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PARTICLE STOCHASTICITY DUE TO MAGNETIC PERTURBATIONS OF AXISYMMETRIC GEOMETRIES

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Particle Stochasticity Due to Magnetic Perturbations of Axisymmetric Geometries

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ABSTRACT

The quasilinear theory of collisionless test particle diffusion in stochastic magnetic fields is extended to include the effects of finite gyroradius ρ and particle drifts (including magnetic trapping). A canonical framework is used, in which both the criterion for onset of stochasticity and the diffusion tensor scale with field-particle coupling coefficients $g_{\underset{\sim}{\ell}}$. The $g_{\underset{\sim}{\ell}}$ contain all the information about a given particle's unperturbed orbit and the perturbation fields with which it interacts. The modification of transport due to finite ρ and drifts is thus found by comparison of the $g_{\underset{\sim}{\ell}}$ including these effects to their driftless, $\rho \to 0$ limit. It is found that runaway electron confinement is substantially improved over earlier, driftless estimates, and that trapped particles in microturbulence ought not be stochastic. The perturbations from proposed ripple injection schemes are large enough to induce stochasticity for certain classes of particles.

I. INTRODUCTION

This paper deals with the effects of finite gyroradius, particle drifts, and magnetic trapping on particle diffusion due to magnetic perturbations of axisymmetric toroidal configurations. Previous authors[1,2,3] have made the approximation in which particles exactly follow stochastic magnetic field lines. We find that inclusion of realistic orbit characteristics can substantially reduce the transport rate from that found by those previous "line-following" theories.

We consider two types of magnetic perturbations: those arising from microturbulence,[1] e.g. from drift or tearing modes, and those arising from a coherent magnetic "ripple" field, due either to coil errors or introduced intentionally as in ripple injection schemes.[4] We also consider two types of particle orbits, trapped and untrapped, and three general classes of particles, thermal electrons, thermal ions, and runaway electrons (species labels $s = e, i$, and r, respectively). In principle, the formalism is applicable to that class of particles in the intermediate region between trapped and passing, where the rapid change in the bounce frequency Ω_b with bounce action J_b is crucial to understanding stochastic effects. However, similar problems have been treated elsewhere,[5-7] and the present work excludes this regime.

The principle results are[8] :

(a) The diffusion of passing particles in turbulence is reduced by three effects. In order of decreasing importance, these are

(i) an averaging over the mode profile due to guiding-center drifts,

(ii) a shift due to drifts of the radius at which a particle is resonant with a given mode, and

(iii) an averaging over the mode profile due to finite gyroradius.

(b) Trapped particles in turbulence are not expected to be stochastic, for reasonable turbulence levels.

(c) In a ripple field, passing particles not too far from the separatrix separating trapped from passing can be stochastic, for perturbation fields of strength exceeded by proposed ripple injection schemes. (Trapped particles in ripple are not explicitly considered here, but preliminary indications are that they are at least as stochastic as the class of passing particles just mentioned.) This calculation is totally collisionless, and thus studies a regime different from those considered previously[9-12] for ripple-induced transport.

The problem is treated using a Hamiltonian framework, which deals succinctly with the unperturbed motion, and isolates the resonances due to the perturbation simply and explicitly. The quasilinear diffusion tensor $\underset{\sim}{D}$ we use was developed in this framework by Kaufman,[13] and the overlap criterion for onset of stochasticity is that used by Chirikov.[7] Here the general abstract quantities in those developments are explicitly evaluated, for the various specific cases we study.

Section II describes the toridal coordinate system we shall use in the subsequent development. In Sec. III the canonical formalism, in terms of which $\underset{\sim}{D}$ and the overlap criterion are phrased, is described, and the form[13] for $\underset{\sim}{D}$ is given. Formal expressions for the overlap criterion in this framework[7] are developed in Sec. IV.

Both $\underset{\sim}{D}$ and the overlap criterion involve a set of field-particle coupling coefficient $g_{\underset{\sim}{\ell}}$, which succinctly express all the information about a given particle's trajectory and the perturbation fields with which it interacts. The modifications of particle transport due to realistic orbit characteristics (hence the contribution of the present work beyond that in Refs. 1 and 2) may be seen by comparison of the expression for $g_{\underset{\sim}{\ell}}$ including these characteristics, to the expression for $g_{\underset{\sim}{\ell}}$ in the zero gyroradius, line-following limit. Accordingly, in Sec. V we evaluate $g_{\underset{\sim}{\ell}}$, and compare it to the line-following limit assumed in previous theories. Further comparison is made in Sec VII.

In Sec. VI various quantities of the canonical formalism, abstractly represented in Refs. 7 and 13, are explicitly evaluated, and their physical content discussed. This readies the canonical machinery to make physical statements. This is done in Sec. VII, where the results already noted are demonstrated and elaborated upon.

II. GEOMETRY

The formalism to be employed in this paper is in principle applicable to any axisymmetric equilibrium configuration, but we shall chiefly have in mind the tokamak goemetry, illustrated in Fig. 1. We parametrize real space by the orthogonal curvilinear coordinates $q^{\mu} \equiv (\alpha,\beta,\phi)$, where ϕ is the toroidal angle, α is the radial coordinate, constant on a given flux surface, and β corresponds to the poloidal angle, generalized to apply to noncircular poloidal cross-sections, reducing to the usual poloidal angle in the particular case of circular cross-sections, (we do not refer to this angle coordinate by the usual θ, to avoid confusion of this symbol with the canonical angle variables θ, to be introduced in Sec. III.) In terms of the covariant components $A^{o}_{\mu} \equiv \underset{\sim}{A}^{o} \cdot \partial \underset{\sim}{x}/\partial q^{\mu}$ of the unperturbed vector potential $\underset{\sim}{A}^{o}$, and in a gauge in which $A^{o}_{\alpha} = 0$, the poloidal and toroidal components of the magnetic field $\underset{\sim}{B}$ are given by

$$B_p = -(g^{\alpha} g^{\phi})^{\frac{1}{2}} \; \partial A^{o}_{\phi}/\partial\alpha \, , \; B_t = (g^{\alpha} g^{\beta})^{\frac{1}{2}} \; \partial A^{o}_{\beta}/\partial\alpha \; , \tag{1}$$

where the $g^{\mu} \equiv |\underset{\sim}{\nabla} q^{\mu}|^{2}$ are the diagonal elements of the metric tensor. In particular, $g^{\phi} = R^{-2}$ (R is the major radius), and, generalizing the definition of minor radius r to noncircular cross sections, $g^{\beta} \equiv r^{-2}$. Fully specifying α by taking $A_{\phi} = \alpha$, one has

$$B_p = -R^{-1}(\partial\alpha/\partial r)\, , \text{ or } \alpha = -\int^{r} d r' \, R B_p \, , \tag{2}$$

and

$$q - r B_t / R B_p = - \partial A^o_\beta / \partial \alpha \ . \qquad (3)$$

It is convenient to further define $B \equiv |\underset{\sim}{B}|$, $\hat{B} \equiv \underset{\sim}{B}/B$, $b_p \equiv B_p/B$ and $b_t \equiv B_t/B$.

III. DIFFUSION TENSOR, COUPLING COEFFICIENTS

In this section we present the form for the diffuison tensor $\underset{\approx}{D}$ developed in Ref. 13, and introduce the canonical quantities in terms of which the present work is expressed. We do not rederive $\underset{\approx}{D}$ here, but instead only sketch the origin of its form, indicating its structural similarity to more familiar forms. The expression for $\underset{\approx}{D}$ involves the square of field-particle coupling coefficients $g_{\underset{\sim}{\ell}}$, which succinctly express all the information about the interaction of a given particle with the perturbing spectrum, including the full nature of the particle trajectory (e.g. finite gyroradius and particle drifts). The $g_{\underset{\sim}{\ell}}$ play a central role in determining both $\underset{\approx}{D}$ and the stochasticity criterion, and in seeing the modification by the present work of previous results.

Following Ref. 13, we consider the diffusion of a particle in the space $\underset{\sim}{I} \equiv (\mu, J_b, P_\phi)$ of canonical momenta which are invariants in the absence of the perturbing fields. For the axisymmetric geometries we consider here, these invariants are:

1) the gyroaction $\mu \equiv mv_\perp^2/2\Omega_c$ (where $\Omega_c \equiv eB/mc$), i.e. $\mu = (mc/e)\tilde{\mu}$, where $\tilde{\mu}$ is the usual magnetic moment,

(2) the longitudinal invariant ("bounce action") J_b, and

(3) the canonical angular momentum P_ϕ. It is P_ϕ which determines the flux surface α_b (the "banana center") about which the particle moves, and it is thus chiefly diffusion in P_ϕ which determines radial particle transport.

