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Particle Aspect Analysis of Drift Instability

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A physical mechanism of drift instability is studied by investigating the trajectories of the particles. The method is the same as was used by Dawson in his theory of the Landau damping. The plasma particles are divided into two groups, the resonant and non-resonant particles. A drift wave is assumed to have started at $t=0$, when the resonant particles are undisturbed. The trajectories of the particles are then computed within the framework of the linear theory. Only those longitudinal waves which propagate obliquely to a static magnetic field in the plane normal to a density gradient are considered. The phase velocity along the magnetic field is assumed to be greater than the mean velocity of the ions, but smaller than that of the electrons. The resonant particles in this case are those electrons which have the velocities near to the phase velocity in the direction along the mean magnetic field. It is shown that the rate of energy change of the resonant electrons consists of two terms: one is the usual term arising from the Landau damping and the other the term due to the drift motion of the electrons in the crossed electric and magnetic fields. The latter term is responsible for instability of the wave. The major part of the wave energy is contained in the form of an oscillatory motion of the non-resonant electrons. Discussion of the stabilization by ion's Landau damping and of the drift instability due to impurity ions are given.

§ 1. Introduction

The so-called drift instabilities in plasmas with density gradient under a static magnetic field have been studied extensively in the literature.[1] However, the physical mechanism of this type of instability is not well understood. The purpose of this paper is to investigate the dynamical details of this type of instability from the particle point of view. Our analysis is based on a physical model similar to that used by Dawson[2] in his theory of Landau damping. The plasma particles are divided into two groups, the resonant and non-resonant particles. It is assumed that the main plasma consisting of the non-resonant particles supports the oscillatory motion of a drift wave while the resonant particles participate in energy exchange with the wave. Our main task is to calculate the net energy transfer between the wave and resonant particles. Only first-order quantities are used in the analysis.

§ 2. Basic assumptions and calculations of the particle trajectories

Let us consider a drift wave of the form

$$\boldsymbol{E}=\boldsymbol{E}_{\perp}+\boldsymbol{E}_{\parallel}=-\nabla\varphi,\qquad \varphi=\varphi_1\cos(k_{\perp}y+k_{\parallel}z-\omega t),\tag{1}$$

where φ_1 is assumed to be a slowly varying function of t, the frequency ω is real, $k_\perp$ and $k_\parallel$ are the components of the wave vector $\boldsymbol{k}$ across and along the static magnetic field $\boldsymbol{B}_0$, and the density gradient is taken in the direction of the x-axis (Fig. 1). For definiteness, $k_\perp$ and $k_\parallel$ are taken to be positive. The wave is assumed to start at time $t=0$ when resonant particles are not disturbed yet.

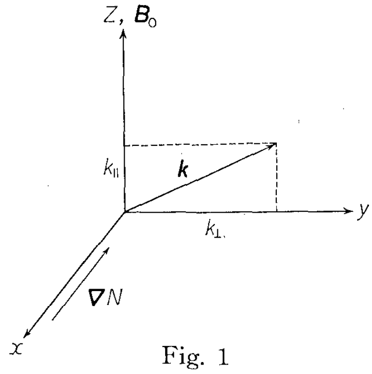

Fig. 1

The validity of the assumption of a pure electrostatic mode is assured in the case of low β-ratio,[1] to which the present analysis is limited. Further, we are mostly interested in the behaviour of those waves which satisfy the conditions

$$\bar{v}_{\parallel i} \ll \omega/k_\parallel \ll \bar{v}_{\parallel e}, \qquad \omega \ll \Omega_i,\ |\Omega_e|, \qquad k_\perp{}^2\rho_e{}^2 \ll k_\perp{}^2\rho_i{}^2 < 1\,, \tag{2}$$

where $\bar{v}_{\parallel i}$ and $\bar{v}_{\parallel e}$ are the mean velocities of the ions and the electrons along the magnetic field, $\Omega_{i,e}$ and $\rho_{i,e}$ are the gyration frequencies and the mean gyration radii of the respective species.

