

# Particle Trajectories in Model Current Sheets

## 1. Analytical Solutions

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**Abstract.** Approximate analytical solutions are found for two model current sheets. In the first the magnetic field is linear and reverses across a neutral sheet, and the electric field is everywhere uniform, perpendicular to the magnetic field and parallel to the neutral sheet. Charged particles of either sign never come out of the neutral sheet, and their energies increase without bound. In the second model a small component of the magnetic field perpendicular to the neutral sheet is added. This component not only serves to bring particles out of the sheet, but turns protons and electrons toward the same direction, 90° away from the accelerating electric field. The particles are accelerated and then ejected when they have been turned 90°, and the emergent pitch angles to a magnetic line of force will be small if the perpendicular magnetic field component is small.

**Introduction.** Ness [1965] has reported the discovery of a magnetically neutral sheet in the earth's geomagnetic tail, with measurements taken on board the Imp 1 satellite. A somewhat uniform magnetic field of about 20  $\gamma$  ( $\gamma = 10^{-8}$  gauss =  $10^{-9}$  weber/m<sup>2</sup>) is found from about 10 $R_e$  (earth radii) to at least 30 $R_e$  in the antisolar direction. This field is predominantly in the solar direction above a plane (roughly identified as the magnetic equatorial plane), and in the anti-solar direction below this plane. The magnetic field reverses across a sheet of thickness about 0.1 $R_e$  and goes to zero within this neutral sheet.

Other neutral, or more generally current, sheets may occur in other situations, such as a day-side magnetospheric current sheet, neutral sheets occurring in interplanetary space, and neutral sheets associated with solar flares.

It is of interest to look at charged particle trajectories about such sheets. Adiabatic theory cannot be used across such a neutral sheet, because the magnetic field changes significantly in distances much less than a gyroradius. The charged particle equations of motion must therefore be either solved analytically or computed numerically.

This paper is concerned with analytical solutions in two model current sheets. Part 2 [Speiser, 1965] applies these results to a magnetospheric

tail model and also discusses numerical results.

**A simple linear model.** The simplest model for the fields about a neutral sheet that can be discussed analytically is

$$\mathbf{B} = -b(x/d)\hat{e}_y \tag{1}$$

$$\mathbf{E} = -a\hat{e}_z \tag{2}$$

where  $b$  is the strength of the magnetic field when  $x = d$ , the sheet half-thickness, and  $a$  is the strength of the electric field. The physical significance of such an electric field will be discussed in part 2 [Speiser, 1965]. This field will merely be assumed for the present treatment.

The coordinate system being used is sketched in Figure 1, as are the magnetic and electric fields.

The equations of motion for a particle in the neutral sheet, using these fields, are

$$\ddot{x} = C_1 \dot{x}x \tag{3}$$

$$\ddot{y} = 0 \tag{4}$$

$$\ddot{z} = -C_2 - C_1 x \dot{z} \tag{5}$$

where  $C_1 = (q/m)(b/d)$ , and  $C_2 = (q/m)a$ ; mks units are used.

Speiser [1964a, b] has given the solution to these equations for large time (see appendix A). The result is that the particle executes a damped oscillation about  $x = 0$  (the amplitude going as  $1/t^{1/2}$ ) while accelerating in the  $+\hat{e}_z$  direction

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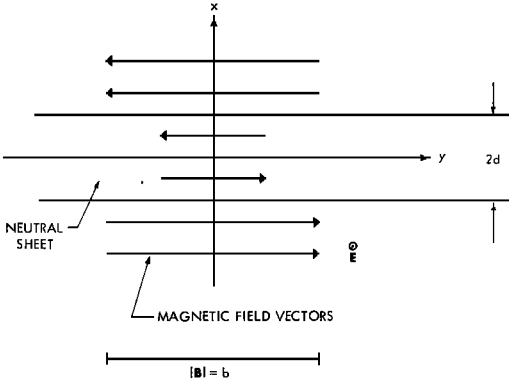


Fig. 1. Simple model.

for electrons and the  $-\hat{e}_z$  direction for protons. This can be understood as follows. The electric field in (5) accelerates a proton in the  $-\hat{e}_z$  direction. Then, ignoring the term  $-C_1 x\dot{x}$  for the moment, we see that  $\dot{z}$  becomes proportional to  $-t$ . Equation 3 then becomes

$$\ddot{x} = -kx \tag{6}$$

where  $k = C_1 |\dot{z}|$ , or  $k = (q/m)^2 (ab/d) t$ .

