JOURNAL OF MODERN OPTICS, 1998, VOL. 45, NO. 3, 539–554

Partially coherent sources with helicoidal modes

F. GORI, M. SANTARSIERO, R. BORGHI
Dipartimento di Fisica, Università 'Roma Tre', Via della Vasca Navale, 84; 00146 Rome, Italy

and S. VICALVI
Dipartimento di Fisica, Università 'La Sapienza', P. le Aldo Moro, 2; 00185 Rome, Italy

(*Received 9 April 1997*)

Abstract. On the basis of the modal theory of coherence, we study partially coherent sources whose modes belong to the class of Laguerre–Gauss functions for which the Laguerre polynomial has zero order. These modes present a phase profile with a helicoidal structure, which is responsible for notable phenomena, such as the propagation of optical vortices, beam twisting, and the presence of dislocations in interference patterns. By suitably choosing the eigenvalues associated with such modes, different partially coherent sources are obtained: sources with a flattened Gaussian profile, twisted Gaussian Schell-model sources with a saturated twist, and a new class of sources having an annular profile. Owing to the shape-invariance property of the underlying modes, the fields radiated by these sources do not change their transverse profile through propagation, except for scale and phase factors. We also prove that, if any such source is covered by a circularly symmetric filter, the new modal structure can be found in a straightforward manner.

1. Introduction

Since its introduction in 1982, Wolf's [1] modal theory of coherence has played an important role in our understanding of partially coherent sources and of the fields that they generate [2]. Explicit knowledge of the modes is not required for certain applications [3] whereas it is essential in other cases [4]. For a prescribed cross-spectral density (CSD) across the source the mode evaluation entails solution of a Fredholm integral equation and this can be a formidable task from the mathematical standpoint. On the other hand, we can ask what type of source is obtained by superposing modes of a family of orthogonal functions with certain weights. This is the same as passing from an analysis to a synthesis problem. In doing this we are free to choose any set of orthogonal functions. As a useful choice, we can refer to modes whose propagation features are known because this makes it easy to describe how the overall field radiated by the source behaves in the course of propagation. Procedures of this type have been used with reference to Hermite–Gaussian modes [5].

In recent times, interest has arisen in helicoidal fields. By this term we mean fields whose wave fronts are endowed with vortex structures [6–8]. Peculiar twisting phenomena are then observed under propagation [9–13]. Further, use of vortices for practical applications has been proposed [7, 13, 14].

Little is known about the role of vortices in partially coherent fields. The most significant results are to be found in twisted Gaussian Schell-model (TGSM) sources [15], whose modal structure, synthesis procedures and propagation features have been discussed [16–20].

In this paper we examine some general features of partially coherent planar sources whose modes are the simplest type of coherent vortex fields belonging to the class of Laguerre–Gauss beams [21]. After some preliminary remarks about the modal theory of partial coherence (section 2), in section 3 we give the general expression for the CSD of sources with these kinds of mode. In the following three sections we specialize our analysis to some cases, namely sources with a flattened Gaussian profile [22], TGSM sources, and a new class of sources with an annular profile. Finally, in section 7 we show that, if any of the previous sources is covered by a filter with a circularly symmetric transmission function, the modal expansion for the resulting source can always be determined explicitly.

2. Preliminaries

Let us recall that the CSD across a typical planar (primary or secondary) source can be written [2]

$$W(\mathbf{r}_1, \mathbf{r}_2) = \langle V^*(\mathbf{r}_1) V(\mathbf{r}_2) \rangle, \tag{1}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of two points in the source plane, V is the optical disturbance, the angular brackets denote average over a suitable ensemble of monochromatic fields, and the asterisk stands for complex conjugate. We do not indicate explicitly the dependence of W on the frequency because we assume the latter to be fixed. Wolf's modal expansion of W is [2]

$$W(\mathbf{r}_1, \mathbf{r}_2) = \sum_n \beta_n \phi_n^*(\mathbf{r}_1) \phi_n(\mathbf{r}_2), \tag{2}$$

where β_n and ϕ_n are eigenvalues and eigenfunctions respectively of the following integral equation:

$$\int W(\mathbf{r}_1, \mathbf{r}_2) \phi_n(\mathbf{r}_1) \, \mathrm{d}^2 r_1 = \beta_n \phi_n(\mathbf{r}_2), \tag{3}$$

integration being extended to the (possibly infinite) source domain. It is assumed that W belongs to the class of Hilbert–Schmidt kernels. It turns out that W is semidefinite positive so that all its eigenvalues β_n are non-negative. The summation index in equation (2) stands for the set of numbers needed to specify an eigenfunction. Generally, for a two-dimensional source a pair of integers is required. The eigenfunctions ϕ_n are known as the modes of the source and represent coherent fields distributions to be assumed as normalized and mutually orthogonal. In physical terms they correspond to different contributions to the total field across the source that oscillate in a completely uncorrelated manner (on ensemble average). The eigenvalue β_n then accounts for the fraction of the source power to be associated to the nth mode. We shall loosely refer to the β_n values as the weights of the modes.

Let us now recall that in the set of Laguerre–Gauss beams there exists a subset in which the Laguerre polynomial has order zero and therefore is a constant. Using cylindrical coordinates r, ϑ, z the normalized field distributions associated with these beams are [21]

$$\Psi_s(r,\vartheta,z) = \left(\frac{2}{\pi|s|!}\right)^{1/2} \frac{\exp\{i[kz - (|s|+1)\Phi_z]\}}{w_z} \times \left(\frac{2^{1/2}r}{w_z}\right)^{|s|} \exp(is\vartheta) \exp\left[\left(\frac{ik}{2R_z} - \frac{1}{w_z^2}\right)r^2\right] \quad (s = 0, \pm 1, \pm 2, \ldots), \quad (4)$$

where k is the wavenumber and w_z, R_z and Φ_z have the well known expressions

$$w_z = w_0\left[1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2\right]^{1/2}, \quad R_z = z\left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right], \quad \Phi_z = \tan^{-1}\left(\frac{\lambda z}{\pi w_0^2}\right). \quad (5)$$

Here λ is the wavelength and w_0 is the spot size at the waist. The term $\exp(is\vartheta)$ accounts for the vortex structure of the wave fronts, which are helicoidally shaped. The sign of s determines the sense of rotation of the helicoid. We note that the fields represented by equation (4) are shape invariant because the corresponding intensity distributions across any plane $z = \text{constant}$ vary only for a scale factor with respect to those seen at $z = 0$. In the following, for brevity, we shall denote by $\Psi_{0s}(\mathbf{r})$ the field distributions corresponding to equation (4) at $z = 0$, or

$$\Psi_{0s}(\mathbf{r}) = \frac{1}{w_0}\left(\frac{2}{\pi|s|}\right)^{1/2}\left(\frac{2^{1/2}r}{w_0}\right)^{|s|} \exp(is\vartheta)\, \exp\left(-\frac{r^2}{w_0^2}\right) \quad (s = 0, \pm 1, \pm 2, \ldots). \quad (6)$$

