

Particle acceleration at oblique shock fronts

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Summary. We calculate the spectral index of particles accelerated at oblique shocks by a combination of the first-order Fermi mechanism and the shock drift mechanism. By assuming that particles conserve their magnetic moment on crossing the shock, and are scattered in pitch angle by turbulence both upstream and downstream of the shock, we show that oblique shocks lead to flatter spectra than parallel shocks. Synchrotron spectral indices as low as $\alpha \sim 0$ are predicted for highly oblique shocks. Only for extremely non-relativistic shocks is the spectral index independent of the angle between the magnetic field and the shock normal.

1 Introduction

Since the pioneering works of the late 1970s (Krimsky 1977; Axford, Leer & Skadron 1977; Bell 1978; Blandford & Ostriker 1978), many papers have addressed the problem of Fermi acceleration at shock waves (for a review see Drury 1983 or Blandford & Eichler 1987). For the most part, the mechanism has been discussed in the context of a simple parallel shock front, in which the magnetic field and the direction of the plasma flow both lie along the direction of the normal to the shock surface. Pitch-angle scattering by magnetic field fluctuations (Alfvén waves) causes the distribution of energetic particles to be almost isotropic, leading to diffusion in space of these particles. It is then possible to understand the acceleration process in terms of the competition between the gain in momentum by a particle as it diffuses from the downstream plasma into the upstream plasma and back, and the probability that it is advected away from the shock front by the the plasma flow. One of the main attractions of the theory is that, given some plausible restrictions, a power-law spectrum of accelerated particles is predicted with an index which does not depend on the microphysics involved in the scattering process, but is solely a function of the compression ratio of the shock.

At oblique shock fronts, the motion of a particle is more complicated, and it may either be transmitted through the shock or reflected from it. In addition, oblique shocks are expected to accelerate particles in the absence of scattering by the so-called ‘shock drift’ mechanism. On the other hand, it was noticed quite early (Bell 1978; Axford 1980) that, provided the motion

of the particles could be considered diffusive, the same result connecting the power-law index of accelerated particles with the compression ratio applies (see also Drury 1983). Nevertheless, the extension of the theory to oblique shocks involves several new features. Webb (1983) noticed that the derivation of the correct matching conditions to be applied at the shock front is more complicated in the case of oblique shocks than had previously been supposed. Jokipii (1982) was able to incorporate the phenomenon of particle drifts into the cosmic ray transport equation and obtained a relationship between the distance a particle ‘drifts’ along a shock and the energy to which it is accelerated. As a result, he obtained an expression for the rate at which a particle is accelerated. This can be several orders of magnitude quicker at an oblique shock than at a parallel one, in accordance with the findings of Ostrowski (1988). Oblique shocks also lead to damping of scattering waves, but not in the long-wavelength region of interest here (Achterberg & Blandford 1986).

Oblique shock fronts may be conveniently classified into two categories: subluminal and superluminal. In the former it is possible to find a Lorentz transformation to a frame of reference in which the electric field is zero in both the upstream and the downstream regions, and the shock front is stationary. This frame of reference is called the de Hoffmann–Teller frame (de Hoffmann & Teller 1950) and in it the energy of a particle remains constant provided it does not suffer scattering. Superluminal shocks, however, do not admit a transformation to such a frame of reference. They correspond to shock fronts in which the point of intersection of the front with a magnetic field line moves at a speed greater than c . Only subluminal shocks are considered in this paper.

Conflicting points of view appear in the literature concerning which kinds of shock front are able to accelerate particles diffusively. Thus, Bell (1978) and Axford (1980) advance the view that diffusive acceleration cannot occur at superluminal shocks, in agreement with Drury (1983). On the other hand, Jokipii (1982) shows that diffusive acceleration is extremely rapid for superluminal shocks, provided the particle distribution remains nearly isotropic. Clearly, the resolution of this question is connected with the extent to which the distribution may be assumed to be isotropic, and the extent to which particles may move across field lines. For subluminal shocks whose speeds in the de Hoffmann–Teller frame are not very small in comparison with c , the anisotropy of the distribution is also of crucial importance in determining the spectral index of the accelerated particles. To attack this problem, we extend, in this paper, the method developed by Kirk & Schneider (1987, hereafter KS), which was originally applied to the problem of acceleration at parallel relativistic shock fronts. This method (used also by Heavens & Drury 1988) explicitly calculates the degree of anisotropy possessed by the distribution function, and advances the earlier approximate treatment of Peacock (1981).

In Section 2 we discuss the various frames of reference used in the remainder of the paper. Section 3 is devoted to deriving the equations to be solved and presents the extension of the method of KS. Numerical results are presented in Section 4, followed by a discussion of our conclusions in Section 5.

2 Frames of reference

Consider a shock front travelling at speed u_{sh} into a stationary plasma containing a uniform magnetic field B_{up} at an angle Φ_{up} to the direction of shock propagation (Fig. 1a). We choose the coordinate system such that the shock front lies in the y - z plane and the magnetic field is in the x - y plane. This frame of reference will be called the upstream rest frame. We look for solutions to the shock acceleration problem which are independent of time in a frame in which the shock is at rest. If the plasma has infinite conductivity, it is most convenient to work in the

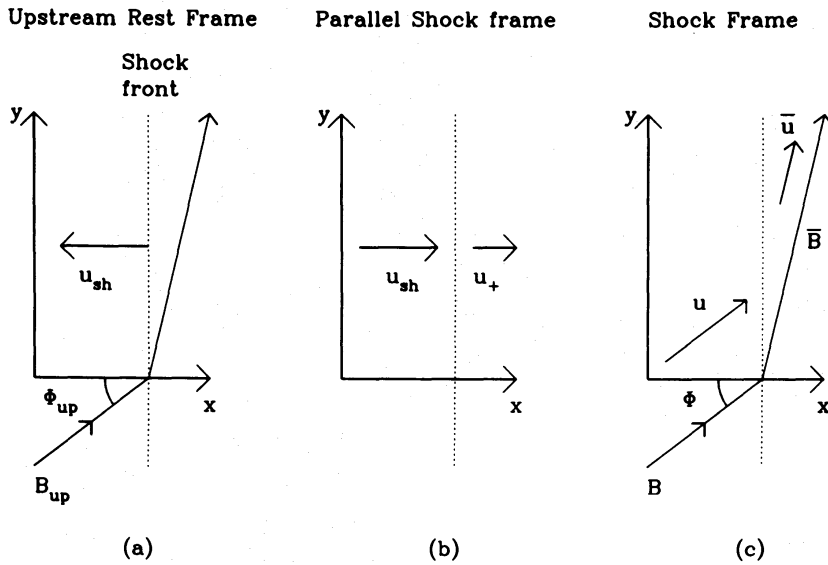


Figure 1. The coordinate systems used in the discussion of oblique shock fronts. In (a) the upstream rest frame is shown, in (b) the parallel shock frame, and in (c) the de Hoffmann–Teller frame, or ‘shock frame’.