Conjugate to these momenta are the coordinates $\underset{\sim}{\theta} \equiv (\theta_g, \theta_b, \Phi)$, with θ_g the gyrophase, θ_b the phase of the bounce motion, and Φ the bounce-averaged value of toroidal angle ϕ. (Note that the concept of "bounce motion" applies to a particle which is passing, as well as to one which is trapped. For passing particles the bounce time τ_b is given by the connection length qR divided by the parallel velocity $v_\parallel$.) In the absence of the perturbation, the Hamiltonian H_o is a function only of the invariants $\underset{\sim}{I}$, and the $\underset{\sim}{\theta}$ thus evolve linearly in time, $\dot{\underset{\sim}{\theta}} = \underset{\sim}{\Omega}(\underset{\sim}{I}) \equiv \partial H_o/\partial \underset{\sim}{I} \equiv (\Omega, \Omega_b, \Omega_\phi)$. Here Ω is the bounce-averaged gyrofrequency, Ω_b is the bounce frequency, and Ω_ϕ is the bounce-averaged toroidal drift (the "banana drift").

The diffusion tensor in $\underset{\sim}{I}$ space is given by [13]

$$\underset{\approx}{D}(\underset{\sim}{I}) = \sum_a \sum_{\underset{\sim}{\ell}} |g_{\underset{\sim}{\ell}}(\underset{\sim}{I}, a)|^2 \underset{\sim}{\ell}\,\underset{\sim}{\ell}\,\pi\,\delta(\omega_a - \underset{\sim}{\ell} \cdot \underset{\sim}{\Omega}) . \tag{4}$$

Here a labels the components of the perturbing field, with component a having frequency ω_a. Each of the components of the vector $\underset{\sim}{\ell} \equiv (\ell_g, \ell_b, \ell_\phi)$ may assume any integral value. From the δ-function in Eq. (4), we read off the resonance condition

$$\Omega = \omega_a - \underset{\sim}{\ell} \cdot \underset{\sim}{\Omega} \quad . \tag{5}$$

Finally, the field-particle coupling coefficients $g_{\underset{\sim}{\ell}}$ are defined by

$$g_{\underset{\sim}{\ell}}(\underset{\sim}{I}, a) = -(e/c)(2\pi)^{-3}\int d\underset{\sim}{\theta}\, e^{-i\underset{\sim}{\ell}\cdot\underset{\sim}{\theta}}\, \underset{\sim}{v}(z)\cdot \underset{\sim}{A}^a[\underset{\sim}{r}(z)] \quad , \tag{6}$$

where $z = (\underset{\sim}{\theta}, \underset{\sim}{I})$ is a particle's phase-space position, $\underset{\sim}{r}(z)$ is its real-space position, given z, and $\underset{\sim}{v}(z)$ is its velocity. $\underset{\sim}{A}^a(x)$ is the vector potential describing both the electric and magnetic parts of contribution a to the perturbation (we work in radiation gauge, $\phi^a = 0$). One sees that $g_{\underset{\sim}{\ell}}$ is just the Fourier coefficient of the first-order perturbing Hamiltonian $H_1 = -\sum_a c^{-1}\, \underset{\sim}{j}\cdot A^a$, i.e.

$$H_1(z,t) = \sum_a \sum_{\underset{\sim}{\ell}} g_{\underset{\sim}{\ell}}(\underset{\sim}{I}, a)\exp i(\underset{\sim}{\ell}\cdot\underset{\sim}{\theta} - \omega_a t) \quad . \tag{7}$$

One notes the structural similarity of $\underset{\approx}{D}$ in Eq. (4) to the more familiar expression for the quasilinear diffusion coefficient in linear momentum space for an unmagnetized plasma, with purely electrostatic perturbations:

$$\underset{\approx}{D}^{ql}(\underset{\sim}{p}) = (2\pi)^{-3}\int d^3\underset{\sim}{k}\, |e\,\phi(k)|^2\, \underset{\sim}{k}\,\underset{\sim}{k}\,\pi\,\delta(\omega_{\underset{\sim}{k}} - \underset{\sim}{k}\cdot\underset{\sim}{v}) \quad . \tag{8}$$

The analog to $g_{\underset{\sim}{\ell}}$ here is $e\,\phi(\underset{\sim}{k})$, again the Fourier coefficient of the perturbing Hamiltonian.

If interpreted literally, expression (4) is singular at each of the wave-particle resonances, and zero elsewhere. However, the non-vanishing Kolmogorov entropy in the stochastic state and the consequent nonlinear mixing of orbits ensures that the resonances are smoothed, so that for perturbation strength sufficiently large that the motion is stochastic, the $\underset{\sim}{\ell}$ sum is to be interpreted as a suitable integral, as discussed in Ref. 1. In the next section we consider the perturbation strength required for the onset of stochasticity.

IV. STOCHASTICITY CRITERIA (FORMAL)

In order that expression (4) for the diffusion tensor be valid, the perturbation strength must be large enough that the motion of a particle in $\underset{\sim}{I}$ space is stochastic in nature. If the perturbation is smaller than this, $\underset{\approx}{D}$ will equal zero instead of the value given by Eq. (4). In this section, we develop general expressions for the required perturbation strength for the onset of stochasticity, similar to those of an analysis by Chirikov,[7] employing the widely used resonance overlap criterion.

One proceeds by using Hamilton's equation for a system with unperturbed Hamiltonian $H_o(\underset{\sim}{I})$, and perturbation of the form of Eq. (7). We assume that the particle has momentum $\underset{\sim}{I} \simeq \underset{\sim}{I}_{\underset{\sim}{\ell}}$, where $\underset{\sim}{I}_{\underset{\sim}{\ell}}$ is a value of $\underset{\sim}{I}$ satisfying resonance condition (5). We first consider the particle motion keeping only the ($\underset{\sim}{\ell}$, a) component of H_1 and its complex conjugate, in which case the perturbed problem is exactly soluble. One has

$$\dot{\underset{\sim}{I}} = - i \underset{\sim}{\ell} g_{\underset{\sim}{\ell}} \exp i(\underset{\sim}{\ell} \cdot \underset{\sim}{\theta} - \omega_a t) + c.c. \quad , \tag{9}$$

and

$$\underset{\sim}{\ell} \cdot \dot{\underset{\sim}{\theta}} - \omega_a = \underset{\sim}{\ell} \cdot \underset{\sim}{\Omega}(\underset{\sim}{I}) - \omega_a \approx \underset{\sim}{\ell} \cdot \partial \underset{\sim}{\Omega}/\partial \underset{\sim}{I} \cdot \delta \underset{\sim}{I} \quad , \tag{10}$$

where $\delta \underset{\sim}{I} \equiv \underset{\sim}{I} - \underset{\sim}{I}_{\underset{\sim}{\ell}}$, and we have expanded $\underset{\sim}{\Omega}(\underset{\sim}{I})$ about $\delta \underset{\sim}{I} = 0$ and used (5) in obtaining (10). Defining $\psi_{\underset{\sim}{\ell}} \equiv \underset{\sim}{\ell} \cdot \underset{\sim}{\theta} - \omega_a t$ (absorbing the a-dependence into the $\underset{\sim}{\ell}$ when used as a subscript), we may combine Eqs. (9) and (10) to give

$$\ddot{\psi}_{\underset{\sim}{\ell}} = M_{\underset{\sim}{\ell}}^{-1} |2 g_{\underset{\sim}{\ell}}| \sin \psi_{\underset{\sim}{\ell}} \quad , \tag{11}$$

where $M_{\underset{\sim}{\ell}}^{-1} \equiv \underset{\sim}{\ell} \cdot (\partial \underset{\sim}{\Omega}/\partial \underset{\sim}{I}) \cdot \underset{\sim}{\ell}$. This is just the equation for a particle of mass $M_{\underset{\sim}{\ell}}$ moving in a one-dimensional sinusoidal potential of amplitude $|g_{\underset{\sim}{\ell}}|$. Particles well-trapped in the sinusoidal wells oscillate at frequency $\omega_{\underset{\sim}{\ell}}$, given by

$$\omega_{\underset{\sim}{\ell}} = |2 g_{\underset{\sim}{\ell}} M_{\underset{\sim}{\ell}}^{-1}|^{\frac{1}{2}} \quad . \tag{12}$$

Using (9) and (12), one sees that the phase points z corresponding to such particles make a maximum excursion $\Delta I_{\underset{\sim}{\ell}}$ in momentum space given by

$$\Delta \underset{\sim}{I}_{\underset{\sim}{\ell}} = \underset{\sim}{\ell} |2 g_{\underset{\sim}{\ell}}/\omega_{\underset{\sim}{\ell}}| = \underset{\sim}{\ell} |2 g_{\underset{\sim}{\ell}} M_{\underset{\sim}{\ell}}|^{\frac{1}{2}} \quad , \tag{13}$$

and corresponding excursion $\Delta\underset{\sim}{\Omega}_{\ell}$ in $\underset{\sim}{\Omega}$-space,

$$\Delta\underset{\sim}{\omega}_{\ell} = (\partial\underset{\sim}{\Omega}/\partial \underset{\sim}{I}) \cdot \Delta \underset{\sim}{I}_{\ell} = (\partial\underset{\sim}{\Omega}/\partial \underset{\sim}{I}) \cdot \underset{\sim}{\ell}\,|2\, g_{\ell}\, M_{\ell}|^{\frac{1}{2}} \quad . \tag{14}$$

From Eqs. (12) and (14), one notes that

$$\omega_{\ell} = \underset{\sim}{\ell} \cdot \Delta\underset{\sim}{\Omega}_{\ell} \quad . \tag{15}$$

Turning now to consideration of motion under the influence of all the components $(\ell\ , a)$, one expects that the motion will become stochastic when the excursion $\Delta \underset{\sim}{I}_{\ell}$ (or $\Delta\underset{\sim}{\Omega}_{\ell}$) due to one component is large enough to put the phase point within a distance $\Delta \underset{\sim}{I}_{\ell'}$ (or $\Delta\underset{\sim}{\omega}_{\ell'}$) of the resonance point $\underset{\sim}{I}_{\ell'}$ of another component.