Equation 6 is just the equation for oscillatory motion, with spring constant  $k$ . However,  $k$  gets larger with time, implying that the spring gets stiffer with time; thus the amplitude of oscillation decays.

The term  $-C_1 x\dot{x}$  in (5) is approximately constant ( $x$  goes like  $\dot{z}^{-1/4}$ ;  $\dot{x}$  goes like  $\dot{z}^{+1/4}$ ) and can modulate the oscillation but will not stop it.  $k$  is proportional to  $q^2$ , and so particles of either sign will oscillate.

The oscillation in  $x(t)$  is due to the  $\mathbf{V} \times \mathbf{B}$  force that is always toward  $x = 0$  for  $x$  either positive or negative, because of the reversal of the magnetic field.

The net result of this simple model is that charged particles of either sign never come out of the neutral sheet, and their energy increases without bound. Figure 2 is a sketch of the results of this simple model.

*The linear model with small perpendicular field added.* Consider now the addition of a small magnetic field component perpendicular to the neutral sheet, i.e. in the  $+\hat{e}_z$  direction in the coordinate system of Figure 1. The magnetic field of (1) now becomes

$$\mathbf{B} = b \left( \eta \hat{e}_x - \frac{x}{a} \hat{e}_y \right) \tag{7}$$

where  $\eta$  is assumed to be small.

The equations of motion become

$$\ddot{x} = C_1 x \dot{z} \tag{8}$$

$$\ddot{y} = C_2 \eta \dot{z} \tag{9}$$

$$\ddot{z} = -C_3 - C_1 x \dot{x} - C_2 \eta \dot{y} \tag{10}$$

where  $C_1 = (q/m)(b/d)$ ,  $C_2 = (q/m)b$ ,  $C_3 = (q/m)a$ ,  $C_1$  and  $C_3$  being the same as for the simple model. When  $\eta = 0$ , the simple model results.

Even without solving these equations, the particle motion can be understood qualitatively as was done for the simple model.

Consider a proton (the arguments also hold for electrons, with appropriate changes in sign) incident on this neutral sheet with small velocity. (Strictly speaking, the sheet is not now a neutral sheet.) The proton will be accelerated initially in the negative  $z$  direction (see equation 10), gaining a velocity  $\dot{z}$  proportional to  $-t$  as for the simple model. From (9) there will then be an acceleration in the  $-y$  direction proportional to  $\dot{z}$  or  $-t$ , and thus  $\dot{y}$  will be proportional to  $-t^2$ . (Note that  $\ddot{y} \propto C_2 C_3 \propto (q/m)^2$ , so that either protons or electrons are turned toward the  $-\hat{e}_y$  direction.)

As long as  $\dot{z}$  is negative, (8) will imply oscillatory motion in  $x(t)$  by the same arguments in the previous section on the simple model. Thus, as long as  $\dot{z}$  is negative, the term  $-C_1 x\dot{x}$  in (10) will be assumed to be small as a first-order approximation. The oscillations can now be either damped or growing, depending on whether  $\dot{z}$  increases or decreases with time. The last term in (10) grows as  $+t^2$ , so that  $\dot{z}$  will grow negatively until  $\dot{z}$  goes to zero, and  $\dot{z}$  will then diminish in absolute value going to zero and even becoming positive after some time. Thus  $x(t)$  will execute damped oscillatory motion until  $\dot{z} = 0$  (the spring gets stiffer). After  $\dot{z}$  becomes positive and until  $\dot{z} = 0$ ,  $x(t)$  will execute growing oscillatory motion. After  $\dot{z}$  goes positive, however,  $x(t)$  will no longer oscillate but will increase exponentially, thus ejecting the particle from the neutral sheet. Figure 3 is a sketch of the particle trajectories in this model.

