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«Partial Truth, Fringes and Motion: Three Applications of a Contradictorial Logic»

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PARTIAL TRUTH, FRINGES AND MOTION: THREE APPLICATIONS OF A CONTRADICTORIAL LOGIC

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§1.-Dialectics and the existence of contradictory truths

As conceived of by Hegel, dialectics belongs to negative reason which, as against the dogmatic onesidedness characterizing **reflection** or **understanding**, puts forward the emergence of contradictions, thus dissolving the apparently hard and solid constructions of understanding. Positive reason instead claims as its own task that of sublating both understanding and the dialectical criticism thereof by positing the contradictions as both true and not true, on a higher level wherein both moments are at the same time kept and yet overcome or cancelled.

The foregoing is of course the traditional mainstream construal of Hegel's views on contradiction. A number of authors have challenged such an interpretation, trying to show that Hegel did not in fact assert the existence of contradictory truths in any straightforward sense of the word.[1]

A similar hermeneutic controversy concerns the thought of the founders of Marxism, esp. Engels's and Lenin's. Both of them seem to maintain the thesis that there are contradictory truths. Still, a number of interpreters argue that what they call 'contradiction' is no contradiction in the logical sense of the word. It has been alleged that mistaking dialectical contradictions for logical ones was a confusion pertaining to the vulgarized "DIAMAT" of the Stalinist era.

I refrain from going into those exegetical issues in this paper. Frankly, my belief is that, as in most if not all cases, the traditional construal is the best one. Charity is more often than not misapplied or overstretched in philosophical interpretations. This is likely to be a matter of opinion, but I think it better to construe any philosophical **corpus** as literally as possible, if only because by so doing we grant it a greater interest. Charitable interpretations succeed in rendering history of philosophy dull, with a Parmenides who would not have denied

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[1]. Among them are the following: Josef Simon, *Das Problem der Sprache bei Hegel*. Stuttgart: W. Kohlhammer V., 1966; Michael Wolff, *Der Begriff des Widerspruchs: Eine Studie zur Dialektik Kants und Hegels*, Königstein: Hain, 1981; Rüdiger Bubner, *Zur Sache der Dialektik*, Stuttgart: Philipp Reclam jun., 1980; Franz Grégoire, *Etudes hégéliennes*, Louvain-Paris: Nauwelaerts, 1958. My own (traditional) more literal construal coincides with those of Nicolai Hartmann, Heinz Heimsoeth, or, among marxists, Georg Lukács (at least as I understand him) and Jacques D'Hondt.

the existence of plurality, a Berkeley who would not have denied that of bodies and a Hegel who would have said nothing falling afoul of Aristotelian logic.

Be it as it may, I have already, in a number of previous papers, dealt with problems of Hegelian and Engelsian interpretation.[2] Thus I hope I can afford to skirt those controversial issues in this paper. Anyway I want to point out that, if by 'dialectical' we mean any conception asserting the existence of contradictory truths, then the dialectical tradition can be cogently argued to be much, much wider than is usually supposed, comprising — as I've tried to show elsewhere[3] — thinkers such as Heracleitus, Plato himself (not just in the *Parmenides* and *Sophist* dialogues but even in the *Phaedo, Republic* etc.), Ænesidemus, most — if not all — neoPlatonists including St Augustine and, most of all, Nicholas de Cusa, with his copulative theology. And nowadays there are outside Marxism at least two philosophical conceptions which claim to be dialectical in that very same sense, namely the neoenergetism proposed by Stephane Lupasco and Marc Beigbeder[4] and the **ontophantic** system of metaphysics and epistemology proposed by the present writer[5]. Furthermore, some philosophers who, needless

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[2]. My contribution to Hegelian interpretation is put forward in «Dialéctica, lógica y formalización: de Hegel a la filosofía analítica», *Cuadernos Salmantinos de Filosofía*, vol. XII (1986), pp. 149-71. As for problems related to how to understand Engels on dialectical contradictions, see «Engels y las nuevas perspectivas de la lógica dialéctica», ap. *Estudios sobre Filosofía moderna y contemporánea* (ed. by Isabel Lafuente). León: CEMI (University of León), 1984, pp. 163-218. See also: *Formalización y lógica dialéctica*, prepublished, Quito: Pontifical University, 1980.; «Negación dialéctica y lógica transitiva», *Crítica* [México], Nº 43 (april 1983), pp. 51-77.; «Significación filosófica de la lógica transitiva», *Ideas y Valores* [Bogotá], Nº 63 (dec. 1983), pp. 59-101. About the evolution of the notion of **dialectics** and the relationship between traditional dialectics and the kind of approach proposed in [Sects. 2&3 of] this paper, see the article «Dialectics and Inconsistency», apud ***Handbook of Metaphysics and Ontology***, ed. by H. Burkhardt & Barry Smith, Munich: Philosophia Verlag, 1991, pp 216-8.

[3]. On Plato, see my paper «Dos sentidos de la preposición 'προς' en algunos pasajes de Platón», *Estudios Humanísticos - Filología*, Nº 8 (1986), pp. 39-58, and «El tratamiento de los comparativos en el FEDON» scheduled to appear in *Nova Tellus*. That St. Augustine, too, is at least committed to espouse the existence of contradictory truths and the relative falseness of the Aristotelian principle of noncontradiction I've tried to show in my two papers «El significado de 'nihil' en diversos escritos de S. Agustín», *Estudios Humanísticos*, Nº 9 (1987), pp. 155-68, and «La identificación agustiniana de verdad y existencia: una defensa filosófica», *La ciudad de Dios* CCII/1 (january-april 1989), pp. 149-72. On Nicholas de Cusa, see my 3 papers: «Au-delà de la coïncidence des opposés: Remarques sur la théologie coupative chez Nicholas de Cuse», *Revue de Théologie et de Philosophie,* 121 (1989), pp. 57-78; «La superación de la lógica aristotélica en el pensamiento del Cusano», *La ciudad de Dios* (1988), pp. 573-98; «La concepción de Dios en la filosofía del Cardenal Nicolás de Cusa», *Revista de la Universidad Católica* (Quito, Ecuador), Nº 47 (1987), pp. 301-28.

[4]. See: Stephane Lupasco, *Logique et contradiction*, Paris: PUF, 1947, and *Du devenir logique et de l'affectivité*, Paris: Vrin, 1973 (2 vols., 2d. edit.: the 1st edit. appeared in 1935); and Marc Beigbeder, *Contradiction et nouvel entendement*, Paris: Bordas, 1972.

[5]. That metaphysical system, which also comprises a treatment of some main issues of philosophical theology, has been proposed in my 3 books: *La coincidencia de los opuestos en Dios*, Quito: Educ, 1981; *El ente y su ser: un estudio lógico-metafísico*. León: Universidad de León, 1985; *Fundamentos de ontología dialéctica*. Madrid: Siglo XXI, 1987; and in a number of papers dealing with epistemological issues, among which are my Belgian Ph.D. diss. *Contradiction et vérité: étude sur les fondements et la portée épistémologique d'une logique contradictorielle* (Liège, 1979), «Conocimiento y justificación epistémica», *Revista de la Universidad Católica*, Nº 28 (nov. 1980), pp. 35-67, and «Naturalized Epistemology and Degrees of Knowledge» (paper delivered to an international Conference held in Tepoztlán, México, in aug. 1988). More accessible presentations of some parts and motivations of the system are to be found in «Verum et ens conuertuntur: The Identity between Truth and Existence within the Framework of a Contradictorial Modal Set-Theory», ap. *Paraconsistent Logic* (ed. by G. Priest, R. Routley & J. Norman), Munich: Philosophia Verlag, 1989, pp. 563-602; «Identity, Fuzziness and Noncontradiction», *Noûs*, XVIII/2 (may 1984), pp. 227-59, «Dialectical Arguments, Matters of Degree, and Paraconsistent Logic», ap. *Argumentation: Perspectives and Approaches* (ed. by F.H. van Eemeren *et al,* Dordrecht: Foris Publications, 1987, pp. 426-33. The ontophantic approach is now being extended to other philosophical fields, as ethics (see, e.g., my papers «El conflicto de valores: Reflexión desde una perspectiva lógico-filosófica», ap. *Crisis de valores*

to say, would have balk at the idea of contradictory truths can nonetheless be cogently argued to be committed to it all the same. Thus, even if Leibniz held the principle of noncontradiction to be undeniable, he espoused a principle of continuity which in fact yields contradictions.[6] And, of course, outside philosophy proper there are lots of writers, esp. poets, who literally assert contradictions and say that the world is contradictory. I have argued elsewhere that more often than not they can be taken to mean what they say since not every contradiction must perforce be absurd[7].

Hence, there is no shortage of philosophical and nonphilosophical approaches which seem to countenance the existence of true contradictions, i.e. which can be termed 'dialectical' in a straightforward sense of the word which can be claimed to stem from Hegel's work. Whether each of them does in fact commit itself to a contradictory reality or not is a matter for interpreters to discuss, but anyway there are reasonable grounds for assigning them such a commitment. Yet my present concern is not an exegetical one but only that of showing that we can make good sense of the idea of contradictory truths in at least three fields, namely those of partial truth, fringes of application of sundry predicates, and motion. All those three fields have of course been claimed to be amenable to contradictorial treatments by the dialectical tradition, but, since I want to keep clear of interpretive matters here, I'll abstain from commenting on any previous attempt at a dialectical treatment of those issues. To my own proposal on them is the remaining of this paper to be given over.