To write down explicit expressions for this verbally described criterion, one must know the spacing between the resonance points $\underset{\sim}{I}_{\ell}$, for the particular perturbation being considered. As noted in the Introduction, we shall consider two types of perturbations here, a turbulent spectrum, consisting of many incoherent, radially localized modes, and a ripple spectrum, consisting of a single, totally coherent, time-independent perturbation. In both cases, the physical mechanism of radial transport comes from the change $\Delta\Omega_{b\,\ell}$ of the bounce frequency with change Δr_{ℓ} in radial position being large enough to allow the particle to come into resonance with another component (ℓ', a') . For the turbulent spectrum the spacing δ_t

between successive resonances is given by the physical radial distance between the surfaces on which the modes are localized, $\delta_t \sim \rho_i/m$. (Here ρ_i is a typical ion gyroradius, and m is a typical poloidal mode number.) The criterion for stochasticity for the turbulent spectrum may thus be written

$$1 < (\Delta r_{\underset{\sim}{\ell}}/\delta_t)^2 \quad . \tag{16}$$

For the ripple spectrum, which is radially unlocalized and has only a single component a, the radial resonance spacing δ_r is determined differently. The resonance spacing $\Omega_\zeta \Delta\ell_\zeta$ in the ℓ_ζ direction of $\underset{\sim}{\Omega}$ space is given by $\Delta\ell_\zeta = n_0 \sim 10$-20. This is wider than the spacing $\Omega_b \Delta\ell_b = \Omega_b$ for the ℓ_b direction. Thus, a particle moves along a chain of successive resonances $0 = \underset{\sim}{\ell}' \cdot \underset{\sim}{\Omega}(r_{\ell'})$, where $\underset{\sim}{\ell}' = \underset{\sim}{\ell}\, , \underset{\sim}{\ell} \pm \hat{\beta}\, , \underset{\sim}{\ell} \pm 2\hat{\beta}\, , \ldots$, with $\hat{\beta}$ the unit vector in the ℓ_b or $\nabla\beta$ direction. Using this condition for two adjacent resonances, viz. $\underset{\sim}{\ell} \cdot \underset{\sim}{\Omega}(r_{\underset{\sim}{\ell}}) = 0$, $(\underset{\sim}{\ell} + \hat{\beta}) \cdot \underset{\sim}{\Omega}(r_{\underset{\sim}{\ell}} + \delta_r) = 0$, and writing $\underset{\sim}{\Omega}(r_{\ell} + \delta_r) = \underset{\sim}{\Omega}(r_{\underset{\sim}{\ell}}) + \Delta\underset{\sim}{\Omega}_{\underset{\sim}{\ell}}$, one obtains the stochasticity criterion $|\underset{\sim}{\ell} \cdot \Delta\underset{\sim}{\Omega}_{\underset{\sim}{\ell}}| > |\Omega_b|$, or squaring both sides for convenience and using Eq. (15),

$$1 < (\omega_{\underset{\sim}{\ell}}/\Omega_b)^2 \quad . \tag{17}$$

Equivalently, given an expression for $\underset{\sim}{\Omega}(r)$, one can expand $\underset{\sim}{\Omega}(r_{\underset{\sim}{\ell}} + \delta_r)$ about $r_{\underset{\sim}{\ell}}$ and obtain criterion (17) in a form involving δ_r explicitly. The expression so obtained has the same form as

Eq. (16),

$$1 \cdot (\Delta r_{\ell} / \cdot_r)^2 \quad . \tag{18}$$

V. FIELD-PARTICLE COUPLING COEFFICIENTS

In the past two sections, we have seen that the coupling coefficients g_{ℓ} play a central role in both the stochasticity criteria (through $\cdot$ or $\Delta r_{\ell} + \Delta P_{\phi\ell}$) and in the form for $\underset{\sim}{D}$. We now adopt forms for the phase functions $\underset{\sim}{r}(z)$ and $\underset{\sim}{v}(z)$ which include finite gyroradius and particle drifts, and use them in expression (6) to obtain a more explicit expression for the $g_{\underset{\sim}{\ell}}$. Comparison of this expression to its zero-gyroradius, driftless limit will show the modifications by these effects of previous results,[1,2] in situations to which those results apply (viz. turbulent spectrum, passing particles).

1. Particle Trajectories

We make the usual separtation of $\underset{\sim}{r}$ and $\underset{\sim}{v}$ into the contributions from guiding-center motion and gyromotion:

$$\underset{\sim}{r} = \underset{\sim}{R} + \underset{\sim}{\rho} \quad , \quad \underset{\sim}{v} = \dot{\underset{\sim}{R}} + \dot{\underset{\sim}{\rho}} \quad . \tag{19}$$

The gyromotion is described by

$$\underset{\sim}{\rho}(\Theta_g) = \rho(\hat{\alpha} \sin \Theta_g + \hat{B} \times \hat{\alpha} \cos \Theta_g) \quad , \tag{20}$$

$$\dot{\underset{\sim}{\rho}}(\Theta_g) = \Omega \rho(\hat{\alpha} \cos \Theta_g - \hat{B} \times \hat{\alpha} \sin \Theta_g) \quad ,$$

and the guiding-center position $\underset{\sim}{R}$ is modeled by

$$\underset{\sim}{R}(\Theta_b, \Phi) = \hat{\alpha}(\alpha_b + \alpha_1 \cos \Theta_b) \qquad (21)$$

$$+ \hat{\beta}(b_o \Theta_b + b_1 \sin \Theta_b) + \hat{\phi}(\Phi + \phi_1 \sin \Theta_b) \quad .$$

(From this, $\dot{\underset{\sim}{R}}$ too amy be written down directly, if desired.) The projection of $R(\Theta_b)$ onto the poloidal plane is illustrated in Fig. 2. Here, α_b is the flux surface about which a particle drifts in the course of its bounce motion, and α_1 is the "banana width", the size of the excursion from α_b which the particle makes, in units of α.

The secular motion of the particle is described by the terms $b_o \Theta_b$ and Φ. For a trapped particle (Fig. 2a), $b_o = 0$, correctly modeling the fact that the only secular drift for such particles is the toroidal banana drift $\Omega_\phi \equiv \dot{\Phi}$. For passing particles (Fig. 2b), $b_o = 1$, so that a particle makes one complete circuit poloidally each bounce period.

The terms in b_1 and ϕ_1 model both drifts normal to $\hat{B}$, and the modulation of $v_{\parallel}$ due to the mirroring effect of the $\tilde{\mu} B$-well. The separation of the parallel from the perpendicular effects may be explicitly accomplished, decomposing the vector $\underset{\sim}{R}_1 \equiv \hat{\beta}\, r b_1 + \hat{\phi}\, R \phi_1$, into its parallel and perpendicular components. Defining $R_{1\parallel} \equiv \hat{B} \cdot \underset{\sim}{R}_1$, $\underset{\sim}{R}_{1\perp} \equiv (\hat{B} \times \hat{\alpha}) \cdot \underset{\sim}{R}_1$, one obtains

$$R_{1\parallel} = b_p\, r\, b_1 + b_t\, R\phi_1 \, , \; R_{1\perp} = b_t\, r\, b_1 - b_p\, R\, \phi_1 \quad . \qquad (22)$$

Thus, in cases where ∇B effects dominate those of perpendicular drifts, setting $\dot{R}_{1\perp} = 0$ yields $\ell_1/b_1 = r\, b_t/R\, b_p \approx q$.