*Results of the approximate theory.* The detailed calculations are contained in appendix B (equation numbers in the appendixes bear the prefix A). First integrals of the equations of motion (equations 8, 9, and 10) are obtained exactly (equations A9, A10, and A14). A first approxi-

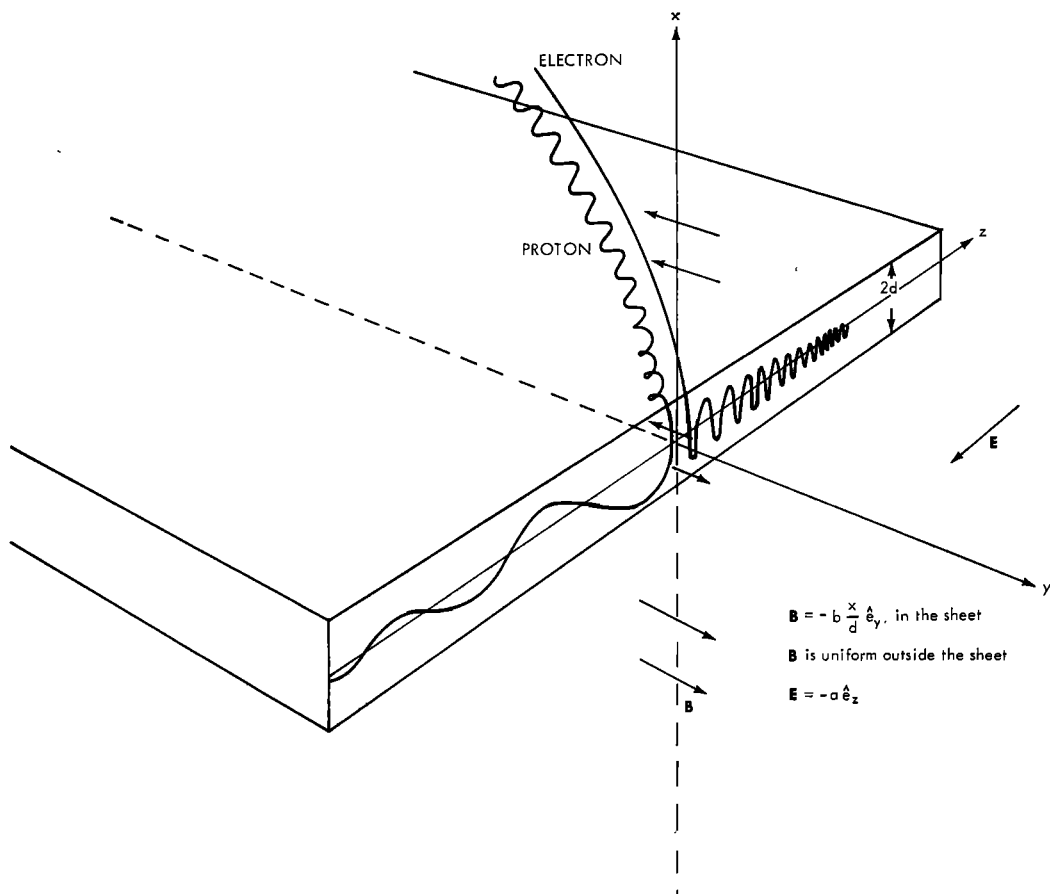


Fig. 2. Sketch of particle trajectories using the fields of the simple model (linear reversal of the magnetic field across the neutral sheet, no magnetic field component perpendicular to the sheet, and a uniform electric field in the  $-\hat{e}_z$  direction). Electrons are accelerated in the  $+\hat{e}_z$  direction, protons in the  $-\hat{e}_z$  direction. The amplitude of oscillation decays, and so particles never come out, and their energy goes to infinity.