We shall first seek the trajectories of the particles in a general form. The equation of motion for a particle is

$$m\frac{d\boldsymbol{v}}{dt} = q\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}_0\right).$$

Throughout this paper, the Gaussian system of units is adopted and the collisions between particles are neglected. If $\boldsymbol{E}$ on the right-hand side is considered to be a small perturbation, $\boldsymbol{v}$ can be expressed as a sum of the unperturbed velocity $\boldsymbol{V}$ and the perturbed one $\boldsymbol{u}$, $\boldsymbol{v}=\boldsymbol{V}+\boldsymbol{u}$. $\boldsymbol{u}$ is determined by the following set of equations:

$$\frac{du_+}{dt} + i\Omega u_+ = \frac{iq}{m}(k_\perp\varphi_1)\sin(k_\perp y + k_\parallel z - \omega t)\,,$$

$$\frac{du_\parallel}{dt} = \frac{q}{m}(k_\parallel\varphi_1)\sin(k_\perp y + k_\parallel z - \omega t)\,, \tag{3}$$

where

$$u_+ = u_x + iu_y, \qquad \Omega = qB_0/mc\,.$$

We solve Eq. (3) in the approximations in which the coordinates of the particle on the right-hand sides are replaced by those of a free gyration and in which the slowly varying quantity φ_1 is treated as a constant. This procedure corresponds to the linearization of the Vlasov equation.

The trajectory of free gyration is

$$
\begin{aligned}
x(t) &= -\frac{V_\perp}{\Omega}[\sin(\theta-\Omega t)-\sin\theta]+x_0,\\
y(t) &= \frac{V_\perp}{\Omega}[\cos(\theta-\Omega t)-\cos\theta]+y_0,\\
z(t) &= V_\parallel t+z_0,
\end{aligned}
\tag{4}
$$

where $\boldsymbol{r}_0\equiv(x_0, y_0, z_0)$ is the position of the particle at $t=0$ and the unperturbed velocity $\boldsymbol{V}$ is taken to be

$$
\begin{aligned}
V_x(t) &= V_\perp\cos(\theta-\Omega t),\\
V_y(t) &= V_\perp\sin(\theta-\Omega t),\\
V_z &= V_\parallel.
\end{aligned}
\tag{5}
$$

Substituting (4) into (3), we find easily an oscillatory solution of $\boldsymbol{u}(t)$ for the non-resonant particles. For the resonant particles it is necessary to take into account the initial condition $\boldsymbol{u}(t=0)=0$ inferred from the assumption stated above. The solution of (3) is

$$
\begin{aligned}
u_x(t) &= -\frac{qk_\perp\varphi_1}{m}\sum_{n=-\infty}^{+\infty}\mathrm{J}_n(\mu)\left\{\frac{\Omega}{a_n}\sin\Psi_n+\delta\frac{\sin(\Psi_n^0-\Omega t)}{2\Lambda_{n-1}}-\delta\frac{\sin(\Psi_n^0+\Omega t)}{2\Lambda_{n+1}}\right\},\\
u_y(t) &= -\frac{qk_\perp\varphi_1}{m}\sum_n\mathrm{J}_n(\mu)\left\{\frac{\Lambda_n}{a_n}\cos\Psi_n-\delta\frac{\cos(\Psi_n^0-\Omega t)}{2\Lambda_{n-1}}-\delta\frac{\cos(\Psi_n^0+\Omega t)}{2\Lambda_{n+1}}\right\},\\
u_z(t) &= -\frac{qk_\parallel\varphi_1}{m}\sum_n\mathrm{J}_n(\mu)\frac{1}{\Lambda_n}\{\cos\Psi_n-\delta\cos\Psi_n^0\},
\end{aligned}
\tag{6}
$$

where $\delta=0$ for the non-resonant particles and $\delta=1$ for the resonant ones, and

$$
\mu=k_\perp V_\perp/\Omega,\quad a_n=(k_\parallel V_\parallel-\omega-n\Omega)^2-\Omega^2\equiv\Lambda_n^2-\Omega^2,
$$

$$
\Psi_n=\Lambda_n t+n\left(\frac{\pi}{2}-\theta\right)+k_\perp y_0+k_\parallel z_0-\mu\cos\theta,
$$

$$
\Psi_n^0\equiv\Psi_n(t=0).
$$

Also use was made of

$$
\exp(i\mu\cos(\theta-\Omega t))=\sum_{n=-\infty}^{+\infty}\mathrm{J}_n(\mu)\exp\left(in\left(\frac{\pi}{2}-\theta+\Omega t\right)\right).
$$

It is easy to calculate the true trajectory of the particle to first order by first integrating $\boldsymbol{u}(t)$ and then adding (4) to it. It should be noted that in view of (2) the dominant contributions come from the terms with $n=0$; it then follows that $u_z(t)$ with $\delta=1$ increases linearly in t if $k_\parallel V_\parallel-\omega=0$. Therefore the resonant condition now is $k_\parallel V_\parallel-\omega=0$ for the electrons, which means that these electrons see the wave independent of t on the particle frame. The resonant particles are thought to be those electrons which nearly satisfy this condition.