For particles near the transition from trapped to passing, the higher harmonics (i.e. terms like $\sin m\,\Theta_b$, $\cos m\,\Theta_b$) of the bounce motion becomes appreciable, and the model (21) for $\underset{\sim}{R}$ may be inadequate. We shall henceforth exclude particle in this transitional, "separatrix" region from consideration. Related problems dealing with this regime have been treated by Smith and Kaufman,[5,6] and by Chirikov.[7]

2. Evaluation of $g_{\underset{\sim}{\ell}}$

We now evaluate $g_{\underset{\sim}{\ell}}$. For the turbulent spectrum, $\underset{\sim}{A} \simeq A_{\parallel}\,\hat{B}$, so we neglect the contribution from the term $\dot{\underset{\sim}{\rho}} \cdot \underset{\sim}{A}$. For the ripple spectrum, because $k_\perp \rho \lesssim \rho/a << 1$ (a is the minor radius at the limiter), $\oint d\,\Theta_g\ \dot{\underset{\sim}{\rho}} \cdot \underset{\sim}{A} \simeq \underset{\sim}{A} \cdot \oint d\,\Theta_g\ \dot{\underset{\sim}{\rho}} = 0$, so again the $\dot{\underset{\sim}{\rho}} \cdot \underset{\sim}{A}$ contribution is negligible. Now writing $\underset{\sim}{A}(\underset{\sim}{R} + \underset{\sim}{\rho}) \simeq \underset{\sim}{A}(\underset{\sim}{R})\, e^{i\,\underset{\sim}{k} \cdot \underset{\sim}{\rho}}$, where $\underset{\sim}{k}$ is the local wavevector, we perform the integral over gyrophase Θ_g :

$$g_{\underset{\sim}{\ell}} \simeq -\,e(2\pi)^{-3} \int d\,\Theta_b \int d\,\phi\ \dot{\underset{\sim}{R}} \cdot \underset{\sim}{A}\,(\underset{\sim}{R}) \int d\,\Theta_g\, e^{-i\,\underset{\sim}{\ell} \cdot \underset{\sim}{\Theta}}\, e^{\,i\,\underset{\sim}{k} \cdot \underset{\sim}{\rho}}$$

$$= -\,e(2\pi)^{-2} \int d\,\Theta_b\, e^{-i\,\ell_b\,\Theta_b} \int d\,\phi\, e^{-i\,\ell_\phi \phi}\, \dot{\underset{\sim}{R}} \cdot \underset{\sim}{A}\,(R)\; J_{\ell_g}(k_\perp \rho)\, e^{-i\,\ell_g\,\Theta_k} \ . \tag{23}$$

Here and henceforth, we set $m = c = 1$ for notational simplicity. The phase Θ_k , defined by $\underset{\sim}{k} \cdot \rho \equiv k_\perp \rho \sin\,(\Theta_g - \Theta_k)$, is unimportant, since it is $|g_{\underset{\sim}{\ell}}|$ which appears in quantities of interest to us here.

We therefore drop it from explicit notation. In obtaining (23), we have used the familiar Bessel identity

$$J_{\ell}(y) = (2\pi)^{-1} \oint d\theta \, e^{-i \ell \theta} e^{i y \sin\theta} \quad . \tag{24}$$

Due to the axisymmetry, the only quantities in (23) dependent upon ϕ are $\underset{\sim}{A}(\underset{\sim}{R})$ and $e^{-i \ell_\phi \phi}$. The integral over ϕ is thus simply the Fourier transform of $\underset{\sim}{A}(\underset{\sim}{R})$. Writing $\underset{\sim}{A}(\underset{\sim}{R}) = \underset{\sim}{A}(\alpha, b, \phi) = \sum_{\ell_\phi} \underset{\sim}{A}(\alpha, b, \ell_\phi)\, e^{i \ell_\phi \phi}$ (b is the β - coordinate of the guiding $\underset{\sim}{R}$), and $\phi = \bar{\phi} + \delta\phi(\theta_b)$ [where from (21), $\delta\phi(\theta_b) = \phi_1 \sin\theta_b$], one has

$$\underset{\sim}{g}_{\ell} = - e\,(2\pi)^{-1} \oint d\theta_b \; e^{-i \ell_b \theta_b} \, \dot{\underset{\sim}{R}} \cdot \underset{\sim}{A}(\alpha, b, \ell_\phi)\, e^{i \ell_\phi \bar{\phi}} \, J_{\ell_g}(k_\perp \rho) \quad . \tag{25}$$

Because we are considering perturbations which are either low or zero frequency ($\omega \ll \Omega_i$), in order that condition (5) be satisfied and also that $\underset{\sim}{g}_{\ell}$ appreciable, we henceforth always take

$$\ell_g = 0 \quad . \tag{26}$$

Since $\underset{\approx}{D} \sim \underset{\sim}{\ell}\,\underset{\sim}{\ell}$, $\ell_g = 0$ implies that $\tilde{\mu}$ is still a good invariant under the perturbation.

For the ripple problem, $k_\perp \rho \ll 1$ for all species s = e,r,i, so the factor $J_{\ell_g} = J_0$ in Eq. (25) is essentially equal to one. For the turbulent spectrum, for both s = r and i, one may have $k_\perp \rho \sim 1$. Thus one sees that finite particle gyroradius may appreciably reduce $\underset{\sim}{g}_{\ell}$, and hence $\underset{\approx}{D} \cdot |\underset{\sim}{g}_{\ell}|^2$. This mechanism was alluded to in Ref. 2.

We now turn to the integral over α_b appearing in (25). We neglect the dependence of $k_\perp$ on α_b, taking the factor $J_0(k_\perp\rho)$ outside the integral. If we also neglect the mode localization width w_a in comparison with the particle banana width $r_1 \approx \alpha_1(\partial r/\partial\alpha)$, we have $\underset{\sim}{A}(\alpha = \alpha_b + \delta\alpha) \approx \underset{\sim}{A}(\alpha_b)\, e^{i k_\alpha \delta\alpha}$. Then using our model expression (21) for $\underset{\sim}{R}$, we obtain

$$g_j = - e J_0(k_\perp\rho) \sum_m \Big\{ \Big[\ell_b b_0 A_\beta + \ell_\phi A_\phi \Big] J_{\ell_b - b_0 m}(y_1) + \tfrac{1}{2} \ell_b \Big[b_1 A_\beta + \phi_1 A_\phi \Big] \Big[J_{\ell_b - b_0 m - 1}(y_1) + J_{\ell_b - b_0 m + 1}(y_1) \Big] \Big\} \tag{27}$$

Here we denote by A_β the component $A^a_\beta(\alpha_b, m, \ell_\phi)$ of the perturbation, where $A^a_\beta(\alpha, m, \ell_\phi) \equiv (2\pi)^{-1} \oint d\beta\, e^{-im\beta} A^\alpha_\beta(\alpha, \beta, \ell_\phi) = (2\pi)^{-2} \oint d\beta \oint d\phi\, e^{-i(m\beta + \ell_\phi \phi)} A^a_\beta(\alpha, \beta, \phi)$, and similarly for A_ϕ. For the individual modes $\underset{\sim}{A}^a(\underset{\sim}{r})$ in the turbulent spectrum, $A^a(r) \sim A^a(\alpha) \exp i(m\beta - n\phi)$, and the sum over m in (27) consists of a single term. Similarly the ripple field from field-coil errors may also be approximated by a single term, with $m = 0$. Ripple fields for particle injection schemes, which are strongest at $\beta = -\pi/2$ and weakest at $\beta = -\pi/2$, may be approximated by three terms, $m = 0, \pm 1$.

The argument y_1 of the Bessel functions is given by

$$y_1^2 \equiv (m b_1 + \ell_\phi \phi_1)^2 + (k_\alpha \alpha_1)^2 ; \tag{28}$$

we have suppressed notation of an accompanying phase factor, as done for $e^{-i \ell_g \Theta_k}$.

3. Discussion and Estimates

The first line in Eq. (27) comes from the non-oscillatory portion of the velocity $\dot{\underset{\sim}{R}}_o \equiv \hat{\theta}\, b_o\, \Omega_b + \hat{\phi}\, \Omega_\phi$, and the second line from the oscillatory portion. We recover the result of the zero-gyroradius, driftless theories by considering passing particles ($b_o = 1$) with the drifts "turned off" ($b_1 = \phi_1 = y_1 = 0$), setting $k_{\perp\rho}$ to zero, and taking $\underset{\sim}{A}^a(\underset{\sim}{r})$ of the $\exp i\,(m\,\theta - n\,\phi)$ form of the turbulent spectrum. Then using the fact that $J_\ell(y = 0) = \delta(\ell)$ (δ here is the Kronecker-delta) , Eq. (27) reduces to

$$g_{\underset{\sim}{\ell}} = -\,e\,\delta(\ell_\phi + n)\,\delta(\ell_b - m)\,\dot{\underset{\sim}{R}}_o \cdot \underset{\sim}{A}^a \,. \tag{29}$$

Including the effects of drifts, one has $y_1 \neq 0$, in general, so the Bessel functions $J_\ell(y_1)$ in Eq. (27), which in the driftless limit acted like a δ-function, will for $y_1 \neq 0$ introduce a spread $\Delta\ell \sim 2\,y_1$ in the effective spectrum which a particle sees. Using the large-and-small-argument limits for $J_\ell(y)$,

$$J_\ell(y) \simeq \begin{cases} (y/2)^\ell/\ell! & (y < \ell), \\ \\ (2/\pi\,y)^{\frac{1}{2}} \cos\,(y - \ell\,\pi/2 - \pi/4) & (y > \ell)\,, \end{cases} \tag{30}$$

in Fig. 3 we illustrate this spreading, sketching $J_\ell(y)$ versus its index ℓ for fixed y. [Eq. (30) and Fig. 3 are strictly valid only when ℓ is an integer, which is always the case here.]

We now consider the size of y_1, for both the turbulent and ripple spectra. We shall see shortly that for the turbulence problem, y_1 is a number on the order of or smaller than 2 or 3, so that the spreading of the spectrum through the terms $J_\ell(y_1)$ in g_ℓ is small and not a dominant effect of particle drifts. The small value of y_1 is due to the small value of $k_{||}$, and the fact that guiding center motion is predominantly parallel to $\hat{\beta}$. For the ripple case, however $k_{||} \sim n/R$ is appreciable, so one finds $y_1 >> 1$ here. Because the ripple spectrum consists of a small number of components, with resonance points I_ℓ widely separated in $\underset{\sim}{I}$ space, the spectrum-spreading effect of $y_1 >> 1$ is crucial to understanding how the coherent ripple field can induce stochasticity. (An analogous problem, in which a purely coherent field induces particle motion, is studied in Refs. 5 and 6.)