3. Sources with twisting modes

In this section we want to discuss some general features of partially coherent planar sources whose modes are of the form (6). We then assume that the source lies in the plane $z = 0$ and that its cross-spectral density has the form

$$W_0(\mathbf{r}_1, \mathbf{r}_2) = \sum_{s=-\infty}^{\infty} \beta_s \Psi_{0s}^*(\mathbf{r}_1)\Psi_{0s}(\mathbf{r}_2), \quad (7)$$

where β_s are non-negative numbers such that the series converges. Written in a more explicit form, equation (7) is

$$W_0(\mathbf{r}_1, \mathbf{r}_2) = \frac{2}{\pi w_0^2} \exp\left(-\frac{r_1^2 + r_2^2}{w_0^2}\right) \sum_{s=-\infty}^{\infty} \frac{\beta_s}{|s|!}\left(\frac{2r_1 r_2}{w_0^2}\right)^{|s|} \exp[is(\vartheta_2 - \vartheta_1)]. \quad (8)$$

The expression for the CSD across a plane $z = \text{constant} > 0$, say $W_z(\mathbf{r}_1, \mathbf{r}_2)$, is obtained by equation (7) on replacing the field distributions $\Psi_{0s}(\mathbf{r})$ at $z = 0$ by the propagated fields $\Psi_s(r, \vartheta, z)$ given by equation (4). We then obtain

$$W_z(\mathbf{r}_1,\mathbf{r}_2) = \sum_{s=-\infty}^{\infty} \beta_s \Psi_s^*(r_1,\vartheta_1,z)\Psi_s(r_2,\vartheta_2,z)$$

$$= \frac{2}{\pi w_z^2} \exp\left(\frac{\mathrm{i}k}{2R_z}(r_2^2 - r_1^2)\right) \exp\left(-\frac{r_1^2 + r_2^2}{w_z^2}\right)$$

$$+ \sum_{s=-\infty}^{\infty} \frac{\beta_s}{|s|!}\left(\frac{2r_1r_2}{w_z^2}\right)^{|s|} \exp\left[\mathrm{i}s(\vartheta_2 - \vartheta_1)\right]. \quad (9)$$

On comparing equations (8) and (9) we see that apart from scale and curvature terms the CSD maintains the same structure across any plane $z =$ constant. This is briefly expressed by saying that the partially coherent fields radiated by our sources are shape invariant. By virtue of this property we can refer most of our considerations to the plane $z = 0$. It should also be noted that equation (9) is rotationally invariant.

In order to distinguish between contributions of opposite helicity we shall set

$$\beta_s^{(+)} = \beta_s, \quad \beta_s^{(-)} = \beta_{-s}, \quad (s = 1, 2, \ldots). \quad (10)$$

Then equation (8) can be written

$$W_0(\mathbf{r}_1,\mathbf{r}_2) = \frac{2}{\pi w_0^2}\exp\left(-\frac{r_1^2 + r_2^2}{w_0^2}\right)\left[\beta_0 + f^{(+)}\left(\frac{2r_1r_2\exp\left[\mathrm{i}(\vartheta_2 - \vartheta_1)\right]}{w_0^2}\right)\right.$$

$$\left. + f^{(-)}\left(\frac{2r_1r_2\exp\left[-\mathrm{i}(\vartheta_2 - \vartheta_1)\right]}{w_0^2}\right)\right], \quad (11)$$

where the functions $f^{(+)}$ and $f^{(-)}$, defined as

$$f^{(+)}(\zeta) = \sum_{s=1}^{\infty} \frac{\beta_s^{(+)}}{s!}\zeta^s, \quad f^{(-)}(\zeta) = \sum_{s=1}^{\infty} \frac{\beta_s^{(-)}}{s!}\zeta^s, \quad (12)$$

account for incoherent superpositions of modes with positive helicity and negative helicity respectively.

Some particular cases can be considered. First, let us refer to the case in which only one of the two functions $f^{(+)}$ and $f^{(-)}$ is different from zero. In such a case all the underlying modes have the same type of helicoidal structure. In the course of propagation they all twist in the same sense and we can expect that some twisting property is inherited by the corresponding cross-spectral density. While this is not immediately evident from equation (9) because of its rotational invariance, the twisting properties can be exhibited when such an invariance is broken by the insertion of astigmatic optical elements, for example a cylindrical lens, on the path of the field [11, 15, 18, 19].

On the other hand, if neither $f^{(+)}$ nor $f^{(-)}$ vanishes, the overall field radiated by the partially coherent source has no *a priori* obvious twisting properties. A particular case occurs when $f^{(+)}$ and $f^{(-)}$ coincide. As can be seen from equation (12), this implies that $\beta_s^{(+)} = \beta_s^{(-)}$. The CSD can be written

$$W_0(\mathbf{r}_1,\mathbf{r}_2)=\frac{2}{\pi w_0^2}\exp\left(-\frac{r_1^2+r_2^2}{w_0^2}\right)\left[\beta_0+2\sum_{s=1}^{\infty}\frac{\beta_s^{(+)}}{s!}\left(\frac{2r_1r_2}{w_0^2}\right)^s\cos\left[s(\vartheta_2-\vartheta_1)\right]\right] \quad (13)$$

and can be thought of as the sum of contributions having angular amplitude modulation but no twist. Then no twisting phenomena are to be expected for the field radiated by the source.

In the sections to follow we shall examine some particular sources belonging to the above classes.