de Hoffmann–Teller frame, in which the electric field vanishes everywhere (de Hoffmann & Teller 1950). There are two straightforward ways of transforming to this frame. In the first, which is also that used originally by de Hoffmann & Teller, a Lorentz transformation is made from the upstream rest frame in the direction of the magnetic field with a velocity equal to that of the point of intersection of the shock and a magnetic field line, namely $u_{sh}/\cos \Phi_{up}$. In the new frame of reference, the shock front no longer lies in the y - z plane, but this procedure has the advantage that it immediately gives the condition under which the transformation is possible, i.e. under which the shock front is subluminal:

$$u_{sh}/\cos \Phi_{up} < 1. \quad (1)$$

(Speeds are expressed in units of c .) In the following, we consider only shock fronts which fulfil this condition. The second way of transforming to the de Hoffmann–Teller frame is, however, more convenient for our purposes. From the upstream rest frame one first of all applies a Lorentz boost along the normal to the shock front (the x -axis) to the frame in which the shock is stationary (the ‘parallel shock frame’ – Fig. 1b). In this frame there exists an electric field, since infinite conductivity implies that $\mathbf{E} = -\mathbf{u} \wedge \mathbf{B}$. A second boost is then applied with speed $u_t = -u_{sh} \tan \Phi_{up} / \sqrt{1 - u_{sh}^2}$ along the shock front (in the y -direction) which eliminates the electric field, leaving the situation as in Fig. 1c. Clearly such a transformation is possible only if $|u_t| < 1$, which translates to the same condition as equation (1). The resulting frame of reference – the de Hoffmann–Teller frame – will be called simply the ‘shock frame’; it is related to the frame found by the first method by a simple rotation. Further details of the transformations may be found in the papers by de Hoffmann & Teller (1950), Hudson (1965) and Terasawa (1979). From now on we shall work in the shock frame, unless noted otherwise.

It is preferable to use the second method of reaching the de Hoffmann–Teller frame for two reasons: first, the shock front remains in the y - z plane. Secondly, in this paper we shall employ, for illustration, a simplified form of the shock jump conditions in order to focus our attention on the method of describing the accelerated particles. For the method we develop, the particular form of the jump conditions is irrelevant, but the simplified ones are most conveniently specified in the parallel shock frame. Assuming the magnetic field to be so weak that it has no influence on the plasma motion, it follows that the motion of the downstream plasma

is, like the upstream motion, directed along the shock normal. The upstream and downstream fluid speeds in this frame can be obtained from the fits of Heavens & Drury (1988). Three quantities suffice to determine the kinematics: the shock speed u_{sh} , the angle Φ_{up} and the downstream plasma speed, measured in the parallel shock frame, u_+ .

In the de Hoffmann–Teller or ‘shock frame’ (which should not be confused with the ‘parallel shock frame’), unadorned quantities such as \mathbf{B} , u and Φ will denote upstream quantities. A superior bar indicates downstream quantities. Sometimes it will be necessary to distinguish whether a quantity is measured in the shock frame or the rest frame of the fluid. Where confusion could arise, a subscript s will indicate that the quantity is measured in the shock frame, and a circumflex will indicate the fluid rest frame.

From the constancy of the component of the magnetic field normal to the shock front, it is straightforward to show that the magnitudes of the up- and downstream magnetic fields are related by

$$b \equiv \frac{B}{\bar{B}} = \frac{u}{r\bar{u}} \leq 1, \quad (2)$$

where r is the compression ratio of the shock. In our simplified shock we have

$$r = u_{\text{sh}}/u_+ \quad (3)$$

and

$$b = [r^2 - \Gamma_{\text{sh}}^2 (r^2 - 1)(\cos^2 \Phi_{\text{up}} - u_{\text{sh}}^2)]^{-1/2}, \quad (4)$$

where $\Gamma_{\text{sh}} = 1/\sqrt{1 - u_{\text{sh}}^2}$. The only quantities which enter the discussion of acceleration are u , \bar{u} and b , but for completeness we indicate how to obtain the full expressions for the transformation of the magnetic field. In the upstream plasma, the magnitude of the magnetic field in the shock frame equals the magnitude in the upstream rest frame (since the frames are connected by a Lorentz transformation along the direction of the field). The angle made with the shock normal in the shock frame is $\Phi = \arctan(\Gamma_{\text{sh}}\Gamma_t \tan \Phi_{\text{up}})$, where $\Gamma_t = 1/\sqrt{1 - u_t^2}$. Then, using the constancy of the normal component of the magnetic field, and the fact that the plasma flow is along the field in the shock frame, it is easy to find the magnitude and direction of the downstream field, using equations (2) and (4).

3 Method

3.1 TRANSPORT EQUATION WITH OBLIQUE FIELD

In this section, we calculate the transport equation in terms of the orthogonal spatial coordinate system shown in Fig. 1(c), and the particle momentum, expressed in spherical polar coordinates with axis along the magnetic field.

In the local fluid frame, the transport equation may be written:

$$\hat{p}^a \frac{\partial F_0}{\partial \hat{x}^a} - \frac{d\hat{p}^a}{d\tau} \frac{\partial F_0}{\partial \hat{p}^a} = \hat{p}^0 (\hat{C} + \hat{S}) \quad (5)$$

(Lindquist 1966; Riffert 1986; Kirk 1988b), where F_0 is the (Lorentz invariant) distribution function, \hat{p}^a is the four-momentum as measured in the local fluid frame, τ is the proper time, and \hat{C} and \hat{S} are collision and source respectively. We have assumed a Minkowski metric. In an

electromagnetic field, the force term is

$$\frac{d\hat{p}^a}{d\tau} = \frac{e}{m} \mathcal{F}^{ab} \hat{p}_b \quad (6)$$

where \mathcal{F}^{ab} is the electromagnetic four-tensor. We now mix frames, by changing the spacetime coordinates in (5) to the shock frame, but continuing to measure the momentum in the rest frame of the fluid. If \hat{L}_b^a represents the Lorentz transformation from the fluid frame to the shock frame, then

$$\hat{L}_b^a \hat{p}^b \frac{\partial F_0}{\partial x^a} + \frac{e}{m} \mathcal{F}_d^{\lambda} \hat{p}^d \frac{\partial F_0}{\partial \hat{p}^\lambda} = \hat{p}^0 (\hat{C} + \hat{S}), \quad (7)$$

where the Greek index λ runs from 1 to 3 because the particle 4-momentum contains only three independent components ($p^a p_a = \text{constant}$). This equation is the special relativistic form of equation (2.3) in Kirk (1988b). Unpacking it involves, in the case of the shock geometry introduced in Section 2, the Lorentz transformation for a boost $\mathbf{u} = (u \cos \Phi, u \sin \Phi, 0)$ which may be found in Jackson (1975, p. 541). Expressing the momentum (in the fluid frame) in a spherical coordinate system (p, μ, ϕ) with axis along the magnetic field one obtains:

$$\begin{aligned} & \Gamma(p^0 + u\mu p) \frac{\partial F_0}{\partial t} + \Gamma \cos \Phi (up^0 + \mu p) \frac{\partial F_0}{\partial x} + \Gamma \sin \Phi (up^0 + \mu p) \frac{\partial F_0}{\partial y} \\ & + p\sqrt{1-\mu^2} \left[-\sin \Phi \sin \phi \frac{\partial F_0}{\partial x} + \cos \Phi \sin \phi \frac{\partial F_0}{\partial y} + \cos \phi \frac{\partial F_0}{\partial z} \right] + \frac{e}{m} B \frac{\partial F_0}{\partial \phi} = p^0 (\hat{C} + \hat{S}), \quad (8) \end{aligned}$$

where Γ is the Lorentz factor corresponding to the speed u (which in this equation is the up- or downstream fluid speed measured in the shock frame), μ is the pitch angle cosine.

