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Particle paths of general relativity as geodesics of an affine connection

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This paper deals with the motion of incoherent matter, and hence of test particles, in the presence of fields with an arbitrary energy-momentum tensor. The equations of motion are obtained from Einstein's field equations and are written in the form of geodesic equations of an affine connection. The special cases of the electromagnetic field, the Proca field and a scalar field are discussed.

1. Introduction

In general relativity, matter can be described by an energy-momentum tensor, or by singularities in the field with the empty space field equations applicable outside the singularities [1]. With the former description, it is easily shown that the equations of motion of incoherent matter follow from the field equations; in the absence of non-gravitational fields, the particles follow geodesics of the Riemannian space of general relativity. In particular, this result applies to a single test particle, as is seen by taking the density to be proportional to a delta function [9, pp. 19-20]. Einstein, Infeld and Hoffman derived the equations of motion of gravitating particles in their total gravitational field by using the second description.

In general relativity, the equations of motion of a test charge in an electromagnetic field can be written as the equations of geodesics in a Finsler space or in a five dimensional Riemannian space [9, p. 85]. They can also be expressed, by suitably choosing an affine connection, in the

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form of the geodesic equations of that connection [2]; the affinity is the sum of the usual Riemann-Christoffel symbol and a third-rank tensor which depends on the electromagnetic field and the particle's 4-velocity. In the present paper, this result is generalized to particles in fields having an arbitrary energy-momentum tensor. Some particular fields are then discussed.

2. The equations of motion

Consider a four dimensional Riemannian space with coordinates x^α and interval ds given by $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, $(g_{\mu\nu})$ being the metric tensor. The contracted curvature tensor and the curvature scalar are written $(R_{\mu\nu})$ and R. Covariant differentation is denoted by a semi-colon.

The field equations of general relativity are

$$R^{\mu\nu} + (\lambda - \tfrac{1}{2}R)g^{\mu\nu} = -\kappa\left(M^{\mu\nu} + S^{\mu\nu}\right) \tag{1}$$

where λ is the cosmological constant and κ is a constant, while $(M^{\mu\nu})$ and $(S^{\mu\nu})$ are the energy-momentum tensors of matter and the (non-gravitational) fields, respectively. For incoherent matter (dust) $M^{\mu\nu} = \rho u^\mu u^\nu$, ρ being the proper density and (u^μ) the 4-velocity (cdx^μ/ds), c being the speed of light in empty space.

Since the divergence of the left side of the set (1) vanishes,

$$K^\mu = M^{\mu\nu}{}_{;\nu} \tag{2a}$$

$$= u^\mu\left(\rho u^\nu\right)_{;\nu} + \rho u^\mu{}_{;\nu}u^\nu \tag{2b}$$

where

$$K^\mu \equiv -S^{\mu\nu}{}_{;\nu}\,. \tag{3}$$

It follows from $u_\mu u^\mu = c^2$, that $\left(u_\mu u^\mu\right)_{;\nu}$ and hence $u_\mu u^\mu{}_{;\nu}$ vanish. So multiplying through by u_μ and summing over μ; (2b) leads to

(4) $$\left(\rho u^{\nu}\right)_{;\nu} = \frac{1}{c^2} u_{\mu} K^{\mu} .$$

Substituting (4) into (2b) gives

(5) $$u^{\nu} u^{\mu}{}_{;\nu} = \frac{1}{\rho}\left(K^{\mu} - \frac{1}{c^2} u_{\alpha} K^{\alpha} u^{\mu}\right)$$

- the equation of motion of the matter.

From the general expression for a covariant derivative

(6) $$u^{\nu} u^{\mu}{}_{;\nu} = u^{\nu}\left(u^{\mu}{}_{,\nu} + \left\{ {}_{\tau}{}^{\mu}{}_{\nu} \right\} u^{\tau}\right) ,$$

where a comma denotes ordinary partial differentiation and the braces denote the Christoffel symbol of the second kind. Thus, since $u^{\nu} = c dx^{\nu}/ds$, (5) has the alternative form

(7) $$\frac{d^2 x^{\mu}}{ds^2} + \left\{ {}_{\tau}{}^{\mu}{}_{\nu} \right\} \frac{dx^{\tau}}{ds} \frac{dx^{\nu}}{ds} = \frac{1}{\rho c^2}\left(K^{\mu} - \frac{1}{c^2} u_{\alpha} K^{\alpha} u^{\mu}\right) .$$

If the right side of (5) or (7) vanishes, the particles follow geodesics in the Riemannian space of general relativity. This occurs if (K^{μ}) vanishes or if (K^{μ}) and (u^{μ}) are parallel.