mation is used to obtain the time of ejection  $\tau$ , which turns out to be inversely proportional to  $\eta b(q/m)$  (equation A19). Electrons are therefore ejected from the neutral sheet much sooner than protons. The velocity in the  $-\hat{e}_z$  direction at the time of ejection is found to be independent of  $(q/m)$  (equation A23), and the maximum pitch angle of the emergent particles (equation A26) is proportional to  $\eta$ , and to  $|\dot{x}_0/u - c|$ , where  $u$  is the bulk flow velocity exterior to the neutral sheet, and  $c$  is a number that is not very different from 1. This result implies that  $\alpha = 0$  for  $\eta = 0$ , which means that the particles would never come out of the neutral sheet if the small perpendicular field were not added. This agrees with the result of the simple model, and implies

that  $\alpha$  will be small if  $\eta$  is small. Since  $\alpha$  is also proportional to  $|\dot{x}_0/u - c|$ , particles incident on the current sheet with  $\dot{x}_0 \approx u$  will all emerge with pitch angles close to zero.

J. W. Dungey (private communication, 1965) suggested that the electric field can be transformed away in this model. Calculations using such a transformation are made in appendix C. The particle motion in the moving (transformed) system is easily visualized, since only a magnetic field is involved. The perpendicular magnetic field component causes a circular drift in the current sheet while the  $x$  coordinate is oscillating due to the magnetic field reversal ( $B_y$ ) (see, for example, equations A29, A37, and A38). The ejection time is seen to be just a half-period of