At this point we introduce the two major simplifications which we shall make in our treatment. First, we assume that the collision operator \hat{C} contains only a pitch angle scattering term. Any wandering of the particle from field line to field line is thus prohibited, and changes in momentum which occur on scattering are neglected. Secondly, we introduce a phase-averaged distribution function

$$f = \frac{1}{2\pi} \int_0^{2\pi} d\phi F_0 \quad (9)$$

and assume F_0 averages to zero when multiplied by $\sin \phi$:

$$\int_0^{2\pi} d\phi \sin \phi F_0 = 0. \quad (10)$$

These simplifications, which will be discussed further below and in Section 5, enable the transport equation to be written in a relatively simple form, provided we adopt the usual restrictions that a time-independent solution is demanded, which has no dependence on the coordinates y and z :

$$\Gamma \cos \Phi (u + v\mu) \frac{\partial f}{\partial x} = \hat{C} = \frac{\partial}{\partial \mu} \left(D_{\mu\mu} \frac{\partial f}{\partial \mu} \right), \quad (11)$$

where we have written the momentum of a particle in terms of its speed v and pitch-angle cosine μ , and $D_{\mu\mu}(p, \mu, x)$ is the pitch-angle diffusion coefficient.

3.2 PARTICLE TRAJECTORIES AT THE SHOCK FRONT

The transport equation (11) is to be solved in both the upstream and downstream regions, and the results then matched to each other at the shock front, in just the same way as for parallel shocks (KS). However, whereas in parallel shocks it is assumed that the distribution function as seen in the shock frame is a continuous function of pitch angle, this assumption cannot be made for oblique shocks. Instead, the matching procedure we employ depends on the details of the particle trajectories which cross the shock front.

Consider a region close to the shock front, extending roughly one gyro-radius into the upstream and downstream zones. We assume that the thickness of this region is small compared to the mean free path of a particle for pitch angle scattering. We will use the term ‘scatter-free zone’ to indicate that it is a good approximation to assume the particle trajectories in this region are *unperturbed*. Several investigations of such trajectories have been conducted (Hudson 1965; Terasawa 1979). It has been shown (see also Drury 1983) that particles incident on the scatter-free zone from upstream may either be transmitted through the downstream boundary or reflected. In either case, the trajectory may cross the shock front a number of times before escaping from the scatter-free zone. Particles incident from downstream are always transmitted through the upstream boundary. In the shock frame, the energy of a particle on an unperturbed trajectory remains constant – even upon crossing the shock front, although its pitch angle and phase do change abruptly when the shock is crossed. Numerical integration of the equations of motion seems to be the only way in which these changes can be calculated exactly.

In order to avoid complicating the matching conditions we adopt the ‘adiabatic’ approximation, in which particle trajectories are assumed to conserve their magnetic moment on crossing the shock front or on being reflected from it (see, for example, Chandrasekhar 1960; Wentzel 1964). The numerical investigations referred to above and the analytic work of Alekseyev & Kropotkin (1970) indicate that this approximation is accurate for highly oblique shocks. For nearly parallel shocks, it is still good in an average sense: although the spread in magnetic moment after reflection or transmission is quite large (depending on the phase of the incoming trajectory), the mean value is close to the incoming magnetic moment. Several authors have used this approximation (Alekseyev & Kropotkin 1970; Toptygin 1980; Webb, Axford & Terasawa 1983) to discuss the shock drift acceleration mechanism. Using Liouville’s theorem (which applies to unperturbed trajectories), Drury (1983) has shown that exact conservation of magnetic moment follows if it is assumed that the process of reflection or transmission is independent of the phase of the trajectory.

Conservation of magnetic moment leads directly to the following relationship between the pitch angle cosine μ_s and a trajectory at the upstream boundary of the scatter-free zone and that at the downstream boundary, $\bar{\mu}_s$:

$$1 - \mu_s^2 = b(1 - \bar{\mu}_s^2). \quad (12)$$

Note that the pitch angles are measured in the shock rest frame. Conservation of energy in the shock frame gives, for highly relativistic particles, $p_s = \bar{p}_s$. Clearly, then, there is a critical pitch angle cosine

$$\mu_0 \equiv \sqrt{1 - b} \quad (13)$$

which separates particles which are transmitted ($|\mu_s| \geq \mu_0$), from those which are reflected

$|\mu_s| \leq \mu_0$. The reflected particles have the sign of μ_s reversed. All particles incident on the shock from downstream are transmitted to the upstream side. In the scatter-free zone, Liouville's theorem applies, so that the distribution function is constant along unperturbed trajectories. Therefore, the distribution function $f_s(\mu_s, p_s)$ of particles at the upstream boundary of the scatter-free zone is related to that at the downstream boundary $\bar{f}_s(\bar{\mu}_s, \bar{p}_s)$ by

$$f_s(\mu_s, p_s) = \bar{f}_s(\bar{\mu}_s, \bar{p}_s) \text{ for transmitted particles } (|\mu_s| > \mu_0). \quad (14)$$

Similarly, for reflected particles a condition on the distribution function at the upstream boundary of the scatter-free zone results:

$$f_s(\mu_s, p_s) = f_s(-\mu_s, p_s) \text{ for } |\mu_s| < \mu_0. \quad (15)$$

Although a good approximation, conservation of magnetic moment is not exact, and our results will be correspondingly inexact. A full treatment would involve the particle gyrophase, which will influence the condition for reflection, and blur out the division into reflected and transmitted particles at $\pm \mu_0$. As we shall see later, our method leads to slight inaccuracies in the distribution function at $\mu_s = \pm \mu_0$, which are unphysical, and result from the artificially sharp division between reflected and transmitted particles.

