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Partially Balanced Incomplete Block Designs

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PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

by

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CHAPTER I

INTRODUCTION

In the randomized-block design, the size of the block of experimental units must be a multiple of the number of the treatments to be compared. Various types of experimental designs have been introduced since 1936[1] for testing a large number of treatments. They are designed to suit the requirements of the experimenter, with the object of achieving maximum efficiency for a limited amount of the experimental material. Sometimes it is desirable or necessary to have the block size smaller than the total number of treatments. These designs are referred to as incomplete block designs.

A balanced incomplete block design has the property that any pair of treatments appears together equally often within the same block. Thus, as shown below, every pair of treatments appears together twice in the same block, where the numbers represent the treatments and the columns represent the block.

$$\begin{matrix} 3 & 1 & 1 & 1 & 2 & 1 & 2 \\ 5 & 4 & 2 & 2 & 3 & 3 & 4 \\ 6 & 6 & 5 & 3 & 4 & 4 & 5 \\ 7 & 7 & 7 & 6 & 7 & 5 & 6 \end{matrix} \qquad (1.1)$$

This property insures that the same standard error may be used for comparing every pair of treatments.

These designs have found fruitful applications in experiments in

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[1]F. Yates, "Incomplete Randomized Blocks," *Annals of Eugenics*, VII (1936), 121-40.

which individuals are asked to make a comparative rating of different objects that are presented to them. Some examples are the rating of different ways of preparing a food as to palatability, different colors in which some article is made as to acceptability, and different occupations as to their social status. This is taste and preference testing.[2] Scheffé[3] has also indicated that these designs can be applied to the following:

1. Many varieties of a crop are to be compared in one field experiment in which yields will be measured, but blocks of this number of plots are undesirable because they would inevitably display soil heterogeneity.

2. Different makes of automobile tires are to be compared. The natural block consists of four wheels of a car.

In balanced incomplete block designs, each pair of treatments is compared with equal precision, and each treatment is paired with every other treatment an equal number of times within a common block; λ is a constant for all treatments. There is one associate class for each treatment in balanced incomplete block designs. These are the most important balanced incomplete block designs, but the need sometimes arises for others; either because no suitable balanced incomplete block design exists, or because, for example, it is necessary to make some comparisons more precisely than others. We can see balanced incomplete block designs form that special case of partially balanced designs in which there is one associate class for each treatment. In

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[2]W. G. Cochran and G. M. Cox, *Experimental Designs,* (2nd ed.; New York: John Wiley & Sons, 1957), p. 440.

[3]Henry Scheffé, *The Analysis of Variance* (New York: John Wiley & Sons, 1958), p. 161.

this paper, the author will discuss the partially balanced incomplete block design with two associate classes.

Partially balanced incomplete block designs (PBIB designs) were first introduced by Bose and Nair. Bose and Shimamoto introduced the concept of association schemes of a PBIB design. Then a considerable quantity of research was carried out on the combinatorial properties of association schemes and the designs obtained from them. Rao[4] introduced the analysis of yields of a tomato trial using the PBIB design (twenty varieties and sixteen blocks). Bose and Shimamoto[5] have also used the PBIB design in work on cotton experiments (ten treatments and ten blocks).

It is the object of this paper to describe the most important properties of partially balanced designs and to give both the intra-block and inter-block analysis. Thus, the scope of this paper includes the following:

1. The definition of PBIB designs.
2. Relations between the parameters.
3. Some theorems for partially balanced designs.
4. Five types of association schemes in PBIB designs and examples.

Many useful designs belonging to each type have been published by the Institute of Statistics, University of North Carolina.[6]

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[4]C. R. Rao, "General Methods of Analysis for Incomplete Block Designs," *Journal of the American Statistical Association*, XLII (1947), 541-61.

[5]R. C. Bose and T. Shimamoto, "Classification and Analysis of Partially Balanced Incomplete Block Designs with Two Associate Classes," *Journal of the American Statistical Association*, XLVII (1952), 151-84.

[6]R. C. Bose, W. H. Clatworthy, and S. S. Shrikhande, *Tables of Partially Balanced Designs with Two Associate Classes*, North Carolina Agricultural Experiment Station, Technical Bulletin No. 107 (Raleigh, North Carolina: North Carolina Agricultural Experiment Station, 1954).

CHAPTER II

THE DESCRIPTION OF PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

Definition

An incomplete block design is said to be partially balanced if it satisfies the following conditions:[7]

1. The experimental material is divided into b blocks of k units each, different treatments being applied to the units in the same block.

2. There are v treatments, each of which occurs in r blocks (times).

3. There can be established a relation of association between any two treatments, satisfying the following requirements:

   a. Two treatments are either first, second,..., or mth associates.

   b. Each treatment has exactly n_i, ith associates ($i = 1,2, \ldots, m$).

   c. Given any two treatments which are ith associates, the number of treatments common to the jth associates of the first, and the kth associates of the second, is P_{jk}^{i} and is independent of the pair of treatments with which we start. Also, $P_{jk}^{i} = P_{kj}^{i}$.

4. Two treatments which are ith associates occur together in exactly λ_i blocks.

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[7]Bose and Shimamoto, "Classification and Analysis," pp. 151-84.

The numbers $v, b, r, k, \lambda_1, \lambda_2, \ldots, \lambda_m, n_1, n_2, \ldots, n_m$ are called the parameters of the first kind and the numbers $P^i_{jk}(i,j,k = 1,2,\ldots,m)$ the parameters of the second kind, belonging to the design. Thus, there are $2m + 4$ parameters of the first kind, and $m^2(m + 1)/2$ parameters of the second kind (since $P^i_{jk} = P^i_{kj}$). Originally, Bose and Nair[8] had imposed the condition that $\lambda_1, \lambda_2, \ldots, \lambda_m$ were unequal, but Nair and Rao[9] found that this condition was not necessary.

<u>The relations between the parameters</u>

It is known that the following relations are satisfied by the parameters of the designs:

$$bk = vr \tag{2.1}$$

$$\sum_{i=1}^{m} n_i = v - 1 \tag{2.2}$$

$$\sum_{i=1}^{m} n_i \lambda_i = r(k - 1) \tag{2.3}$$

$$\begin{aligned} \sum_{k=1}^{m} P^i_{jk} &= n_j \quad (\text{if } i \neq j) \\ &= n_j - 1 \ (\text{if } i = j) \end{aligned} \tag{2.4}$$

$$n_i P^i_{jk} = n_j P^j_{ik} = n_k P^k_{ij} \tag{2.5}$$

If $m = 2$, then clearly

$$n_1 + n_2 = v - 1 \tag{2.6}$$

$$n_1\lambda_1 + n_2\lambda_2 = r(k - 1) \tag{2.7}$$

$$P^2_{11} + P^2_{12} = P^1_{11} + P^1_{12} + 1 = n_1 \tag{2.8}$$

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[8] R. C. Bose and K. R. Nair, "Partially Balanced Incomplete Block Designs," *Sankhya,* IV (1939), 337.

[9] K. R. Nair and C. R. Rao, "A Note on Partially Balanced Incomplete Block Designs," *Science and Culture,* VII (1942), 568.

$$P^1_{21} + P^1_{22} = P^2_{21} + P^2_{22} + 1 = n_2 \qquad (2.9)$$

$$n_1P^1_{12} = n_2P^2_{11} \text{ and } n_1P^1_{22} = n_2P^2_{12} \qquad (2.10)$$

There are eight parameters of the first kind and six parameters of the second kind. When m = 2, we shall require that $\lambda_1 \neq \lambda_2$, for if $\lambda_1 = \lambda_2$, the design becomes a balanced incomplete block design, which we do not wish to consider.

These relationships establish constraints on the choice of the parameter values. Altogether, given the n_i, only $m(m^2 - 1)/6$ of the P^i_{jk} are independent.

Proof: To begin with, consider the matrix (P^i_{jk}) for some fixed i, say for i = 1. Since the matrix is symmetric, it is defined by $\binom{m+1}{2}$ values; since the row and column totals are fixed, only $\binom{m}{2}$ of these are independent.

If the P^1_{2k} are known, then the relation $n_1P^1_{2k} = n_2P^2_{1k}$ determines all $P^2_{1k} = P^2_{k1}$. Hence, in the matrix (P^2_{ji}), only the second to mth columns and rows remain free to be determined, and these form a matrix in which $\binom{m-1}{2}$ values are independent, since row and column totals are again fixed.