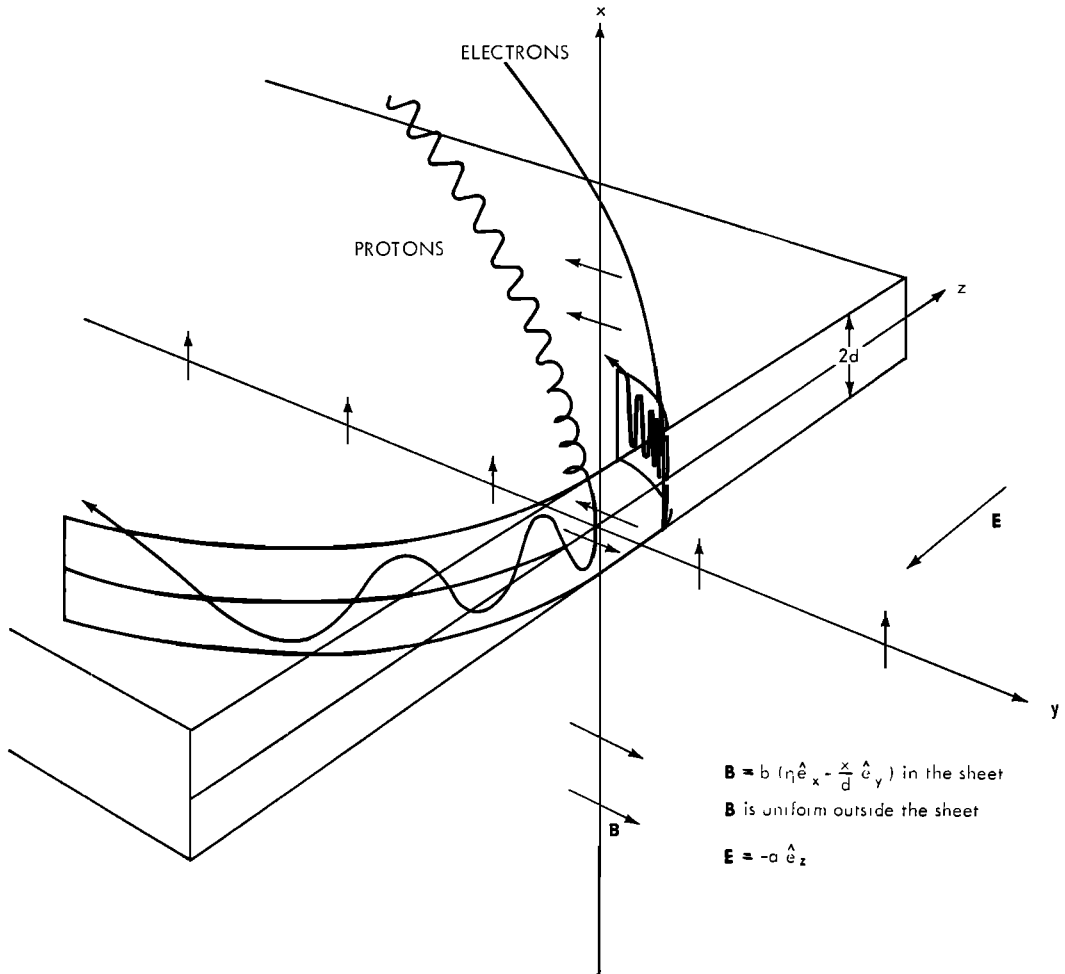


Fig. 3. Sketch of particle trajectories using the fields of the simple model plus a small component perpendicular to the neutral sheet. Both protons and electrons oscillate about the sheet accelerating in opposite directions and are turned toward the same direction by the small magnetic field component perpendicular to the sheet. When the particles are turned  $90^\circ$ , they are ejected from the neutral sheet. Electrons come out much sooner than protons, hence gain less energy. Electrons also make fewer oscillations than protons before ejection. The above sketch is illustrative and not to scale.

the circular drift (equation A39) and corresponds closely to the approximate result in appendix B.

The particle energy in the transformed system is a constant, with the initial velocity given by the transformation velocity (in the simple case where the initial velocities are approximately zero in the stationary system). It is therefore apparent that protons and electrons will be ejected with the same velocity, a result which was found before (appendix B) only for the first and second approximations.

An oscillation frequency about the neutral sheet is found approximately (equation A42), and the number of oscillations before ejection is determined (equation A43). It is seen that this number is proportional to the square root of the mass, and so electrons will execute about  $1/40$  the number of proton oscillations before ejection.

*Summary.* Two models of possible current sheets are considered, and approximate analytical results of charged particle trajectories about these sheets are found.

In the simplest model, where the magnetic field varies linearly across a neutral sheet, and there is an electric field perpendicular to the magnetic field and parallel to the current sheet, charged particles of either sign execute damped oscillations about the sheet, accelerating along the sheet. Thus for this simple model particles never come out of the sheet and their energies go to infinity.

A new model is constructed by adding a small component of the magnetic field perpendicular to the sheet. The addition of this small component not only turns both protons and electrons toward the same direction 90° away from the accelerating electric field, but serves to eject the particles from the sheet. Protons and electrons are thus accelerated and are then ejected with the same velocity, the electrons being turned much faster and being ejected sooner than the protons.

The pitch angle of ejected particles about a line of force is proportional to the size of the small perpendicular component of magnetic field. Thus, if this component is small, all particles will be ejected nearly along lines of force.

The energy gained is inversely proportional to the square of the perpendicular component of magnetic field (see equation A23). Therefore, in agreement with the simple model, the energies go to infinity when this component is zero. Moreover, the energies of the ejected particles can be large if this component is small.

In the moving frame (where  $\mathbf{E}' = D$ ) neglect of the term  $-C_1 x' \dot{x}'$  essentially decouples the circular motion due to  $B_x$  from the oscillatory motion about the neutral sheet. The oscillatory motion about the neutral sheet is, however, still coupled to  $z'$ , and indeed the particle is ejected when  $z'$  changes sign.

A few computer solutions of equations 8, 9, and 10 have been made, and the results agree in general with these analytical results. The velocity at ejection may have a larger  $x$ -component than indicated in (A24) and the  $z$ -component may be non-zero. In two cases, the emergent pitch angles were about a factor of 10 larger than (A26) predicts. ( $\alpha \approx 5^\circ$  rather than  $\approx \frac{1}{2}^\circ$  for  $\eta \approx 0.01$ .)