Let me conclude this Section by pointing that the formal system which is sketched out below has already been presented in similar if not quite identical ways in some previous papers, which however are not easily accessible[8].

§2.- **An infinite-valued paraconsistent approach**

§2.1. *Paraconsistent logics*

Paraconsistent logics are by now gaining recognition in the community. There are a number of alternative ways of laying down requirements for acceptable paraconsistency. The discrepancy among the several paraconsistent schools hinges mainly upon what is regarded as required for a functor to qualify as a negation. My own requirements are not uncontroversial. Still, I think they are highly plausible.

(ed. by J. González López). Quito: Educ, 1982, pp. 133-62, and «Un enfoque no clásico de varias antinomias deónticas», *Theoria* (S. Sebastián) Nº 7-8-9 (1988), pp. 67-94), so as to become a comprehensive account of all major problems of philosophy, which is going to be offered in a new book, viz. *Hallazgos filosóficos* (*Philosophical Findings*).

6. See my paper «Armonía y continuidad en el pensamiento de Leibniz: Una ontología barroca», ***Cuadernos Salmantinos de Filosofía*** vol. XVI (1989), pp. 19-55.

7. See my paper «La ruptura del sistema lógico en la teoría poética de Carlos Bousoño» *Anthropos*, Nº 73 (junio 1987), pp. 43-50.

8. See, for instance, «Tres enfoques en lógica paraconsistente» I & II, *Contextos* Nºs 3 & 4, pp. 81-130 & 49-72, resp.; «(Quasi)Transitive Algebras», *Proceedings of the XIII International Symposium on Multiple-Valued Logic* (Kyoto, may 1983), Los Angeles, Ca: IEEE Computer Society, pp. 129-35.; «Consideraciones filosóficas sobre la teoría de conjuntos», ***Contextos*** 11 & 12 (University de León, 1988), pp. 33-62 & 7-43; «Algunos desarrollos recientes en la articulación de lógicas temporales», apud ***Lenguajes naturales y lenguajes formales IV.1***, ed. by Carlos Martín Vide. Barcelona: Universitat de Barcelona, 1989, pp. 413-39.

Intuitively, a paraconsistent logic is one which allows for two theorems of a theory to be such that one is a negation of the other without thereby everything being a theorem of that theory. Following da Costa[9], I also add some further requirements for a paraconsistent logic to be acceptable: (1) It must possess an intuitive appeal; (2) it must recognize the validity of as many usual ways of reasoning as possible; (3) it must contain as much of classical logic as possible keeping short of counternancing the Scotus rule (p, not-p $\vdash$ q). The 2d. requirement leads me to also demand that our chosen logic should be a fuzzy one, since only logics of fuzziness can capture reasonings with words such as 'up to a point', 'more… than' and so on.

Thus, from a proof-theoretic viewpoint my requirements for «correct» paraconsistency are the following ones. (In what follows, a dot written immediately on the right of a functor stands for a left-hand parenthesis with its mate as far to the right as possible. Remaining ambiguities are dispelled by associating leftwards.) A theory T is a 2-tuple $<\mathcal{T},\mathcal{R}>$, where $\mathcal{T}$ is a set included in a set $\mathcal{F}$ of formulae, formulae being such finite sequences of members of some set $\mathcal{V}$ of symbols as comply with a number of stipulations (formation rules) and $\mathcal{R}$ is a set of operations on $\mathcal{F}$ (inference-rules). (Henceforth, while speaking about a theory whose set of inference-rules is $\mathcal{R}$ I'll mean the fact that $\mathcal{R}$ contains a rule of the form $\{p_1,\ldots,p_n\} \vdash q$ simply by saying that in that theory $p_1,\ldots,p_n \vdash q$.) A **healthy** theory is a theory T whose set of symbols, $\mathcal{V}$, comprises two two-place functors, $\vee$, $\wedge$, and one-place functor, ~, such that any p, q, r $\in \mathcal{F}$, $\mathcal{T}$ is closed for every operation (inference-rule) belonging to $\mathcal{R}$, and (writing pΘq whenever s $\vdash$ s', where s' is like s except for containing occurrences of p at zero or more places where s contains respective occurrences of q, or conversely): (1) T is nondeliquescent (i.e. $\mathcal{T} \neq \mathcal{F}$); (2) $p\vee q\vee r$ Θ $q\vee r\vee p$; $p\wedge q\wedge r$ Θ $q\wedge r\wedge p$; (4) $p\vee q\wedge r$ Θ $p\wedge r\vee.q\wedge r$; $p\wedge q\vee r$ Θ $p\vee r\wedge.q\vee r$; (5) $p\vee q\wedge q$ Θ q Θ $p\wedge q\vee q$; (6) p $\vdash$ $p\vee q$; (7) $p\wedge q$ $\vdash$ p; (8) $\sim(p\vee q)$ Θ $\sim p\wedge\sim q$; (9) $p\vee\sim p$ is a theorem (i.e. $\in \mathcal{T}$); (10) $\sim(p\wedge\sim p) \in \mathcal{T}$; (11) $\sim\sim p$ $\vdash$ p; (12) p Θ $\sim\sim p\wedge p$.

Let T be a healthy theory, ~ being a one-place functor thereof satisfying the above requirements: then T is also said to be negationally right as regards ~. A theory $<\mathcal{T},\mathcal{R}>$ negationally right is said to be negationally inconsistent for ~ iff $\mathcal{T}$ comprises two theorems p, q such that q = ~p. If T = $<\mathcal{T},\mathcal{R}>$ is a theory, a theory T' = $<\mathcal{T}',\mathcal{R}>$ is said to be **a taut extension** of T iff $\mathcal{T}'$ includes $\mathcal{T}$ and $\mathcal{R}'$ includes $\mathcal{R}$. A theory which is negationally right for one of its symbols, ~, is said to be **paraconsistent** as regards ~ iff there is a taut extension thereof which is both **nondeliquescent** (i.e. it's not closed for the rule p $\vdash$ q, for any p and q) and yet negationally inconsistent for ~. A theory is paraconsistent iff it is paraconsistent for at least one of its symbols.

[9]. Newton C.A. da Costa has laid down those condition in his paper «On the Theory of Inconsistent Formal Systems» (*Notre Dame Journal of Formal Logic*, 15/4 (1974), pp. 497-510). See also a paper by N.C.A. da Costa and E.H. Alves, «Relations between Paraconsistent and Many-Valued Logics», *Bulletin of the Section of Logic* 10/4 (1981), 185-191. An interesting discussion of those conditions from a technical viewpoint is to be found in Igor Urbas's Ph.d. diss., *On Brazilian Paraconsistent Logics*, Canberra: Australian National University, 1987, pp. 17ff, 73ff. Notice that da Costa also lays down that for a system to be paraconsistent it must not have noncontradiction as a theorem. But there da Costa and I part company. On the relationship between those general requirements for paraconsistency and the particular kind of logical framework I am going to develop in this section, see another paper, also by Newton da Costa: «Aspectos de la filosofía de la lógica de Lorenzo Peña», *Arbor* Nº 520 (Madrid, april 1989), pp. 9-32.

A healthy theory T = <$\mathcal{T},\mathcal{R}$> will be said to be **proficuous** iff its vocabulary comprises a symbol ~ such that T is negationally right for ~ and the following rule belongs to $\mathcal{R}$: p∨q, ~p ├ q. (Any proficuous theory is a taut extension of classical logic.)

Nedless to say, if a theory is negationally inconsistent as regards a symbol N, then that very same symbol cannot be one which renders it proficuous (i.e. disjunctive syllogism cannot hold for N). Still a theory can be both proficuous and negationally inconsistent for different negations, one strong negation ~ and another, simple, negation N. Such is the case as regards the system **Aq** I'm now going to set forth.

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§2.2. *System **Aq** : Syntactic approach*

Aq = <$\mathcal{T},\mathcal{R}$> where $\mathcal{T}$ is the smallest subset of $\mathcal{F}$ containing all instances of each axiom schema A01 through A08 below and closed for every rule belonging to $\mathcal{R}$; where $\mathcal{F}$ is a set comprising an element, α, and closed for this formation-rule: whenever p, q ∈ $\mathcal{F}$, then so do Bp, Hp, p↓q, pIq, p•q, ∀xp (for any variable instead of x).