If we again substitute (4) into (6), $\boldsymbol{u}(t)$ can be written down in the form of $\boldsymbol{u}(\boldsymbol{r}, t)$:

$$\begin{aligned}
u_x(\boldsymbol{r}, t) &= -\frac{qk_\perp\varphi_1}{m}\sum_{n=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} \mathrm{J}_n(\mu)\mathrm{J}_l(\mu)\left\{\frac{\Omega}{a_n}\sin\chi_{nl}\right. \\
&\qquad\left. +\delta\frac{\sin(\chi_{nl}-\Lambda_{n-1}t)}{2\Lambda_{n-1}}-\delta\frac{\sin(\chi_{nl}-\Lambda_{n+1}t)}{2\Lambda_{n+1}}\right\}, \\
u_y(\boldsymbol{r}, t) &= -\frac{qk_\perp\varphi_1}{m}\sum_n\sum_l \mathrm{J}_n(\mu)\mathrm{J}_l(\mu)\left\{\frac{\Lambda_n}{a_n}\cos\chi_{nl}\right. \\
&\qquad\left. -\delta\frac{\cos(\chi_{nl}-\Lambda_{n-1}t)}{2\Lambda_{n-1}}-\delta\frac{\cos(\chi_{nl}-\Lambda_{n+1}t)}{2\Lambda_{n+1}}\right\}, \\
u_z(\boldsymbol{r}, t) &= -\frac{qk_\parallel\varphi_1}{m}\sum_n\sum_l \mathrm{J}_n(\mu)\mathrm{J}_l(\mu)\frac{1}{\Lambda_n}\{\cos\chi_{nl}-\cos(\chi_{nl}-\Lambda_n t)\},
\end{aligned} \tag{7}$$

where $\delta=0$ for the non-resonant particles and $\delta=1$ for the resonant particles, and

$$\chi_{nl}=k_\perp y+k_\parallel z-\omega t+(n-l)\left(\frac{\pi}{2}+\theta-\Omega t\right).$$

Next problem is to find the density perturbation associated with the velocity perturbation $\boldsymbol{u}(\boldsymbol{r}, t)$. Let us consider a group of the particles with the same initial conditions and let its number density be

$$n(\boldsymbol{r}, t\,;\boldsymbol{V})=N(x, \boldsymbol{V})+n_1(\boldsymbol{r}, t\,;\boldsymbol{V}) \tag{8}$$

where N is the zeroth order distribution and n_1 is the perturbed one. Since $x+V_y/\Omega$ is a constant of motion, the zeroth order distribution can be expressed as $N(x+V_y/\Omega, \boldsymbol{V})$ and satisfy

$$V_x\frac{\partial N}{\partial x}+\frac{q}{mc}(\boldsymbol{V}\times\boldsymbol{B}_0)\cdot\frac{\partial N}{\partial \boldsymbol{V}}=0\,. \tag{9}$$

The conservation of the particle number is

$$n[\boldsymbol{r}(t'),\ t'=t+\Delta t\,;\boldsymbol{V}(t')]d\boldsymbol{r}'=n[\boldsymbol{r}(t),\ t\,;\boldsymbol{V}(t)]d\boldsymbol{r}\,, \tag{10}$$

where $\boldsymbol{r}(t)$ is the trajectories of the particles, that is $d\boldsymbol{r}(t)/dt=\boldsymbol{V}(t)+\boldsymbol{u}(t)$. The change in volume element is given by

$$d\boldsymbol{r}'=\left|\frac{\partial(x', y', z')}{\partial(x, y, z)}\right|d\boldsymbol{r}\cong[1+(\boldsymbol{\nabla}\cdot\boldsymbol{u})\Delta t]d\boldsymbol{r}\,, \tag{11}$$

where the solution of $\boldsymbol{u}$ in the form of (7) has been used. By using a Taylor expansion of (10) together with (8) and (9), we find an equation for the perturbed density n_1:

$$\frac{dn_1}{dt} = -(\boldsymbol{\nabla}\cdot\boldsymbol{u})N - u_x\frac{\partial N}{\partial x}. \tag{12}$$