Continuing, we find $\binom{m-2}{2}$ independent values among the P^2_{jk}, and so on, until $\binom{m-(m-2)}{2} = 1$ value remains among the P^{m-1}_{jk}. The values of the P^m_{jk} are then implicitly fixed and we have in all $\sum_{i=o}^{m-2} \binom{m-i}{2} = m(m^2 - 1)/6 \cdot$ independent values of the parameters of the second kind.

For m = 2, we have $m(m^2 - 1)/6 = 1$, and this will be used later.

Association schemes and plans

Most of the research carried out so far has been concerned with designs of two associate classes. These designs depend on eight parameters of the first kind $v,b,r,k,\lambda_1,\lambda_2,n_1,n_2$ and six parameters of the second kind $P^i_{jk}(i,j,k = 1,2)$. The parameters of the second kind may be exhibited as elements of two symmetric matrices.

$$P^1_{jk} = \begin{bmatrix} P^1_{11} & P^1_{12} \\ P^1_{21} & P^1_{22} \end{bmatrix}, \quad P^2_{jk} = \begin{bmatrix} P^2_{11} & P^2_{12} \\ P^2_{21} & P^2_{22} \end{bmatrix} \qquad (2.11)$$

As an illustration, let v = 10, m = 2. Arrange the ten treatments 1,2,3,4,5,6,7,8,9,10 in the scheme.

$$\begin{array}{ccccc} * & 1 & 2 & 3 & 4 \\ 1 & * & 5 & 6 & 7 \\ 2 & 5 & * & 8 & 9 \\ 3 & 6 & 8 & * & 10 \\ 4 & 7 & 9 & 10 & * \end{array} \qquad (2.12)$$

Suppose that the rule of association is that two treatments are first associates if and only if they occur together in the same column of the scheme; otherwise, they are second associates. Let us consider treatment 1. The other treatments that appear in the same block as treatment 1 are treatments 2, 3, and 4 (block (1)), and treatments 5, 6, and 7 (block (2)). There are three other treatments (8, 9, and 10) which do not appear in the same block as treatment 1. Similarly, if we start with any other treatment, we find six treatments that are in the same block with it, and three that are not. Now, that means each treatment has six first associates and three second associates. Thus, condition (2.2) is satisfied with $n_1 = 6$, $n_2 = 3$.

Take a pair of treatments that are first associates, say 1 and 2, and consider the relationships of the other treatments to these two. Treatments 2, 3, 4, 5, 6, and 7 are in the same block as 1, while 8, 9, and 10 are not. For treatment 2, the other treatments that are in the same block are 1, 3, 4, 5, 8, and 9; while 6, 7, and 10 do not appear in the same block; namely,

Treatment	First associates	Second associates	
1	2, 3, 4, 5, 6, 7	8, 9, 10	
2	1, 3, 4, 5, 8, 9	6, 7, 10	(2.13)

These relationships may be presented in a 2 x 2 table (Table 1).

Table 1. Relationship of the other treatment to 1 and 2

		Relation to 1	
		1st associate	2nd associate
Relation to 2	1st associate	3, 4, 5	8, 9
	2nd associate	6, 7	10

Treatments 3, 4, and 5 are first associates of both 1 and 2; treatments 6 and 7 occur in the same block with 1 but not with 2, and hence are first associates of 1 and second associates of 2, and so on. The same holds for other P_{jk}^{i}'s. The parameters P_{jk}^{i}, when exhibited in equation (2.11) are

$$P_{jk}^{1} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}, \quad P_{jk}^{2} = \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix} \qquad (2.14)$$

Remembering that $n_1 = 6$, $n_2 = 3$, and $m = 2$, we can now verify the relations (4) and (5) as follows:

$$\left.\begin{array}{l} P_{11}^{2} + P_{12}^{2} = 4 + 2 = 6(= n_1) \\ P_{21}^{1} + P_{22}^{1} = 2 + 1 = 3(= n_2) \end{array}\right\} \quad (\text{if } i \neq j)$$

$$\left.\begin{array}{l} P_{11}^{1} + P_{12}^{1} = 3 + 2 = 5(= n_1 - 1) \\ P_{21}^{2} + P_{22}^{2} = 2 + 0 = 2(= n_2 - 1) \end{array}\right\} \quad (\text{if } i = j)$$

$$\left.\begin{array}{ll} n_1 P_{21}^{1} = n_2 P_{11}^{2} & 6 \times 2 = 3 \times 4 \\ n_2 P_{11}^{2} = n_1 P_{12}^{1} & 3 \times 4 = 6 \times 2 \end{array}\right\} \quad (\text{if } i,k = 1,\ j = 2)$$

$$n_2P^2_{12} = n_1P^1_{22} \qquad 3 \times 2 = 6 \times 1$$
$$n_1P^1_{22} = n_2P^2_{21} \qquad 6 \times 1 = 3 \times 2 \qquad \text{(if } i,k = 2,\ j = 1)$$

As the scheme (2.12) pictorially summarizes the association relations between the treatments, we may call it an "association scheme." To get a design based on this association scheme, we now have to arrange the ten treatments into blocks, satisfying the requirements (1), (2), and (4) of the definition. These blocks then give the plan of the design. It may happen that there is more than one design based on the same association scheme. For example, there are four different designs based on the association scheme discussed here. These are the design numbers 1, 2, 3, and 4 of Tables 3 and 4. Table 3 gives the parameters v, b, r, k, λ_1, λ_2, n, c_1, c_2, Δ, H, where n in the side of the square in the association scheme; and c_1, c_2, Δ, and H are certain constants useful for analysis of the results of the design. Their use is explained later. Table 4 gives the association scheme and the plans (the blocks are given by the columns of the plan). As an illustration, consider the plan for design number 4. Since $b = 10$, $k = 4$, $r = 4$, there are ten blocks each containing four treatments, and each treatment occurring in four blocks. This design has two associate classes. For example, the first associates of treatment 1 are 2, 3, 4, 5, 6, and 7; treatment 1 appears with each of these treatments in $\lambda_1 = 1$ block. The second associates of treatment 1 are 8, 9, and 10; treatment 1 appears with each of these treatments in $\lambda_2 = 2$ block.

Table 2. Associate classes for design

Treatment	First associates	Second associates
1	2, 3, 4, 5, 6, 7	8, 9, 10
2	1, 3, 4, 5, 8, 9	6, 7, 10
3	1, 2, 4, 6, 8, 10	5, 7, 9
4	1, 2, 3, 7, 9, 10	5, 6, 8
5	1, 2, 6, 7, 8, 9	3, 4, 10
6	1, 3, 5, 7, 8, 10	2, 4, 9
7	1, 4, 5, 6, 9, 10	2, 3, 8
8	2, 3, 5, 6, 9, 10	1, 4, 7
9	2, 4, 5, 7, 8, 10	1, 3, 6
10	3, 4, 6, 7, 8, 9	1, 2, 5

CHAPTER III

TYPES OF ASSOCIATION SCHEMES

All partially balanced incomplete block designs with two associate classes can be divided into a small number of types according to the nature of the association relations among the treatments. The following types of association schemes will be discussed.

1. The group divisible type.
2. The triangular block type.
3. The Latin square type.
4. The cyclic type.
5. The simple type.