4. Partially coherent flattened Gaussian beams

Equation (11) for the modal expansion assumes a very simple form when only the function $f^{(+)}$ (or, equivalently, $f^{(-)}$), defined in equation (12), is different from zero and reduces to a finite sum of powers of the variable ζ. If we let, for example,

$$\beta_0=A,\quad \beta_s^{(+)}=\begin{cases}A & (s=1,2,\ldots,N),\\ 0 & (s>N),\end{cases}\qquad \beta_s^{(-)}\equiv 0\,(s=1,2,\ldots), \quad (14)$$

with $A>0$, equation (11) reduces to

$$W_0(\mathbf{r}_1,\mathbf{r}_2)=\frac{2A}{\pi w_0^2}\exp\left(-\frac{r_1^2+r_2^2}{w_0^2}\right)\sum_{s=0}^{N}\frac{1}{s!}\left(\frac{2r_1r_2\exp\left[\mathrm{i}(\vartheta_2-\vartheta_1)\right]}{w_0^2}\right)^s. \quad (15)$$

Recalling that [2]

$$G_0(\mathbf{r})=W_0(\mathbf{r},\mathbf{r}),\quad \mu_0(\mathbf{r}_1,\mathbf{r}_2)=\frac{W_0(\mathbf{r}_1,\mathbf{r}_2)}{\left[G_0(\mathbf{r}_1)G_0(\mathbf{r}_2)\right]^{1/2}}, \quad (16)$$

where G_0 is the spectral density or optical intensity at the chosen frequency and μ_0 the degree of spectral coherence, both at the plane $z=0$, we easily arrive at the following expressions:

$$G_0(\mathbf{r})=\frac{2A}{\pi w_0^2}\exp\left(-\frac{2r^2}{w_0^2}\right)\sum_{s=0}^{N}\left[\frac{1}{s!}\left(\frac{2r^2}{w_0^2}\right)^s\right], \quad (17)$$

$$\mu_0(\mathbf{r}_1,\mathbf{r}_2)=\left[\sum_{s=0}^{N}\frac{1}{s!}\left(\frac{2r_1r_2\exp\left[\mathrm{i}(\vartheta_2-\vartheta_1)\right]}{w_0^2}\right)^s\right]\Bigg/\left\{\sum_{s=0}^{N}\left[\frac{1}{s!}\left(\frac{2r_1^2}{w_0^2}\right)^s\right]\sum_{s=0}^{N}\left[\frac{1}{s!}\left(\frac{2r_2^2}{w_0^2}\right)^s\right]\right\}^{1/2}. \quad (18)$$

The expression for the optical intensity in equation (17) coincides with that for a flattened Gaussian profile of order N [22, 23]. It presents a plateau for values of r ranging from 0 up to approximately $w_0(N/2)^{1/2}$ and goes continuously to zero for higher values (figure 1). For $N=0$ it coincides with a Gaussian function, while it tends to a constant for increasing values of N.

In figure 2(*a*) the intensity distribution of a partially coherent flattened Gaussian beam (FGB) of order $N=4$ as a function of the normalized variable $\rho=r/w_0(N/2)^{1/2}$ is represented. With this choice, the transition to low values of the function occurs approximately around $\rho=1$. In figures 2(*b*), (*c*) and (*d*),

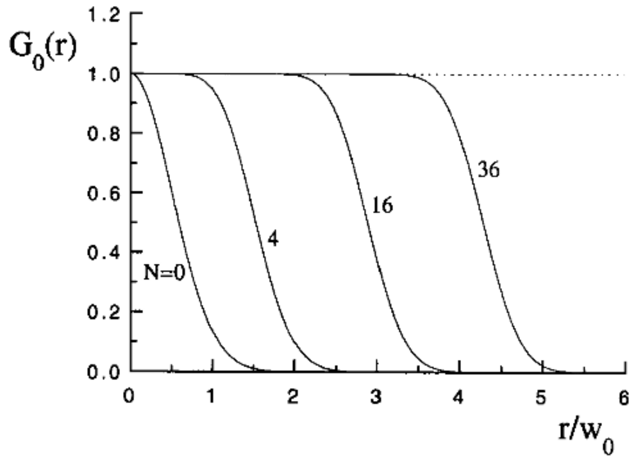

Figure 1. Optical intensity (arbitrary units) of flattened Gaussian sources (see equation (17)) for several values of N.

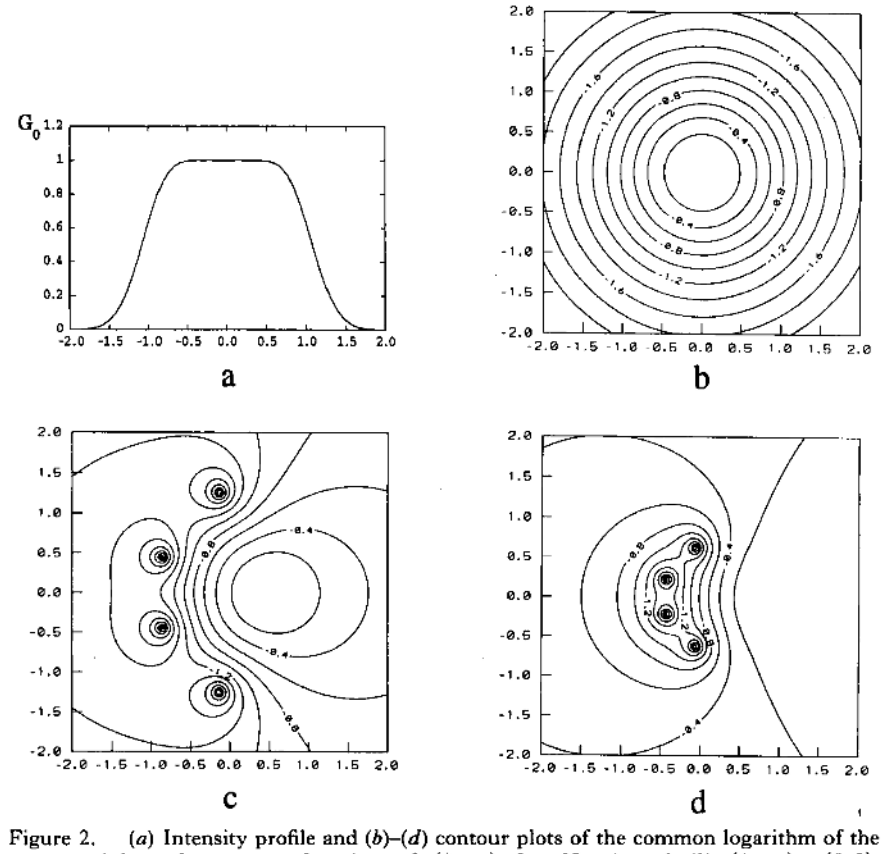

Figure 2. (*a*) Intensity profile and (*b*)–(*d*) contour plots of the common logarithm of the modulus of μ_0 as a function of (ξ_1, η_1) for $N = 4$ and (*b*) $(\xi_2, \eta_2) = (0, 0)$, (*c*) $(\xi_2, \eta_2) = (0{\cdot}5, 0)$ and (*d*) $(\xi_2, \eta_2) = (1, 0)$.