The quantities on the right-hand side of (12) can be expressed as functions only of the variable t; for instance, $\boldsymbol{\nabla}\cdot\boldsymbol{u}$ can be written down in terms of t and the initial parameters with the aid of (4) after differenciating (7) with respect to $\boldsymbol{r}$. (This procedure of the change of the variables will be repeated below.) By integration we obtain the following solutions. For the non-resonant particles,

$$n_1(t) = \frac{qk_\perp\varphi_1}{m}\sum_{n=-\infty}^{+\infty}\frac{J_n(k_\perp V_\perp/\Omega)}{\Lambda_n^2-\Omega}\cos\Psi_n\left\{k_\perp N - \frac{\Omega}{\Lambda_n}\frac{\partial N}{\partial x}\right\}$$
$$+\frac{qk_\parallel\varphi_1}{m}\sum_n J_n(k_\perp V_\perp/\Omega)\cos\Psi_n\frac{k_\parallel N}{\Lambda_n^2}$$

or

$$n_1(\boldsymbol{r},t) = \frac{qk_\perp\varphi_1}{m}\sum_{n=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}\frac{1}{\Lambda_n^2-\Omega^2}J_n\left(\frac{k_\perp V_\perp}{\Omega}\right)J_l\left(\frac{k_\perp V_\perp}{\Omega}\right)\cos\chi_{nl}$$
$$\times\left\{k_\perp N - \frac{\Omega}{\Lambda_n}\frac{\partial N}{\partial x}\right\} + \frac{qk_\parallel\varphi_1}{m}\sum_n\sum_l J_n\left(\frac{k_\perp V_\perp}{\Omega}\right)J_l\left(\frac{k_\perp V_\perp}{\Omega}\right)$$
$$\times\cos\chi_{nl}\frac{k_\parallel N}{\Lambda_n^2}$$
$$\cong\frac{q\varphi_1 N}{m}\left\{-\frac{k_\perp^2}{\Omega^2}+\frac{m}{T_\perp}\frac{k_\perp v_d}{k_\parallel V_\parallel-\omega}+\frac{k_\parallel^2}{(k_\parallel V_\parallel-\omega)^2}\right\}$$
$$\times\sum_{l=-\infty}^{+\infty}J_0\left(\frac{k_\perp V_\perp}{\Omega}\right)J_l\left(\frac{k_\perp V_\perp}{\Omega}\right)\cos\chi_{0l}. \tag{13·a}$$

Similarly for the resonant particles under the assumption that $n_1(t=0)=0$,

$$n_1(\boldsymbol{r},t)\cong\frac{q\varphi_1 N}{m}\left\{-\frac{k_\perp^2}{\Omega^2}+\frac{m}{T_\perp}\frac{k_\perp v_d}{k_\parallel V_\parallel-\omega}+\frac{k_\parallel^2}{(k_\parallel V_\parallel-\omega)^2}\right\}$$
$$\times\sum_{l=-\infty}^{+\infty}J_0\left(\frac{k_\perp V_\perp}{\Omega}\right)J_l\left(\frac{k_\perp V_\perp}{\Omega}\right)\{\cos\chi_{0l}-\cos(\chi_{0l}-\Lambda t)+\Lambda t\sin(\chi_{0l}-\Lambda t)\}, \tag{13·b}$$

where $\Lambda\equiv\Lambda_0=k_\parallel V_\parallel-\omega$. Here v_d denotes the pressure drift[3] which is defined by

$$v_d = \frac{T_\perp}{m\Omega}\frac{1}{N}\frac{\partial N}{\partial x}, \tag{14}$$

where $T_\perp$ is the perpendicular temperature, or the mean kinetic energy in the direction perpendicular to the magnetic field. One sees that the term proportional to $k_\perp v_d$ in n_1 denotes the density variation due to the $\boldsymbol{E}_\perp \times \boldsymbol{B}_0$ drift, the one proportional to $k_\perp^2/\Omega^2$ is due to the oblique propagation of the wave and the one proportional to $k_\parallel^2$ is caused directly by $\boldsymbol{E}_\parallel$.

Without loss of generality, we can hereafter take the zeroth order distribution $N(x, \boldsymbol{V})$ of the form

$$N(x, \boldsymbol{V}) = N_0[1-\epsilon(x+V_y/\Omega)]f_\perp(V_\perp^2)f_\parallel(V_\parallel),$$

where ϵ is a small parameter, and

$$f_\perp(V_\perp) = (m/2\pi T_\perp)\exp[-mV_\perp^2/2T_\perp],$$
$$f_\parallel(V_\parallel) = (m/2\pi T_\parallel)^{1/2}\exp[-mV_\parallel^2/2T_\parallel]. \tag{15}$$