The group divisible type

Group divisible incomplete block designs are an important subclass of partially balanced designs with two associate classes, and the simplest type of partially balanced designs with two associate classes is the group divisible, denoted by GD. There are $v = mn$ treatments divided into m groups of n treatments each. The treatments of the same group occur together in λ_1 blocks and the treatments of the different groups occur together in λ_2 blocks. If $\lambda_1 = \lambda_2 = \lambda$ (say), then every pair of treatment occurs in λ blocks, and the design becomes a balanced incomplete block design. We shall therefore confine outselves to the case $\lambda_1 \neq \lambda_2$. The association scheme can be exhibited by placing the treatments in an nxm rectangle, where the columns form the groups. If, in a GD design with parameters v, b, r, k, λ_1, λ_2, m, n, treatments belonging to the same group are considered as first associates, and treatments belonging to different groups are considered as second associates, then it is easy to see that it is a partially balanced design with two associate classes for which

$$n_1 = n - 1 \quad , \quad n_1 = n(m - 1) \tag{3.1}$$

$$P^1_{jk} = \begin{bmatrix} n - 2 & 0 \\ 0 & n(m - 1) \end{bmatrix} \quad , \quad P^2_{jk} = \begin{bmatrix} 0 & n - 1 \\ n - 1 & n(m - 2) \end{bmatrix} \tag{3.2}$$

Conversely, suppose for a partially balanced design, $P^1_{12} = o$. Then from (2.8), $P^1_{11} = n_1 - 1$. The relation of the first association between two treatments is by definition commutative. We shall show that in the present case, it is also transitive. Let the treatments θ_0 and θ_1 be first associates. Let the other first associates of θ_0 be θ_2, θ_3, $\ldots,\theta_{n_1}$. Now, since θ_0 and θ_1 have $n_1 - 1$ common first associates, they can be no other than $\theta_2, \theta_3, \ldots, \theta_{n_1}$. Also, since θ_1 has exactly n_1 first associates, all its first associates are θ_0, $\theta_2, \theta_3, \ldots, \theta_{n_1}$. This shows that any first associate of θ_0 (other than θ_1) is also a first associate of θ_1. These conditions are sufficient to insure that the v treatments can be divided into groups of $n_1 + 1$ such that two treatments in the same group are first associates, and two treatments in different groups are second associates. Hence, the design is a GD design where the treatments of the same group are first associates. Similarly, if $P^2_{12} = 0$, we can show that the partially balanced design is a GD design, the treatments of the same group being second associates. We can therefore state:

Theorem 1. The necessary and sufficient condition for a partially balanced design, to be group divisible, is vanishing of P^1_{12}. If $P^1_{12} = 0$, then the treatments in the same group are associates i = 1.

Any given treatment occurs in r blocks. Since each of these blocks contains k - 1 other treatments, there are r(k - 1) pairs of which one member is θ. But θ must form λ_1 pairs with each of the n - 1 treatments belonging to the same group as θ, and λ_2 pairs with each of the n(m - 1) treatments not in the same group as θ. Hence,

$$v = mn \ , \ bk = vr \tag{3.3}$$

$$(n - 1)\lambda_1 + n(m - 1)\lambda_2 = r(k - 1) \ , \ n_1 = n - 1 \ , \ n_2 = n(m-1) \tag{3.4}$$

also,

$$r \geqq \lambda_1 \quad , \quad r \geqq \lambda_2 \tag{3.5}$$

The eight parameters v, b, r, k, λ_1, λ_2, m, n are therefore connected by the three relations (3.3) and (3.4), so that only five parameters are free.

Let N be the incidence matrix of the general PBIB design; that is,

$$N = \begin{bmatrix} n_{11} & n_{12} \ldots n_{1b} \\ n_{21} & n_{22} \ldots n_{2b} \\ \ldots\ldots & \ldots\ldots \\ n_{v1} & n_{v2} \ldots n_{vb} \end{bmatrix} \tag{3.6}$$

where the rows represent treatments, the column represents blocks, and n_{ij} = 1 or 0 according as the ith treatment (i = 1,2,..., v) does or does not occur in the jth block (j = 1,2,...,b). Since every treatment is replicated r times,

$$\sum_{j=1}^{b} n_{ij}^2 = r \tag{3.7}$$

and since every treatment must occur in λ_s blocks with each of its sth associates (s = 1, 2,..., m), if treatments i and u are sth associates, then

$$\sum_{j=1}^{b} n_{ij}n_{uj} = \lambda_s \qquad (i \neq u; i, u = 1, 2, \ldots, v) \tag{3.8}$$

Hence, the elements of the symmetric matrix NN' are r in the principal diagonal and λ_s's elsewhere. We can write

$$NN' = \begin{bmatrix} A & B & \ldots B \\ B & A & \ldots B \\ \ldots & & \\ \ldots & \ldots & \ldots \\ B & B & \ldots A \end{bmatrix} \quad (3.9)$$

where N' is the transpose of the matrix N, and A and B are nxn matrices defined by

$$A = \begin{bmatrix} r & \lambda_1 & \ldots & \lambda_1 \\ \lambda_1 & r & \ldots & \lambda_1 \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_1 & \lambda_1 & \ldots & r \end{bmatrix} \quad , \quad B = \begin{bmatrix} \lambda_2 & \lambda_2 & \ldots & \lambda_2 \\ \lambda_2 & \lambda_2 & \ldots & \lambda_2 \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_2 & \lambda_2 & \ldots & \lambda_2 \end{bmatrix} \quad (3.10)$$

Each row or column in the matrix on the right-hand side in (3.9) contains A in the diagonal position and contains B in the other m-1 positions.

Bose and Conner[10] have shown that

$$|NN'| = rk(rk - v\lambda_2)^{m - 1}(r - \lambda_1)^{m(n - 1)} \quad (3.11)$$

For every GD design, the following inequalities can be shown:

$$r \geqq \lambda_1 \ , \ rk - v\lambda_2 \geqq 0 \quad (3.12)$$

According to these relations, we can divide all GD designs into three exhaustive and mutually exclusive classes:

1. Singular GD designs, characterized by $r = \lambda_1$.
2. Semi-regular GD designs, characterized by $r > \lambda_1$, $rk - v\lambda_2 = 0$.
3. Regular GD designs, characterized by $r > \lambda_1$, $rk - v\lambda_2 > 0$.

The main combinatorial properties of each of these three classes are as follows:

Singular GD designs. A singular GD design is always derivable from a corresponding balanced incomplete block design on replacing each

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[10] R. C. Bose and W. S. Connor, "Combinatorial Properties of Group Divisible Incomplete Block Designs," *Annals of Mathematical Statistics*, XXIII (1952), 367-383.

treatment by a group of n treatments. Consider a balanced incomplete block design with v^* treatment, each replicated r^* times in b^* blocks of size k^*, such that any two treatments occur together in λ^* blocks. Now there are $v = nv^*$ treatments divided into v^* groups (each group corresponding to one of the original treatments). Two treatments belonging to the same group now occur together r^* times and two treatments belonging to different groups occur together λ^* times. Thus, we get a GD design with parameters.

$$v = nv^*, \quad b = b^*, \quad r = r^*, \quad k = nk^*$$
$$\lambda_1 = r^*, \quad \lambda_2 = \lambda^*, \quad m = v^*, \quad n \tag{3.13}$$

Conversely, consider a singular GD design with parameters v, b, r, k, λ_1, λ_2, m, n, where $r = \lambda_1$. Let θ and ϕ be any two treatments belonging to the same group. θ occurs in r blocks, and since $r = \lambda_1$, ϕ must occur in each of these r blocks and nowhere else. Hence, if a treatment occurs in a certain block, every treatment belonging to the group occurs in that block. Let each group of treatments be replaced by a single treatment in the design; then there are $v^* = m$ treatments in the new design, and because any two treatments belonging to different groups occur together λ_2 times in the original design, the new design is a balanced incomplete block design with parameters.

$$v^* = m, \quad b^* = b, \quad r^* = r, \quad k^* = k/n, \quad \lambda^* = \lambda_2 \tag{3.14}$$

Therefore, we may state:

Theorem 2. If, in a balanced incomplete block design with parameters v^*, b^*, r^*, k^*, λ^*, each treatment is replaced by a group of n treatments, we get a singular GD design with parameters given by (3.13). Conversely, every singular GD design is obtainable in this way from a corresponding balanced incomplete block design.

For example, let us consider the balanced incomplete block design with parameters $v^* = b^* = 7$, $r^* = k^* = 3$, $\lambda^* = 1$. The plan for this is given below, where the columns represent the blocks.

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{matrix} \qquad (3.15)$$

If $n = 2$, then we may replace the treatment 1 by 1 and 8, and do the same for the other treatments. We then get the singular GD design with the parameters

$$v = 14,\ b = 7,\ r = 3,\ k = 6,\ \lambda_1 = 3,\ \lambda_2 = 1,\ m = 7,\ n = 2$$

the plan for which is shown below.

Reps.							
I	1	2	3	4	5	6	7
	8	9	10	11	12	13	14
II	2	3	4	5	6	7	1
	9	10	11	12	13	14	8
III	4	5	6	7	1	2	3
	11	12	13	14	8	9	10

(3.16)

As before, the columns represent the blocks.

The relation $rk - v\lambda_2 \geq 0$ is true by definition for semi-regular and regular GD designs. We shall show that it holds for singular GD designs also.