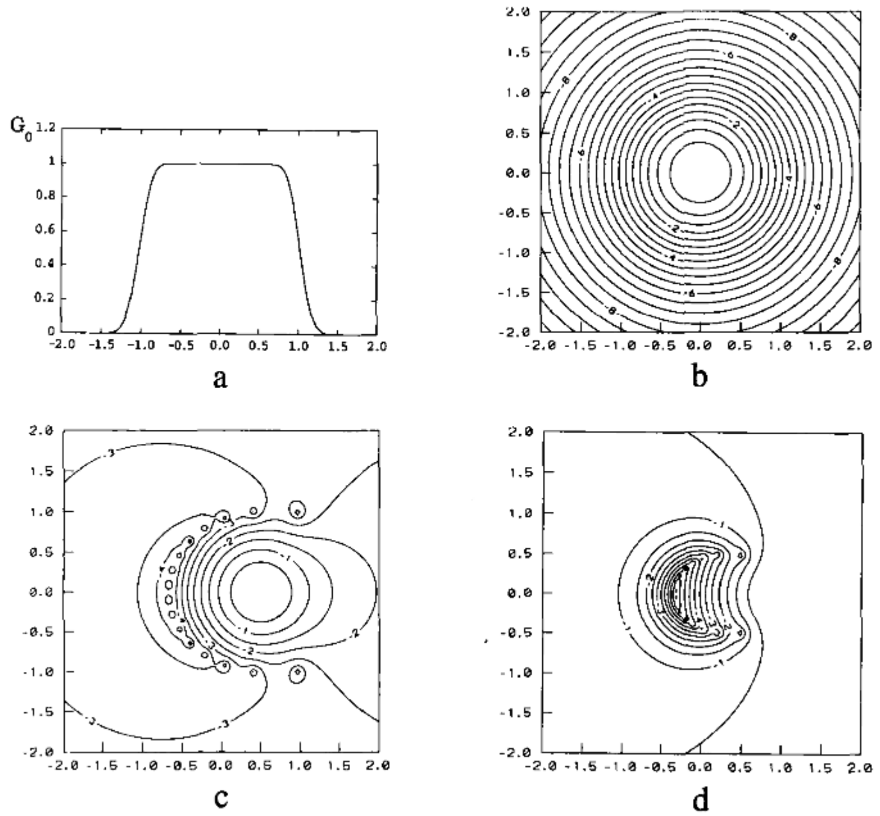

Figure 3. The same as in figure 2, but with $N = 16$.

contour plots of the common logarithm of the modulus of the degree of spectral coherence μ_0 are shown as a function of the normalized variables (ξ_1, η_1),† for $(\xi_2, \eta_2) = (0, 0)$, $(\xi_2, \eta_2) = (0{\cdot}5, 0)$ and $(\xi_2, \eta_2) = (1, 0)$ respectively. In figure 3 the same quantities as in figure 2 are shown, but with $N = 16$. The analysis of these figures leads to some remarks. First, the coherence area decreases on increasing N, and this is ascribed to the presence of a greater number of uncorrelated modes in the total field. Second, the coherence area increases when the point with coordinates (ξ_1, η_1) is far from the origin. This is because, owing to the spatial distribution of the Laguerre–Gauss functions, in the outside regions only higher-order modes contribute to the total field. Finally, we note that, except when $(\xi_2, \eta_2) = (0, 0)$, the degree of spectral coherence is not radially symmetric and presents a number of zeros equal to the order of the beam. In fact, if the vector $\mathbf{r}_2$ (i.e. the pair (ξ_2, η_2)) is kept fixed, the numerator of equation (18) is a polynomial of order N with respect to the complex variable $r_1 \exp(-\vartheta_1)$ and then it admits exactly N complex roots.

† By ξ and η we mean the Cartesian coordinates of the normalized vector $\boldsymbol{\rho}$.

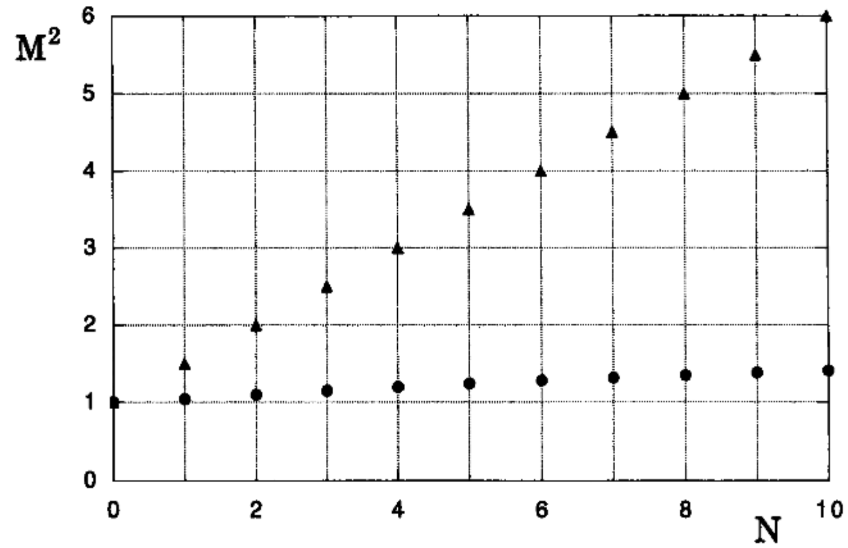

Figure 4. Behaviour of M^2 as a function of N for coherent (●) and partially coherent (▲) FGBs.

Propagation of coherent beams having a flattened Gaussian profile at a given transverse plane were studied in [24–26]. While, in the coherent case, flattening of the waist profile is lost on free-space propagation, partially coherent FGBs exhibit an intensity profile that is shape invariant, apart from a scale factor, as we remarked in the previous section. This is a consequence of the mutual uncorrelation between the modes of the source.

Since flat-top profiles [27–30] are desirable in several applications, one might argue that the partially coherent case should be preferred to the coherent case. While it is true that partially coherent beams can exhibit behaviours that could not be obtained by coherent beams, it is to be borne in mind that this does not occur without paying some price. In the present case, the width of the partially coherent FGB increases at a higher pace with respect to the coherent case.

For a better specification of this point let us refer to the M^2 quality factor [31], which indicates the diverging attitude of a beam once its spot size at the waist, that is its minimum width, has been fixed. M^2 can be evaluated for both coherent [24] and partially coherent FGBs. In the latter case it is simply given by the sum of the M^2 factors of the underlying modes, weighted by the coefficients of the modal expansion and normalized with respect to the energy carried out by the beam [31]. The final result, that is

$$M^2 = 1 + \frac{N}{2}, \tag{19}$$

is compared in figure 4 (full triangles) with the data obtained in the case of coherent FGBs (full circles). On increasing N, the difference between the two quality factors becomes greater and greater. This is because the M^2 of a partially coherent FGB is a linear function of N while in the coherent case it behaves as $N^{1/4}$ [24].