DEFINITIONS:

«Np» abbr. «p↓p»

«p∨q» abbr. «N(p↓q)»

«p∧q» abbr. «Np↓Nq»

«¬p» abbr. «HNp»

«½» abbr. «αIα»

«Lp» abbr. «N¬p»

«0» abbr. «½Iα∨¬(½IN½)»

«Xp» abbr. «p•p»

«1» abbr. «N0»

«p⊃q» abbr. «¬p∨q»

«Sp» abbr. «p∧Np»

«np» abbr. «p•Nα»

«mp» abbr. «NnNp»

«p→q» abbr. «q∧pIq»

«p≡q» abbr. «p⊃q∧.q⊃p»

«Yp» abbr. «pIα∧p»

«fp» abbr. «¬Yp∧p»

«p&q» abbr. «Lp∧q»

«p\q» abbr. «p→q∧¬(q→p)»

«¥p» abbr. «np\p&fSp»

«∃xp» abbr. «N∀x(1•Np)»

«p⇒q» abbr. «B(p→q)»

«Jp» abbr. «¬B¬p»

«Kp» abbr. «NXNp»

AXIOM SCHEMATA:

A01 p∧q⊃p

A02 r∧sIp⊃(p↓qI.q↓s∨.q↓r)

A03 pIq⊃(rIqI.pIr)∧.KXpIp∧.Yp∨Yq∨¬Y(p•q)∧.fSp∧fSq⊃(p•q\p)∧.p∧q⊃.p•q

A04 q∧p∨pIp∧.Hp∧HqILH(p∧q)∧.pIq⊃(Hp∨HrIH(q∨r))∧.p•q→p∧.p•1Ip

A05 pINqI(NpIq)∧.pIpI½∧.p'∧pIq⊃(q•r•sI.s•r•p∧.s•p'•r)∧.¥p∧fNq⊃¬¥N(p•mq)

A06 pIq⊃(q⊃p)∧.mp→mnp∨Hp∧.mp→np≡(Yp∨YNp)∧.q→np∨(pImq)∧Lp∨.p→q

A07 Bp∨B¬BLp∧.BpIp∨¬Bp∧.p⇒q&Bp→Bq

A08 ∃x(∀xq•p)I∀x(∃xp•q)∧.∀x(p•q)→(∀xp•q)∧.∀xs\r⊃∃x(s\r)∧.∀xp∧∃xq→∃x(p∧q)∧. ∀x¬p→¬∃xp∧.nr\r⊃∃x(r→∃xp→.r→p)

(In A08 «r» is to be a formula with no free occurrence of variable 'x'.)

INFERENCE RULES:

rinf01 (modus ponens) p , p⊃q ⊦ q

rinf02 p ⊦ Bp

rinf03 (Universal generalization) p ⊦ q if q is nothing else but the result of prefixing to p any finite string of universal quantifiers

rinf04 (alphabetic variation) p ⊦ q if q is nothing else but the result of replacing within p a formula r by another r', where r' is an alphabetic variant of r.

rinf05 (shift of variables) p ⊦ q if q is nothing else but the result of uniformly replacing all free occurrences of some variable with respective free occurrences of another variable.

READINGS of those symbols: 'α': 'The least true of truths (is true)' (or 'That which is just infinitesimally true in all respects (is true)'; '1': 'The wholly true (is true)'; '0': 'The wholly false (is true)'; '½': 'That which is just as true as false (is true)'; '↓': 'neither… nor'; '•': 'not only… but also'; 'I': 'as… as' (or: 'to the same extent as'); 'H': '(It's) wholly (true that)'; 'B': 'It's truthfully assertable that' (or 'It's in every way the case that' or 'In all respects'); 'N': 'not'; '∨': 'or'; '∧': 'and'; '¬': 'not at all' (or 'by no means'); 'L': 'more or less' or '(at least) up to a point' (or 'to some extent', 'in some degree'); 'X': '(It's) very (true that)'; '⊃': 'only if'; 'n': 'It's overtrue that'; 'm': '(It's) much like (true that)'; '→': 'insomuch only as' (or: 'only to the extent that'); '≡': 'iff'; 'Y': '(It's) (just) infinitesimally (true that)' (or: 'It's just in the smallest degree true that'); 'f': '(It's) somewhat (true that)'; 'K': '(It's) a little (true that)’; ‘\’: ‘It’s less true that… than that’; ‘J’: ‘(It’s) (at least) relatively true that’; «p&q»: «While p, q». ‘∀’ (‘∃’) is the universal (existential) quantifier prefix.

It seems safe to lay down a «transformation rule» — or «surface-structure» generating rule — to the effect that whenever a one-place functor, g, is prefixed to a formula, p, p being a predicative formula of the form «x is so-and-so», an alternative reading is available, viz. «x is g so-and-so».

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§2.3 *MODELS. Semantic treatment*

I am going to set forth the notion of **quantificational transitive algebras**. A **quasi transitive algebra** is an algebra <A,Γ> where Γ = <1,N,H,n,∨,•,I> where 1 is a zero-ary operation, N, H, n are unary operations, •, ∨, I are binary operations, all of then satisfying the postulates (01) through (24) below. Let’s first introduce some abbreviations. 0 is N1; x≤y means that y = y∨x; x<y means that, while x≤y, xIy = 0. Let 𝒟 = {x∈A: HNx = 0} be called the set of **dense** elements of A. Furthermore:

/x∧y/ eq /N(Ny∨Nx)/

/Sx/ eq /x∧Nx/

/x→y/ eq /x∧yIx/

/¬x/ eq /HNx/

/Xx/ eq /x•x/

/mx/ eq /NnNx/

/Kx/ eq /NXNx/

/α/ eq /m0/

/fx/ eq /¬(xIa)∧x/

/Lx/ eq /N¬x/

/½/ eq /1I1/

/¥x/ eq /¬(nxIx)∧fSx/

/x\y/ eq /(x→y)∧¬(y→x)/

POSTULATES (for any x,y,z,u,v, ∈A):

(01) y∧x∨x = x

(02) xIy ≤ x.u∨zI(y∨z∧.u∨z)

(03) Hx∧Hy = LH(y.x)

(04) zIy ≤ Hx∨HzIH(x∨y)

(05) vIy ≤ v•(x.u)•zI(u•z∧(x•z)•y)

(06) x•1 = x

(07) $x \bullet y \leq y \wedge x$

(08) $x \wedge y \wedge \neg(x \bullet y) = 0$

(09) $xIy \in \mathcal{D}$ iff $x=y$

(10) $½ = N½$

(11) $xIy \leq zIyI(xIz)$

(12) $xIy \wedge \neg x \wedge y = 0$

(13) $\neg(xI0 \vee x) = 0$

(14) $xIyI½ \vee (xIyI0) = ½$

(15) $xINy = NxIy$

(16) $x \rightarrow y \vee (y \rightarrow nx) \vee (xImy) = ½$

(17) $\neg(nmxInx) \wedge x = 0$

(18) $x \bullet yI\alpha \leq xI\alpha \vee (yI\alpha)$

(19) $x = XKx$

(20) $nx = x \bullet n1$

(21) $nxImx = xI\alpha \vee (xIN\alpha)$

(22) $\alpha < ½$

(23) $fSx \wedge fSy \leq \neg(x \wedge yI(x \bullet y)$

(24) $¥x \wedge fNz \wedge ¥N(x \bullet mz) = 0$

A **transitive algebra**, t.a., is an algebra <A,Γ> such that Γ = γ∪B}, <A,γ> being a q.t.a., and B being a unary operation which satisfies this further postulate:

(25) For any $x \in A$, either $x \vee \alpha = x = Bx$, or else $\neg x \neq 0 = Bx$ (the 2d. disjunct means that, while $\neg x \neq 0$, $0 = Bx$).

A **quantificational transitive algebra**, tQa, is a t.a. enlarged sith two (infinitary) operations, $\bigwedge$, $\bigvee$, carrying non-empty subsets of A (where A is the carrier of the tQa under consideration) into members of A, and such that for any elements x,z,y of A and any non-empty sets A_1, A_2 included in A:

(26) $\bigvee\{\bigwedge A_1 \bullet z : z \in A_2\} = \bigwedge\{\bigvee A_2 \bullet y : y \in A_1\}$

(27) $\bigwedge\{x \bullet z : x \in A_1, z \in A_2\} \leq \bigwedge A_1 \bullet y$ (if $y \in A_2$)

(28) $\bigwedge\{xIz : x \in A_1, z \in A_2\} \leq \bigwedge A_1 I \bigwedge A_2$

(29) $\bigwedge A_1 \wedge \bigvee A_2 \leq \bigvee\{y \wedge x : y \in A_1, x \in A_2\}$

(30) $nz \backslash z \leq \bigvee\{z \rightarrow \bigvee A_1 \rightarrow (z \rightarrow y) : y \in A_1\}$