§ 3. Dispersion relation

The ions are supposed to have the unit charge, then the Poisson equation is

$$\nabla\cdot\boldsymbol{E} = k^2\varphi = 4\pi e(\tilde{n}_i - \tilde{n}_e), \tag{16}$$

where $\tilde{n}_{i,e}$ is the integrated perturbed density of the non-resonant particles

$$\tilde{n}_{i,e} = \int_0^\infty 2\pi V_\perp dV_\perp \int_{-\infty}^{+\infty} dV_\parallel n_{1i,e}(\boldsymbol{r}, t).$$

From (13·a) and (15) we find

$$\tilde{n}_i = -\frac{N_0 e\varphi}{T_{\perp i}}\left\{b_i - \frac{k_\perp v_d^i}{\omega}\zeta_i Z_p(\zeta_i) + \frac{T_{\perp i}}{T_{\parallel i}}(1+\zeta_i Z_p(\zeta_i))\right\}$$
$$\times \mathrm{I}_0(b_i)\exp(-b_i)$$
$$\cong -\frac{N_0 e\varphi}{T_{\perp i}}\left\{b_i + \frac{k_\perp v_d^i}{\omega}(1-b_i)\right\}, \tag{17}$$

$$\tilde{n}_e \cong +\frac{N_0 e\varphi}{T_{\parallel e}}, \tag{18}$$

where the conditions in (2) are taken into account and the following formulae are used:

$$b_i \equiv \frac{1}{2}k_\perp^2\rho_i^2 = \frac{k_\perp^2 T_{\perp i}}{m_i\Omega_i^2}, \quad \zeta_i \equiv \omega/k_\parallel(2T_{\parallel i}/m_i)^{1/2} \gg 1,$$

$$\int_0^\infty \mathrm{J}_0^2\left(\frac{k_\perp V_\perp}{\Omega_i}\right) f_{\perp i}(V_\perp)2\pi V_\perp dV_\perp = e^{-b_i}\mathrm{I}_0(b_i) \cong 1-b_i,$$

$$\int_0^\infty J_0^2\left(\frac{k_\perp V_\perp}{\Omega_e}\right) f_{\perp e}(V) 2\pi V_\perp dV_\perp \cong 1\,,$$

$$Z_p(\zeta) = \frac{1}{\sqrt{\pi}} \mathcal{P}\int_{-\infty}^{+\infty} dx \frac{e^{-x^2}}{x-\zeta} \cong -\frac{1}{\zeta} - \frac{1}{2}\frac{1}{\zeta^3} \quad \text{for } \zeta \gg 1.$$

Substituting (17) and (18) into (16), we obtain the dispersion relation

$$\omega \cong \frac{T_{\parallel e}}{T_{\perp e}} k_\perp v_d^e \left[1 - \frac{1}{2} k_\perp^2 \rho_i^2 \left(1 + \frac{T_{\parallel e}}{T_{\perp i}}\right)\right], \tag{19}$$

where besides (2) we have assumed

$$k^2 d_i^2 \equiv k^2 (T_{\perp i}/4\pi N_0 e^2) \ll 1\,. \tag{20}$$

Note that $v_d^e > 0$ for $\partial N/\partial x < 0$ as was defined in (14). The phase velocity of the drift wave across the magnetic field, $\omega/k_\perp$, is nearly equal to the electron's pressure-drift velocity.

§ 4. Energy balance and growth rate

The wave energy density per unit wavelength, W_w, is the sum of the pure field energy and the changes in energy of the non-resonant particles. That is,

$$W_w = \frac{\lambda k^2 \varphi_1^2}{8\pi} + W_i + W_e\,, \tag{21}$$

$$W_{i,e} = \int_0^\lambda ds \int_0^\infty 2\pi V_\perp dV_\perp \int_{-\infty}^{+\infty} dV_\parallel \frac{m_{i,e}}{2} \{(N+n_1)(\boldsymbol{V}+\boldsymbol{u})^2 - NV^2\}_{i,e}$$

where (7) with $\delta = 0$, (13·a) and (15) are to be substituted and

$$ks = \boldsymbol{k}\cdot\boldsymbol{r}, \qquad \lambda = 2\pi/k.$$

Under the assumptions of (2), the results of calculation are

$$W_i \cong \frac{\lambda k_\parallel^2 \varphi_1^2}{16\pi} \frac{\omega_{pi}^2}{\omega^2} (1 - k^2 \rho_i^2) \left(1 - 4\frac{T_{\parallel i}}{T_{\perp i}} \frac{k_\perp v_d^i}{\omega}\right),$$