Denoting by $\delta v_{||}$ the amplitude of modulation of the parallel velocity by the μ B well (hence $\delta v_{||} \tau_b \sim R_{1||} \simeq R\,\phi_1$) and by v_d the perpendicular drift velocity, from the origin of y_1 in the integral of Eq. (25) one sees that we many approximate the size (and physical interpretation) of y_1 by the formula

$$y_1 \sim (k_{||}\, \delta v_{||}/\Omega_b) + (k_\perp v_d/\Omega_b) \equiv y_{1||} + y_{1\perp} \quad . \tag{31}$$

One has that $v_d \sim v\,(\rho/R)$, where $v \equiv |\underset{\sim}{v}|$ is the magnitude of the particle velocity. For a trapped or barely passing particle, $\delta v_{\parallel} \sim \varepsilon v^2/v_{\parallel} \sim \varepsilon v$. The size of $y_{1\parallel}$, or $\phi_1 \simeq q b_1$, is greatest for the former class of particles, for which

$$\phi_1 \simeq \delta v_{\parallel} \tau_b/R \simeq \varepsilon^{\frac{1}{2}} q \ , \tag{32}$$

and thus

$$y_{1\parallel} \simeq k_{\parallel} R \phi_1 \simeq \varepsilon^{\frac{1}{2}} q R k_{\parallel} \ , y_{1\perp} \simeq q \rho k_{\perp} \ . \tag{33}$$

Putting in the values $k_{\parallel} \sim L_S^{-1} \sim (q R)^{-1}$, $k_{\perp} \sim \rho_i^{-1}$ for the turbulent spectrum, and $k_{\parallel} \sim n/R$, $k_{\perp} \sim b_p n/R$ for ripple, one finds the estimates

$$y_{1\parallel} \lesssim \varepsilon^{\frac{1}{2}}, \quad y_{1\perp} \simeq q(\rho/\rho_i) \tag{34}$$

for turbulence, and

$$y_{1\parallel} \lesssim \varepsilon^{\frac{1}{2}} q n, \quad y_{1\perp} \simeq \varepsilon(\rho/R) n << y_{1\parallel} \tag{35}$$

for ripple.

4. Effect of Finite (r_1/w_a)

For the turbulent spectrum and for $s = r, i$, one may have the particle banana width r_1 comparable to the width w_a of the mode a with which the particle is resonant. There are two effects to be considered here.

First, the approximation $\underset{\sim}{A}(\alpha) \approx \underset{\sim}{A}(\alpha_b)\, e^{ik_\alpha \delta\alpha}$ made in obtaining Eq. (27) from (25) is not strictly valid, and the size of g_ℓ may accordingly be modified. One can obtain an analytic expression for this modification by writing $\underset{\sim}{A}(\alpha) = \tilde{\underset{\sim}{A}}(\alpha)\, e^{i k_\alpha \delta_\alpha}$, where $\tilde{\underset{\sim}{A}}$ is a slowly-varying mode amplitude, and expanding $\tilde{\underset{\sim}{A}}$ about $\alpha = \alpha_b$: $\tilde{\underset{\sim}{A}}(\alpha) = \tilde{\underset{\sim}{A}}(\alpha_b) + \delta\alpha\, \tilde{\underset{\sim}{A}}^1(\alpha_b) + \ldots$. Then, noting that $(\delta\alpha)^n e^{i k_\alpha \delta\alpha} = (-i\, \partial/\partial k_\alpha)^n e^{i k_\alpha \delta_\alpha}$, one may take the derivatives $(\partial/\partial k_\alpha)$ outside the integral in (25), yielding these derivatives acting on the same form as (27), with $A_{\beta,\phi}$ there replaced by derivatives of $A_{\beta,\phi}$ to the appropriate order.

While such an approach may be useful for subsequent numerical analysis, it does not give much physical insight. We therefore make the rough approximation that the effect of this excursion in α is to average the mode amplitude over the range α_1 about the point α_b. The form of (27) is then unchanged, if one interprets $A_{\beta,\phi}$ there to include this averaging effect.

The second effect of finite (r_1/w_a) is to shift the value α_{res} which a particle's α_b must equal in order to make it resonant with a given mode a, localized at α_a. For simplicity, and because it is the most important instance of this effect, we consider runaway electrons, $s = r$. Then $\omega_a \sim \omega_*$ may be neglected in the resonance condition, which appears as

$$0 = \ell_b \Omega_b + \ell_\phi \Omega_\phi = k_\parallel v_\parallel + k_\perp v_d \ . \qquad (36)$$

With v_d set to zero, (36) says $k_{\parallel}(\alpha_{res}) = 0$, i.e. a particle is resonant with a wave at that α_{res} where the wave has $k_{\parallel} = 0$. For the turbulent spectrum, $\alpha_{res} = \alpha_a$, the position of maximum amplitude of the mode. For finite v_d, however, one has instead $|k_{\parallel}/k_{\perp}| = |v_d/v_{\parallel}|$. Using $k_{\parallel} = k_{\perp}(\delta r/L_S)$, where $\delta r \equiv r - r_a = (\partial r/\partial \alpha)(\alpha - \alpha_a)$, we are led to the estimate

$$\delta r_{res} \equiv r_{res} - r_a \sim q \rho_s \ . \tag{37}$$

Because $q\rho_s$ is comparable to the mode width $w_a \sim \rho_i$ for $s = r$, a runaway electron will interact resonantly with a mode at a position where the mode amplitude is appreciably reduced from its value at $r = r_a$.

VI. HAMILTONIAN $H_o(\underset{\sim}{I})$ AND AUXILIARY QUANTITIES

1. $H_o(\underset{\sim}{I})$

The formalism of the preceding sections calls for the unperturbed Hamiltonian H_o in terms of the invariants $\underset{\sim}{I}$, both in evaluating $\underset{\sim}{\Omega} \equiv \partial H_o / \partial \underset{\sim}{I}$ for the resonance condition (5), and for $\partial \underset{\sim}{\Omega}/\partial \underset{\sim}{I}$, used in determining the stochasticity threshold. In this section we obtain approximate expressions for $H_o(\underset{\sim}{I})$, for the two types of particle trajectories modeled by Eq. (21).

We begin from the guiding-center Hamiltonian K_o, valid for tokamak geometries, for which $b_t >> b_p$ [13]:

$$K_o(\mu\,;\,b\,,\,P_b\,;\,P_\phi) = \mu\,\Omega + \tfrac{1}{2}\,R^{-2}\,(P_\phi - e\,\alpha_G)^2\ . \tag{38}$$

Here Ω and R are evaluated at the particle guiding-center position $(\alpha_G\,,\,b)$ (the toroidal angle ϕ does not enter), and α_G is determined by the guiding-center condition

$$P_b = e\,A^o_\beta(\alpha_G\,,\,b)\ . \tag{39}$$

From Hamilton's equation $\dot\phi = R^{-2}(P_\phi - e\,\alpha_G)$, one sees that in the course of a bounce period, α_G executes a single oscillation, as does P_b . For trapped particles, the oscillation is about the point where $\dot\phi = 0$, hence where $e\,\alpha_G = P_\phi$. For this reason, it is appropriate to define α_b by

$$e\,\alpha_b \equiv P_\phi\ . \tag{40}$$

(For passing particles, we may also adopt this form for α_b , adequate for purposes of estimation.)

We want to transform from the guiding-center variables $(b\,,\,P_b)$ in terms of which K_o is expressed, to action-angle variables $(\Theta_b\,,\,J_b)$ used in H_o , where

$$J_b \equiv (2\pi)^{-1}\oint d\,b\,P_b\ . \tag{41}$$

For passing particles not in the immediate vicinity of the separatrix between passing and trapped, P_b is roughly constant over a bounce period, so from (41),

$$J_b \simeq P_b - e A^o_\beta(\alpha_G) \ , \quad \theta_b \simeq b \quad . \tag{42}$$

[The dependence of A^o_β on b, which is weak in any case, has been dropped in (42), since we have averaged over b in obtaining J_b .]

We now define A_G as the functional inverse of A^o_β , i.e. $A_G[A^o_\beta(\alpha_G)] = \alpha_G$. Thus

$$\partial A_G/\partial A_\beta = (\partial A_\beta/\partial \alpha_G)^{-1} = -q^{-1} \quad . \tag{43}$$

Using (42) in (38), therefore, H_o for passing particles is approximately given by

$$H_o \simeq \mu\Omega + \tfrac{1}{2} R^{-2}[P_\phi - e A_G(J_b/e)]^2 \quad . \tag{44}$$

(Here Ω and R are understood as bounce-averaged quantities.)