Then, using the parameters of a singular GD design given by (3.13), $r = r^*$, $k = nk^*$, $\lambda_1 = r^*$, $\lambda_2 = \lambda^*$, $m = v^*$. From (3.4) we know

$$(n - 1)\lambda_1 + n(m - 1)\lambda_2 = r(k - 1)$$

$$n(m - 1)\lambda_2 = r(k - 1) - (n - 1)\lambda_1$$

$$n(v^* - 1)\lambda^* = r^*nk^* - r^* - nr^* + r^*$$

$$\therefore \lambda^*(v^* - 1) = r^*(k^* - 1) \qquad (3.17)$$

The relation (3.17) holds for a balanced incomplete block design.

$$rk - v\lambda_2 = n(r^*k^* - v^*\lambda^*)$$
$$= n(r^* - \lambda^*)$$
$$\geqq 0$$

Hence, we may state:

Theorem 3. For any GD design, $rk - v\lambda_2 \geqq 0$.

Semi-regular GD design. For a semi-regular GD design, we have by definition

$$r - \lambda_1 > 0, \quad rk - v\lambda_2 = 0$$
$$\because (n - 1)\lambda_1 + n(m - 1)\lambda_2 = r(k - 1)$$
$$(n - 1)\lambda_1 = r(k - 1) - n(m - 1)\lambda_2$$
$$r + (n - 1)\lambda_1 = v\lambda_2 - nv^*\lambda_2 + n\lambda_2$$
$$= nv^*\lambda_2 - nv^*\lambda_2 + n\lambda_2$$
$$= n\lambda_2 \qquad (3.18)$$

Hence,

$$r + (n - 1)\lambda_1 = n\lambda_2 \qquad (3.19)$$

For a semi-regular GD design, these hold the inequality.[11]

$$b \geqq r - m + 1 \qquad (3.20)$$

Also, each block must contain the same number of treatments from each group so that k must be divisible by m. If $k = cm$, then every block must contain c treatments from every group.

Proof: Let e_j treatments from the first group occur in the jth block ($j = 1, 2, \ldots, b$).

Then,

$$\sum_{j=1}^{b} e_j = nr \qquad (3.21)$$

$$\sum_{j=1}^{b} e_j(e_j - 1) = n(n - 1)\lambda_1 \qquad (3.22)$$

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[11] *Ibid.*

since each treatment from the first group occurs in r blocks, and every pair of treatments from the first group occurs in λ_1 blocks. Using (3.19), (3.21), and (3.22)

$$\sum_{j=1}^{b} e_j^2 = n^2\lambda_2 \qquad (3.23)$$

Let

$$\bar{e} = \frac{1}{b}\sum_{j=1}^{b} e_j = \frac{nr}{b}$$

$$\because bk = vr = mnr$$

$$\frac{bk}{m} = nr$$

$$\frac{nr}{b} = \frac{k}{m}$$

$$\therefore \bar{e} = \frac{k}{m}$$

from (3.3), hence,

$$\sum_{j=1}^{b} (e_j - \bar{e})^2 = \sum_{j=1}^{b} e_j^2 - b\bar{e}^2$$

$$= n^2\lambda_2 - bk^2/m^2$$

$$= n^2\lambda_2 - vrk/(v^2/n^2)$$

$$= n^2\lambda_2 - n^2rk/v$$

$$= n^2(\lambda_2 - rk/v)$$

$$= 0 \; (\because rk - v\lambda_2 = 0)$$

from (3.3) and (3.18); therefore,

$$e_1 = e_2 = \ldots = e_b = \bar{e} = k/m \qquad (3.24)$$

Since e_j must be integral, k must be divisible by m. If k = cm, then $e_j = e$ (j = 1, 2,..., b). The same argument applies to treatments of any other group.[12]

[12] D. Raghavarao, "A Generalization of Group Divisible Designs," *Annals of Mathematical Statistics*, XXXI (1960), 756-771.

For example, if $v = 6$, $b = 9$, $r = 6$, $k = 4$, $\lambda_1 = 3$, $\lambda_2 = 4$, $m = 2$, $n = 3$, the plan for which is shown below:

Reps.									
I	1	1	1	5	5	5	3	3	3
II	2	4	6	2	4	6	2	4	6
III	3	3	3	1	1	1	5	5	5
IV	4	6	2	4	6	2	4	6	2

(3.25)

As before, the columns represent the blocks.

Regular GD designs. For a regular GD design, we have, by definition:

$$r - \lambda_1 > 0, \; rk - v\lambda_2 > 0 \tag{3.26}$$

For a regular GD design, the following inequality holds:

$$b \geqq v \tag{3.27}$$

For example, if $v = 6$, $b = 6$, $r = 4$, $k = 4$, $\lambda_1 = 3$, $\lambda_2 = 2$, $m = 2$, $n = 3$.

Reps.						
I	1	3	5	2	4	6
II	2	4	6	1	3	5
III	4	6	2	3	5	1
IV	6	2	4	5	1	3

(3.28)

As before, the columns represent the blocks.

The triangular block type

Another important type of partially balanced design is the triangular type denoted by T. A partially balanced design with two associate classes is said to be triangular if the number of treatments is $v = n(n - 1)/2$ and the association scheme is an array of n rows and n columns with the following properties.[13]

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[13] S. S. Shrikhande, "On a Characterization of the Triangular Association Scheme," *Annals of Mathematical Statistics*, XXX (1959), 39-47.

1. The positions in the principal diagonal (running from upper left hand to lower right hand corner) are left blank.

2. The $n(n - 1)/2$ positions above the principal diagonal are filled by the numbers $1, 2, \ldots, n(n - 1)/2$ corresponding to the treatments.

3. The $n(n - 1)/2$ positions below the principal diagonal are filled so that the array is symmetrical about the principal diagonal.

4. For any treatment i, the first associates are exactly those treatments which lie in the same row (or in the same column) as i.

$$n_1 = 2n - 4, \; n_2 = (n - 2)(n - 3)/2 \qquad (3.29)$$

$$P^1_{jk} = \begin{bmatrix} n - 2 & n - 1 \\ n - 3 & (n - 3)(n - 4)/2 \end{bmatrix}$$

$$P^2_{jk} = \begin{bmatrix} 4 & 2n - 8 \\ 2n - 8 & (n - 4)(n - 5)/2 \end{bmatrix} \qquad (3.30)$$

It is easy to write down the association scheme for the triangular block type. Shrikhande pointed out the general rule, as follows:

*	1	2	3	4	...	n - 2	n - 1
1	*	n	n - 1	n - 2	...	2n - 4	2n - 3
2	n	*	2n - 2	2n - 1	...	3n - 7	3n - 6
3	n - 1	2n - 2	*	3n - 5	...	4n - 11	4n - 10
4	n - 2	2n - 1	3n - 5	*	...	...	...
....							
n - 2	2n - 4	3n - 7	4n - 11	....		*	v
n - 1	2n - 3	3n - 6	4n - 10	....		v	*

(3.31)

* represents the empty treatment which lies in the principal diagonal. The array is symmetric about the principal diagonal.

It can happen that there is more than one design based on the same association scheme. For example, there are four different designs based on the same association scheme in Table 4. Table 3 gives the parameters.

Table 3. Parameters of some triangular type designs

Reference no.	v	b	r	k	λ_1	λ_2	n_1	n_2	n	c_1	c_2	Δ	H
1	10	5	2	4	1	0	6	3	5	2/5	-1/5	10	13/2
2	10	10	3	3	1	0	6	3	5	3/10	-3/20	40/9	13/3
3	10	5	3	6	2	1	6	3	5	2/3	1/3	15/2	11/2
4	10	10	4	4	1	2	6	3	5	4/15	8/15	45/4	27/4

Theorem 4. It has been pointed out by Raghavarao[14] that if in a PBIB design with two associate classes having a triangular association scheme

$$rk - v\lambda_1 = n(r - \lambda_1)/2 \tag{3.32}$$

then 2k is divisible by n. Further, every block of the design contains 2k/n treatments from each of the n rows of the association scheme.