(31) $\bigwedge A_1 = N\bigvee\{Nx : x\in A_1\}$

Let's now go about finding out a tQa as follows. Let's take the set of standard reals; for each of them, x, let's take x, x+α and x—α, α being some given (nonstandard) infinitely small number, with '+' standing for **addition** and '—' standing for **subtraction**. All those numbers will be called **hyperreals**. An alethic number is a hyperreal h such that 0≤h≤1. We now define these algebraic operations. If x is a standard alethic number, then: (1) Nx = $2^{\log_x 2}$ if 0<x<1, whereas n0=1 and N1=0; (2) N(x—α) = Nx+α; (3) N(x+α) = Nx—α. Furthermore, x∨z = **max**(x,z); H1=1; if x≠1, Hx=0; if x is either =z or =z+α, where z is a standard ≠0, then nx = z—α; otherwise nx = x. If x, z are standard, then x•z = x×z; if one among x, z, is either a standard u≠0 or u+α (u, standard), while the other is v+α, v being standard, then x•z = (u•v)+α; 0•x = 0 = x•0; if z≠0, z•Nn1 = Nn•z = Nn1 = α; last, if one among x, z is u—a, u being a standard, while the other is either v or v—α or v+α, v being a standard ≠0, then x•z = z•x = n(u•v) = (u•v)—α. Finally, xIz = ½ if x=z, and else xIz = 0. The set of alethic numbers provided with those operations is a q.t.a., Ã. The relation of numeric order ≤ coincides with the algebraic order relation ≤. Let's now take the set of **alethic tensors**, which is nothing else but the direct product of Ã with itself infinitely many times. On that direct product we define: 0 as the element which is mapped by every projection function into the alethic number 0; likewise for, 1, ½, α. The other operations are defined in the usual way (for instance: x•z is the element y such that, for any projection function *p*, *p*x•*p*z = *p*y, where of course • is the algebraic operation on alethic numbers as defined hereinabove). We now add the operation B as follows: for any alethic tensor x, Bx=0 if some projection function maps x into 0 (0 here being the alethic **number** zero); otherwise Bx=x.

(A most convenient narrowing-down of the set of alethic tensors would be to leave out those which have at least a component which does not recur infinitely many times, i.e. such tensors x as for some projection function *p* are such that there are only finitely many projection-functions *p'* such that *p'*x = *p*x. That restriction will turn out to be most helping in order to deal with continuous change and the like.)

VALUATIONS

A valuation is a mapping, *v*, from the set of formulae of an **Aq** theory (i.e. of an extension of **Aq**) into members of the carrier of a tQa complying with these conditions (for any formulae p, q): *v*(p↓q) = N*v*p∧N*v*q; *v*(pIq) = *v*pI*v*p; *v*(Bp) = B*v*p; *v*(p•q) = *v*p•*v*q; *v*(Hp) = H*v*p; *v*(α) = Nn1. In those equations of course a sign which within a left-hand member stands for a symbol of **Aq** does in the right-hand member stand for an operation of the algebra under consideration.)

Let v' be an x-variant of v iff every formula q with no occurrence of x is such that v(q) = vq. Then: v(∀xp) = $\bigwedge$\{u : some x-variant of v, v', is such that v(p)=u\}

It's quite easy (if cumbersome or tedious) to prove that a formula p is a theorem of **Aq** iff all valuations, v, are such that v(p) is a dense element of the algebra whose carrier includes the range of v.[10] In particular, let's enlarge the transitive algebra whose carrier is the set of alethic tensors with the infinitary operations $\bigwedge$ and $\bigvee$ defined in the usual way ($\bigwedge$ is the greatest lower bound, while $\bigvee$ is the least upper bound); the result is a tQa; and it's quite straightforward to demostrate that any valuation of an **Aq**-theory, v, whose range is included in the set of alethic tensors is such that for any formula ⌜p⌝ of that theory, if ⌜p⌝ is a theorem of **Aq**, then v(p) is a dense element (a dense alethic tensor) — the converse implication may well fail to obtain. It's also a straightforward task to "construct" a Tarski algebra of formulae of an **Aq**-theory and to prove that it is a tQa.

Therefore, **Aq**is a system both sound and complete — and of course nondeliquescent. (Nondeliquescence immediately ensues upon there being tQas with nondense elements. As a matter of fact, there is only one tQa with no nondense element, viz. the trivial algebra whose carrier comprises just one element, 0=1.)

Before proceeding into the following, let me remark that other models isomorphic to the algebra whose carrier is the set of alethic numbers as introduced above are algebras whose respective carriers are:

(1) A set of **swap-sets**, a swap-set being a set **s** of real numbers such that either **s** is the unit set of a standard real, r, or else it is the set of such nonstandard reals as are greater (smaller) than but infinitely close to r; the ordering relations and operations are defined thereon in a straightforward way so as to allow us to discard all swap-sets which are either greater than {1} or smaller than {0};

(2) A set of two-tuples <r,n> provided that n is one of three [given, fixed] numbers greater than 1 and 0≤r≤1;

(3) A set D defined as follows: Let R be the set nonnegative reals and D' = R∪{∞}. Now D = {x : for some y∈D' : x=y; or x = y+α and y∈R; or x = y—α and y>0}, where '+' stands for addition and 'α' stands for some (arbitrarily chosen) infinitesimal. Now we define these operations:

nx =: y+α, if y∈R is such that y=x or x = y—α; x, else

mx =: y—α, if y∈D'—{0} is such that y=x or y+α = x; x, else

Nx =: 0, if x=∞; ∞, if x=0; 1/x, if x∈R—{0}; mNy, if x=ny; nNy, if x=my

x↓y = max(Nx, Ny)

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[10]. Those results are proved in my paper «Caraterísticas técnicas y significación filosófica de un cálculo lambda libre», ap. *Lógica y filosofía del lenguaje* (ed. by S. Alvarez, F. Broncano, M.A. Quintanilla), Salamanca: University of Salamanca, 1984, pp. 89-114, and. more fully, in ***Rudimentos de lógica matemática***, Madrid: Servicio de Publicaciones del CSIC, 1991. Pp. vi+324.

$x \bullet y = x+y$, if both $x,y \in R$

$x \bullet \infty = \infty \bullet x = \infty$;

$mx \bullet y = y \bullet mx = m(x \bullet y)$ if $y \nparallel ny$

$nx \bullet y = y \bullet nx = n(x \bullet y)$

Hx =: 0, if x=0; ∞, else

x→y =: 1, if x≥y; ∞, else

Finally we extend those operations to tensors in the same way as the one referred to above. (Notice that this structure is mapped into Ã by a homomotphism h such that for every $x \in R$ $hx = 2^{-x}$, while h∞=0, and, for any $z \in D$ such that $z = x - \alpha$ (= $x+\alpha$), with $x \in R$, $hz = hx+\alpha$ (= $hx - \alpha$).)

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§3.- **Applications**

I am going to show in this Section how the above technical system can cope with several problems which have been deemed to constitute serious challenges to [the application of] classical logic, namely those of partial truth, fringes and motion. The first problem is raised by the fact that when a predicate correctly or truthfully applies to a zone, or part, or area, of some object but not to other parts thereof, it can only be said with partial truth that the object satisfies that predicate or has the property it denotes. Thus, e.g., if some large part of a country is barren while another part is fertile, is the country fertile or is it not? Likewise, is it true that it rains in Madrid on 29th. August if on that day it rains there but only for seven hours on end?

The second problem arises because of the fact that many predicates can be neither completely assigned to some things nor completely withheld from them. Thus, predicates like ‘young’, ‘kind’ ‘poor’, ‘red’, ‘healthy’, ‘near’, ‘old’, ‘new’, ‘mountanous’, and the like, apply in degrees. But so do other predicates such as ‘table’, ‘car’, and all implement-terms, of course. And so are species terms bound to do, if evolution theory is correct as most of us nowadays think: there is bound to have been something or other to which the predicate ‘monkey’ applies (or applied, if you prefer) neither wholly nor yet not at all — some creature which was only to some extent a monkey; or a mammal, or a vertebrate, or… In any such case is the sentence to the effect that the predicate applies true? Or is it not?

The 3d. problem is nothing else but Zeno’s paradox of the arrow. Is the travelling body here or is it there, or both, or neither here nor there?

A number of accounts have been proposed. One of them is that terminological distinctions allow to dispel all those perplexities. Thus, the predicate ‘red’ could be applied, e.g., only to extensionless points, but in such a way that to say that point x is red would amount to what in our ordinary way of speaking is expressed, more or less, as ‘x belongs to a uniformly red surface’. I regard that [Aristotelian] account as merely programmatic and in fact probably impossible to carry out. It would cripple the role of logic and sever the links between language and reality. Yet without resorting to some such procedure, classical logic is unlikely to be able to cope with any of those problems.

An alternative approach is that implemented by the kind of fuzzy set theories proposed by Lofti Zadeh and other researchers working within similar frameworks (Łukasiewicz's logics or other logics of a similar kind). The logical systems they favour give up excluded middle (and noncontradiction). Thus what they in the end advise is for us to renounce the very same question «Is it or isn't it?» Not that according to them it neither is nor fails to be, no. What they advise is that you cannot say that it is, nor that it is not, not that it neither is nor is not, nor that it both is and is not. You can say nothing of the sort. So, the question «Yes or no?» is — according to them — misapplied, misasked. No answer, no question. That is that simple.

However, that approach seems to me to be in serious trouble. It amounts to a kind of ineffabilism. Moreover, it requires, for any predicate to be assigned to an object, that the object should **completely** have the property the predicate denotes, which seems to me to constitute an implausible alethic maximalism.