$$W_e \cong \frac{\lambda k_\parallel^2 \varphi_1^2}{16\pi} \frac{\omega_{pe}^2}{k_\parallel^2 (T_{\parallel e}/m_e)}\,, \tag{22}$$

where

$$\omega_{pi,e}^2 = 4\pi N_0 e^2 / m_{i,e}\,.$$

By comparing (22) with the first term on the right-hand side of W_w, one sees that the major part of the wave energy is contained in the form of the oscillatory motions of the non-resonant electrons. Hence

$$W_w \cong W_e \cong \frac{\lambda k_\parallel^2 \varphi_1^2}{16\pi} \frac{\omega_{pe}^2}{k_\parallel^2 (T_{\parallel e}/m_e)} . \tag{23}$$

Next, we shall calculate the change in energy of the resonant electrons per unit wavelength

$$W_r = \int_0^\lambda ds \int_0^\infty 2\pi V_\perp dV_\perp \int_{\omega/k_\parallel - \Delta V}^{\omega/k_\parallel + \Delta V} dV_\perp \frac{m_e}{2} \{(N+n_1)(\boldsymbol{V}+\boldsymbol{u})^2 - NV^2\}_e , \tag{24}$$

where (7) with $\delta=1$, (13·b) and (15) are to be substituted. It is sufficient to take only the main terms with $n=0$ in (7). The calculation is rather lengthy; however, the course of it will be presented below to show which terms give the main contribution. If we retain only the main terms, then

$$W_r \cong \int \cdots\cdots \int \left\{ \frac{1}{2} N m_e u_z^2 + n_1 m_e u_z V_\parallel \right\} .$$

Expanding the integrand around $V_\parallel = \omega/k_\parallel$, we have

$$W_r = \frac{\lambda \varphi_1^2}{8\pi} \omega_{pe}^2 k_\parallel t \left\{ f_\parallel\left(\frac{\omega}{k_\parallel}\right) I_1 - 2\left[f_\parallel\left(\frac{\omega}{k_\parallel}\right) + \frac{\omega}{k_\parallel} f_\parallel'\left(\frac{\omega}{k_\parallel}\right)\right] I_2 \right\}_e H$$

$$- \frac{\lambda \varphi_1^2}{4\pi} \omega_{pe}^2 \frac{k_\perp v_d^e \omega t}{k_\parallel (T_{\perp e}/m_e)} f_{\parallel e}\left(\frac{\omega}{k_\parallel}\right) I_2 H .$$

Here the extent of the resonant electrons in $V_\parallel$ is plausibly chosen to be

$$k \Delta V t \gtrsim 2\pi ,$$

then the quantities $I_{1,2}$ in the above are estimated with $x = (k_\parallel V_\parallel - \omega)t/2$:

$$I_1 = \int_{-\pi}^{\pi} dx \frac{\sin^2 x}{x^2} \cong \int_{-\infty}^{+\infty} dx (\cdots) = \pi ,$$

$$I_2 = \int_{-\pi}^{\pi} dx \frac{\sin x}{x^2} (\sin x - x \cos x) \cong \int_{-\infty}^{+\infty} dx (\cdots) = \frac{\pi}{2} .$$

Also approximately

$$H = \int_0^\infty 2\pi V_\perp dV_\perp f_{\perp e}(V_\perp) J_0^2\left(\frac{k_\perp V_\perp}{\Omega_e}\right) \cong 1 .$$

Hence

$$W_r = -\pi \frac{\lambda \varphi_1^2}{8\pi} \omega_{pe}^2 \left\{ \frac{k_\perp v_d^e}{k_\parallel (T_{\perp e}/m_e)} f_{\parallel e}\left(\frac{\omega}{k_\parallel}\right) + f'_{\parallel e}\left(\frac{\omega}{k_\parallel}\right) \right\} \omega t . \tag{25}$$

If we use the explicit form of $f_{\parallel e}$ as in (15), the above turns out to be

$$W_r = \sqrt{\pi} \frac{\lambda k_\parallel^2 \varphi_1^2}{8\pi} \frac{\omega_{pe}^2 \omega t}{k_\parallel^2 (T_{\parallel e}/m_e)} \frac{\omega - (T_{\parallel e}/T_{\perp e}) k_\perp v_d^e}{k_\parallel (2T_{\parallel e}/m_e)^{1/2}} \exp\left[-\frac{m_e \omega^2}{2T_{\parallel e} k_\parallel^2} \right] . \tag{25'}$$