From (40) and (42), and noting that $\alpha_G \simeq \alpha_b$, we see that P_ϕ and J_b play essentially the same role for passing particles, that of a radial coordinate, with

$$\partial/\partial J_b = (\Omega r b_t)^{-1} \partial/\partial r = -q^{-1} \partial/\partial P_\phi \ , \ \partial/\partial P_\phi = -(\Omega R b_p)^{-1} \partial/\partial r \quad . \tag{45}$$

For trapped particles, it is precisely the variation of P_b over a bounce period (finite banana width) which gives a nonzero value for J_b in (41). Hence, $J_b = (2\pi)^{-1} \oint db\, \delta P_b(b)$, where from Eq. (39), $\delta P_b(b) \simeq (\partial A^o_\beta/\partial \alpha)\delta(e\alpha_G)$. We solve (38) for $\delta(e\alpha_G) \equiv e(\alpha_G - \alpha_b)$,

$$\delta(e\,\alpha_G) = R\left[2(K_o - \mu\,\Omega)\right]^{1/2} \quad , \tag{46}$$

and so evaluate J_b:

$$J_b \simeq -(2\pi)^{-1} \oint d\,b\;q\,R\;2(K_o - \mu\,\Omega)^{1/2} = (2\pi)^{-1} \oint d\,\ell\,v_{\parallel} \quad . \tag{47}$$

Here $d\,\ell = -q\,R\,d\,b$ is a differential length element along the field, so the last form in (47) is the usual definition of the longitudinal invariant.

Expanding $\Omega(b)$ about its $b=0$ value, one evaluates (47) explicitly and solves for $H_o = K_o$, obtaining

$$H_o = \mu\,\Omega + (q\,R)^{-1} J_b(\varepsilon\,\mu\,\Omega)^{1/2} \tag{48}$$

for well-trapped particles. In Fig. 4 we sketch $H_o - \mu\,\Omega$ verses J_b, using the forms (44) and (48) in their domains of validity, and interpolating between them to give the proper plateau behavior ($\Omega_b = \partial\,H_o/\partial\,J_b \to 0$) in the separatrix region.

2. Auxiliary Quantities, Physical Interpretation

Now we compute the frequencies $\underset{\sim}{\Omega}$ and their derivatives $\partial\,\underset{\sim}{\Omega}/\partial\,\underset{\sim}{I}$, using Eqs. (44) and (48) for $H_o(\underset{\sim}{I})$, and check that these expressions give physically reasonable results. For passing particles, Eq. (44) yields

$$\Omega_\phi = \partial H_o/\partial P_\phi = R^{-2}(P_\phi - e A_G) \quad ,$$

$$\Omega_b = \partial H_o/\partial J_b = q^{-1} R^{-2}(P_\phi - e A_G) \quad . \tag{49}$$

Noting from (44) that $v_{||}^2 = R^{-2}(P_\phi - e A_G)^2$,we find from (49) that

$$\Omega_b^2 = (v_{||}/q R)^2 \quad , \tag{50}$$

i.e. the bounce time for passing paricles is just the time required to travel a connection length $q R$.

From (49) one also sees that

$$\Omega_\phi/\Omega_b = q \quad , \tag{51}$$

showing that passing particles basically follow field lines.

Similarly, for trapped particles, one has

$$\Omega_\phi = \mu \partial \Omega/\partial P_\phi = -(\kappa_B v_\perp^2/2 \Omega R b_p) \quad ,$$

$$\Omega_b = (q R)^{-1}(\epsilon \mu \Omega)^{\frac{1}{2}} \quad , \tag{52}$$

where $\kappa_B \equiv \partial \ln \Omega/\partial r \simeq |\nabla B|/B$. For the second form given for Ω_ϕ , we have used the second of Eqs. (45), and that $\mu\Omega = 1/2\ v_\perp^2$. We see that Ω_ϕ is just $-b_p v_B/R$, where $v_B \equiv \kappa_B v_\perp^2/\Omega$ is the usual ∇B drift. The "amplification" of this drift by the factor $-b_p^{-1}$ comes from the fact that the predominantly poloidal ∇B drift puts the particle on new field lines, which arrive after one poloidal transit considerably displaced in toroidal angle.[12]

The factor $(\varepsilon\mu\Omega)^{1/2}$ in Ω_b in (52) is equal to the maximum $v_\parallel$ which the particle attains bouncing in the $\tilde{\mu} B$ well. Hence the interpretation of Ω_b is about the same as for passing particles. From these physical interpretations, we obtain the estimate

$$\Omega_\phi/\Omega_b \sim (q\,\kappa_B\,\rho/b_p\,\varepsilon^{1/2}) \sim \varepsilon^{-1/2}\,(\rho/r)q^2 \quad . \tag{53}$$

For $s = i$ this ratio may be on the order of 1/5.

We now calculate $\partial\underset{\sim}{\Omega}/\partial\underset{\sim}{I}$. For passing particles,

$$\partial\,\Omega_\phi/\partial\,P_\phi = R^{-2} \;,\;\; \partial\,\Omega_\phi/\partial\,J_b = \partial\,\Omega_b/\partial\,P_\phi = q^{-1}\,R^{-2} \quad , \tag{54}$$

and

$$\partial\,\Omega_b/\partial\,J_b = q^{-2}\,R^{-2} - (\Omega_\phi\,R/r^2\,b_t\,\Omega\,L_s) \quad , \tag{55}$$

where $L_s = qR/(\partial \ln q/\partial \ln r) = -\varepsilon/(\partial q^{-1}/\partial r)$ is the shear scale length. We have used the first of Eqs. (45) in obtaining the last term in Eq. (55). This term, expressing the change in Ω_b with r due to shear, is critical in determining the overlap criterion.

The components of $\partial \underset{\sim}{\Omega}/\partial \underset{\sim}{I}$ for trapped particles may be similarly computed using Eqs. (52). However we shall be able to find the desired results using quantities already computed, so we do not display these additional formulae here.

Finally, we use $\partial \underset{\sim}{\Omega}/\partial \underset{\sim}{I}$ to compute $M_{\underset{\sim}{\ell}}^{-1} \equiv \underset{\sim}{\ell} \cdot \partial \underset{\sim}{\Omega}/\partial \underset{\sim}{I} \cdot \underset{\sim}{\ell}$ for passing particles. Neglecting ω_a in Eq. (5), one has

$$\ell_b = -\ell_\phi(\Omega_\phi/\Omega_b) = n(\Omega_\phi/\Omega_b) \sim q\,\varepsilon^{-1/2}(\rho/\rho_i) \quad . \tag{56}$$

Neglect of ω_a is not justified only for trapped ions in turbulence for which $\ell_\phi \Omega_\phi/\omega_a \sim \varepsilon$, so that $\omega_a \approx \ell_b \Omega_b$ is an appropriate approximation to the resonance condition. In this case,

$$\ell_b \approx \omega_a/\Omega_b \approx \omega_*/\Omega_b \sim q\,\varepsilon^{-3/2}(\rho/\rho_i) \quad . \tag{57}$$

These expressions for ℓ_b are understood to be approximations to its nearest resonant value, which must be integral.

Using relations (56) and (51) with Eqs. (54) and (55), one finds a cancellation of all contributions to $M_{\underset{\sim}{\ell}}^{-1}$ for passing particles except the second term in (55):

$$M_{\ell}^{-1} = - \ell_b^2 (\Omega_{\theta} R/r^2 b_T \Omega L_s) = \ell_b \ell_{\phi} (\Omega_{\phi}/b_p \Omega r L_s) \quad . \tag{58}$$

The results needed to study the central problem of this paper are now in hand. We utilize them in the following section.

VII. Results of the Analysis

Now we are ready to obtain explicit expressions for the formal criteria of Sec. IV for the onset of stochasticity, as well as to see the modifications due to drifts and finite gyroradius on the diffusion tensor.

We consider first the case studied in Refs. 1,2, passing particles in a turbulent spectrum. Then the factor $\Omega_{\phi} A_{\phi}$ dominates g_{ℓ} in (27). Using this and Eq. (58) in Eq. (13) to compute $\Delta r_{\ell} = (\partial r/\partial e \alpha) \Delta P_{\phi \ell}$, one finds that criterion (16) becomes, after some algebra,

$$1 < |B_1 (L_s/k_{\beta} \delta_t^2)| \sim | B_1 (m^2 L_s/k_{\beta} \rho_i^2) | \quad . \tag{59}$$

This expression is formally the same as that in Ref. 3, but with the ratio $B_{1,0} \equiv B_{1r}(r_a)/B$ [where $B_{1r}(r_a)$ is the radial field of the component a with which the particle is resonant, evaluated at the radius r_a at which B_{1r} is greatest] there replaced by

$$B_1 \equiv B_{1,0} J_o(k_{\perp} \rho) J_{\ell_b - b_o m}(y_1) [B_{1r}(r_{res})/B_{1r}(r_a)] \quad . \tag{60}$$

Here r_{res} is the radius at which a particle is resonant with mode a, and $B_{1r}(r_{res})$ is to be regarded as an average of the mode amplitude over a "banana width" $r_1 \sim q\rho$ about r_{res}. The ratio $\Gamma = B_{1r}(r_{res})/B_{1r}(r_a)$ then accounts for both effects described in Sec. V.4. Assuming a Gaussian form for $B_{1r}(r)$, one has $\Gamma = e^{-(r_1/w_a)^2}$. Since $r_1 \sim w_a$ for $s = r, i$, Γ is strongly dependent upon the value (r_1/w_a).