Proof: Let e_j^i treatment occur in the jth block from the ith row of the association scheme ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, b$). Then we have

$$\sum_{j=1}^{b} e_j^i = (n - 1)r$$

$$\sum_{j=1}^{b} e_j^i(e_j^i - 1) = (n - 1)(n - 2)\lambda_1 \tag{3.33}$$

since each of the treatments occurs in r blocks and every pair of treatments from the same row of the association scheme occurs together in λ_1 blocks. From (3.5), we get

[14] D. Raghavarao, "On the Block Structure of Certain PBIB Designs with Two Associate Classes Having Triangular and L_2 Association Scheme," *Annals of Mathematical Statistics*, XXXI (1960), 787-791.

Table 4. Association schemes and plan for some triangular type designs

Reference no.	Association scheme	Reps.	Plan
	* 1 2 3 4	Reps.	
	1 * 5 6 7	I	1 5 8 10 4
1	2 5 * 8 9		2 6 9 3 7
	3 6 8 * 10	II	3 7 2 6 9
	4 7 9 10 *		4 1 5 8 10
		Reps.	
2	Same as 1	I	1 8 2 7 9 5 3 10 4 6
		II	2 9 8 5 4 6 10 7 1 3
		III	5 10 3 9 2 8 4 6 7 1
		Reps.	
		I	5 10 4 7 1
			6 9 3 2 8
3	Same as 1	II	7 4 1 9 3
			8 2 10 5 6
		III	9 3 7 1 5
			10 8 6 4 2
		Reps.	
		I	2 10 7 6 1 5 8 3 9 4
4	Same as 1	II	10 1 3 2 9 4 7 5 6 8
		III	6 2 8 9 10 3 4 7 1 5
		IV	7 5 2 4 8 10 1 9 3 6

$$\sum_{j=1}^{b} (e_j^i)^2 = (n - 1)\{r + (n - 2)\lambda_1\} \tag{3.34}$$

Define

$$e_.^i = \frac{1}{b} \sum_{j=1}^{b} e_j^i = (n - 1)r/b = 2k/n.$$

Then

$$\sum_{j=1}^{b} (e_j^i - e_.^i)^2 = (n - 1)\{r + (n - 2)\lambda_1\} - 4bk^2/n^2$$
$$= 2(n - 1)[\{n(r - \lambda_1)/2\} - (rk - v\lambda_1)]/2$$
$$= 0$$

from (3.4). Therefore, $e_1^i = e_2^i = \ldots = e_b^i = e_.^i = 2k/n$. Since e_j^i ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, b$) must be integral, 2k is divisible by n.

For example, if $v = 15$, $b = 15$, $r = 3$, $k = 3$, $n_1 = 8$, $n_2 = 6$, $\lambda_1 = 0$, $\lambda_2 = 1$. We have

Association scheme

*	1	2	3	4	5
1	*	6	7	8	9
2	6	*	10	11	12
3	7	10	*	13	14
4	8	11	13	*	15
5	9	12	14	15	*

Reps.	Plan														
I	1	11	12	2	14	13	15	8	3	6	4	9	5	7	10
II	10	1	13	15	2	9	3	12	11	14	7	4	6	5	8
III	15	14	1	7	8	2	6	3	9	4	12	10	13	11	5

This triangular association scheme is $rk - v\lambda_1 = n(r - \lambda_1)/2 = 6$, then 2k is divisible by n.

The Latin square type

There are $v = n^2$ treatments (n a positive integer). This type is denoted by LS. The number of treatments is a perfect square, say $v = n^2$. The treatment may be set forth in a square scheme. For the case i = 2, two treatments are first associates if they occur in the same row or column, and second associates otherwise. For the general case, we take a set of i - 2 mutually orthogonal Latin squares and superimpose them on the array. Then, two treatments are first associates if they occur in the same row or column of the array, or if they correspond to the same letter in one of the squares.

For this scheme,

$$n_1 = i(n - 1), \qquad n_2 = (n - i + 1)(n - 1) \tag{3.35}$$

$$P^1_{jk} = \begin{bmatrix} i^2 - 3i + n & (i - 1)(n - i + 1) \\ (i - 1)(n - i + 1) & (n - i)(n - i + 1) \end{bmatrix}$$

$$P^2_{jk} = \begin{bmatrix} i(i - 1) & i(n - i) \\ i(n - i) & (n - 1)^2 + i - 2 \end{bmatrix} \tag{3.36}$$

For example, if $v = 9$, $b = 9$, $r = 4$, $k = 4$, $n_1 = 4$, $n_2 = 4$, $\lambda_1 = 1$, $\lambda_2 = 2$, $i = 2$

$$P^1_{jk} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \quad P^2_{jk} = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$$

Association scheme

1	2	3
4	5	6
7	8	9

Plan

Reps.									
I	1	6	8	9	2	4	5	7	3
II	6	8	1	2	4	9	7	3	5
III	9	2	4	5	7	3	1	6	8
IV	2	4	9	7	3	5	6	8	1

The cyclic type

For this case, instead of taking the treatments to be 1, 2, 3, ..., v. The first associates of the treatment i are $i + d_1$, $i + d_2$, ..., $i + d_{n_1}$, where $d_1, d_2, \ldots, d_{n_1}$ are integers satisfying the following conditions:

1. The d_j are all different, and $0 < d_j < v$ for each $j (j = 1, 2, \ldots, n_1)$.

2. Among the $n_1(n_1 - 1)$ differences $d_j - d_j'(j, j' = 1,2,\ldots,n_1, j \neq j')$, each of the integers $d_1, d_2, \ldots, d_{n_1}$ occurs g times, and each of the other n_2 positive integers less than v occurs h times.

Then $n_1 g + n_2 h = n_1(n_1 - 1)$, $g = P^1_{11}$, $h = P^2_{11}$.

The parameters P^i_{jk} are given by

$$P^1_{jk} = \begin{bmatrix} g & n_1 - g - 1 \\ n_1 - g - 1 & n_2 - n_1 + g + 1 \end{bmatrix}$$

$$P^2_{jk} = \begin{bmatrix} h & n_1 - h \\ n_1 - h & n_2 - n_1 + h - 1 \end{bmatrix} \tag{3.37}$$

As an example, consider the following design

$v = 13$, $b = 13$, $r = 3$, $k = 3$, $n_1 = 6$, $n_2 = 6$, $\lambda_1 = 1$, $\lambda_2 = 0$, $g = 2$, $h = 3$

$$P^1_{jk} = \begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix} , \quad P^2_{jk} = \begin{bmatrix} 3 & 3 \\ 3 & 2 \end{bmatrix}$$

Association scheme

Variety	1st associates
I	3 6 7 8 9 12

Plan

Reps.													
I	1	2	3	4	5	6	7	8	9	10	11	12	13
II	3	4	5	6	7	8	9	10	11	12	13	1	2
III	9	10	12	12	13	1	2	3	4	5	6	7	8

The simple type

A partially balanced design with two associate classes is said to be simple if either: (a) $\lambda_1 \neq 0$, $\lambda_2 = 0$, or (b) $\lambda_1 = 0$, $\lambda_2 \neq 0$. The case (b) can be reduced to (a) by interchanging the designation of first and second associates. Hence, case (a) will be taken as the standard. It may happen that a design of the simple type also belongs to one of the other types; viz., group divisible, triangular, Latin square, or cyclic. For a simple design, a separate association scheme is not necessary since the plan itself serves as an association scheme. Any two treatments which occur together in the same block must necessarily be first associates (it being assumed that the design is in the standard form $\lambda_1 \neq 0$, $\lambda_2 = 0$).

For example, if $v = 19$, $b = 19$, $r = 3$, $k = 3$, $n_1 = 6$, $n_2 = 12$, $\lambda_1 = 1$, $\lambda_2 = 0$.

$$P^1_{jk} = \begin{bmatrix} 1 & 4 \\ 4 & 8 \end{bmatrix} , \quad P^2_{jk} = \begin{bmatrix} 2 & 4 \\ 4 & 7 \end{bmatrix}$$

Plan

Reps.																			
I	1	4	7	15	19	5	16	8	12	14	17	11	2	3	18	6	10	9	13
II	3	1	6	8	4	11	5	10	13	16	18	2	14	9	15	12	19	17	7
III	2	5	1	4	13	18	10	9	11	15	19	8	17	12	3	14	6	7	16

CHAPTER IV

ANALYSIS OF PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

When partially balanced incomplete block designs are used for procuring experimental data, it may be possible to make two independent analyses on the data. The usual analysis, termed the intra-block analysis, depends only on comparisons within the block; whereas the second analysis, termed the inter-block analysis makes use of the block totals only. This latter analysis is sometimes referred to as "the recovery of inter-block information," and depends on the assumption that the block effects are random variables.