5. Twisted Gaussian Schell-model sources

Another simple choice for the expansion coefficients in equation (11) is the following:

$$\beta_0 = A, \quad \beta_s^{(+)} = Aq^s, \quad \beta_s^{(-)} = 0 \qquad (s = 1, 2, \ldots), \tag{20}$$

with $A > 0$ and $0 < q < 1$. In fact the function $f^{(+)}$ becomes in this case (see equation (12))

$$f^{(+)}(\zeta) = A[\exp(q\zeta) - 1], \tag{21}$$

while $f^{(-)}$ vanishes. Taking equation (11) into account, it is seen that the corresponding CSD is

$$W_0(\mathbf{r}_1, \mathbf{r}_2) = \frac{2A}{\pi w_0^2} \exp\left(-\frac{r_1^2 + r_2^2}{w_0^2}\right) \exp\left(\frac{2q}{w_0^2} r_1 r_2 \exp\left[i(\vartheta_2 - \vartheta_1)\right]\right). \tag{22}$$

By simple algebraic manipulations, equation (22) can be written

$$W_0(\mathbf{r}_1, \mathbf{r}_2) = \frac{2A}{\pi w_0^2} \exp\left(-\frac{1-q}{w_0^2}(r_1^2 + r_2^2)\right) \times \exp\left(-\frac{q}{w_0^2}(\mathbf{r}_1 - \mathbf{r}_2)^2\right) \exp\left(i\frac{2q}{w_0^2}(x_1 y_2 - x_2 y_1)\right), \tag{23}$$

where (x_j, y_j) are the Cartesian components of $\mathbf{r}_j$ $(j = 1, 2)$. The source described by equation (23) is characterized by the following quantities:

$$G_0(\mathbf{r}) = B_0 \exp\left(-\frac{r^2}{2\sigma_{G0}^2}\right), \quad \mu_0(\mathbf{r}_1, \mathbf{r}_2) = \exp\left(-\frac{(\mathbf{r}_1 - \mathbf{r}_2)^2}{2\sigma_{\mu 0}^2}\right) \exp\left[iu_0(x_1 y_2 - x_2 y_1)\right], \tag{24}$$

where

$$B_0 = \frac{2A}{\pi w_0^2}, \quad \sigma_{G0}^2 = \frac{w_0^2}{4(1-q)}, \quad \sigma_{\mu 0}^2 = \frac{w_0^2}{2q}, \quad u_0 = \frac{2q}{w_0^2}. \tag{25}$$

It belongs to the class of TGSM sources [15]. The only difference with respect to the ordinary Gaussian Schell-model source [2] is constituted by the twist term $\exp\left[iu_0(x_1 y_2 - x_2 y_1)\right]$.† The quantity u_0 is called the twist parameter and, for TGSM sources of the most general type described in [15], it has to satisfy the inequality

$$|u_0| \leqslant \frac{1}{\sigma_{\mu 0}^2}. \tag{26}$$

In the present case it is seen from equation (25) that the equality sign holds. One then speaks of saturated twist. In conclusion, the elementary procedure explained above has led us to a particular case of TGSM source. It should be noted that finding the modal expansion for the general case required an impressive amount of analytical work [16, 17]. It is a gratifying result that the case of saturated

† The sign in front of u_0 is minus, as in [15], or plus, as in the present case, depending on whether one defines the CSD in its original form [32] or in its complex conjugate form (equation (1)).

twist can be dealt with in such a simple way. We note that the expression of the cross-spectral density across any plane $z =$ constant could be easily written down using the shape-invariance property discussed in relation to equation (9). The reader is referred to [15, 19, 20] for a discussion of the effects of twist on the propagation of the field radiated by this type of sources. An experimental realization of TGSM sources using an acousto-optic coherence control technique has been reported in [18].

A further type of source can be obtained by superposing in an incoherent way two TGSM sources with opposite helicities. The resulting source is characterized by the following relations:

$$G_0(\mathbf{r}) = 2B_0 \exp\left(-\frac{r^2}{2\sigma_{G0}^2}\right), \quad \mu_0(\mathbf{r}_1, \mathbf{r}_2) = \exp\left(-\frac{(\mathbf{r}_1 - \mathbf{r}_2)^2}{2\sigma_{\mu 0}^2}\right) \cos\left[u_0(x_1 y_2 - x_2 y_1)\right]. \tag{27}$$

This is no longer a twisted source because the opposite twists of the two underlying TGSM sources cancel each other. In fact, it is not a Schell-model source because of the presence of the cosine term in the degree of spectral coherence. It is interesting to note, however, that the two TGSM sources with opposite twists as well as the source specified by equation (27) give rise to the same optical intensity distribution throughout the whole space. This is a further example of fields endowed with distinct coherence properties that cannot be distinguished from one another on the basis of intensity measurements [33–35].

Finally, it should be noted that the modes (6) constitute a one-parameter family, being in fact dependent only on s. On the contrary, when $|u| < \sigma_{\mu 0}^2$, eigenfunctions and eigenvalues depend on two indices [15, 17]. This is true in particular for vanishing twist ($u = 0$), in which case the TGSM source reduces to an ordinary GSM source [36, 37]. Therefore the presence of saturated twist simplifies the modal structure.

6. Annular twisted Gaussian Schell-model sources

In this section we introduce a new class of Gaussian correlated partially coherent sources† whose intensity distributions have annular shapes. The pertaining CSDs include a saturated twist term and, as we shall see in a moment, can be derived from that of a TGSM source with saturated twist.