A second effect of drifts is contained in the factor $J_{\ell_b - b_o m}(y_1)$. For passing particles $b_o = 1$. We determine ℓ_b from Eq (56), $\ell_b = -\ell_\phi \Omega_\phi/\Omega_b = n\,q\,(r_a)$. For the turbulent spectrum, one also has $m = n\,q\,(r_a)$, so $J_{\ell_b - b_o m}(y_1) = J_o(y_1)$. Using (34), we see that for $s = i, r$, $y_1 \sim 2$ or 3, hence J_o may be considerably reduced from its driftless, $y_1 = 0$ value. For small (r_1/w_a), the separation of $A(r)$ into an oscillatory ($\sim e^{ik_r \delta a}$) and amplitude portion is not uniquely determined, so there is some exchange of information possible between the factors Γ and $J_o(y_1)$. However, they are not the same. In particular, from (28) one sees that even for $k_r = 0$ and a constant mode amplitude, y_1 would still be of the same order of magnitude, due to drifts in the $\hat{B} \times \hat{\alpha}$ direction.

We estimate the size of $B_1/B_{1,0}$ for the present case (turbulence, passing particles). If one takes $k_\perp \rho \sim 1$, $y_1 \sim 2$, $r_1/w_a \sim 1$, then $J_o(k_\perp\rho) \sim 2/3$, $J_o(y_1) \sim 1/3$, and $\Gamma \sim 1/3$, so that $B_1/B_{1,0} \sim 1/13$. The stochasticity criterion (62) is then about 13 times more difficult to satisfy than the driftless, zero gyroradius result, from roughly $B_{1,0} > 2 \times 10^{-7}$ to $B_{1,0} > 2.5 \times 10^{-6}$. One notes, however, that this estimate is highly sensitive to the parameters $k_\perp\rho$, y_1, and r_1/w_a, which are not well-known. For example, if one instead takes $k_\perp\rho \sim 1/2$, $y_1 \sim 1$, and $r_1/w_a \sim 1/2$, one has $J_o(k_\perp\rho) \sim 9/10$, $J_o(y_1) \sim 2/3$, and $\Gamma \sim 4/5$, hence $B_1/B_{1,0} \sim 1/2$.

The diffusion tensor $\underset{\sim}{D}$ is correspondingly reduced by these effects. For comparison to previous results, we first remove these effects by mathematically "turning off" the drifts and setting $k_\perp\rho$ to zero. Then $g_{\underset{\sim}{\ell}}$ is given by Eq. (29). Radial transport comes from the component $D_{rr} \approx D_{p_\phi p_\phi}(\partial r/\partial p_\phi)^2$ of $\underset{\sim}{D}$ in Eq. (4). In this driftless limit, one recovers the result of Refs. 1 and 2,

$$D_{rr}^{\ o} = \sum_{m,n} (R\,\Omega_\phi)^2 \, B_{1,0}^{\ \ 2} \; \pi\,\delta(m\,\Omega_b - n\,\Omega_\phi) \ . \tag{61}$$

Restoring the new effects, D_{rr} is given by Eq. (61), but with $B_{1,0}$ replaced by B_1. Radial diffusion is therefore reduced from the expectations of previous theories by a factor $D_{rr}/D_{rr}^{\ o} \sim (B_1/B_{1,0})^2$. For runaway electrons, the estimates just made show that this factor may range from 1/4 to as much as two orders of magnitude. In Ref. 2 it is noted that the simple line-following estimate $D_{rr}^{\ o}$ predicts that the confinement time for runaway electrons should be reduced from that for thermal electrons by a factor $v_e/c \sim 1/15$, whereas experimentally the confinement times for these two particle classes seem to be comparable. One sees that the reduction of D_{rr} from $D_{rr}^{\ o}$ by $(B_1/B_{1,0})^2$ provides a possible explanation for this discrepancy (though alternative explanations may also exist).

The analysis is similar for the other cases covered by the theory. For ripple, we may take $A_\phi = 0$. For passing particles in the ripple field, we evaluate criterion (17) or (18), finding

$$1 < |B_1 (q^3 R \ell_\psi / \epsilon L_s)| \, . \qquad (62)$$

Now $J_o(k_\perp \rho) = 1 \simeq \Gamma$, and in $J_{\ell_b - m}(y_1)$, one has $\ell_b \simeq q n$ as before. Now however, $m \ll q n \sim 30$, and from (28), (35), $y_1 \simeq |\ell_\phi \phi_1| \simeq |n q b_1| \lesssim n q$. Thus $B_1/B_{1,0} \simeq J_{q n}(y_1 \lesssim q n) \lesssim (q n)^{-1/3} \simeq 1/3$. Using this in (62), one obtains the estimate

$$B_{1,0} > 1/50 \quad , \qquad (63)$$

which current ripple injection schemes satisfy. Eq. (63) assumes, however, $b_1 \sim 1$. For more strongly-passing particles, whose trajectories are less affected by the $\tilde{\mu} B$ well, one should instead use the small argument value in (30) for $J_{q n}(y_1)$, making criterion (63) more difficult to satisfy by a factor $J_{q n}(q n)/J_{q n}(y_1)$ $\simeq (q n/y_1)^{q n} \simeq (b_1)^{-q n}$.

We now consider the case of trapped particles. The dominant contribution to $g_{\underline{\ell}}$ (27) is now from the factor $\phi_1 A_\phi$ for turbulence, and $b_1 A_\beta$ for ripple. We thus redefine B_1 slightly, letting $J_{\ell_b - b_o m}$ in (60) be replaced by $\frac{1}{2}\left[J_{\ell_b - b_o m - 1} + J_{\ell_b - b_o m + 1}\right] = \frac{1}{2}\left[J_{\ell_b - 1} + J_{\ell_b + 1}\right]$. For $s = e$, Eq. (56) says $\ell_b \simeq 0$. For this resonance, however, $B_1 \propto J_1(y_1) + J_{-1}(y_1) = 0$. This zero coupling arises because an electron stays so close to its original field line in a bounce period that on the return half of the bounce motion it follows almost the same path along which it came.

Since no stochasticity arises from the nearest resonance, one may look at the next nearest ones, $\ell_b = \pm 1$. For these to be effective, the electron must make an excursion δ_t to the next resonant surface in less than half a bounce period, in order that the particle not retrace its steps, as just described. For such perturbation strengths, the electron effectively "doesn't know" if it is trapped or passing, and so one may use expressions derived for passing particles. In a bounce period, an electron makes an excursion δr which is a fraction ω_{ℓ}/Ω_b of its full excursion Δr_{ℓ}. For stochasticity, one must have $\delta r > \delta_t$, i.e.

$$1 < (\omega_{\ell}/\Omega_b)(\Delta r_{\ell}/\delta_t) \quad . \tag{64}$$

From expressions (12), (13) and (58), one may compute the ratio of the two factors in (64), finding $(\Delta r_{\ell}/\delta_t)(\Omega_b/\omega_{\ell}) \simeq \epsilon q^{-1} L_s/\delta_t \sim (r/\rho)^2 \cdot 10^4$. Therefore condition (66) is a factor of 10^4 more difficult to satisfy that (16) or (59), requiring $B_{1,0} > 2.5 \times 10^{-2}$, a regime not considered here. We conclude that trapped electrons should not be stochastic.

Since there are no trapped runaway electrons, the only remaining species is the ions. For these, from (57) and (34), $\ell_b \simeq \omega_a/\Omega_b \simeq q\epsilon^{-3/2} \sim 12$, and $y_1 \simeq q$. Thus the small-argument expansion of $J_{\ell_b \mp 1}$ is appropriate, reducing g_{ℓ} by a factor $\lesssim (y_1/\ell_b)^{\ell_b} \simeq (\epsilon^{3/2})^{\ell_b} \sim (\tfrac{1}{4})^{12}$. This factor in g_{ℓ} overwhelms the others in criterion (16), and so one expects no stochasticity from trapped ions in turbulence, for any reasonable size of $B_{1,0}$. The physical origin here is that because ω_a is large compared to ω_b for $s = i$, an ion cannot resonate with the wave, which moves basically across field lines.

The final case to be discussed would be trapped particles in a ripple field. However, since the present theory assumes integration along unperturbed trajectories is valid, it may not apply well to trapped particles, which will be strongly affected by the ripple fields as they approach the turning points of their unperturbed orbits. The proper study of this case, removing this limitation of the formalism, is thus left to future work.

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References

[1]J.A. Krommes, R.G. Kleva, and C.R. Oberman, Princeton Plasma Physics Laboratory Report 1389 (1978) (to be published).

[2]A.B. Rechester and M.N. Rosenbluth, Phys. Rev. Lett. 40, 38 (1978).

[3]T.H. Stix, Nucl. Fusion 18, 353 (1978).

[4]R.J. Goldston et al., Princeton Plasma Physics Laboratory Report 1396 (1977).

[5]G.R. Smith, Ph.D. dissertation, 1977, University of California at Berkeley; Lawrence Berkeley Laboratory Report No. 6824.

[6]G.R. Smith and A.N. Kaufman, Phys. Fluids 12, 2230 (1978).