The law of energy conservation is

$$\frac{d}{dt}(W_w + W_r) = 0\,. \tag{26}$$

From this with (23), (25′) and (19), the growth rate of the drift wave is finally found to be

$$\begin{aligned}\gamma \equiv \frac{1}{\varphi_1}\frac{d\varphi_1}{dt} &= \pi\omega\frac{T_{\parallel e}}{m_e}\left\{\frac{k_\perp v_d^e}{k_\parallel (T_{\perp e}/m_e)} f_{\parallel e}\left(\frac{\omega}{k_\parallel}\right) + f'_{\parallel e}\left(\frac{\omega}{k_\parallel}\right)\right\} \\ &= \frac{\sqrt{\pi}}{2}\frac{\omega^2 k_\perp{}^2 \rho_i{}^2}{|k_\parallel|(2T_{\parallel e}/m_e)^{1/2}}\left(1 + \frac{T_{\parallel e}}{T_{\perp i}}\right)\exp\left[-\frac{m_e\omega^2}{2T_{\parallel e}k_\parallel{}^2}\right].\end{aligned} \tag{27}$$

The result coincides with that derived from the Vlasov equation.

§ 5. Concluding remarks

(i) The physical mechanism of drift instability seems to be clear in the above considerations. There are two compensating terms which contribute to the growth (or damping) of the drift wave of interests. The first term in (25) or (27) means the enhancement of the wave due to the $\boldsymbol{E}_\perp \times \boldsymbol{B}_0$ drift while the second term denotes the Landau damping. The detailed mechanism of the $\boldsymbol{E}_\perp \times \boldsymbol{B}_0$ drift will be seen in the phase relation between the perturbed velocity $\delta v_\parallel = u_z$ given in (7) and the perturbed density due to the $\boldsymbol{E}_\perp \times \boldsymbol{B}_0$ drift, n_{drift}, which is the term proportional to $k_\perp v_d$ in (13·b). We always have the negative value of $\langle n_{\text{drift}}\,\delta(mv_\parallel^2/2)\rangle \cong \langle n_{\text{drift}}\, mV_\parallel u_z\rangle$ for either $v_\parallel > \omega/k_\parallel$ or $v_\parallel < \omega/k_\parallel$, where the bracket denotes to take the average over the phase seen from the particles. This means that for any value of $v_\parallel$ near to $\omega/k_\parallel$, the resonant electrons spend more time in losing energy than in gaining. This mechanism is different from the Landau damping, in which the density perturbation due to $\boldsymbol{E}_\parallel$ takes part and the resonant electrons with $v_\parallel > \omega/k_\parallel$ lose their energies while those with $v_\parallel < \omega/k_\parallel$ gain some energy.

It is also remarked that the wave energy is contained mainly in the ordered motions of the non-resonant electrons in the directions along the magnetic field. The ions give less contribution to the wave energy simply because their velocity perturbation is small owing to their large mass. However, the density perturbation of the ions in the plane normal to $\boldsymbol{B}_0$ is comparable in magnitude to that of the electrons along $\boldsymbol{B}_0$ and it should be included in the dispersion relation.

(ii) As was argued by Krall and Rosenbluth,[4] the unstable motion of a drift wave with short wavelength is possibly stabilized by ion Landau damping. One will readily see that this effect can be treated on the same ground as above. When $\omega/k_\parallel$ becomes comparable to $\bar{v}_{\parallel i}$, we must add a term to (25), which expresses the contribution of the resonant ions. Namely

$$\frac{dW_r}{dt} = (\text{electron's term}) + (\text{ion's term})$$

$$\cong \sqrt{\pi}\,\frac{\lambda\varphi_1{}^2}{8\pi}\,\frac{\omega_{pe}^2}{T/m_e}\,\frac{\omega^2}{k_\parallel(2T/m_e)^{1/2}}\left\{\left(1-\frac{k_\perp v_d}{\omega}\right)\exp(-\zeta_i{}^2 m_e/m_i)\right.$$

$$\left.+\mathrm{I}_0(b_i)e^{-b_i}\left(1+\frac{k_\perp v_d}{\omega}\right)\left(\frac{m_i}{m_e}\right)^{1/2}e^{-\zeta_i{}^2}\right\}, \tag{28}$$

where b_i and ζ_i are given below Eq. (18), and also we set for brevity $T=T_{\parallel e,i}=T_{\perp e,i}$ and $v_d=v_d{}^e=-v_d{}^i$. The stability condition is $dW_r/dt>0$, that is, letting $\exp(-\zeta_i{}^2 m_e/m_i)\cong 1$,

$$\left(1-\frac{k_\perp v_d}{\omega}\right)+\mathrm{I}_0(b_i)e^{-b_i}\left(1+\frac{k_\perp v_d}{\omega}\right)\left(\frac{m_i}{m_e}\right)^{1/2}e^{-\zeta_i{}^2}>0 \quad \text{for stability.} \tag{29}$$