The customary use of the inter-block information is to combine this additional information with the intra-block analysis so as to estimate the treatment effects with greater precision than if the intra-block estimates had been used alone.

Intra-block analysis

Let there be v treatments whose effects it is required to compare by using a partially balanced incomplete block design with two associate classes. We may divide the N(= bk) experimental units to which the treatments are to be applied into b blocks, each block containing k units. The design divides the treatments into b sets, such that each set contains k treatments, and each treatment occurs in r sets, other conditions for a partially balanced design with two associate classes being satisfied. The b sets of treatments are assigned randomly to the b blocks, and the individual treatments of a set are assigned randomly to the units of the corresponding block. Let y_{ij} represent the observation of the ith treatment in the jth block. Furthermore, we assume that the mathematical model can be written as[15]

$$y_{ij} = \mu + t_i + b_j + \varepsilon_{ij} \qquad (i = 1,2,\ldots,v;\ j = 1,2,\ldots,b) \tag{4.1}$$

where y_{ij} is only defined when the ith treatment occurs in the jth block. The parameter μ represents a general component common to all observations, and component t_i and b_j denote the constant effects of the ith treatment and the jth block, respectively. In order to obtain the minimum variance unbiased estimates of the unknown parameters, it is only necessary to assume that the ε_{ij} are a sequence of uncorrelated random variables having a mean zero and variance σ^2. We also assume that the ε_{ij} follow a normal distribution.

Let

$$n_{ij} = \begin{cases} 1 \text{ if the ith treatment appears in the jth block.} \\ 0 \text{ otherwise.} \end{cases}$$

and denote by T_i the total of the observations for the ith treatment; i.e., T_i is the sum of the observations from the r experimental units to which the ith treatment has been applied. Let B_i be the sum of the totals for blocks in which treatment i occurs. Denote by kQ_i the adjusted treatment totals, where Q_i is obtained by subtracting from T_i the sum of the block averages for those blocks in which the ith treatment occurs. Thus, we have

$$kQ_i = kT_i - B_i \tag{4.2}$$

Further, let $kS_1(Q_i)$ be the sum of the adjusted treatment totals for

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[15]Bose and Shimamoto, pp. 167-170.

all the first associates of the ith treatment; and likewise, $kS_2(Qi)$ is the sum of the adjusted treatment totals for all the second associates of the ith treatment. We denote the grand total of the N observations by G.

Bose[16] has shown that the ith intra-block equation can be written

$$kQ_i = r(k - 1)\hat{t}_i - \sum_{s \neq i} \lambda_{is}\hat{t}_s \qquad i = 1, 2, \ldots, v \tag{4.3}$$

where λ_{is} is the number of blocks in which treatment i and s occur together, and $\hat{t}_i$ is the intra-block estimate of treatment effect i. The side condition is $\sum_i \hat{t}_i = 0$.

Now let $S_j(Q_i) = \sum_s Q_s$ and $S_j(\hat{t}_i) = \sum_s \hat{t}_s$, where the summation is made over all treatments s, such that i and s are jth associates. Then (4.3) becomes

$$kQ_i = r(k - 1)\hat{t}_i - \sum_{j=1} \lambda_j S_j(\hat{t}_i) \tag{4.4}$$

In the case of two associate classes, we have[17]

$$kS_1(Q_i) = - \lambda_1 n_1 \hat{t}_i + S_1(\hat{t}_i)\{r(k - 1) - \lambda_1 P^1_{11} - \lambda_2 P^1_{12}\}$$
$$+ S_2(\hat{t}_i)\{ - \lambda_1 P^2_{11} - \lambda_2 P^2_{12}\} \tag{4.5}$$

$$kS_2(Q_i) = - \lambda_2 n_2 \hat{t}_i + S_1(\hat{t}_i)\{ - \lambda_1 P^1_{12} - \lambda_2 P^1_{22}\}$$
$$+ S_2(\hat{t}_i)\{r(k - 1) - \lambda_1 P^2_{12} - \lambda_2 P^2_{22}\} \tag{4.6}$$

There are two methods of continuing the analysis. They are given by Rao[18] and by Bose and Shimamoto.[19]

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[16] R. C. Bose, "Least Square Aspects of the Analysis of Variance," Series 9 (Raleigh, North Carolina: University of North Carolina, 1949), pp. 10-12. (Mimeographed.)

[17] Z. Marvin, "Analysis for Some Partially Balanced Incomplete Block Designs Having a Missing Block," *Biometrics,* X (1954), 274.

[18] Rao, pp. 550-553.

[19] Bose and Shimamoto, p. 169.

Rao's solution.

$$\lambda_2\sum_i \hat{t}_i = 0 \qquad (\because \sum_i \hat{t}_i = 0)$$

$$\lambda_2\sum_i \hat{t}_i = \lambda_2\{\hat{t}_i + S_1(\hat{t}_i) + S_2(\hat{t}_i)\} = 0$$

$$\lambda_2\{\hat{t}_i + S_1(\hat{t}_i) + S_2(\hat{t}_i)\} = 0$$

$$\therefore - \lambda_2 S_2(\hat{t}_i) = \lambda_2 S_1(\hat{t}_i) + \lambda_2 \hat{t}_i \qquad (4.7)$$

Substituting from (4.7), then (4.4) becomes

$$kQ_i = r(k - 1)\hat{t}_i - \lambda_1 S_1(\hat{t}_i) + \lambda_2 S_1(\hat{t}_i) + \lambda_2 \hat{t}_i$$

$$= (r(k - 1) + \lambda_2)\hat{t}_i + (\lambda_2 - \lambda_1)S_1(\hat{t}_i)$$

$$\because \hat{t}_i + S_1(\hat{t}_i) + S_2(\hat{t}_i) = 0 \qquad (4.8)$$

We can simplify (4.5) as follows:

$$kS_1(Q_i) = -\lambda_1 n_1 \hat{t}_i + S_1(\hat{t}_i)\{r(k - 1) - \lambda_1 P^1_{11} - \lambda_2 P^1_{12}\}$$
$$+ (\hat{t}_i + S_1(\hat{t}_i))\{\lambda_1 P^2_{11} + \lambda_2 P^2_{12}\}$$
$$= - \lambda_1 n_1 \hat{t}_i + S_1(\hat{t}_i)\{r(k - 1) - \lambda_1 P^1_{11} - \lambda_2 P^1_{12}\}$$
$$+ \hat{t}_i\{\lambda_1 P^2_{11} + \lambda_2 P^2_{12}\} + S_1(\hat{t}_i)\{\lambda_1 P^2_{11} + \lambda_2 P^2_{12}$$
$$= \{ - \lambda_1 n_1 + \lambda_1 P^2_{11} + \lambda_2 P^2_{12}\}\hat{t}_i + S_1(\hat{t}_i)\{r(k - 1)$$
$$- \lambda_1 P^1_{11} - \lambda_2 P^1_{12} + \lambda_1 P^2_{11} + \lambda_2 P^2_{12}\}$$

Using (2.8) and (2.9),

$$kS_1(Q_i) = \{ - \lambda_1 n_1 - \lambda_1(n_1 - P^2_{12}) + \lambda_2 P^2_{12}\}\hat{t}_i + S_1(\hat{t}_i)\{r(k - 1)$$
$$- \lambda_1 P^1_{11} - \lambda_2(n_1 - 1 - P^1_{11}) + \lambda_1 P^2_{11} + \lambda_2(n_1 - P^2_{11})\}$$

$$kS_1(Q_i) = (\lambda_2 - \lambda_1)P^2_{12}\hat{t}_i + S_1(\hat{t}_i)$$
$$\{r(k - 1) + \lambda_2 + (\lambda_2 - \lambda_1)(P^1_{11} - P^2_{11})\} \qquad (4.9)$$

Let

$$A_{12} = r(k - 1) + \lambda_2$$

$$B_{12} = \lambda_2 - \lambda_1$$

$$A_{22} = (\lambda_2 - \lambda_1)P^2_{12}$$

$$B_{22} = r(k - 1) + \lambda_2 + (\lambda_2 - \lambda_1)(P^1_{11} - P^2_{11})$$

$$\Delta_r = A_{12}B_{22} - A_{22}B_{12}$$

Then (4.8) and (4.9) become

$$kQ_i = A_{12}\hat{t}_i + B_{12}S_1(\hat{t}_i) \quad (4.10)$$

$$kS_1(Q_i) = A_{22}\hat{t}_i + B_{22}S_1(\hat{t}_i) \quad (4.11)$$

Solving two equations (4.10) and (4.11), we get

$$\Delta_r\hat{t}_i = k\{B_{22}Q_i - B_{12}S_1(Q_i)\} \quad (4.12)$$

Rao has also shown that the variance of the estimate of the difference between two treatments is

$$Var(\hat{t}_i - \hat{t}_u) = 2k(B_{22} + B_{12})\sigma^2/\Delta_r \quad (4.13)$$

if treatment i and u are first associates

or

$$2kB_{22}\sigma^2/\Delta_r$$

if treatment i and u are second associates.