APPENDIX A

The first integrals of equations (3) and (5) are

$$\dot{z} = \dot{z}_0 - C_3 t - \frac{1}{2} C_1 (x^2 - x_0^2) \quad (A1)$$

$$\frac{1}{2} (\dot{x}^2 + \dot{z}^2) + C_3 z = \frac{1}{2} (\dot{x}_0^2 + \dot{z}_0^2) + C_3 z_0 \quad (A2)$$

Equation A2 is just the equation of conservation of energy. The zero subscripted values refer to initial values. Equation 3 becomes, using (A1),

$$\ddot{x} = -C_1 x (-\dot{z}_0 + \frac{1}{2} C_1 (x^2 - x_0^2) + C_3 t) \quad (A3)$$

For large enough time, the quantity in parenthesis on the right-hand side of (A3) will be positive and monotonically increasing, implying oscillatory, bounded motion in  $x(t)$ . Thus for large time (A3) is approximately

$$\ddot{x} \approx -C_1 C_3 x t = -\left(\frac{q}{m}\right)^2 \frac{ab}{d} x t \quad (A4)$$

The solution to (A4) is given by *Jahnke and Emde* [1945, p. 147] as

$$x = \sqrt{t'} Z_{1/3}(\frac{2}{3} t'^{3/2}) \quad (A5)$$

where

$$t' = \left(\frac{ba}{d}\right)^{1/3} \left(\frac{q}{m}\right)^{2/3} t$$

and  $Z_{1/3}$  is a linear combination of Bessel functions of the first and second kinds, of order one-third. Approximating (A5) for large time [*Jahnke and Emde*, 1945, p. 138],

$$x \approx \frac{t^{-1/4}}{\left(\frac{ba}{d}\right)^{1/12} \left(\frac{q}{m}\right)^{1/6}} \cdot \left[ A \cos\left(\frac{2}{3} t'^{3/2} \left(\frac{ba}{d}\right)^{1/2} \left(\frac{q}{m}\right)\right) + B \sin\left(\frac{2}{3} t'^{3/2} \left(\frac{ba}{d}\right)^{1/2} \left(\frac{q}{m}\right)\right) \right] \quad (A6)$$

$A$  and  $B$  are constants depending on the initial values.

We can obtain  $z(t)$  as a function of time by integrating (A1):

$$z(t) \approx -\left(\frac{q}{m}\right) \frac{at^2}{2} + \left(\dot{z}_0 + \left(\frac{q}{m}\right) \left(\frac{b}{d}\right) \frac{x_0^2}{2}\right) t + z_0 + \text{smaller oscillatory terms} \quad (A7)$$

From (A6), after large time the amplitude of oscillation decays as  $1/t^{1/4}$ , and from (A7) and

(A2) it is seen that the kinetic energy increases where as  $t^2$ .

## APPENDIX B

First integrals of (8), (9), and (10) are easily obtained. They are:

$$\frac{1}{2}(\dot{x}^2 - \dot{x}_0^2) = -C_3(z - z_0) - \frac{1}{2}(\dot{z}^2 - \dot{z}_0^2) - C_2\eta \int_0^t \dot{y}\dot{z} dt \quad (\text{A8})$$

$$\dot{y} - \dot{y}_0 = C_2\eta(z - z_0) \quad (\text{A9})$$

$$(\dot{z} - \dot{z}_0) = -C_3t - \frac{C_1}{2}(x^2 - x_0^2) - C_2\eta(y - y_0) \quad (\text{A10})$$

Equation A10 becomes, on another integration,

$$(z - z_0) = -\frac{C_3t^2}{2} - \frac{C_1}{2} \int_0^t x^2 dt - C_2\eta \int_0^t y dt + C_4t \quad (\text{A11})$$

where

$$C_4 = \dot{z}_0 + \frac{C_1x_0^2}{2} + C_2\eta y_0$$

Using (A11) with (A9), we have

$$\dot{y} = \dot{y}_0 + \eta \left( -C_5t^2 + C_6t - C_7 \int_0^t x^2 dt - C_2^2\eta \int_0^t y dt \right) \quad (\text{A12})$$

where

$$C_5 = \frac{C_2C_3}{2} \quad C_6 = C_2C_4 \quad C_7 = \frac{C_1C_2}{2}$$

From (9) we have

$$C_2\eta \int_0^t \dot{y}\dot{z} dt = \frac{1}{2}(\dot{y}^2 - \dot{y}_0^2) \quad (\text{A13})$$