Let us start from the modal expansion established in the preceding section. We shall write it in the form

$$\frac{2A}{\pi w_0^2} \exp\left(-\frac{r_1^2 + r_2^2}{w_0^2} + \frac{2q}{w_0^s} r_1 r_2 \exp\left[\mathrm{i}(\vartheta_2 - \vartheta_1)\right]\right) = A \sum_{s=0}^{\infty} q^s \Psi_{0s}^*(\mathbf{r}_1) \Psi_{0s}(\mathbf{r}_2), \tag{28}$$

obtained from equations (7), (10), (20) and (22). The series on the right-hand side is uniformly convergent with respect to q. Consequently we can take the derivative term by term. Taking the nth derivative of both sides of equation (28) (with respect to q) we obtain the identies

† Following [15], by Gaussian correlated source we mean a partially coherent source for which the modulus of the degree of spectral coherence is Gaussian.

$$\frac{2A}{\pi w_0^2}\left(\frac{2r_1r_2}{w_0^2}\exp\left[\mathrm{i}(\vartheta_2-\vartheta_1)\right]\right)^n \exp\left(-\frac{r_1^2+r_2^2}{w_0^2}+\frac{2q}{w_0^2}r_1r_2\exp\left[\mathrm{i}(\vartheta_2-\vartheta_1)\right]\right)$$
$$= A\sum_{s=n}^{\infty}\frac{s!}{(s-n)!}q^{s-n}\Psi_{0s}^*(\mathbf{r}_1)\Psi_{0s}(\mathbf{r}_2), \quad (29)$$

where the case $n = 0$ gives the same result as equation (28). For conciseness, in equation (29) and throughout this section we omit the specification $(n = 0, 1, \ldots)$. The right-hand sides can be thought of as modal expansions because the weights remain non-negative. This ensures that the kernels on the left-hand sides are positive semidefinite, and hence that they represent possible CSDs. Therefore we have the new class of partially coherent sources whose CSDs are

$$W_0^{(n)}(\mathbf{r}_1,\mathbf{r}_2;q) = B_0\left(\frac{2r_1r_2}{w_0^2}\exp\left[\mathrm{i}(\vartheta_2-\vartheta_1)\right]\right)^n$$
$$\times\exp\left(-\frac{r_1^2+r_2^2}{w_0^2}+\frac{2q}{w_0^2}r_1r_2\exp\left[\mathrm{i}(\vartheta_2-\vartheta_1)\right]\right), \quad (30)$$

the quantity B_0 being defined in equation (25). Here, for convenience, the dependence of W_0 on q has been explicitly indicated.

The main features of such sources are demonstrated by considering the optical intensity $G_0^{(n)}(\mathbf{r})$ and the spectral coherence degree $\mu_0^{(n)}(\mathbf{r}_1,\mathbf{r}_2)$. The following expressions are derived immediately:

$$G_0^{(n)}(\mathbf{r}) = B_0\left(\frac{2r^2}{w_0^2}\right)^n\exp\left(-\frac{2(1-q)}{w_0^2}r^2\right), \quad (31)$$

$$\mu_0^{(n)}(\mathbf{r}_1,\mathbf{r}_2) = \exp\left[\mathrm{i}n(\vartheta_2-\vartheta_1)\right]\exp\left(-\frac{q}{w_0^2}(\mathbf{r}_1-\mathbf{r}_2)^2+\mathrm{i}\frac{2q}{w_0^2}(x_1y_2-x_2y_1)\right). \quad (32)$$

We note that the modulus of the spectral coherence degree is Gaussian and depends on the difference $\mathbf{r}_1-\mathbf{r}_2$ only. Furthermore, the cross-spectral densities (30) have circular symmetry and contain a saturated twist term. Accordingly, the present sources constitute a generalization of the TGSM source with saturated twist, to which they reduce when $n = 0$.

An interesting feature of this class of sources is the following. From equations (31) and (32) we obtain

$$G_0^{(n)}(\mathbf{r}) = C_0\left(\frac{2r^2}{\rho_0^2}\right)^n\exp\left(-\frac{2r^2}{\rho_0^2}\right), \quad (33)$$

$$|\mu_0^{(n)}(\mathbf{r}_1,\mathbf{r}_2)| = \exp\left(-\frac{(\mathbf{r}_1-\mathbf{r}_2)^2}{\tau_0^2}\right), \quad (34)$$

where C_0 is independent of $\mathbf{r}$ and the quantities ρ_0^2 and τ_0^2, defined as

$$\rho_0^2 = \frac{w_0^2}{1-q}, \quad \tau_0^2 = \frac{w_0^2}{q}, \quad (35)$$

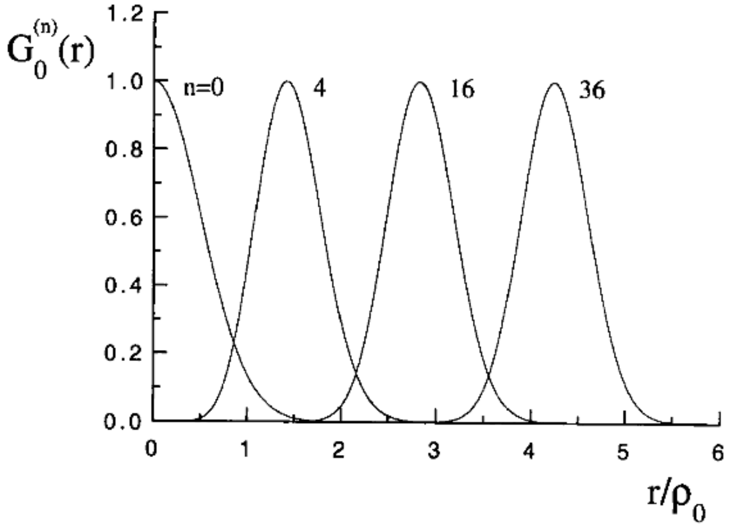

Figure 5. Normalized optical intensities of annular beams (see equation (33)) for several values of n.

give account of the source size and the coherence area respectively. We stress that, for fixed n, the intensity profile is completely determined by the value of ρ_0^2. By inverting equation (35) the following relations are easily obtained:

$$w_0^2 = \frac{1}{1/\rho_0^2 + 1/\tau_0^2}, \quad q = \frac{1}{1 + \tau_0^2/\rho_0^2}. \tag{36}$$

In particular we see that, for any choice of ρ_0^2, the quantity τ_0^2 may assume any value from zero to infinity, the corresponding values of w_0 and q being given by equation (36). This means that, for any given size of the source, the coherence area can be chosen at will.