The term in $S^{\mu\nu}$ in (1) could be regarded as modifying Einstein's field equations, and thus being appropriate to the left side of (1), rather than as being part of the energy-momentum tensor. Such a modification was introduced by Hoyle [4, 5] in his original treatment of continuous creation. Equation (7) is equivalent to one stated by Hoyle [5].

3. The affinity

Consider the set of quantities defined by

(8) $$\Gamma_{\alpha}{}^{\mu}{}_{\beta} = \left\{ {}_{\alpha}{}^{\mu}{}_{\beta} \right\} + A_{\alpha}{}^{\mu}{}_{\beta}$$

in which the $A_{\alpha}{}^{\mu}{}_{\beta}$ are components of a third rank tensor. The sum of an affinity and a tensor of the appropriate type is always an affinity, so Γ is one. Covariant differentiation with respect to Γ will be denoted by a

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dot. Hence

(9) $$u^{\nu}u^{\mu}{}_{.\nu} = u^{\nu}\left(u^{\mu}{}_{;\nu} + A_{\tau}{}^{\mu}{}_{\nu}u^{\tau}\right) .$$

Substituting (5) into (9), it is seen that

(10) $$u^{\nu}u^{\mu}{}_{.\nu} = 0$$

provided A satisfies

(11) $$A_{\tau}{}^{\mu}{}_{\nu}u^{\tau}u^{\nu} = \frac{-1}{\rho}\left(K^{\mu} - \frac{1}{c^2}u_{\alpha}K^{\alpha}u^{\mu}\right) .$$

Equation (10) can be written in the alternative form

(12) $$\frac{d^2x^{\mu}}{ds^2} + \Gamma_{\tau}{}^{\mu}{}_{\nu}\frac{dx^{\tau}}{ds}\frac{dx^{\nu}}{ds} = 0 ,$$

which can be obtained directly by using (8) and (11) to eliminate the Christoffel symbol from (7). Equation (12), like (10), is the equation of the geodesics of the affine connection Γ : the paths of test particles can be described by such geodesics.

From the general expression for a covariant derivative, it follows that

(13) $$g_{\mu\nu.\rho} = g_{\mu\nu;\rho} - A_{\mu\nu\rho} - A_{\nu\mu\rho} .$$

Hence, if A is chosen so that $A_{\alpha\beta\gamma} = -A_{\beta\alpha\gamma}$,

(14) $$g_{\mu\nu.\rho} = 0 .$$

Given an affinity Γ , consider the set of quantities defined by

(15) $$\Gamma'_{\alpha}{}^{\mu}{}_{\beta} = \Gamma_{\alpha}{}^{\mu}{}_{\beta} + \delta_{\alpha}{}^{\mu}V_{\beta} + \delta_{\beta}{}^{\mu}V_{\alpha} + B_{\alpha}{}^{\mu}{}_{\beta}$$

where V is an arbitrary vector and B an anti-symmetric third rank tensor; these form an affinity Γ' with the same geodesics as Γ [7 , p. 55]. If $V_{\alpha} = 0$ and the torsion vector of B vanishes, (14) holds with respect to the affinity Γ' , if it does with respect to Γ . Equation (12) shows that the geodesics of Γ are unaffected by its anti-symmetric part. The same geodesics will be obtained if A above is replaced by its symmetric part; but for (14) to hold, Γ will in general be non-symmetric.

A "dual affinity" [2] can be introduced by defining

$$(16) \qquad {}^*\Gamma_{\alpha}{}^{\mu}{}_{\beta} = \left\{ {\mu \atop \alpha\ \beta} \right\} + {}^*A_{\alpha}{}^{\mu}{}_{\beta}$$

where

$$(17) \qquad {}^*A_{\alpha\beta\tau} = \tfrac{1}{2}\, \eta_{\alpha\beta\mu\nu} A^{\mu\nu}{}_{\tau} ,$$

η being the permutation tensor. Covariant differentiation with respect to ${}^*\Gamma$ will be denoted by an asterisk. Since $({}^*A_{\alpha\beta\gamma})$ is anti-symmetric in the first two indices, $g_{\mu\nu *\rho}$ vanishes. The geodesics of ${}^*\Gamma$ are not those of Γ .