3.3 GENERAL SOLUTION

The transport equation (11) is identical to that obtained by KS for parallel shocks, except for the multiplicative $\cos \Phi$ factor. This may be absorbed into the length scale x . If the diffusion coefficient is separable in x , p and μ , $D_{\mu\mu}(\mu, p, x) = D(\mu)\kappa(p)W(x)$, then the spatial dependence can also be absorbed into the spatial coordinate. Separating the variables as in KS leads to the general solution

$$f(p, \mu, x) = \sum_{i=-\infty}^{\infty} g_i(p) Q_i(\mu; u) \exp[\Lambda_i \kappa(p) \tilde{x}/\Gamma], \quad (16)$$

where the spatial coordinate is

$$\tilde{x} \equiv \frac{1}{\cos \Phi} \int_0^x \frac{dx'}{W(x')} \quad (17)$$

and $Q_i(\mu; u)$ and Λ_i are the eigenfunctions and eigenvalues of the equation

$$\left[\frac{\partial}{\partial \mu} D(\mu) \frac{\partial}{\partial \mu} - \Lambda_i (u + \mu) \right] Q_i(\mu; u) = 0. \quad (18)$$

We have set the particle speed v to unity in this equation, because we discuss below only the application to highly relativistic particles. Downstream, the general solution is the same but with u replaced by \bar{u} , etc. There is a trivial eigenfunction $Q_0 = \text{constant}$, with $\Lambda_0 = 0$. The remaining eigenvalues are positive for $i > 0$, negative for $i < 0$. The eigenfunctions obey the orthogonality relation

$$\int_{-1}^{+1} Q_i(\mu; u) Q_j(\mu; u) (u + \mu) d\mu = 0 \quad i \neq j. \quad (19)$$

A discussion of the eigenfunction equation and its properties may be found in Kirk (1988a, b). In this paper, we wish to concentrate on the effects peculiar to oblique shocks, and so use the simplest form of the pitch angle diffusion coefficient:

$$D(\mu) = 1 - \mu^2, \quad (20)$$

corresponding to isotropic pitch angle diffusion. The functions $g_i(p)$ in equation (16), which express the momentum dependence of the distribution function at the shock, are as yet unspecified. For highly relativistic particles with momenta much greater than the injection momentum, there is no preferred momentum scale, and the distribution is a power law:

$$\begin{aligned} g_i(p) &= a_i p^{-s} \\ \bar{g}_i(\bar{p}) &= \bar{a}_i \bar{p}^{-s}. \end{aligned} \quad (21)$$

In this case, the problem reduces to one of determining the spectral index s .

In order for the distribution to remain finite as $\tilde{x} \rightarrow \pm \infty$, the coefficients $a_{i < 0}$ and $\bar{a}_{i > 0}$ must be zero. (We assume there exist scattering centres throughout the upstream and downstream plasma, i.e. - cf. equation (17) - $\tilde{x} \rightarrow \pm \infty$ as $x \rightarrow \pm \infty$). In addition, if there are no pre-existing particles far upstream, then $a_0 = 0$. If we write the distribution function at the upstream boundary of the scatter-free zone as

$$f(p, \mu) = p^{-s} h(\mu) = p_s^{-s} h_s(\mu_s) \quad (22)$$

(the second equality follows from the Lorentz invariance of f), then the above conditions on the a_i may be expressed as

$$\int_{-1}^1 Q_i(\mu; u) h(\mu) (u + \mu) d\mu = 0 \quad \forall i \leq 0. \quad (23)$$

Transforming to the shock frame by means of the relations $\mu = (\mu_s - u)/(1 - u\mu_s)$ and $p = \Gamma p_s (1 - u\mu_s)$, (23) may be written as

$$E_i \equiv \int_{-1}^1 Q_i[\mu(\mu_s); u] (1 - u\mu_s)^{s-3} h_s(\mu_s) \mu_s d\mu_s = 0 \quad \forall i \leq 0. \quad (24)$$

These equations are the key to solving the problem. We need now to devise suitable forms for the particle distribution, which satisfy the conditions on the \bar{a}_i as well as the matching conditions (15) and (14).

3.4 REPRESENTATION OF THE PARTICLE DISTRIBUTION

Transmitted particles occupy all of the phase space on the downstream side of the shock front. Thus, it is straightforward to write an *Ansatz* for their distribution which automatically fulfils the boundary condition far downstream by omitting from the general solution (16) those terms with divergent behaviour (i.e. $i > 0$). Setting $\tilde{x} = 0$ at the downstream boundary of the scatter-free zone leads to

$$\begin{aligned} \tilde{f}(\bar{p}, \bar{\mu}) &\equiv \bar{p}^{-s} \bar{h}(\bar{\mu}) \\ &= \bar{p}^{-s} \sum_{i=-\infty}^0 e_i \bar{Q}_i(\bar{\mu}; \bar{u}), \end{aligned} \quad (25)$$

which determines $h_s(\mu_s)$ of equation (24) in the range $|\mu_s| > \mu_0$. For $|\mu_s| < \mu_0$, however, it is necessary to represent the distribution of reflected particles. These particles must fulfil equation (15), i.e. they must have a distribution function which is even in μ_s . As there is no additional restriction, we choose to represent the distribution as a sum of even Chebychev polynomials T_{2j} :

$$h_s(\mu_s) = \sum_{j=0}^{\infty} d_j T_{2j}(\mu_s/\mu_0) \quad \text{for } -\mu_0 < \mu_s < \mu_0. \quad (26)$$

There is nothing special about the representation except that the Chebychev polynomials always take on the value unity at their maxima and minima, which makes it easier to decide whether a numerical solution involving only a finite number of polynomials has converged.

3.5 METHOD OF SOLUTION

The solution is obtained by approximating the reflected and transmitted distributions by a finite number of terms in the series (26) and (25). Splitting the range of integration in (24) in the obvious way, we have $E_i \equiv R_i + T_i$, with

$$R_i = \int_{-\mu_0}^{\mu_0} Q_i(\mu; u) (1 - u\mu)^{s-3} \left[\sum_{j=0}^{JR} d_j T_{2j}(\mu_s) \right] \mu_s d\mu_s. \quad (27)$$

For the transmitted particles, we change the integral to extend over the downstream variable $\bar{\mu}$, by using $\mu_s d\mu_s = b\bar{\mu}_s d\bar{\mu}_s$, Liouville's theorem, which requires that $h_s(\mu_s) = \bar{h}_s(\bar{\mu}_s)$, and the invariance of f , which implies $\bar{f}(\bar{p}, \bar{\mu}) = f_s(\bar{p}_s, \bar{\mu}_s)$. This leads to the following:

$$T_i = b\bar{\Gamma}^{4-s} \sum_{j=0}^{-JT} e_j \int_{-1}^1 Q_i(\mu; u) \bar{Q}_j(\bar{\mu}, \bar{u}) (1 - \bar{u}\bar{\mu}_s)^{3-s} (1 - u\mu_s)^{s-3} (\bar{u} + \bar{\mu}) d\bar{\mu}. \quad (28)$$

Numerical evaluation of the E_i follows in the usual manner (KS), by calculating the eigenfunctions of the scattering operator using the Galerkin method. The spectral index is obtained by the following method: we write the coefficients d_j , e_j in a vector c_j , with $JR + JT + 2$ elements. The conditions $E_i = 0$ then read

$$E_i \equiv A_{ij} c_j = 0. \quad (29)$$

This equation defines the matrix \mathbf{A} , which is square when i is allowed to range from 0 to $JR + JT + 1$. Each element of \mathbf{A} contains a term from the R_i and one from the T_i , which are functions of the unknown spectral index s . The requirement that $\|\mathbf{A}\| = 0$ then determines s . Note that the distribution function is not explicitly assumed to be continuous at $\mu_s = \pm \mu_0$.