For further characterization of our sources, note that, except for $n = 0$, they have annular shapes (figure 5). The dark central region can be traced back to the fact that the first $n - 1$ modes are absent in the modal expansion, as seen from equation (29). The value of r at which the maximum intensity is attained, say $r_0^{(n)}$ (figure 6), is easily found to be

$$r_0^{(n)} = \rho_0 \left(\frac{n}{2}\right)^{1/2}. \tag{37}$$

If we define the width $\Delta r_0^{(n)}$ of the annulus as the distance between the values of r where the second derivative of $G_0^{(n)}$ vanishes (figure 6) we find the expression

$$\Delta r_0^{(n)} = \frac{\rho_0}{2} \frac{[4n + 1 + (16n + 1)^{1/2}]^{1/2} - [4n + 1 - (16n + 1)^{1/2}]^{1/2}}{2^{1/2}}. \tag{38}$$

Since the size of the source can be chosen arbitrarily, a significant parameter is the relative width of the annulus, that is the ratio $\Delta r_0^{(n)}/r_0^{(n)}$. By taking equations (37) and (38) into account, this quantity turns out to be

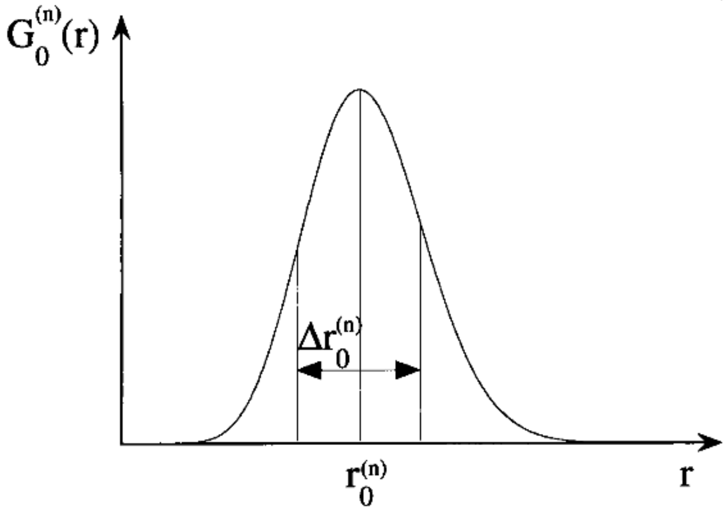

Figure 6. Definition of parameters used to characterize an annular profile.

$$\frac{\Delta r_0^{(n)}}{r_0^{(n)}} = \frac{[4n + 1 + (16n + 1)^{1/2}]^{1/2} - [4n + 1 - (16n + 1)^{1/2}]^{1/2}}{2n^{1/2}}, \tag{39}$$

or, for large values of n,

$$\frac{\Delta r_0^{(n)}}{r_0^{(n)}} \approx \frac{1}{n^{1/2}}. \tag{40}$$

Then, the relative width of the annulus can be controlled through the sole order n of the sources. For practical purposes, it is convenient to note that the estimate given in equation (40) differs from the exact value by less than 1% if $n > 3$.

So far we considered CSDs at the source plane. As already remarked in the previous section, the extension to plane $z = \text{constant} > 0$ is straightforward. We only stress that, because of the shape invariance of the underlying modes, the beams produced by such sources are themselves shape invariant, that is they maintain the annular transverse intensity distribution, apart from a scale factor, at any plane $z = \text{constant}$.

We recall that annular beams with different analytical expressions have been studied for the coherent case [38–40].

7. Effect of circularly symmetric filters on the modal structure

An important feature of the sources considered above is that the orthogonality of the underlying eigenfunctions stems from their angular dependence (see equation (6)). This implies that any two such functions remain orthogonal under multiplication by an arbitrary function of the radius r. This in turn allows us to find the pertinent modal expansion when any such source is covered by a circularly symmetric filter. The same technique was used for the determination of the modal structure of J_0-correlated Schell-model sources [41].

Suppose that the source is covered by a filter possessing a (possibly complex) transmission function, say $\tau(r)$, endowed with circular symmetry. The CSD of the field emerging from the source now becomes

$$W_0^{(\tau)}(\mathbf{r}_1, \mathbf{r}_2) = \tau^*(r_1)\tau(r_2)W_0(\mathbf{r}_1, \mathbf{r}_2) = \tau^*(r_1)\tau(r_2)\sum_{s=0}^{\infty}\beta_s\Psi_{0s}(\mathbf{r}_1)\Psi_{0s}(\mathbf{r}_2), \tag{41}$$

where W_0 has been expressed through its modal expansion (see equation (7)).

Let us define the functions

$$U_s(\mathbf{r}) = N_s\tau(r)\Psi_{0s}(\mathbf{r}), \quad (s = 0, 1, \ldots), \tag{42}$$

where N_s are normalization coefficients to be found in the following way. Taking into account the explicit form of the functions $\Psi_{0s}(\mathbf{r})$ (see equation (6)) the following relation is easily proved:

$$\int U_s^*(\mathbf{r})U_{s'}(\mathbf{r})\,\mathrm{d}^2r = \frac{4N_s^2\delta_{ss'}}{w_0^2 s!}\int_0^{\infty}|\tau(r)|^2\left(\frac{2r^2}{w_0^2}\right)^s\exp\left(-\frac{2r^2}{w_0^2}\right)r\,\mathrm{d}r, \tag{43}$$

where $\delta_{ss'}$ is the Kronecker symbol. The functions U_s are then orthonormal if we let

$$N_s = \left[\frac{4}{w_0^2 s!}\int_0^{\infty}|\tau(r)|^2\left(\frac{2r^2}{w_0^2}\right)^2\exp\left(-\frac{2r^2}{w_0^2}\right)r\,\mathrm{d}r\right]^{-1/2}. \tag{44}$$

Equation (41) can now be written

$$W_0^{(\tau)}(\mathbf{r}_1, \mathbf{r}_2) = \sum_{s=0}^{\infty}\beta_s' U_s^*(\mathbf{r}_1)U_s(\mathbf{r}_2). \tag{45}$$

This shows that the functions U_s are the modes of the modified source and the weights β_s' are given by

$$\beta_s' = \beta_s\left[\frac{4}{w_0^2 s!}\int_0^{\infty}|\tau(r)|^2\left(\frac{2r^2}{w_0^2}\right)^2\exp\left(-\frac{2r^2}{w_0^2}\right)r\,\mathrm{d}r\right]. \tag{46}$$

It should be stressed that the new modes (42) differ from the original modes. In particular, in the course of propagation they generate fields that are no longer shape invariant and therefore the study of propagation of the field generated by the present sources can be less immediate than for the cases in the previous sections.

8. Conclusions

In this work, the properties of some classes of partially coherent sources have been studied. Such sources are obtained as incoherent superpositions of Laguerre–Gauss beams having Laguerre polynomials of zero order. A notable characteristic of these beams is to present a vortex phase structure, which can be demonstrated through interference experiments. Fields of this type can be realized, for example, by converting Hermite–Gauss into Laguerre–Gauss beams by means of cylindrical lenses or, alternatively, by using masks with suitable transmission functions [42–45]. As a consequence of the shape invariance of the Laguerre–Gauss beams under paraxial propagation, beams generated by such partially coherent sources retain this property.