Example: See reference number 4 of Tables 3 and 4.

There have

$$v = b = 10,\ r = k = 4,\ \lambda_1 = 1,\ \lambda_2 = 2,\ n_1 = 6,\ n_2 = 3$$

$$P^1_{ij} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix},\quad P^2_{ij} = \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix}$$

$$A_{12} = 14,\ B_{12} = 1,\ A_{22} = 2,\ B_{22} = 13,\ \Delta_r = 180$$

using (4.12), then

$$45\hat{t}_1 = 13Q_1 - S_1(Q_1) = 13Q_1 - \sum_{i=2}^{7} Q_i$$

$$45\hat{t}_2 = 13Q_2 - S_1(Q_2) = 13Q_2 - \sum_{i=1,3,4,5,8,9} Q_i$$

$Var(\hat{t}_1 - \hat{t}_2) = 28\sigma^2/45$ if treatment 1 and 2 are first associates.

$Var(\hat{t}_1 - \hat{t}_8) = 26\sigma^2/45$ if treatment 1 and 8 are second associates.

The solution of Bose and Shimamoto. Rewriting the equations (4.5) and (4.6) as

$$kS_1(Q_i) = -\lambda_1 n_1 \hat{t}_i + a_{11}S_1(\hat{t}_i) + a_{12}S_2(\hat{t}_i) \tag{4.14}$$

$$kS_2(Q_i) = -\lambda_2 n_2 \hat{t}_i + a_{21}S_1(\hat{t}_i) + a_{22}S_2(t_i) \tag{4.15}$$

Where

$$a_{11} = r(k-1) - \lambda_1 P^1_{11} - \lambda_2 P^1_{12}$$

$$a_{12} = -\lambda_1 P^2_{11} - \lambda_2 P^2_{12}$$

$$a_{21} = -\lambda_1 P^1_{12} - \lambda_2 P^1_{22}$$

$$a_{22} = r(k-1) - \lambda_1 P^2_{12} - \lambda_2 P^2_{22} \tag{4.16}$$

Consider the linear combination

$$L_i = k^2 Q_i + c_1 k S_1(Q_i) + c_2 k S_2(Q_i) \tag{4.17}$$

Substituting from (4.14), (4.15), and (4.4)

$$\begin{aligned} L_i &= k\{r(k-1)\hat{t}_i - \lambda_1 S_1(\hat{t}_i) - \lambda_2 S_2(\hat{t}_i)\} + c_1\{-\lambda_1 n_1 \hat{t}_i + a_{11}S_1(\hat{t}_i) \\ &\quad + a_{12}S_2(\hat{t}_i)\} + c_2\{-\lambda_2 n_2 \hat{t}_i + a_{21}S_1(\hat{t}_i) + a_{22}S_2(\hat{t}_i)\} \\ &= rk(k-1)\hat{t}_i - (c_1\lambda_1 n_1 + c_2\lambda_2 n_2)\hat{t}_i + (a_{11}c_1 + a_{21}c_2 - \lambda_1 k) \\ &\qquad S_1(\hat{t}_i) + (a_{12}c_1 + a_{22}c_2 - \lambda_2 k)S_2(\hat{t}_i) \end{aligned} \tag{4.18}$$

We choose c_1 and c_2 so that (4.18) becomes

$$k^2 Q_i + c_1 k S_1(Q_i) + c_2 k S_2(Q_i) = rk(k-1)\hat{Q}_i$$

This requires that

$$-(c_1\lambda_1 n_1 + c_2\lambda_2 n_2)\hat{t}_i + (a_{11}c_1 + a_{21}c_2 - \lambda_1 k)S_1(\hat{t}_i)$$

$$+ (a_{12}c_1 + a_{22}c_2 - \lambda_2 k)S_2(\hat{t}_i) = 0$$

$$\because \hat{t}_i + S_1(\hat{t}_i) + S_2(\hat{t}_i) = 0$$

So

$$-\lambda_1 n_1 c_1 - \lambda_2 n_2 c_2 = -\lambda_1 k + a_{11}c_1 + a_{21}c_2 = -\lambda_2 k + a_{12}c_1 + a_{22}c_2$$

giving

$$\lambda_1 k = (a_{11} + \lambda_1 n_1)c_1 + (a_{21} + \lambda_2 n_2)c_2$$

$$\lambda_2 k = (a_{12} + \lambda_1 n_1)c_1 + (a_{22} + \lambda_2 n_2)c_2 \tag{4.19}$$

solving by determinants

$$c_1 = D_1/D$$

$$c_2 = D_2/D \tag{4.20}$$

Where

$$D = (a_{11} + \lambda_1 n_1)(a_{22} + \lambda_2 n_2) - (a_{12} + \lambda_1 n_1)(a_{21} + \lambda_2 n_2)$$

$$D_1 = \lambda_1 k(a_{22} + \lambda_2 n_2) - \lambda_2 k(a_{21} + \lambda_2 n_2)$$

$$D_2 = \lambda_1 k(a_{12} + \lambda_1 n_1) - \lambda_2 k(a_{11} + \lambda_1 n_1) \tag{4.21}$$

Substituting from (4.16) to D, we get

$$\begin{aligned} D = {} & \{r(k-1) - \lambda_1 P^1_{11} - \lambda_2 P^1_{12} + \lambda_1 n_1\}\{r(k-1) - \lambda_1 P^2_{12} - \lambda_2 P^2_{22} + \lambda_2 n_2\} \\ & - \{-\lambda_1 P^2_{11} - \lambda_2 P^2_{12} + \lambda_1 n_1\}\{-\lambda_1 P^1_{12} - \lambda_2 P^1_{22} + \lambda_2 n_2\} \\ = {} & \{r(k-1) - \lambda_1(n_1 - 1 - P^1_{12}) - \lambda_2 P^1_{12} + \lambda_1 n_1\} \\ & \{r(k-1) - \lambda_1 P^2_{12} - \lambda_2(n_2 - 1 - P^2_{21}) + \lambda_2 n_2\} \\ & - \{-\lambda_1(n_1 - P^2_{12}) - \lambda_2 P^2_{12} + \lambda_1 n_1\}\{-\lambda_1 P^1_{12} - \lambda_2(n_2 - P^1_{21}) + \lambda_2 n_2\} \\ = {} & \{r(k-1) + P^1_{12}(\lambda_1 - \lambda_2) + \lambda_1\}\{r(k-1) - P^2_{12}(\lambda_1 - \lambda_2) + \lambda_2\} \\ & + \{\lambda_1 P^2_{12} - \lambda_2 P^2_{12}\}\{\lambda_1 P^1_{12} - \lambda_2 P^1_{21}\} \\ & (\because P^2_{21} = P^2_{12}\ ,\ \ P^1_{21} = P^1_{12}) \end{aligned}$$

$$\begin{aligned} \therefore D = {} & (rk - r + \lambda_1)(rk - r + \lambda_2) + (\lambda_1 - \lambda_2)\{r(k-1)(P^1_{12} - P^2_{12}) \\ & + \lambda_2 P^1_{12} - \lambda_1 P^2_{12}\} \end{aligned}$$

(Bose and Shimamoto write $k^2\Delta$ for D.)