3.6 NON-RELATIVISTIC LIMIT

In the extreme non-relativistic limit, it is sufficient to take $JR = JT = 0$, i.e. we assume that the particle distribution is isotropic upstream (in the shock frame), and isotropic in the downstream fluid frame. Expanding in powers of u , \bar{u} , we get, to leading order

$$R_0 \simeq -\frac{2}{3} d_0 Q_0 u (s-3) \mu_0^3 \quad (30)$$

and

$$T_0 \simeq b e_0 Q_0 \bar{Q}_0 \left[\frac{2}{3} s \bar{u} - u(s-3) \int_{-1}^1 \mu_s \bar{\mu}_s d\bar{\mu} \right]. \quad (31)$$

Transforming the remaining integral to upstream variables gives

$$\int_{-1}^1 \mu_s \bar{\mu}_s d\bar{\mu}_s = (1/b) \int_{|\mu_s| > \mu_0} \mu_s^2 d\mu_s = 2(1 - \mu_0^3)/(3b).$$

Applying continuity of these approximate distributions (to leading order) and the constraint $E_0 = 0$ gives the result

$$(s-3)u = bs\bar{u}. \quad (32)$$

Eliminating b in terms of the compression ratio r , using (2), we get the result $s = 3r/(r-1)$, as obtained by Bell (1978), Drury (1983) and Ostrowski (1988). However, we show in Section 4 that, even for modest shock speeds, ($u \approx 0.1$, say) there are quite large deviations from this result.

4 Numerical results

4.1 RELATIVISTIC SHOCKS

We have applied this method to a number of cases of different shock speeds and angles of the magnetic field, shown in Figs 2 and 3. The spectral index s is easily found using only a few Chebychev polynomials in the expansions (26) and (25). As the speeds in the de Hoffmann–Teller frame increase, more eigenfunctions (and polynomials) are required, and the results become difficult to calculate, in common with the KS method for parallel shocks. We

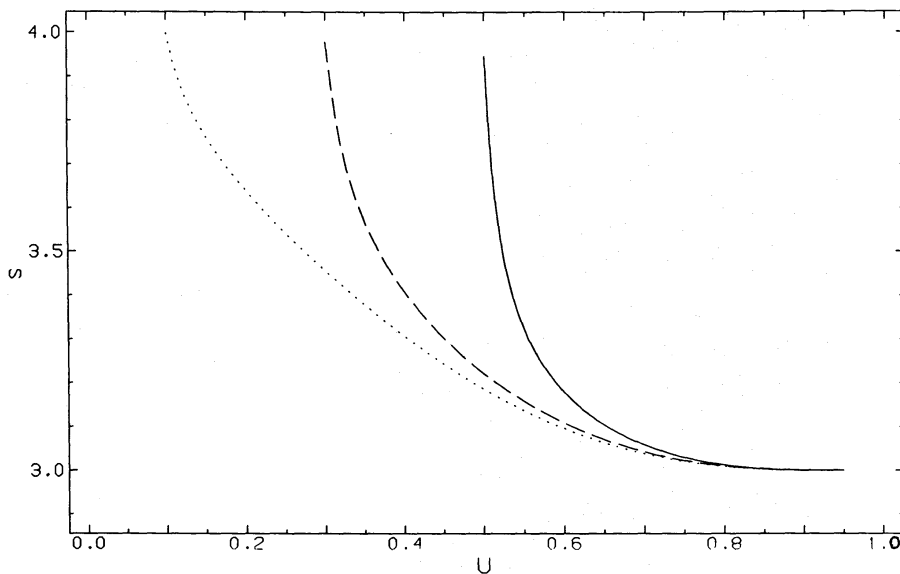


Figure 2. The power-law index s of particles accelerated at three different oblique subluminal shock fronts each of compression ratio 4. The curves show s as a function of the upstream speed u (in units of c) in the de Hoffmann–Teller frame. The angle between the magnetic field and the shock normal measured in the upstream fluid frame Φ_{up} is given by $\cos \Phi_{\text{up}} = u_{\text{sh}}/u$, where u_{sh} , the shock speed, takes on the values 0.1 (dotted line), 0.3 (dashed line) and 0.5 (solid line).

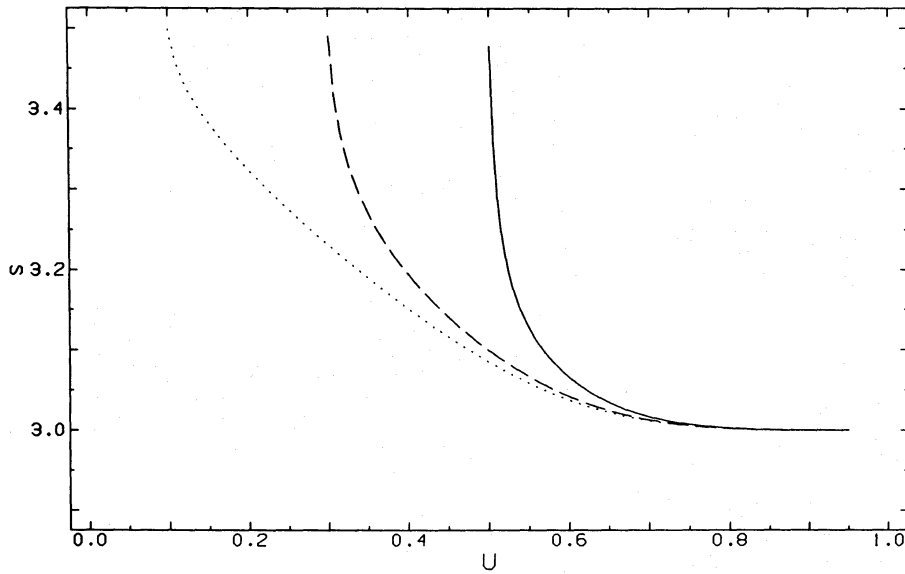


Figure 3. As in Fig. 2, but for three shock fronts of compression ratio 7, corresponding to strong non-relativistic shocks in a gas whose ratio of specific heats is $4/3$.

often find that the use of more eigenfunctions than Chebychev polynomials is most successful: six eigenfunctions and three Chebychev polynomials were employed in the figures, in which the maximum value of u is 0.95.