As a particular case, we presented a model for describing partially coherent beams having a flat-top transverse profile and studied how their divergence features depend on their coherence properties.

Furthermore, we showed how TGSM sources with saturated twist can be obtained through a suitable choice of the weights of the modal expansion, and also a new class of sources with annular intensity profiles can be defined. With reference to the latter class, we proved that, for any given relative width of the annulus, it is possible to tune the parameters in such a way that any value of the coherence area can be chosen. As limiting cases, perfectly coherent and complete incoherent annular sources are obtained.

References

[1] Wolf, E., 1982, *J. opt. Soc. Am.*, **72**, 343.

[2] Mandel, L., and Wolf, E., 1995, *Optical Coherence and Quantum Optics* (Cambridge University Press), chapters 4 and 5.

[3] Wolf, E., and Agarwal, G. S., 1984, *J. opt. Soc. Am.* A, **1**, 541.

[4] Gori, F., and Grella, R., 1984, *Optics Commun.*, **49**, 173.

[5] Gori, F., Guattari, G., Palma, C., and Padovani, C., 1988, *Optics Commun.*, **66**, 255.

[6] Berry, M. 1981, *Physics of Defects*, edited by R. Balian *et al.* (Amsterdam: North-Holland).

[7] Indebetouw, G., 1993, *J. mod. Optics*, **40**, 73.

[8] Harris, M., Hill, C. A., and Vaughan, J. M. R., 1994, *Optics Commun.*, **106**, 161.

[9] Serna, J., Mejías, P. M., Martínez-Herrero, R., 1992, *Opt. quant. Electron.*, **24**, S873.

[10] Abramochkin, E., and Volostnikov, V., 1993, *Optics Commun.*, **102**, 336.

[11] Gori, F., Bagini, V., Santarsiero, M., Frezza, F., Schettini, G., and Schirripa Spagnolo, G., 1994, *Opt. Rev.*, **1**, 143.

[12] Chávez-Cerda, S., McDonald, G. S., and New, G. H. C., 1996, *Optics Commun.*, **123**, 225.

[13] Paterson, C., and Smith, R., 1996, *Optics Commun.*, **124**, 131.

[14] Paterson, C., 1994, *J. mod. Optics*, **41**, 757.

[15] Simon, R., and Mukunda, N., 1993, *J. opt. Soc. Am.* A, **10**, 95.

[16] Simon, R., Sundar, K., and Mukunda, N., 1993, *J. opt. Soc. Am.* A, **10**, 2008.

[17] Sundar, K., Simon, R., and Mukunda, N., 1993, *J. opt. Soc. Am.* A, **10**, 2017.

[18] Friberg, A. T., Tervonen, E., and Turunen, J., 1994, *J. opt. Soc. Am.* A, **11**, 1818.

[19] Ambrosini, D., Bagini, V., Gori, F., and Santarsiero, M., 1994, *J. mod. Optics*, **41**, 1391.

[20] Simon, R., Friberg, A. T., and Wolf, E., 1996, *Pure appl. Optics*, **5**, 331.

[21] Siegman, A. E., 1986, *Lasers* (Mill Valley, California: University Science Books), chapter 16.

[22] Gori, F., 1994, *Optics Commun.*, **107**, 335.

[23] Gori, F., 1994, *Proceedings of the Second Workshop on Laser Beam Characterization*, edited by H. Weber, N. Reng, J. Lüdtke and P. M. Mejías (Berlin: Bundesministerium für Forschung und Technologie), p. 224.

[24] Bagini, V., Borghi, R., Gori, F., Pacileo, A. M., Santarsiero, M., Ambrosini, D., and Schirripa Spagnolo, G., 1996, *J. opt. Soc. Am.* A, **13**, 1385.

[25] Amarande, S.-A., 1996, *Optics Commun.*, **129**, 311.

[26] Santarsiero, M., Aiello, D., Borghi, R., and Vicalvi, S. 1997, *J. mod. Optics*, **44**, 633.

[27] De Silvestri, S., Laporta, P., Magni, V., and Svelto, O., 1988, *IEEE J. quant. Electron.*, **24**, 1172.

[28] Bélanger, P. A., Lachance, R. L., and Paré, C., 1991, *Optics Lett.*, **17**, 739.

[29] Perrone, M. R., Piegari, A., and Scaglione, S., 1993, *IEEE J. quant. Electron.*, **29**, 1423.

[30] Sheppard, C. J. R., 1996, *Optics Commun.*, **132,** 144.
[31] Siegman, A. E., 1990, *Optical Resonators*, Proceedings of the SPIE, Vol. 1224, edited by D. A. Holmes (Bellingham, Washington: SPIE), p. 2.
[32] Born, M., and Wolf, E., 1993, *Principles of Optics*, sixth edition (Oxford: Pergamon).
[33] Gori, F., Santarsiero, M., and Guattari, G., 1993, *J. opt. Soc. Am.* A, **10,** 673.
[34] Bagini, V., Gori, F., Santarsiero, M., Guattari, G., and Schirripa Spagnolo, G., 1993, *Optics Commun.*, **102,** 495.
[35] Gureyev, T. E., Roberts, A., and Nugent, K. A., 1995, *J. opt. Soc. Am.* A, **12,** 1942.
[36] Gori, F., 1980, *Optics Commun.*, **34,** 301.
[37] Starikov, A., and Wolf, E., 1982, *J. opt. Soc. Am.*, **72,** 923.
[38] Sheppard, C. J. R., and Wilson, T., 1978, *Microw., Optics Acoustics*, **2,** 105.
[39] Bagini, V., Frezza, F., Santarsiero, M., Schettini, G., and Schirripa Spagnolo, G., 1996, *J. mod. Optics*, **43,** 1155.
[40] Ojeda-Castaneda, J., Escalera, J. C., and Yzuel, M. J., 1995, *Optics Commun.*, **114,** 189.
[41] Gori, F., Guattari, G., and Padovani, C., 1987, *Optics Commun.*, **64,** 311.
[42] Tidwell, S. C., Ford, D. H., and Kimura, W. D., 1990, *Appl. Optics*, **29,** 2234.
[43] Khonina, S., Kotlyar, V., Soifer, V., Shinkaryev, M., and Uspleniev, G., 1992, *Optics Commun.*, **91,** 158.
[44] Beijersbergen, M. W., Allen, L., van der Veen, H. E. L. O., and Woerdman, J. P., 1993, *Optics Commun.*, **96,** 123.
[45] Padgett, M., Arlt, J., Simpson, N., and Allen, L., 1996, *Am. J. Phys.*, **64,** 77.