But what about using a logic which would e.g. be like Łukasiewicz's infinite valued logic but with all values ≥½ designated? It would enforce excluded middle, and so be immune to the objection above. But if we add a strong negation, ¬, to be read 'not… at all' (such that $v(\neg p) = 0$ iff $v(p) \neq 0$ and else $v(\neg p) = 1$), the system would still fail to enforce **strong excluded middle**, i.e.: «p∨¬p», which also seems to me to be a drawback. Moreover, many applications of the system would be desastrous: let's suppose that a predicate «h» applies to an object o to a degree of ½; then the logic under consideration would warrant our asserting both «p» and «not p», with «p» meaning that o has the property denoted by «h»; now, in all Łukasiewicz's systems we have the Scotus rule: p, Np ⊢ q. So any such application would yield a deliquescent theory. In order to avoid that result we would have to change the conditional functor of Łukasiewiczian logics. (Notice that in Łukasiewicz logic proper — with only value 1 designated — neither of these schemata is theorematic: «p→(q→r)→.p∧q→r» [importation], «p→Np→Np» [abduction]; with all values ≥½ designated, only the former becomes theorematic.)

But what would we choose as a correct conditional? For reasons I'm not going to canvass here, I think the only satisfactory [mere] conditional, ⊃, would be defined in a classical way: «p⊃q» as abbreviating «¬p∨q». But then in order for modus ponens to hold, all values are to be designated except 0.

Now, if we have all real numbers in the interval]0,1] designated, then within the kind of Łukasiewiczoid logic we are entertaining the following result can ensue (let's call it a strong ω-oversinconsitency): let's suppose that there is a set of terrains such that for any dry member of that set there is some other member which is less dry, the sequence of them tending towards zero (total lack of dryness), none of those terrains being wholly non-dry, though. Then of each of those terrains it would be truthfully assertable that it was dry but yet it would be **utterly** false (hence **by no means** truthfully assertable) that all them are dry. Separate truth of each and every would not entail conjoint truth of all. That surely is mistaken. Hence the need for hyperreals.

Now, **scalar** semantics, even with hyperreals, are not sufficient either. We need tensors. For otherwise we couldn't have functors such as 'in all respects' (or 'it is truthfully assertable') which are pluri-dimensional. For in some cases — e.g. cases of partial truth — besides an object having (let's say: all in all) a property to some extent, it is true that in some respects it has it to a much higher degree than in other respects. Hence the need for a kind of logic like the one presented above, in Sect. 2.

I am now going to separately examine each of the three domains of applicability of that logical approach which I have listed above.

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§3.1 *Application to the treatment of partial truth*

I am now going to explore a number of alternative ways of dealing with one of the issues I want to tackle namely that of **partial truth**, in the sense of that which, being true of a part of some object, is said to be partly, or partially, true of that object. I also include under this head cases of some predicate being satisfied by an object [only] in some «ways», i.e. of a sentence's being true [only] when expanded with some «circumstantial complement», be it temporal, or locative, or introduced by a prepositional phrase like «with regard to», or whatever.

Our present approach could be expected to handle that issue in the following way. Let's e.g. take as our truth-values the alethic tensors, as defined hereinabove. An alethic tensor, t, will be said to be **encompassed** by another, t', iff there is a mapping ϕ such that for any projection-function p there is a projection-function $p' = \phi p$, with $p \leq p'$ (projection functions are well-ordered, each of them picking out the i-th component of a tensor whatever, for some positive integer i), ϕ being monotonic (i.e. for any p_1, p_2, $\phi p_1 \leq \phi p_2$ iff $p_1 \leq p_2$) and for every p pt = ϕpt'.

We might lay down that, when an area z' was a part of another area z, then the truth-value of «p obtains at z'» would be encompassed by the one of «p obtains at z». That account I'll call **the encompassing approach**. Winsome though it seems to be, I'm not satisfied with it. For it would countenance our drawing from «at z' p» the conclusion «J(at z p)» (i.e. that it's at least in a way, or relatively, true that, at z, p), and, although very often that seems right (e.g., when the southern part of some country is very dry, it seems safe to say that it's relatively true that that country is very dry), I'm far from being convinced that such is always the case. Recall that the "areas" we are thinking of are taken to be any relativizing "respects" whether spatial, temporal or whatever. Now it's far from obvious that, since in 1792 France is undergoing a revolution, it's relatively true that in the 18th century France is undergoing a revolution, if, at least, 'in' means the same as 'through' or 'over'. However, my purported counterexample may well turn out to rest on a confusion since 'through' or 'over' may embody some universal quantifier in their underlying structures: 'through that year' may well mean 'at each time interval included in that year'. So, it may after all be right to say that it's relatively true that in (**not** 'through') the 18th century France is undergoing a revolution.

Accordingly, let's try another tack, **the averaging approach**. I am going to assume that is **some** function μ carrying alethic tensors into alethic tensors and such that for any tensor t there and projection function p there are projection functions p_1, p_2 such that p_1t ≤ pμt ≤ p_2t. Then that function μ (for Greek 'μεταξύ') is an averaging function of sorts. Let's also write 'μ' as the symbol denoting that function (which means that, for a valuation v, and a formula p, v(μp) = μvp, the former μ being the functor, the latter the unary operation on alethic tensors); 'μ' may be read as 'on the whole', 'on an average', 'on balance', 'all in all', 'all things considered', or the like. A postulate will be laid down to the effect that the following schemata are to hold: $H\mu p \to \mu Hp$; $\mu(p \wedge q) \to .\mu p \wedge \mu q$; $\mu p \vee \mu q \to \mu(p \vee q)$; $\mu(p \bullet q) \to .\mu p \bullet \mu q$

Let's now introduce a measure of the distance between two alethic numbers, γ: $u\gamma v$ will be the difference or distance between u and v. The averaging approach will be articulated by laying down that, the greater the overlapping between area z and area z' (with z' being a part of z), the smaller the distance between «at z' p» and «μ(at z p)»; at least other things being equal.

Outfitted with those semantic tools, we now may account for sundry intuitively appealing correlations. In addition to the degree of overlappingness between the areas, other factors may turn out to bear on the result, too; among them: the «importance» of the subarea; how much «population» it contains: the distance between the truth values of 'All in all, in that country people live well' and 'In region R people live well' will be smaller than that between the truth-value of the former sentence and the one of 'In region R' people live well' if e.g. two thirds of the country's population live in R and less than a third in R'. Likewise, supponsing Jacques Coeur had more to do with his Bourges countrymen than with foreign captives, the difference between the truth-values of 'All in all Jacques Coeur was a good man' and 'Jacques Coeur was good towards people in Bourges' will be smaller than that between the value of the former sentence and the one of 'Jacques Coeur was good towards foreign capitves'.

An open question remains, though: what about the distance between the sentences resulting from those considered above when the functor 'all in all' is dropped? I want my present approach to remain neutral on that issue, while it affords means for coping with statements of partial truth through the averaging functor. Should our qualms over the meaning of 'through' above turn out to be groundless — owing to 'through' containing an implicit quantifier, as already hinted at, and so differing in meaning from 'at' or 'in' — then we could wonder if after all 'μ' is a redundant functor, the meanings of «p» and «all in all p» then being one and the same. Even so, the averaging approach would not boil down to the encompassing one, since the latter alone entails, e.g., for Jacques Coeur's being a good man to be in some respects true up to a certain degree if Jacques Coeur was up to that degree good towards his Bourges countrymen.

Nevertheless, I do not think that the averaging approach can be satisfactory with such a qualification (taking μ to be redundant), the trouble being that, as it stands, it applies to all and every sentence, and to all states of affairs those sentences would stand for. But of course even if for a long period a person is completely happy, it may well be [utterly] false that that person is completely happy period — i.e. [if 'all in all' is redundant] that she is all in all completely happy. So what is probably required is to narrow down the field of sentences (or of states of affairs) for which 'μ' is redundant (for which μ is an identical transformation).

But even without such a quelification the averaging account seems to me to face another difficulty. Suppose Helen is for some time most happy and for some time most unhappy. Then the averaging approach would entail that she is all in all most happy and yet also all in all most unhappy. That of course would not entail that she is all in all both most happy and most unhappy (since we would not have as theorematic the schema «$\mu p \wedge \mu q \to \mu(p \wedge q)$»), but anyway it would be an unwanted result even as it sounds. Thus again, the field of sentences to which the averaging account is going to to apply has to be narrowed down.

A *prima facie* likely candidate would be the set of atomic sentences, but, to be sure, that notion has by now rightly become suspect.

Inspite of those difficulties and perplexities, it seems to me that the averaging approach, endowed with careful qualifications of that sort, is the most promising one. However, a safeguard clause of **other things being equal** may turn out to constitute the wisest qualification, even if — as it is always the case with such a clause — it also weakens the thesis to the point of somehow blunting it. Since our present exploration does not aim at finding fully satisfactory results but rather at pondering the **pros** and **cons** of different approaches, we had better stop our discussion of this issue here by provisionally recommending the averaging approach with a number of qualifications and provisos.