In order to simplify the presentation of our results we restrict ourselves to shock fronts at which the magnetic field plays no dynamical role. Thus, there exists a frame of reference in which the shock is stationary and both the upstream and downstream plasma flows along the normal (the parallel shock frame). In this frame the ‘shock speed’ (which is conventionally the speed of the shock front seen from the upstream rest frame) is equal to the speed of the upstream plasma. This quantity is kept constant in each of the curves presented in Figs 2 and 3, as is the compression ratio of the shock front (which is the ratio of upstream speed to downstream speed in the parallel shock frame). However, the angle Φ_{up} between the magnetic field direction and the shock normal, as seen in the upstream fluid frame, is varied between zero and the maximum value permitted by the constraint that the shock be subluminal. The abscissa in Figs 2 and 3 is the upstream speed u seen in the shock frame $u = u_{\text{sh}}/\cos \Phi_{\text{up}}$. For $\Phi_{\text{up}} = 0$, one has $u = u_{\text{sh}}$; the condition for the shock to be subluminal is $u < 1$. In Fig. 2, the compression ratio of the shock front is taken to be 4. A parallel shock front with such a compression ratio produces a power-law index close to 4, as found by KS. When the magnetic field is turned, causing more and more particles to be reflected, the spectrum hardens, (in agreement with the conjecture of Webb 1983) tending to $s = 3$ for a shock which is close to becoming superluminal. This is to be expected, since the spectral index depends on the ratio of the probability for a particle to escape downstream to the gain in energy which occurs if the particle returns upstream. As the shock approaches the critical point at which it becomes superluminal, the parameter b tends to its minimum value: $1/r$. The opening angle of the cone of transmitted particles also reaches its minimum: $\arccos(\sqrt{1 - 1/r})$, which restricts the escape probability of a particle reaching the shock from upstream. The energy gained by a particle on reflection, however, does not tend to a limit, but continues to increase as $u \rightarrow 1$. In the simple model considered here with no loss mechanisms other than escape into the downstream plasma, the power law for the stationary solution tends to the nonphysical value $s = 3$ at which the number of particles with momentum in excess of any specified value diverges.

It is also apparent from both Figs 2 and 3 that s decreases rapidly as the magnetic field

direction is turned slightly away from being exactly parallel to the shock normal. This behaviour may be understood physically, since there is a fundamental difference between the situation with $\cos \Phi_{\text{up}} \neq 1$ in which reflected particles are present and that with $\cos \Phi_{\text{up}} = 1$ in which *no* particles are reflected. Mathematically, the infinite gradient of s as a function of u at constant u_{sh} arises because of the linear dependence of the quantities R_i of equation (27) on the parameter μ_0 (the cosine of the opening angle of the cone containing transmitted particles in the upstream distribution). This parameter is related to the angle Φ_{up} through the parameter b defined in equation (2) by equation (13). Although b has a finite derivative as $\cos \Phi_{\text{up}} \rightarrow 1$ (and $b \rightarrow 1$), the derivative of μ_0 diverges because it contains the factor $1/\sqrt{1-b}$.

4.2 DISTRIBUTION FUNCTION

As indicated above, the spectral index is quite easily obtained by our method, with only a few eigenfunctions and polynomials needed to obtain an accurate solution. The problem of obtaining the distribution function, which requires the numerical solution of the set of linear algebraic equations $A_{ij}c_j = 0$, is more difficult, and usually requires many more eigenfunctions and polynomials.

In Fig. 4 we show results found using 15 eigenfunctions and 15 Chebychev polynomials, for a shock of compression ratio 4 propagating at speed 0.1. The magnetic field angle is $\Phi_{\text{up}} = 60^\circ$, leading to an upstream plasma speed of $u = 0.2$ in the shock frame. The distribution function on the upstream side of the shock front is displayed as seen in the upstream rest frame and the shock frame in Fig. 4(a), whereas Fig. 4(b) shows the distribution on the downstream side of the shock, seen in the shock frame and the frame in which the downstream plasma is at rest. It can be seen that the distribution of reflected particles is almost isotropic in the shock frame for

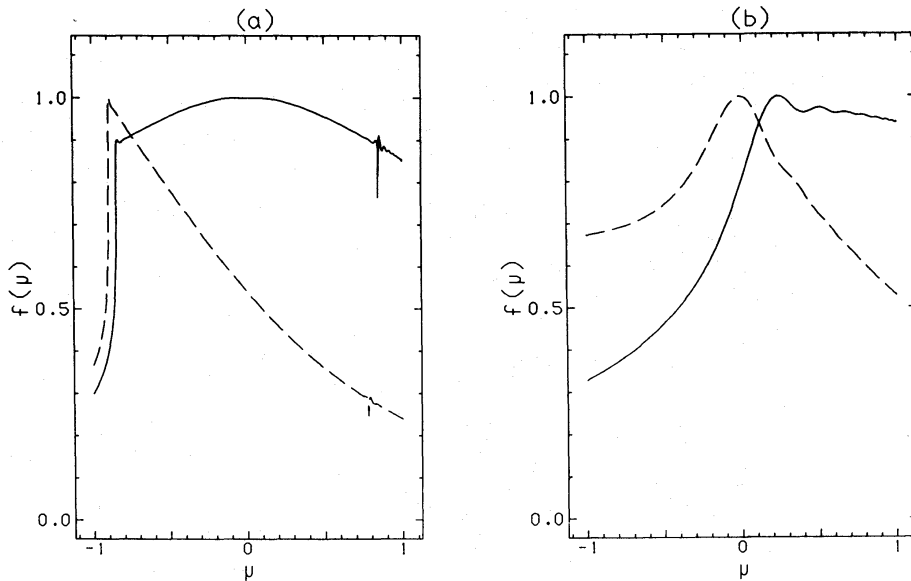


Figure 4. The distribution function of accelerated particles at a shock front with compression ratio $r = 4$, shock speed $u_{\text{sh}} = 0.1$ and angle of magnetic field $\Phi_{\text{up}} = 60^\circ$. The spectral index produced by such a shock is 3.638. In (a), the distribution function on the upstream side of the shock front is shown, the full line indicating f as a function of μ_s , i.e. as seen in the shock frame, and the dashed line shows f as a function of μ , i.e. as seen in the upstream rest frame. Each curve is normalized to have its maximum value unity. In (b) the distribution function on the downstream side of the shock front is displayed. Once again, the solid line indicates the shock frame (f as a function of $\tilde{\mu}_s$) and the dashed line the downstream rest frame (f as a function of $\tilde{\mu}$). The boundaries between reflected and transmitted particles occurs at $\mu_s = \pm \mu_0 = \pm 0.85$, corresponding to $\mu = 0.78, -0.89$.

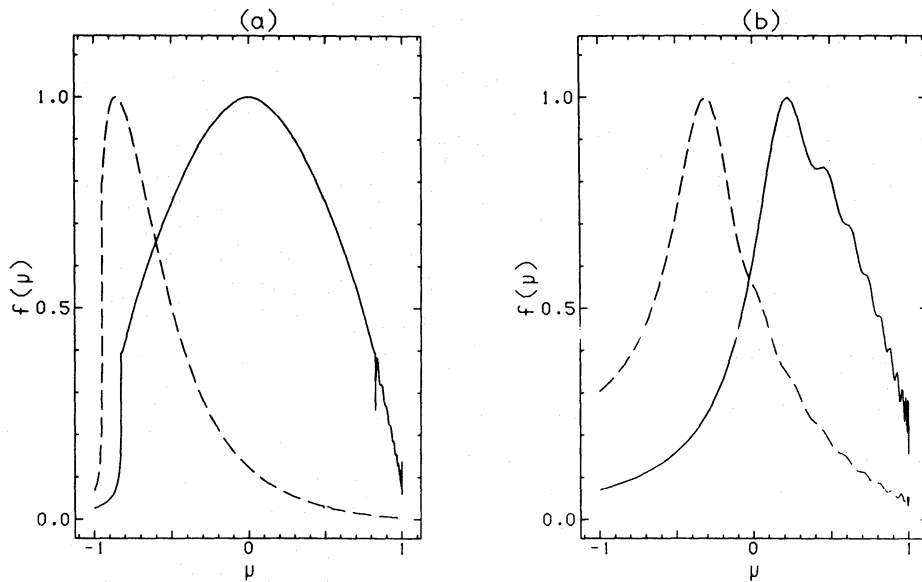


Figure 5. The same as Fig. 4, but for a faster shock front with compression ratio $r=4$, shock speed $u_{sh}=0.4$ and angle of magnetic field $\Phi_{up}=45^\circ$. The spectral index produced by such a shock is 3.171.

this rather non-relativistic shock. The transmitted particles are also almost isotropic in the shock frame when they leave the upstream region ($\mu_s > \mu_0$), but show substantial anisotropy when re-entering it ($\mu_s < -\mu_0$).