The next section will deal with a special form for A .

4. A special choice of affinity

Here, A will be chosen to have the form given by

$$(18) \qquad A_{\tau}{}^{\mu}{}_{\nu} = B_{\tau}{}^{\mu} u_{\nu}$$

where B is a second rank tensor, with $(B_{\alpha\beta})$ anti-symmetric; (14) is satisfied. Since $u_{\nu}u^{\nu} = c^2$, the condition (11) becomes

$$(19) \qquad B_{\tau}{}^{\mu} u^{\tau} = \frac{-1}{\rho c^2}\left(K^{\mu} - \frac{1}{c^2}\, u_{\alpha} K^{\alpha} u^{\mu}\right) .$$

It may be noted that

$$(20) \qquad B_{\mu\nu.\rho} = B_{\mu\nu;\rho} .$$

The torsion vector is given by

$$(21) \qquad \Gamma_{\mu} \equiv \Gamma_{[\mu}{}^{\sigma}{}_{\sigma]} = A_{[\mu}{}^{\sigma}{}_{\sigma]} .$$

So, using (18) and (19),

$$(22) \qquad \Gamma_{\mu} = \tfrac{1}{2}\, B_{\mu}{}^{\sigma} u_{\sigma} = \frac{1}{2\rho c^2}\left(K_{\mu} - \frac{1}{c^2}\, u_{\alpha} K^{\alpha} u_{\mu}\right)$$

since, because of the anti-symmetry of $(B_{\mu\nu})$, B^{σ}_{σ} vanishes. The

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torsion vector vanishes if (K^{μ}) vanishes or if (K^{μ}) and (u^{μ}) are parallel; if (K^{μ}) and (u^{μ}) are perpendicular, (Γ_{α}) is proportional to (K_{α}) .

Two further tensors are now defined by

(23) $$I_{\mu}{}^{\nu} \equiv \tfrac{1}{2} A_{\mu\alpha\beta}A^{\alpha\nu\beta} = \frac{c^2}{2} B_{\mu\alpha}B^{\alpha\nu}$$

and

(24) $$J_{\mu\nu} \equiv A_{\alpha\beta\mu}A^{\beta\alpha}{}_{\nu} = B_{\alpha\beta}B^{\beta\alpha}u_{\mu}u_{\nu} .$$

These can be regarded as geometrical quantities, defined in terms of part of the affinity. Hence the matter tensor can be expressed in terms of geometrical quantities through

(25) $$M_{\mu\nu} = \rho u_{\mu}u_{\nu} = \rho c^2 J_{\mu\nu}/J^{\alpha}_{\alpha} ,$$

provided J^{α}_{α} does not vanish.

In addition to I^{α}_{α} , another invariant can be obtained from B ;

(26) $$\tfrac{1}{2} \eta_{\mu\nu\sigma\rho}A^{\mu\nu\lambda}A^{\sigma\rho}{}_{\lambda} = \frac{c^2}{2} \eta_{\mu\nu\sigma\rho}B^{\mu\nu}B^{\sigma\rho} = c^2 {}^{*}B_{\mu\nu}B^{\mu\nu}$$

where ${}^{*}B$ is the dual of B .

Using (18), (17) becomes

(27) $${}^{*}A_{\alpha\beta\tau} = {}^{*}B_{\alpha\beta}u_{\tau} .$$

The corresponding torsion vector is given by

(28) $${}^{*}\Gamma_{\mu} = \tfrac{1}{2} {}^{*}B_{\mu}{}^{\sigma}u_{\sigma} .$$

Also ${}^{*}B_{\mu\nu{*}\rho} = {}^{*}B_{\mu\nu;\rho}$.