One important point to be highlighted, though, is that, for all we have said hitherto, μ may either be a uniformizing operator, or fail to be so. A uniformizing operator on alethic tensors is a function, ϕ, such that, for any tensor t and projection functions p_1, p_2, $p_1(\phi t) = p_2(\phi t)$. It could be felt to be natural that, while a sentence like «Helen is happy» can be truer in some respects, less true in others, a sentence like «All in all Helen is happy» would have to be as true in any given respect as in any other. I'm far from sure that such is the case, though.

One of the advantages of our choosing alethic tensors — rather than scalar unities like the alethic numbers — as truth values is that by doing so we allow for a great many alternative ways of coping with the just discussed problems. The encompassing approach would by impossible, were it not for the acceptance of tensors. And even the averaging approach would become one-sided and scarcely fruitful within a scalar truth-values framework, no interplay being then possible between the operator 'all in all' on the one hand and, on the other, 'it is truthfully assertable that' or 'it is at least relatively true that'.

Let me bring this subsection to a close by warning the reader. Th present array of approaches may turn out to allow for ways of looking upon historical facts that, natural though they seem to be to the present writer, may well deserve rejection in accordance with other ways of thinking. Thus, for instance, even though both Sulla and Marius ordered many killings, the former being [all in all] murderous may be much more true than the latter being so. Therefore, it is **not only** a question of knowing if one of them had a property (like that of murderousness) but also how much he had it, or if he had it more, or less, than [certain] other personages. I can imagine some people frowning at such results, but I refrain from elaborating on those points here.

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§3.2 *Application to the treatment of fringes (or fuzziness)*

Most predicates we avail ourselves of are fuzzy, or vague. Many philosophers have contended that fuzziness, or vagueness, is not — or even cannot be — a matter of how the world is, but only a feature of our «concepts», of our linguistic usage, speech patterns or whatever; or at most something pertaining to semantics, but nowise to ontology. The view that fuzziness is a real trait of reality itself has been spurned and contemptiously styled 'the ugly view'.

Fuzziness, or vagueness, has often been characterized as failure of excluded middle.[11] I want to challenge such a characterization. Let's take the example of dryness. Desertic lands are dry; steppes are dry; many regions which are neither deserts nor steppes are dry, too. But of course there are degrees of dryness. Now pick out any region which lies in the fringe of the predicate 'dry', i.e. which is neither completely dry nor completely wet; call it 'Alboran'. Does excluded middle «fail» for the sentence 'Alboran is dry'? What does such a failure mean? Perhaps that the sentence 'Either Alboran is dry or it is not dry' does not hold. But that sentence does hold. What we would usually say is that Alboran is neither dry not wet, or that it neither is nor fails to be dry. Now, 'Alboran neither is dry nor fails to be dry' is equivalent to 'It is not the case that Alboran is dry and it is not the case (either) that Alboran is not dry'. Which, in virtue of involutivity of simple negation, means the same as 'Alboran is not dry and (yet) it is dry', i.e. (in virtue of commutativity of conjunction): 'Alboran is dry and it isn't. And — as even people unprepared to countenance true contradictions acknowledge — such sentences are very often used to represent situations consisting in a thing's lying in the fringe of a predicate.

But some people are concerned over our purported unwillingness to accept counterexamples to the principle of noncontradiction while we would be apparently less disinclined to put up with counterexamples to excluded middle.[12] Such a difference seems to me to amount to nothing else than a stylistic preference. For simple negation noncontradiction and excluded middle are strictly equivalent. Counterexamples to the one are to the same extent counterexamples to the other. Yet there are stylistic, pragmatic constraints about how to put a message, whether or not other ways of putting it are semantically as acceptable.

Furthermore, the meaning of 'counterexample' in those sentences ought to be clarified. What emerges is not a case which compels us to jettison either noncontradiction or excluded middle, but quite another thing: those are cases wherein negations of instances of those principles are shown, or at least believed, to be (up to a point) true — which does not completely rule out the truth of the thus negated instances of the principles. Alboran neither is nor fails to be dry; in virtue of DeMorgan, that amounts to its being true that the following is not the case: Alboran is dry or it is not. Still, the latter may also be true; why not? Then, by means of adjunction, we can draw the conclusion that Alboran's being either dry or not

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[11]. One of those who so characterize vagueness is Kenton F. Machina in his «Truth, Belief,, and Vagueness», *Journal of Philosophical Logic* 5/1 (feb. 1976), pp. 47-77. See, for an interesting discussion, Susan Haack, *Deviant Logic*, Cambridge U.P., 1974, chap. 6, pp. 109-25. Akin to the characterization of fuzziness as failure of excluded middle are those which regard vagueness as truth-valuelessness (e.g. Bertil Rolf, «A Theory of Vagueness», *Journal of Philosophical Logic* 9/3 (aug. 1980), pp. 315-26). There are lots of variations on those kinds of approaches, whereas unfortunately I know of no other approach more or less along the lines of the one I am now offering.

[12]. See A.C.H. Wright, «Verificationism and the Principle of Non-Contradiction», *History and Philosophy of Logic* 5 (1984), pp. 195-217. In n.31 (on p. 213) Wright says that vague concepts are not counter-examples against his claim that it is absurd to think that reality is inconsistent; and he adds: 'Speakers may have discretion as to whether, on occasion, something is to be dubbed 'red' or 'orange' but an individual speaker cannot consistently say something is red and orange all over. And the truth of 'It is raining and it is not raining' merely reflects the vagueness of the concept 'drizzle'.' Of course you cannot consistently say something which is contradictory if 'consistently' here means 'complying with negational consistency', as different from absolute or Post consistency (nondeliquescence, nontriviality). As for the truth of 'It is raining and it isn't, well my opinion is that the existence of such truths is reflected by our use of the fuzzy predicate 'to rain'; as for the fuzzy predicate 'to drizzle', it denotes a fuzzy subproperty of the fuzzy property of raining. Concepts I meither know nor need to posit. In the same way as sometimes it both rains and does not rain, some things are both red and orange all over, namely such as are of a reddish orange hue.

dry is and is not the case. You may have other, independent, grounds for rejecting excluded middle; e.g. you may be a constructivist. But our customary ways of using fuzzy predicates does not compel you to do so, far from it. Unless of course you are bent on avoiding contradictions at any cost. But why? Probably because of Scotus' rule (p, not-p $\vdash$ q) and fear of your body of beliefs foundering — becoming deliquescent. No worry! Paraconsistent approaches are devised to ward that danger off.

A paraconsistent account of the use of fuzzy predicates is quite straightforward, if it is carried out within the framework of a logical system like the one put forward in Section 2 above. Neither noncontradiction nor excluded middle need to be waived; in fact both are kept as valid principles, with all instances of either recognized as logical truths. Still, negations of a number of those instances can also be countenanced or stated with no loss of logical coherence (nondeliquescence). All such situations ensue upon some thing's neither wholly failing to exemplify some property nor entirely exemplifying it either. Hence fuzziness has nothing to do with our being unable to say whether a thing exemplifies a property or not. We are unable to say whether King John's head was on 01.01.1201 covered by an odd number of hairs, but that makes 'odd' none the fuzzier. On the other hand, we know what we are to say concerning some people which are neither wealthy nor entirely destitute: we say that they neither are nor fail to be poor — and hence that they are and are not poor; but some of them are less poor than others. Those fringes make up the field where comparatives are in order.

But then, why to remain adamantly loath to considering fuzzy properties real — as real anyway as nonfuzzy properties may be? You may be a nominalist, of course. Then, you'll accept neither real fuzzy properties nor real nonfuzzy ones. Well and good. Or you may be a realist, of whatever variety. Then, in order to rule out fuzzy properties from the world you need some convincing argument. Such arguments as I have thus far come across fail to carry conviction. For instance, that speech or linguistic mechanisms can sufficiently account for fuzziness phaenomena. Well, that is not generally demonstrated in any enlightening way, and, withal, such pragmatic accounts as have been devised (since that is what they are — or at most pragmatic-semantic) lack the smooth, simple, clear straightforwardness ensuant on alethic-semantic accounts like the one I am here putting forward. Another argument: that the world «as such» (?) can be neither fuzzy nor non-fuzzy. Why not? The discussion about whether it is fuzzy or not (that is to say about whether there are fuzzy properties and situations or not) makes clear sense, doesn't it? Another way of putting the same argument: that we lack any verification procedure in order to ascertain whether the world is fuzzy; so… Well, verification procedures we may fail to have, but not so criteria on ways of shaping our worldview according to some epistemological principles; furthermore, that contention seems to me a little dogmatic, since many people would quite sincerely say that they **see** that the man yonder is neither bald nor not bald, and so forth.