A more accurate formula of the dispersion relation than (19) is

$$\frac{k_\perp v_d}{\omega}=\frac{1+k^2d^2+[b_i+1+\zeta_i Z_p(\zeta_i)]\mathrm{I}_0(b_i)e^{-b_i}}{[-\zeta_i Z_p(\zeta_i)]\mathrm{I}_0(b_i)e^{-b_i}}, \tag{30}$$

where $d^2=(T/4\pi N_0 e^2)$. This is essentially in agreement with Eq. (23) in reference 4). By the use of (30), the stability condition (29) turns out to be

$$\{\mathrm{I}_0(b_i)e^{-b_i}[-\zeta_i Z_p(\zeta_i)]-1\}+\mathrm{I}_0(b_i)e^{-b_i}\left(1+\frac{1}{2}k^2d^2\right)e^{-\zeta_i{}^2}\left(\frac{m_i}{m_e}\right)^{1/2}>0\,. \tag{31}$$

The result coincides with Eq. (25) in reference 4).

(iii) Finally we may add a remark on drift instability due to impurity ions, which has recently been discussed by Coppi et al.[5] Let us consider the case under the same conditions imposed by these authors:

$$\bar{v}_I<\omega/k_\parallel\ll\bar{v}_i\ll\bar{v}_e,$$

$$k_\perp^2\rho_e{}^2\ll k_\perp^2\rho_i{}^2<k_\perp^2\rho_I{}^2<1\,.$$

Here the quantities about impurity ions with the charge $Z_I e$ and the mass M_I are indicated by the index "I", and the ions are supposed to have the unit charge. Changing the roles of the particles in the discussions above, we find the integrated density perturbations for the respective species as follows:

$$\tilde{n}_I\cong-\frac{Z_I e\varphi N_I}{T_I}\left\{b_I-\frac{k_\perp v_d{}^I}{\omega}\zeta_I Z_p(\zeta_I)+1+\zeta_I Z_p(\zeta_I)\right\},$$

$$\tilde{n}_i\cong-\frac{e\varphi N_i}{T_i}\quad\text{and}\quad\tilde{n}_e\cong\frac{e\varphi N_e}{T_e}\,. \tag{32}$$

The Poisson equation

$$k^2\varphi=4\pi e(Z_I\tilde{n}_I+\tilde{n}_i-\tilde{n}_e)$$

leads

$$\frac{k_\perp v_d{}^I}{\omega} = -\frac{N_e/T_e + N_i/T_i - (Z_I^2 N_I/T_I)(1/2\zeta_I^2 - b_I)}{Z_I^2 N_I/T_I}, \tag{33}$$

where $\zeta_I = \omega/k_\parallel (2T_I/M_I)^{1/2}$ and $b_I = (1/2) k_\perp{}^2 \rho_I{}^2$, and the following approximations are used:

$$k^2 d_D{}^2 = k^2 (T_e/4\pi e^2 N_e) \ll 1, \quad 1 + \zeta_I Z_p(\zeta_I) \cong -1/2\zeta_I^2.$$

Now only the resonant ions participate in energy exchange with the wave. The rate of energy transfer is

$$\frac{dW_r}{dt} \cong \sqrt{\pi}\,\frac{\lambda \varphi_1{}^2}{8\pi}\,\frac{\omega_{pi}^2}{T_i/m_i}\,\frac{\omega^2}{k_\parallel (2T_i/m_i)^{1/2}}\left(1 - \frac{k_\perp v_d{}^i}{\omega}\right) \mathrm{I}_0(b_i)\, e^{-b_i - \zeta_i{}^2}. \tag{34}$$

The condition for instability is

$$\frac{dW_r}{dt} < 0 \quad \text{or} \quad \omega < k_\perp v_d{}^i.$$

From (33), this can be expressed in the form

$$\frac{Z_I N_i}{N_e \dfrac{T_i}{T_e} + N_i - N_I Z_I^2 \left(\dfrac{1}{2\zeta_I^2} - b_I\right)\left(\dfrac{T_i}{T_I}\right)} + \frac{\nabla N_i}{\nabla N_I} < 0. \tag{35}$$

The criterion is in agreement with that given by the previous authors.

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