These features are enhanced in Fig. 5(a) and (b), which show the same functions as Fig. 4 for the case of a much faster shock: $u=0.57$. Now the reflected particles also show a large anisotropy, the slope of which continues unchanged into the region of transmitted particles with $\mu_s > \mu_0$. The number of eigenfunctions used to obtain this figure is 20, with 10 Chebychev polynomials. Even so, there is a distinct ‘ringing’ of the distribution at $\mu_s > \mu_0$, which is especially noticeable in the distribution on the downstream side of the shock front, Fig. 5(b). Increasing the number of eigenfunctions and polynomials produces a noticeable change only in the frequency of the ‘ringing’, and not in the remainder of the distribution.

Notice that, although continuity at $\mu_s = \pm\mu_0$ is not enforced, the distribution is clearly continuous at $\mu_s = \pm\mu_0$. The numerical method introduces an obviously spurious feature at this point which is reminiscent of the Gibbs phenomenon in the theory of Fourier series (e.g. Mathews & Walker 1970). The same feature appears also at $\mu_s = -\mu_0$, but is less easily distinguished because of the steep gradient of the distribution of transmitted particles. At μ_0 , there is a clear division between the reflected and transmitted parts of the distribution. The difference between the features at $\pm\mu_0$ may be understood as follows: for $\mu_s > 0$, the particles are streaming towards the shock from upstream. They have no prior knowledge of whether they will be reflected or transmitted, so the scattering ensures that the distribution function and its first two derivatives are continuous there. For $\mu_s < 0$, however, the distribution is composed of two distinct populations: for $\mu_s > -\mu_0$, the particles have been reflected from the shock, whereas for $\mu_s < -\mu_0$, the particles have been transmitted from downstream to upstream. We would not, therefore, expect continuity of f and its first two derivatives there. In fact, our numerical results indicate that the distribution on the downstream side of the shock is continuous at the point $\bar{\mu}_s = 0$, i.e. where $\mu_s = \pm\mu_0$. This fact, together with the constraint that the reflected particles have a distribution even in μ_s not only forces f to be continuous at $\mu_s = -\mu_0$ but also forces the first derivative of f with respect to μ_s to be discontinuous at this point.

It should be pointed out, however, that such sharp features are a result of our imposition of exact conservation of magnetic moment. In practice, particles near $\pm\mu_0$ will have a finite probability of reflection and transmission, dependent on gyrophase, and this will presumably smooth out the distribution function near these points. It also is reasonable to expect that the rapid oscillations imposed on the relatively smooth distribution in Fig. 5(b) would disappear in a more physically realistic approach.

4.3 NON-RELATIVISTIC SHOCKS

For shock speeds much less than one, the result of the diffusion approximation equation (32) applies for a large range of angles Φ_{up} . Only when the magnetic field is quite close to being in the plane of the shock front (seen from the upstream rest frame) does the upstream plasma speed u reach an appreciable fraction of the speed of light, and only then is a departure from equation (32) expected. In this case, however, it is possible to simplify our results considerably. For $u_{\text{sh}} \ll 1$, one has

$$b \approx (r^2 \sin^2 \Phi_{\text{up}} + \cos^2 \Phi_{\text{up}})^{-1/2}. \quad (33)$$

For a significant deviation from the diffusion approximation, $\cos \Phi_{\text{up}} \ll 1$, so that $b \approx 1/r$. In general, the power-law index is characterized by three parameters; for example r , u and b . Since, however, b depends only on r in the range of interest, it follows that the spectral index is determined by only two parameters: r and $u (= u_{\text{sh}}/\cos \Phi_{\text{up}})$. This is illustrated in Fig. 6, where s is shown as a function of u for two shocks with compression ratio 4 and (different) non-relativistic shock speeds. The results for different shock speeds are coincident provided the shock speeds are small compared to u , and s deviates significantly from the diffusion approximation.

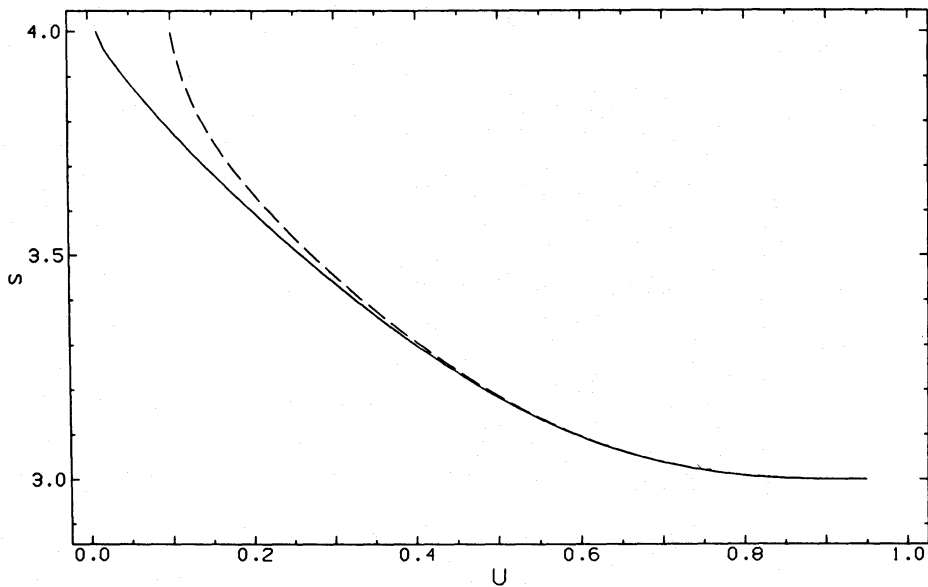


Figure 6. The power-law index for two shock fronts with the same compression ratio $r=4$ but different, non-relativistic shock speeds: $u_{\text{sh}}=0.1$ (dashed line) and $u_{\text{sh}}=0.01$ (solid line). The curves are almost coincident for $u > 0.2$, showing that in this range s is not a function of u_{sh} and Φ_{up} independently, but only in the combination $u_{\text{sh}}/\cos \Phi_{\text{up}} (= u)$.