Thus, our track having thus been blazed and cleared, we can now sketch out our approach to fuzziness. First, a nominalistic, satisfactional account of predicates can be articulated Tarski-ways; only, our truth-values will be the alethic tensors introduced above. Rather than saying that a sequence of things satisfies a predicate *tout court*, we'll be saying that the degree of its satisfying it is the alethic tensor into which the predicate carries the sequence under consideration. We also may relativize the satisfaction relation to some given **respect**, a respect being represented either by some projection function or, better still, by an infinite sequence of projection functions, thus saying that $\mathscr{d}$ = the degree to which a sequence of things,

s, satisfies a predicate, ϕ, in a respect *r* iff, if $<p_i1, \ldots, p_in, \ldots>$ is the sequence of projection functions which represents *r* and ⌜q⌝ is the formula resulting from writing m variables to the right of the m-place predicate ϕ, then, upon the m first components of **s** being respectively assigned to those m variables by a valuation *v*, $<p_i1(v\text{q}), \ldots, p_in(v\text{q}), \ldots> = d \neq <0,0, \ldots>$

In an alternative way, we may prefer a realistic semantic account of predication, e.g. by positing both properties and facts. Let's for simplicity sake conceive of atomic facts as wholes made up by an m-adic property along with m individuals. Then we may claim that, since the characteristic function, χ, of the m-adic predicate ϕ has a range included in the set of alethic tensors, the fact that $\phi x_1,\ldots,x_m$ will have a degree of existence *d* if $d = \chi\phi(x_1,\ldots,x_m)$. We also may, for the sake of ontological economy, identify a property with its characteristic function. Thus, a sentence will be true insomuch as the fact it puports to signify or represent exists; when the sentence's truth-value = 0 (i.e. <0,0,…>) that fact does not exist at all in any respect, so the sentence represents nothing at all — and thus it is absolutely false, completely false, that is, in every respect.

You may still prefer to do without degrees of existence, even if you are patient or kind enough to envisage degrees of truth. Then perhaps you'd better say that all facts are existent (existent to the same extent); only, besides existing, and unrelated to the existence they have, they possess a degree of trueness (or of «obtaining»). Or you may contend that what makes a sentence more, or less, true is not a feature of the fact it represents, but the degree of its representing it. This track seems to me to be fraught with unsurmountable difficulties, which I refrain from elaborating upon here.

I think what I have hitherto said clearly shows that there are viable, workable ways of shaping semantic accounts of fuzziness (or predicate fringes) by means of the logical system set forth above, thus keeping clear of all snags into which certain accounts unavoidably come. For if a thing x is less red than y while being more red than z, what, according to my approach, is happening is not x's redness stretching on an «area» in such a way that redness and nonredness are so tangledly mingled that at each subarea there still are both redness and nonredness, without thereby merging or blending. No, my approach posits precisely a blending of sorts, since what, acccording to it, happens is for x to exemplify redness in a smaller degree than y but in a greater degree than z. If my present approach is correct, fuzziness has nothing to do with uncertainty or indeterminacy (unless by 'indeterminacy' we understand just for something to be determined as neither absolutely obtaining nor absolutely failing to obtain) or with truth-valuelessness, or failure of excluded middle (which would indeed constitute indeterminacy) or anything like that. Fuzziness is ensuant upon **degrees**, such diverse degress in which several things exemplify fuzzy properties. The present approach thus prompts us to set up theories of fuzzy modal, deontic, temporal and doxastic extensions, in order to handle degrees of possibility, obligatoriness, simultaneousness, belief and so on: a bump crop in the fields of ontology, epistemology, ethics, philosophy of mind, philosophy of science, philosophy of language, with novel solutions to sundry puzzles which arise when just the two classical alternatives of **absolutely yes** and **absolutely no** are envisaged, or, at the very most, enriched with supervaluational devices, indeterminacy cases, oscillations or inextricable-intermingling of those two alternatives and so on.[13] As against all those approaches, the present one has it that there is real continuousness, shades or transitions consisting in nothing else but in things exemplifying properties in such a way that they up to a point exemplify them and yet also to some extent or other fail to exemplify them, in different degrees which pass into one another through intermediate degrees — those degrees forming a set on which a partial order is defined as pointed out above: the order ≤ which corresponds to the functor '⇒'.

Notice finally how hard it seems to be for supervaluational, indeterminacy or inextricable-intermingling approaches to fuzziness to account for comparative constructions. If colour fringes lie in some kind of inextricable intermingling of redness and non-redness, or the like, then how to explain that, even within the fringe, some objects are less red than others? Is there more redness in their constitution? Not so since the inextricability in question demands for those elements of redness, or whatever, to be infinitely many in all those cases — and surely the cardinality will be the same in all such cases of the same kind, be it denumerable or not. More obviously still comparative construcions fail to be clearly accountable for by means of supervaluational or indeterminacy approaches. This is why little is left — besides an alethic-semantic account of degrees like the present one — except pragmatic approaches, about which, I feel bound to confess, I have misgivings — lest genuine alethic issues may appear to dissolve into ways of using words on different occasions or «utterance contexts», with nothing in reality corresponding to such variations.[14]

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§3.3. *Motion and change*

Motion is just a particular case or kind of transition fringes. Motion or more generally change is therefore nothing else but such fuzziness as is displayed in a continuous temporal way. There are degrees of redness; they either do or at least could form a continuous transition, e.g. from red to blue, even should nothing undergo a colour change at all. What pertains to the process (or change) of a blue thing's reddening, or becoming red, is that those shades are successively exemplified by one same thing in a continuous manner and during some time-interval.

Let me sketch how the present approach can account for motion as a sample case of change. Let's assume that a body b does length wise and continuously move over an interval i along a stretch s, running through it from one extreme to the other. Let s_1 , s_2,…, be sub-stretches of b's run (i.e. parts of s), at least as long as b, while i_1,i_2,…, are subintervals of i. Then it seems safe to lay down that the following points hold if v is a correct valuation (if, that is to say, for any sentence ⌜q⌝, v(q) is the truth value ⌜q⌝ really has). (1) Any projection-function p is such that (N being — remember! — the negation operator on alethic

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[13]. What I am calling 'an inextricable-intermingling approach' is G.H. von Wright's proposal in a couple of papers, namely «Time, Change, and Contradiction», and «Truth and Logic», both reprinted in *Philosophical Papers* by the same author, resp. vol. II (Blackwell, 1983), pp. 115-31, and vol. III (Blackwell, 1984), pp. 26-41. I shall discuss von Wright's approach in a paper entitled «Von Wright on Truth and Contradiction».

[14]. I have discussed approaches to the semantics of comparative constructions articulated within classical logic in my Ph.D. diss. (quoted above, in n.4) and in my paper «Contribución a la lógica de los comparativos» ap. *Lenguajes naturales y lenguajes formales II* (ed. by c. Martin Vide). Barcelona: University of Barcelona, 1987, pp. 335-49.

tensors) Nu ≥ pv(b is in s_1 at i_1) ≥ u, u being some fixed alethic number — the same for all the inequalities concerned, even when s_1 and i_1 vary — such that $a<u\leq Nu<N\alpha$, α ($N\alpha$) being the infinitesimally true (false). (2) The greater the motion's speed, the smaller the difference between those different values into which $p\circ v$ (i.e. the composition of p and v, p being any projection function) carries the several sentences «b is in s_1 at i_1» for different stretches s_1 and time intervals i_1, those truth-values then tending to gather around ½. (3) The closer s_1 is to s_2 — assuming they do not overlap — the smaller the distance between pv(b lies in s_1 at i_1) and pv(b lies in s_2 at i_1) when the former rises above pv(b lies in s at i_1). (4) The greater the disparity between pv(b lies in s_1 at i_1) and pv(b lies in s_1 at i_2), i_1 and i_2 not overlapping, the more temporally distant i_1 is from i_2. (5) To s_1 there corresponds a subinterval i_1 such that all subintervals i_2 of i which neither are parts of i_1 nor conversely are such that pv(b lies in s_1 at i_2) < pv(b lies in s_1 at i_1), the difference being the greater the larger the temporal distance is between i_1 and i_2. (6) To i_1 there corresponds a substretch s_1 of s such that every substretch s_2 which neither is a part of s_1 nor has s_1 as a part is such that pv(b lies in s_2 at i_1) < pv(b lies in s_1 at in i_1), the disparity being the greater the larger is the spatial distance between s_1 and s_2. (7) Any place r outside s is such that pv(b lies in r at i_1) ≤ u (u bieng as pointed out above, in (1)). The seventh point — which may possibly need weakening or qualifying — captures the idea that, when travelling, the body lies more at every part of its path than anywhere else. The intuitive motivation of the other six postulates ought to be obvious. Point (1) ensures each part of the travelling body's itinerary a degree of that body's lying therein at any time during the travel at least as high as some given degree truer than the merely infinitesimally true but less true than the infinitesimally false (since the infinitesimally false is infinitely true). Point (2) shows that, were the travelling body to move with an infinite speed, it then would lie within each place of its trajectyory to the same extent, viz. in a degree as true as false; there are degrees of motion, and of rest: infinite-speed motion would be the highest degree of motion, which of course we have good reasons to assume does not take place; doubtless, the fixed value u of point (1) depends on the motion's speed, the slower being the motion, the lower u — and hence the higher Nu — a zero-speed motion being sheer quiescence. (I do not know whether pure quiescence takes place or not. We may Leibniz-ways assume that rest is nothing else but an infinitely slow motion, which according to our present lights would mean that u in point (1) would be $=\alpha$. Still, even so I do not want to commit myself to claiming that rest, so conceived, exists.) Point (3) has it that when the body is in some substretch of its itinerary to a greater extent than in the itinerary as a whole, it is rightly said to be more at places closer to that substretch than at places far away from it, even if they are (remote) parts of the itinerary. Point (4) is similar but as regards intervals instead. Points (5) and (6) allow us to say that at any given time interval the travelling body is more lying at a certain place than elsewhere, and that for any given substretch the body lies in it more at a certain time interval than before or afterwards. (As a trivial case i_1 may be i itself, and s_1 may be s.)