[7]B.V. Chirikov, Physics Reports 52, 263 (1979).

[8]H.E. Mynick and J.A. Krommes, Princeton Plasma Physics Laboratory Report No. 1581 (to be published). These results were first reported in Proc. Sherwood Meeting on Theoretical Aspects of Controlled Thermonuclear Research (1979), Abstract 3B 39. At the same meeting, effect (a) and a less general form of effect (b-i) were also discussed by M.S. Chu and C. Chu, Abstract 2B 16.

[9]J.A. Krommes and P.H. Rutherford, Nucl. Fusion 14 695 (1974).

[10]K.T. Stringer, Nucl. Fusion 12, 689 (1972).

[11] K.T. Tsang, Nucl. Fusion 17, 557 (1977).

[12] A.H. Boozer (private communications).

[13] A.N. Kaufman, Phys. Fluids 15, 1063 (1972).

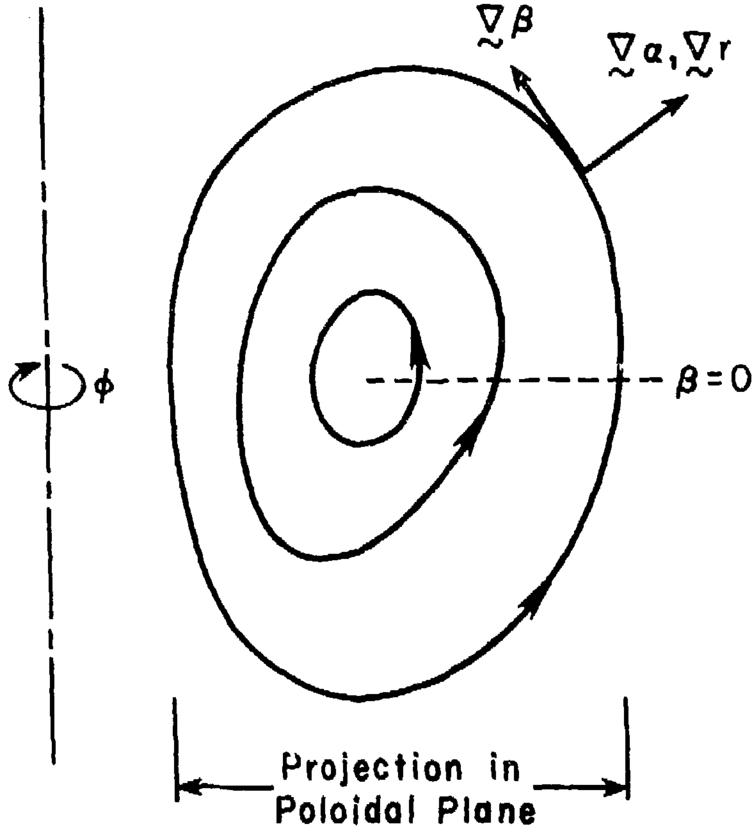

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Fig. 1. Illustration of the toroidal geometry considered in the text, showing the coordinate system (α, β, ϕ) used there.

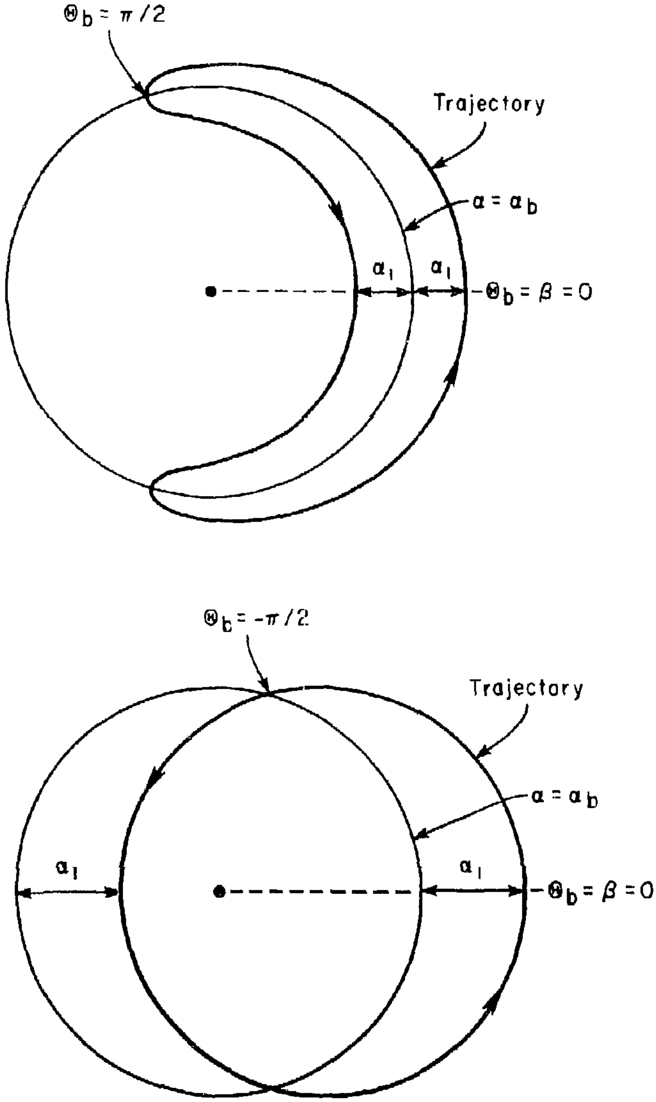

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Fig. 2. The poloidal projection of the unperturbed guiding-center trajectories modeled by Eq. (21), for (a) trapped and (b) passing particles.

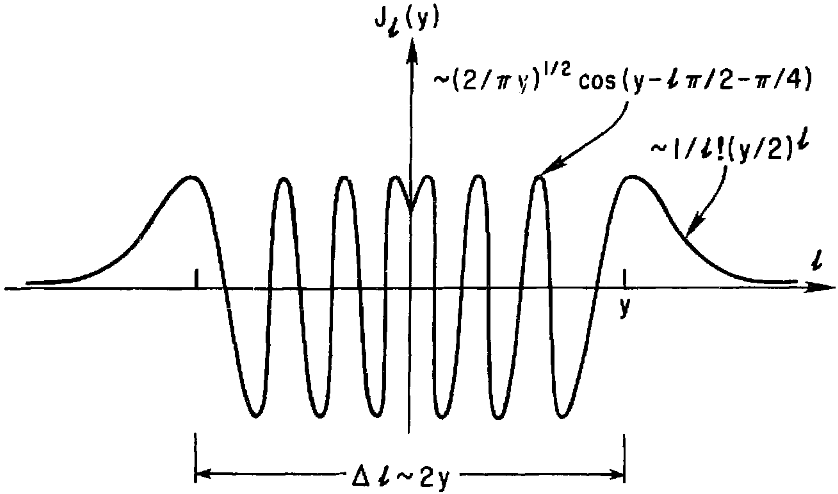

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Fig. 3. Sketch of $J_\ell(y)$ versus ℓ (y fixed), using the limiting forms in Eq. (30), showing the spreading $\Delta\ell\sim 2y$ due to inclusion of drift effects from the driftless (y=0) limit. The sketch, and expressions (30) from which it is drawn, are valid only for integral ℓ, as is always the case in the text.

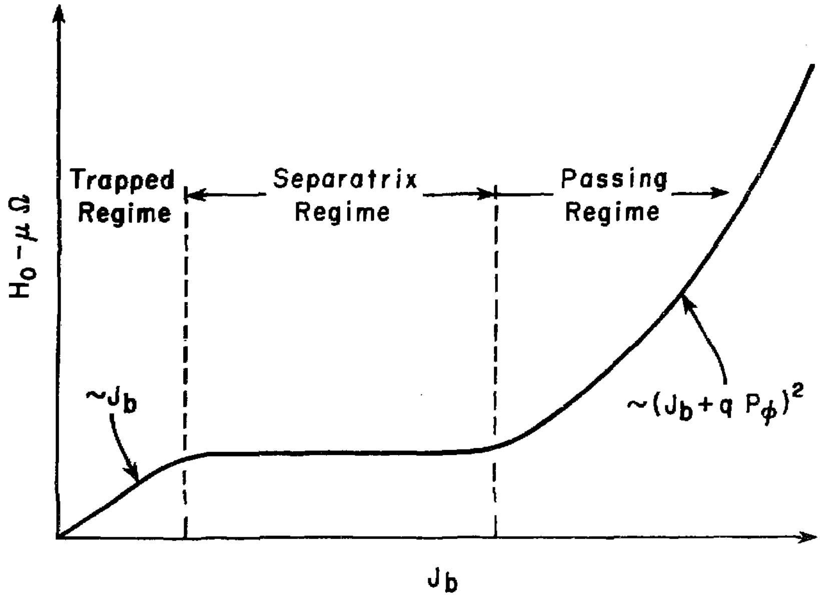

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Fig. 4. Sketch of the parallel Kinetic energy $H_0 - \mu\Omega$ versus bounce action J_b, using forms (44) and (48) for H_0 for passing and trapped particles, respectively, and interpolating in the intermediate separatrix regime, in conformity with the requirement that $\omega_b \to 0$ in this region.