5. Particular fields

5.1 The electromagnetic field

The electromagnetic field in free space can be represented by an anti-symmetric second rank tensor $(\phi_{\mu\nu})$ which, in general relativity, satisfies

(29) $$\phi_{\mu\lambda,\nu} + \phi_{\lambda\nu,\mu} + \phi_{\nu\mu,\lambda} = 0 ,$$

(30) $$\phi^{\mu\nu}{}_{;\nu} = \frac{4\pi}{c} j^{\mu} ,$$

where the current density 4-vector is given by $j^{\mu} = \bar{\rho} u^{\mu}$, $\bar{\rho}$ being the proper charge density. The energy-momentum tensor has components [3, p. 182]

(31) $$S_{\mu}{}^{\nu} = \frac{1}{4\pi}\left(\phi_{\mu\alpha}\phi^{\alpha\nu} + \frac{1}{4}\delta_{\mu}{}^{\nu}\phi_{\sigma\tau}\phi^{\sigma\tau}\right) .$$

Using (31), (29) and (30) in the definition (3):

(32a) $$K_{\mu} = -\frac{1}{4\pi}\phi_{\mu\alpha}\phi^{\alpha\nu}{}_{;\nu}$$

(32b) $$= \frac{1}{c}\phi_{\alpha\mu}j^{\alpha} ,$$

giving the 4-force per unit proper volume acting on the matter.

Choosing

(33) $$B_{\tau}{}^{\mu} = -\frac{\bar{\rho}}{\rho c^{3}}\phi_{\tau}{}^{\mu} ,$$

it is seen, using (32b), that the condition (19) is satisfied. The affinity is now essentially that chosen by Droz-Vincent [2]. Since $u_{\alpha}K^{\alpha} = 0$, the torsion vector is proportional to (K^{μ}) : from (22) and (32b)

(34) $$\Gamma_{\mu} = \frac{1}{2\rho c^{3}}\phi_{\alpha\mu}j^{\alpha} .$$

The energy-momentum tensor components can be expressed, using (23), by

$$S_{\mu}{}^{\nu} = \frac{1}{2\pi}\left(\frac{\rho c^{2}}{\bar{\rho}}\right)^{2}\left(I_{\mu}{}^{\nu} - \frac{1}{4}\delta_{\mu}{}^{\nu}I^{\alpha}_{\alpha}\right) .$$

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Droz-Vincent expressed $S_{\mu\nu}$ and $M_{\mu\nu}$ in terms of geometrical quantities by using both the affinity and its dual.

The invariants I^{α}_{α} and (26) become proportional to $E^2 - H^2$ and E.H , where E and H are the electric and magnetic fields.

Apart from factors involving c , ρ and $\bar{\rho}$, (25) and (35) express the matter tensor and the electromagnetic energy-momentum tensor in terms of I and J , which can be thought of as geometrical quantities, defined in terms of the affinity. Equation (25) fails if $E^2 = H^2$.

The affinity depends on the 4-velocity of the matter and on the ratio of mass and charge densities, as well as on the electromagnetic field. For an individual test charge, the affinity depends on its 4-velocity and its charge to mass ratio: the affinity is not an "external" property of the space, independent of the test particle.

5.2 The Proca field

The Proca field is represented by a 4-vector potential (A_{μ}) in terms of which the field tensor is defined by $\phi_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu}$. The fields satisfy

(36) $$\phi_{\mu\lambda,\nu} + \phi_{\lambda\nu,\mu} + \phi_{\nu\mu,\lambda} = 0$$

and

(37) $$\phi^{\mu\nu}{}_{;\nu} = \frac{4\pi}{c}\left(j^{\mu} - k^2 A^{\mu}\right) ,$$

which reduce to Maxwell's equations when the constant k is zero.

Consider the energy-momentum tensor with components [6]

(38) $$S_{\mu}{}^{\nu} = \frac{1}{4\pi}\left(\phi_{\mu\alpha}\phi^{\alpha\nu} + \frac{1}{4}\,\delta_{\mu}{}^{\nu}\phi_{\sigma\tau}\phi^{\sigma\tau}\right) + \frac{k^2}{c}\left(A_{\mu}A^{\nu} - \frac{1}{2}\,\delta_{\mu}{}^{\nu}A_{\alpha}A^{\alpha}\right) .$$

This gives, using (36) and the definition of $\phi_{\mu\nu}$ in (3),

(39) $$K_{\mu} = \frac{-1}{4\pi}\,\phi_{\mu\alpha}\phi^{\alpha\nu}{}_{;\nu} - \frac{k^2}{c}\left(\phi_{\mu\nu}A^{\nu} + A_{\mu}A^{\nu}{}_{;\nu}\right) .$$