5 Discussion

In this paper we use the ‘ Q_J -method’ (KS) to calculate the spectral index of highly relativistic particles accelerated at oblique (subluminal) shock fronts. In addition, we obtain the full pitch angle dependence of the particle distribution on both the upstream and downstream sides of the shock front. The process of acceleration is a combination of the two widely discussed mechanisms of particle acceleration at shocks: the shock drift mechanism (Webb *et al.* 1983) and the first-order Fermi mechanism (Drury 1983), often referred to as ‘diffusive acceleration’. Pitch angle scattering in both the upstream and downstream media is responsible for returning some particles to the shock front repeatedly, as in the Fermi mechanism. However, the energy gained by a particle at a single encounter with the shock front is of great importance – despite the fact that this quantity depends on the frame of reference from which it is viewed. In particular, reflection of particles incident on the shock from the upstream region results in a spectral index which approaches the theoretical maximum (and unphysical) value $s=3$ in the loss-free case, for shocks which are almost superluminal. Previous investigations of this problem either employ the diffusion approximation (Bell 1978; Axford 1980; Jokipii 1982; Drury 1983; Ostrowski 1988) or simulate the particle motion numerically (Decker & Vlahos 1986), with the exception of Chiueh (1988), who considers the process for non-relativistic ions, in the absence of upstream turbulence.

The results we find are presented in Section 4. We consider only highly relativistic particles, so that the velocity may be approximated by that of light c . In this case, and provided we are interested only in momenta far above that at which particles are injected, the distribution of accelerated particles is a power law in momentum, whose index s is the object of the investigation. There is a substantial deviation of this index from the well-known results of the diffusion approximation only if the velocity of the *fluid* at the shock front is an appreciable fraction of c . This, in fact, is also the case for purely parallel shock fronts, but there the deviation from the results of the diffusion approximation turned out to be very small ($\Delta s \leq 0.06$) even for fluid speeds as high as $c/2$. Our results for oblique shocks are in marked contrast, as shown in Figs 2, 3 and 6. A shock speed of only $0.1c$, for example, which, for a compression ratio $r=4$, accelerates particles into a spectrum with $s=4$ when the magnetic field is parallel to the shock normal, produces a spectrum with $s=3.6$ when the magnetic field is inclined by 60° . There are two reasons for this sensitivity. First, the speed in the de Hoffmann–Teller frame of reference is the most relevant in the discussion of acceleration, and this can be close to c for a sufficiently oblique shock, even if the shock speed is very small. Secondly, the speed of the downstream plasma in the de Hoffmann–Teller frame is, for an oblique shock, not too different from the speed of the upstream plasma in this frame. In the case of a parallel shock, on the other hand, the downstream plasma speed is always less than $c/3$ (for a strong shock), so that relativistic effects play a minor role. In the case of parallel shocks, it was found (KS; Heavens & Drury 1988) that the non-relativistic expression $s=3r/(r-1)$ was a good approximation to the spectral index, even for shocks travelling at an appreciable fraction of c . Clearly, for even mildly relativistic oblique shocks, this approximation is no longer accurate.

The conditions under which our results find application occur in astrophysical objects in which fast-moving shock fronts are thought to create power-law distributions of highly relativistic electrons, giving rise, in turn, to synchrotron emission in the radio to optical bands with a spectral index $\alpha=(s-3)/2$, and intensity $S_\nu \propto \nu^{-\alpha}$. In addition, the acceleration of relativistic ions by fast shocks is affected. Thus, the emerging shock front from a supernova explosion, which may move at a speed of $20\,000\text{ km s}^{-1}$ and which is likely to be oblique (Völk & Biermann 1988), will produce a much harder spectral index ($s \sim 3\text{--}3.5$) than that predicted by the diffusion approximation. Supernova remnants too, despite their lower shock speeds

($\sim 3000 \text{ km s}^{-1}$), can be expected to produce spectral indices harder than predicted on the basis of the diffusion approximation, if they propagate into a plasma containing the wound-up magnetic field in the stellar wind of the progenitor star. There are, however, some problems in interpreting the synchrotron spectra in supernova remnants and extragalactic sources in terms of this model. With the exception of the filled-centre remnants, supernova remnants are not observed to have very flat spectra, with best estimates of α exceeding 0.3 (Green 1988). The filled-centre remnants, whose spectra are significantly harder ($\alpha \sim 0.0\text{--}0.3$, Weiler 1985), are prime candidates for relativistic oblique shocks (see, for example, Blandford 1982). The problem here may be that the Lorentz factor of the fluid with respect to the shock may be so high (Blandford suggests $\gamma \sim 10^4$ for the Crab) that the field/shock intersection may be superluminal. The angle between the field and the shock normal in the upstream frame needs to be less than $\sim 1/\gamma$ for a de Hoffmann–Teller frame to exist. Many extragalactic compact radio sources also have synchrotron spectral indices near zero. Conventionally these are interpreted as the superposition of a number of self-absorbed components with different turnover frequencies, a view which has some observational support (e.g. Marscher & Broderick 1985). Finally, the knots and hot-spots observed in jets in extragalactic radio sources (Meisenheimer *et al.* 1989) are interpreted as shock fronts in plasma moving with at least mildly relativistic bulk velocity (Meisenheimer & Heavens 1986, Appl & Camenzind 1988). Only a very small departure of the magnetic field direction from the normal to the shock front is necessary in this case to produce a strong effect on the spectral index.

We must mention here the limitations of the approach used in this paper, in that we have had to make a number of simplifying assumptions. In addition to those conventionally adopted in the treatment of the test particle problem, we use two peculiar to the problem of oblique shocks. The first, and probably less serious, is the adiabatic approximation, under which a particle is assumed to conserve its magnetic moment on interaction with the shock front. Numerical work has shown this to be true only in a phase-averaged sense. However, in our calculation of the spectral index s , the quantities which enter are all integrals over the angular distribution of accelerated particles, so that we may be fairly confident that relaxation of this assumption will not change the result drastically. The reconstruction of the distribution function (Figs 4 and 5) is a different matter. Here the effects of the assumption of exact conservation of magnetic moment manifest themselves in a small, but nevertheless non-physical feature at the boundary between reflected and transmitted particles.

The second assumption concerns the nature of the scattering operator C . Although diffusion of particles across field lines is expected to be a slow process, we have neglected it completely. At an oblique shock, motion across the field lines enables a particle to take a shorter path back to the shock front than motion along the field. Thus, a simple estimate indicates that our assumption is reasonable only if the distance a particle moves across field lines λ_{\perp} is small compared to the distance λ_{\parallel} it travels along them: $\lambda_{\perp}/\lambda_{\parallel} \ll \cos \Phi_{\text{up}}$. In view of the uncertainty associated with the calculation of the scattering operator, we are unable to assert that this condition is fulfilled for oblique shocks. It is, of course, possible to take the opposite point of view and assume that, at least for non-relativistic shocks, scattering across the field lines is so effective that the angular distribution is maintained close to isotropy. This is the approach adopted by Jokipii (1982). It results in a spectral index equal to that of the standard diffusion approximation. Nevertheless, our approach demonstrates the sensitivity of the expected power-law index to the details of the scattering operator.

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