Substituting for a_{21} and a_{22} in D_1, we get

$$\begin{aligned} D_1 = {} & \lambda_1 k\{r(k-1) - \lambda_1 P^2_{12} - \lambda_2 P^2_{22} + \lambda_2 n_2\} \\ & - \lambda_2 k(-\lambda_1 P^1_{12} - \lambda_2 P^1_{22} + \lambda_2 n_2) \\ = {} & \lambda_1 k\{rk - r - \lambda_1 P^2_{12} - \lambda_2(n_2 - 1 - P^2_{12}) + \lambda_2 n_2\} \\ & - \lambda_2 k\{-\lambda_1 P^1_{12} - \lambda_2(n_2 - P^1_{12}) + \lambda_2 n_2\} \\ = {} & \lambda_1 k(rk - r + \lambda_2) + \lambda_1 k(\lambda_2 P^1_{12} - \lambda_1 P^2_{12}) - \lambda_2 k(\lambda_2 P^1_{12} - \lambda_1 P^2_{12}) \\ = {} & \lambda_1 k(rk - r + \lambda_2) + k(\lambda_1 - \lambda_2)(\lambda_2 P^1_{12} - \lambda_1 P^2_{12}) \end{aligned}$$

Using (4.20), we get

$$k\Delta c_1 = \lambda_1(rk - r + \lambda_2) + k\ (\lambda_1 - \lambda_2)(\lambda_2 P^1_{12} - \lambda_1 P^2_{12}) \qquad (4.22)$$

$$(\because c_1 = D_1/D = D_1/k^2\Delta,\ k\Delta c_1 = D_1/k)$$

Similarly, substituting for a_{11} and a_{12} in D_2, we get

$$D_2 = \lambda_1 k(-\lambda_1 P^2_{11} - \lambda_2 P^2_{12} + \lambda_1 n_1) - \lambda_2 k\{r(k-1) - \lambda_1 P^1_{11} - \lambda_2 P^1_{12} + \lambda_1 n_1\}$$

$$= \lambda_1 k\{-\lambda_1(n_1 - P^2_{12}) - \lambda_2 P^2_{12} + \lambda_1 n_1\}$$

$$- \lambda_2 k\{r(k-1) - \lambda_1(n_1 - 1 - P^1_{12}) - \lambda_2 P^1_{12} + \lambda_1 n_1\}$$

$$= \lambda_2 k(rk - r + \lambda_1) + \lambda_1 k(\lambda_2 P^1_{12} - \lambda_1 P^2_{12}) - \lambda_2 k(\lambda_2 P^1_{12} - \lambda_1 P^2_{12})$$

$$= \lambda_2 k(rk - r + \lambda_1) + k(\lambda_1 - \lambda_2)(\lambda_2 P^1_{12} - \lambda_1 P^2_{12})$$

$$\therefore k\Delta c_2 = \lambda_2(rk - r + \lambda_1) + (\lambda_1 - \lambda_2)(\lambda_2 P^1_{12} - \lambda_1 P^2_{12}) \qquad (4.23)$$

Bose and Shimamoto have also shown that the variance of the estimate of the difference between two treatment effects are

$$Var(\hat{t}_i - \hat{t}_u) = \frac{2(k - c_1)\sigma^2}{r(k-1)}$$

if treatments i and u are first associates, or

$$\frac{2(k - c_2)\sigma^2}{r(k-1)}$$

if treatments i and u are second associates.

In our example $a_{11} = 5$, $a_{12} = -8$, $a_{21} = -4$, $a_{22} = 10$.

Two equations (4.19) to be solved are

$$\begin{cases} 4 = 11c_1 + 2c_2 \\ 8 = -2c_1 + 16c_2 \end{cases}$$

$$c_1 = 4/15,\ c_2 = 8/15,\ D = 16\Delta = 180,\ \Delta = 45/4$$

$Var(\hat{t}_1 - \hat{t}_2) = 28\sigma^2/45$ if tretment 1 and 2 are first associates.

$Var(\hat{t}_1 - \hat{t}_8) = 26\sigma^2/45$ if treatment 1 and 8 are second associates.

In practice, it is convenient to write the solution

$$rk(k-1)\hat{t}_i = k^2 Q_i + c_1 k S_1(Q_i) + c_2 k S_2(Q_i)$$

$$\because Q_i + S_1(Q_i) + S_2(Q_i) = 0$$

$$rk(k-1)\hat{t}_i = k^2 Q_i + c_1 k S_1(Q_i) - c_2 k\{Q_i + S_1(Q_i)\}$$

$$rk(k - 1)\hat{t}_i = (k - c_2)kQ_i + (c_1 - c_2)kS_1(Q_i) \qquad (4.25)$$

In the above example, we get the same result.

$$45\hat{t}_1 = 13Q_1 - S_1(Q_1) = 13Q_1 - \sum_{i=2}^{7} Q_i$$

The combined inter- and intra-block analysis

The general formulae for the recovery of inter-block information were shown by Nair[20] and Rao[21]. Bose and Shimamoto have also defined as additional constant H for the combined inter- and intra-block analysis,

$$kH = \{2r(k - 1) + \lambda_1 + \lambda_2\} + (P^1_{12} - P^2_{12})(\lambda_1 - \lambda_2) \qquad (4.26)$$

and define

$$w_1 = 1/\sigma^2 \qquad (4.27)$$

$$w_2 = \frac{t(r - 1)}{k(b - 1)\sigma_b^2 - (t - k)\sigma^2} \qquad (4.28)$$

$$W = \frac{w_2}{w_1 - w_2} \qquad (4.29)$$

We also require constant d_1 and d_2, which take the place of the c_1 and c_2 in the intra-block analysis.

$$d_1 = (c_1\Delta + r\lambda_1 W)/(\Delta + rHW + r^2W^2) \qquad (4.30)$$

$$d_2 = (c_2\Delta + r\lambda_2 W)/(\Delta + rHW + r^2W^2) \qquad (4.31)$$

The combined intra- and inter-block estimate $\hat{t}'_i$ of the treatment t_i is given by the equation

$$r\{w_2 + w_1(k - 1)\}\ \hat{t}'_i = (k - d_2)P_i + (d_1 - d_2)S_1(P_i) \qquad (4.32)$$

Where

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[20]K. R. Nair, "The Recovery of Inter-Block Information in Incomplete Block Designs," *Sankhya,* VI (1944), 383-390.

[21]Rao, pp. 550-553.

$$Q_i' = T_i - Q_i - rG/N$$

$$P_i = w_1 Q_i + w_2 Q_i'$$

And $S_1(P_i)$ is the sum of the P_i's for those treatments which are the first associates of the ith treatment.

The variance of the estimate of the difference between two treatment effects is given by

$$\mathrm{Var}(\hat{t}_i' - \hat{t}_u') = \{2(k - d_1)\}/[r\{w_2 + w_1(k - 1)\}]$$

if treatment i and u are first associates.

$$\text{or} = \{2(k - d_2)\}/[r\{w_2 + w_1(k - 1)\}] \qquad (4.33)$$

if treatment i and u are second associates.

Computational details for the intra-block analysis and combined inter- and intra-block analysis are given in Cochran and Cox.[22]

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[22]Cochran and Cox, pp. 456-463.

CHAPTER V

SUMMARY

The essential feature of balanced incomplete block designs is that each treatment is paired with every other treatment an equal number of times within a common block; λ is a constant for all treatments. These balanced designs may be considered as special cases of a broader class of designs known as partially balanced incomplete block designs.

In the general case of partially balanced incomplete block designs, each treatment may have m associate classes. Treatments in the first associate class of treatment i are paired with treatment i in λ_1 blocks; treatments in the second associate class of i are paired with i in λ_2 blocks; and treatments in the mth associate class of i are paired with i in λ_m blocks. Balanced incomplete block designs form that special case of partially balanced designs in which there is one associate class for each treatment.

The partially balanced incomplete block designs with two associate classes are classified as: (a) group divisible type; (b) triangular block type; (c) Latin square type; (d) cyclic type; and (e) simple type. Group divisible designs are divided into three types: (a) singular; (b) semi-regular; and (c) regular.

The symmetries imposed upon the structure of partially balanced designs are what permit simplified solution of the normal equations obtained in using the least-square principle in the estimation process. Partially balanced incomplete block designs may be analyzed either with or without the recovery of inter-block treatment information. Extensive tables of partially balanced incomplete block designs have been compiled by Bose, Clatworthy, and Shrikhande.[23] Since all treatments are not paired an equal number of times within a common block, in comparing pairs of treatments, the proper error terms depend upon the kind of association between the pair in question.

[23]Bose, Clatworthy, and Shrikhande, pp. 90-255.

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