Using $k^2 A^{\mu}{}_{;\mu} = j^{\mu}{}_{;\mu}$, which follows from (37), and (37) itself, (39)

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becomes

(40) $$K_\mu = \frac{1}{c}\left(\phi_{\alpha\mu}j^\alpha - A_\mu j^\nu{}_{;\nu}\right) .$$

Choosing

(41) $$B_\tau{}^\mu = \frac{-\bar{\rho}}{\rho c^3}\,\phi_\tau{}^\mu - \frac{1}{\rho c^5}\,j^\nu{}_{;\nu}\left(A_\tau u^\mu - u_\tau A^\mu\right) ,$$

it is seen, using (40), that the condition (19) is satisfied. From (22) and (40), the torsion vector has components

(42) $$\Gamma_\mu = \frac{1}{2\rho c^3}\left[\phi_{\alpha\mu}j^\alpha - j^\nu{}_{;\nu}\left(A_\mu - \frac{1}{c^2}\,u_\mu u^\alpha A_\alpha\right)\right] .$$

5.3 A scalar field

Consider a scalar field ϕ , and let ϕ_α denote $\phi_{,\alpha}$. Choosing

(43) $$B_{\tau\mu} = \frac{1}{\rho c^4}\left(u_\tau\phi_\mu - \phi_\tau u_\mu\right) ,$$

it is seen that the condition (19) is satisfied if $K_\mu = -\phi_\mu$.

Consider the energy-momentum tensor with components

(44) $$S_{\mu\nu} = h\left[\phi_\mu\phi_\nu - \tfrac{1}{2}\,g_{\mu\nu}\left(\phi_\sigma\phi^\sigma + k^2\phi^2\right)\right]$$

where h and k are constants. This gives $K_\mu = -\phi_\mu$ provided ϕ satisfies

(45) $$\phi^\nu{}_{;\nu} - k^2\phi = \frac{1}{h} .$$

If $u^\alpha\phi_\alpha = 0$, use of (43) in (23) leads to

(46) $$I_\mu{}^\nu = \frac{-1}{2(\rho c^2)^2}\left(\phi_\mu\phi^\nu + \frac{1}{c^2}\,u_\mu u^\nu\phi_\alpha\phi^\alpha\right)$$.

and hence

(47) $$I^\sigma_\sigma = \frac{-1}{(\rho c^2)^2}\,\phi_\sigma\phi^\sigma .$$

Thus, using (25) to eliminate $u_\mu u^\nu$,

$$\phi_\mu \phi^\nu = (\rho c^2)^2 \left(\tfrac{1}{2} J_\mu{}^\nu - 2I_\mu{}^\nu \right) . \tag{48}$$

So if $u^\alpha \phi_\alpha = 0$ and $k = 0$, (44) can be written

$$S_{\mu\nu} = h(\rho c^2)^2 \left(\tfrac{1}{2} J_{\mu\nu} - 2I_{\mu\nu} + \tfrac{1}{2} g_{\mu\nu} I^\sigma_\sigma \right) , \tag{49}$$

which can be thought of as expressing the energy-momentum tensor in terms of geometrical quantities.

6. Concluding remarks

This paper has dealt with the motion of incoherent matter, and hence of test particles, in the presence of fields with an arbitrary energy-momentum tensor. By expressing the total energy-momentum tensor as the sum of a part due to the matter and a part due to the (non-gravitational) fields, the equations of motion follow from Einstein's field equations of general relativity. By suitable choice of affine connection, the equations of motion have been expressed as the equations of geodesics of the connection. The affinity is the sum of the usual Christoffel symbol and a part which depends on the fields. If the energy-momentum tensor of the fields is expressed as the sum of parts due to separate fields, the parts will contribute additively to the affinity. A particular form of affinity was chosen with respect to which the covariant derivative of the metric tensor $(g_{\mu\nu})$ vanishes; the affinity is then, in general, non-symmetric. For a test particle, the affinity depends on its 4-velocity and on ratios of coupling constants, such as its charge and mass: the affinity is not an "external" property of space, independent of the test particles.

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