Those postulates may turn out to need some qualifications. For one thing, I have laid down that they apply for any projection-function p. Now it may be argued that a body may be travelling only in some respects or at least more in certain respects than in others. Then the qualification required would be worded somehow as follows: 'When a body is travelling in some respects which are represented by projection-functions p_i1, p_i2,…, then for each of those projection functions, p, the following postulates apply'. Nonetheless, even without any such qualification our postulates allow for a motion to be swifter in some respects than in others, and so u in points (1) and (7) may be different for diverse projection-functions.

It would be rewarding to go into the relationship between the foregoing approach to motion and the accounts of partial truth set forth above, in Sect. 3.1. I am inclined to think that the present approach is like the averaging approach to partial truth (restricted to a certain kind of sentences) but with the ‘all in all’ operator taken to be pleonastic. For the just sketched account of motion assumes that the truth value (alethic tensor) corresponding to the fact that a body lies in some place at some time interval is not made out of those corresponding to the facts consisting in that body lying in the various parts of that place at such times as are encompassed by that interval, but rather that that tensor somehow lies in between the “extremes”. Working out an encompassing approach to motion is going to give us a headade (and I doubt that it should be a rewarding one). Thus I choose to refrain from that task for the time being.

Some of our postulates may well need strengthening. One way of doing so would be to ensure continuity along these lines. Let’s take a set, (s_j), of substretches having all of them the same final point and each of them a particular starting point. A bijective mapping exists between them and their initial points. Then we need to lay down that the function pv(b lies in s_j at i_1) — for variable s_j ranging over that set — is continuous on the set (s_j). Likewise we require continuity on the set of subintervals, to be defined in a similar manner. Those postulates would strengthen points (3) and (4) above. But even without such strengthenings, our approach keeps clear of irksome results ensuant upon other accounts; e.g. their being committed to countenancing some kind of oscillation and inextricable intermingling of being-here and being-there when a body travels from here to there. The present account has it instead that when passing from here to there the body is both here and there, so it both is and is not here (there); but at first it is more here than there, then the other way round, in a continuous process, spanning an infinite and continuous range of degrees.

Before bringing this paper to a close, let me briefly consider an objection which has been raised against fuzzy approaches to change. The objection is connected with Aristotle’s discussion about the first or the last instant of an object’s having a property.[15] Even if an object’s having a property changes by degrees so that from its completely having it to its completely lacking it there is an interval (and notice that the property may be that of being at some place, or that of being at least at to a point such that it is rather lying at that place, or wahtever), there is, by continuity, bound to be either a last instant of its wholly having the property or a first instant of its [utterly] failing to completely have the property. Which would constitute an abrupt, transitionless change. Thus not all changes can be gradual or fuzzy.

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[15]. There are a number of points about motion which will not be dealt with in this paper. So for instance Zeno’s paradox of the arrow, which happens to constitute my main motivation for setting up my present proposal, has — both of old and late — received a great many different treatments, which I here refrain from commenting on. There also is the closely related topic (on which I’ll nonetheless feel bound to say something at the end of this subsection) of whether there is a last instant of rest or a first instant of motion, and so on; see Richard Sorabji, «Aristotle on the Instant of Change» (in *Articles on Aristotle. 3 Metaphysics*, ed. by Jonathan Barnes *et al.*, London: Duckworth, 1979, 159-77). Interesting questions connected with Aristotle’s attempts to wrestle with the contradictions that motion and change seem to give rise to are gone into in Sarah Waterlow’s *Nature, Change, and Agency in Aristotle’s Physics* (Oxford: Clarendon, 1982), esp. on pp. 141ff. My approach is anti-Aristotelian to the utmost and would doubtless be found uncongenial by Waterlow, Sorajbi and many others.

My reply is that, first, there may be no instants at all, and, second, that, even if there are instants, they are not durations, that is to say they are not entities **at** which things happen or states of affairs exist or obtain (except in a stretched way of speaking: to say that exactly at 7 p.m. something happened means that it happened at some short interval around that instant — if there are instants — or something of that sort). But then, is there any transition fringe between its being the case that p and its failing to be the case that p, when «p» is «Object o completely has property ϕ»? Yes, there may be: if t is a time [interval] such that it is truthfully assertable that at t p, and t' is a later time such that it is truthfully assertable that it is [completely] false that at t' p, there is an intermediate time, t_1 (i.e. a time, t_1 starting after t but before t' and ending after t but also before t'), such that it is neither truthfully assertable that at t_1 p nor truthfully assertable that at t_1 not-p [at all].[16] But, what about the case where «p» is «It is truthfully assertable that it is entirely true that o has ϕ»? My answer is that such sentences do change from absolute truth to absolute falseness in an abrupt way, but that such a change is **supervenient** upon a gradual, transitional change.[17]

In order to highlight how closely related the foregoing discussion is to the problems gone into in Sect. 1 above concerning the dialectical tradition let me conclude by remarking that the present treatment seems to be a way of formalizing Hegel's Heraclitean view of motion as set forth in his historical account of Zeno's paradoxes.[18] And of course my approach shares with others (such as von Wright's account referred to in note 13) the aim of affording a nonclassical and, somehow or other, contradictorial solution to Zeno's paradox of the arrow — only, unlike von Wright's, mine is straightforwardly contradictorial.[19]

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[16]. I have developed the solution just sketched here in my paper «Algunos resultados recientes en la articulación de lógicas temporales», quoted above, in n. 8.

[17]. That idea is developed in my new book *Hallazgos filosóficos* (quoted above, at the end of n. 5) and, less fully, in my discussion of Leibniz on the principle of continuity in the paper mentioned in n. 6. The notion of supervenience which is closest to mine is Jaegwon Kim's as put forward in «Supervenience for Multiple Domains», *Philosophical Topics* 16/1 (Spring 1988), pp. 129-50. See also (by the same author) the article «Supervenience» in the *Handbook of Metaphysics and Ontology* (quoted above, at the end of n. 2).

[18]. In his *Naturphilosophie* (2d. Part of the *Enzykopaedie*: Frankfurt, Suhrkamp V. edit., 1970, vol. 9), Hegel, after characterizing time in the following way (§258, p. 48): 'Die Zeit, als die negative Einheit des Aussersichseins, ist gleichfalls … das Sein, das, indem es **ist, nicht** ist, und indem es **nicht** ist, **ist**', proceeds to describing motion (*Zusatz* to §261, p. 58) as a situation wherein the travelling body both is and is not in this place or spot; Hegel views such a situation as one of change being both what it is and yet also unchangeableness, or one of a place becoming another while keeping its own identity; that side of his approach has nothing to do with mine. I find a clearer exposition of Hegel's ideas on Zeno's arguments in the *Vorlesungen ueber die Geschichte der Philosophie*, vol. 18 of the Suhrkamp edition, pp. 300ff, esp. this sentence on p. 314: 'Bewegen heisst aber: an diesem Orte sein und zugleich nicht'.

[19]. I cannot here go into other contradictorial accounts of motion. The best known is Priest's (see his paper «To Be and Not To Bee: Dialectical Tense Logic», *Studia Logica* XLI/2-3 (1982), pp. 246-68). To my knowledge none of those accounts regards motion as a particular sort of **fringe**, or as a matter of fuzziness or graduality. Thus, they fail to realize that when passing from being here to being there the travelling body is not constantly and changelessly in a unique or undifferentiated state of both being here-and-there, and hence being here and not here as well as there and not there, but that what is taking place is a succession of infinitely many such states, each of them being characterized by the [sequence of] degree[s] to which, when in that state, the travelling body lies here and that to which it then lies there. Let me also mention a paper by Frank Jackson and Robert Pargetter entitled «A Question about Rest and Motion» (*Philosophical Studies* 53/1 (jan. 1988), 141-6); the authors try to cope with the problem of the (last-or-first) instant of change — which is of course closely connected with the paradox of the arrow — by saying that motion at a time is relative to earlier and later times; a body is moving at t if it has non-zero velocity at t as regards times both earlier and later than t. Leaving technical details aside, I find two important philosophical difficulties concerning that approach, viz: (1)

if motion is to be defined through velocity — which is the derivative of the function carrying times into positions — rather than the other way round, then there is no motion proper, no passage from a position to another, but just a set of different positions patched together, each of them the image into which the function in question maps some time taken as an argument; but then, if that is the end of the story, there is no need for motion to be defined in any way, since, properly speaking, there is no motion at all (or, put in other words, for b to move at a time, t, is quite compatible with b's occupying some position at t in the same way and to the same extent it would do were it resting at that position); (2) those authors' approach countenances indeterminacy cases, which to myself had better be avoided.