigcup_{i=1}^{\infty} \underline{F}_i$

From (2) : $\mu_f(\underline{F}_1) \leqslant \mu_f(\underline{F}_2) \leqslant \ldots \leqslant \mu_f(\underline{F})$.

The sequence $\mu_f(\underline{F}_i)$ is a monotone increasing and upperly bounded sequence. It is therefore converging to the limit $\sup_{i \geqslant 0} \mu_f(\underline{F}_i)$ and $\sup_{i \geqslant 0} \mu_f(\underline{F}_i) \leqslant \mu_f(\underline{F})$ from the properties of limits and of sup.

We use a reductio ad absurbum proof.
Let us assume strict inequality: $\sup_{i \geqslant 0} \mu_f(\underline{F}_i) < \mu_f(\underline{F})$

By definition: $\mu_f(\underline{F}) = \sup_{x \in \underline{F}} f(x)$
Therefore, $\exists\, x_o \in \underline{F}$ such that $\mu_f(\underline{F}) = f(x_o)$,
or that: $\sup_{i \geqslant 0} \mu_f(\underline{F}_i) < f(x_o)$
Now $\underline{F} = \bigcup_{i=1}^{\infty} \underline{F}_i$. Therefore, $\exists\, j \in I$ such that $x_o \in \underline{F}_j$.

Consequently $\exists\, j \in J$ such that $\mu_f(\underline{F}_j) \geqslant f(x_o)$

Now $\sup_{i \geqslant 0} \mu_f(\underline{F}_i) < f(x_o)$. Whence the contradiction.

Thus we have: $\lim_{i \to \infty} \mu_f(\underline{F}_i) = \sup_{i \geqslant 0} \mu_f(\underline{F}_i) = \mu_f(\underline{F})$.

[Q.E.D.]

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