So that (A8) becomes the energy integral

$$\frac{1}{2}(x^2 + \dot{y}^2 + \dot{z}^2) + C_3z = \frac{1}{2}(x_0^2 + \dot{y}_0^2 + \dot{z}_0^2) + C_3z_0 = \frac{E_0}{m} \quad (\text{A14})$$

Using (A10), we obtain for (8)

$$\ddot{x} = -k(t, x, y, \eta)x \quad (\text{A15})$$

$k(t, x, y, \eta)$

$$= C_3t + \frac{C_1x^2}{2} + C_2\eta y + C_8 \quad (\text{A16})$$

$$C_8 = -\frac{C_1x_0^2}{2} - C_2\eta y_0 - \dot{z}_0 = -C_4$$

$k$  goes to zero when

$$t = -\frac{1}{C_3} \left( \frac{C_1x^2}{2} + C_2\eta y + C_8 \right) = \tau \quad (\text{A17})$$

$\tau$  is therefore the critical time when  $k$  goes to zero and then becomes negative ejecting the particle ( $\dot{z}$  becomes positive).

*Approximations.* Integrating (A12), we obtain

$$y = y_0 + \dot{y}_0t + \frac{C_6t^2}{2} - \frac{\eta C_5t^3}{3} \quad (\text{A18})$$

where the integral over  $x^2$  in (A12) is assumed to be small, since  $x(t)$  is oscillating, and the integral over  $y$  in (A12) is multiplied by  $\eta^2$ , and so it will also be neglected in this first approximation. To facilitate finding the critical time of ejection  $\tau$ , initial conditions are chosen such that  $y_0 = \dot{y}_0 = 0$  and  $\dot{z}_0 = -C_1x_0^2/2$ , which implies that  $C_4 = C_6 = C_8 = 0$ . Using (A17) and (A18) with these initial conditions,  $\tau$  becomes

$$\tau = \frac{\sqrt{6}}{\eta(q/m)b} \quad (\text{A19})$$

From (A18)

$$y(t) = -\eta C_5t^3/3 \quad (\text{A20})$$

$$\dot{y}(t) = -\eta C_5t^2 \quad (\text{A21})$$

At  $t = \tau$ , equations A20 and A21 become

$$y(\tau) = -\frac{\sqrt{6}a}{(\eta b)^2(q/m)} \quad (\text{A22})$$

$$\dot{y}(\tau) = -3a/\eta b \quad (\text{A23})$$

Thus, in this first approximation, the ejection velocity is independent of  $(q/m)$  because electrons are ejected much sooner than protons. (As the next approximation, if  $y(t)$  from (A20) is used in the integral over  $y$  from (A11), it is found that  $\tau$  is increased by about 30%,  $|\dot{y}(\tau)|$  is decreased by about 50%, and the ejection velocity is still independent of  $(q/m)$ . The first approximation should therefore give the right order of magnitude. See also Appendix C.)

*Pitch angle distributions of emergent particles.*  
 In order to calculate the pitch angle of an emergent particle, we must estimate the maximum value  $\dot{x}$  can have at the time of ejection. Since  $x(t)$  oscillates until  $t = \tau$ , it is reasonable to assume that  $\dot{x}_{\max} \approx \dot{x}_0$ . Using this assumption and (A23) with (7) for the magnetic field with  $x = d$ , we have

$$\mathbf{V} = \dot{x}_0 \hat{e}_x - (3a/\eta b) \hat{e}_y \quad (\text{A24})$$

and the cosine of the pitch angle  $\alpha$  becomes

$$\begin{aligned} \cos \alpha &= \frac{(\mathbf{B} \cdot \mathbf{V})}{B V} \\ &= 1 - \frac{2\eta^2[(\dot{x}_0/u) - 3]^2}{36} \end{aligned} \quad (\text{A25})$$

where  $u = a/b$ , the bulk flow velocity exterior to the neutral sheet, and only terms of the order of  $\eta^2$  are kept. For small  $\alpha$ ,  $\cos \alpha \approx 1 - \alpha^2/2$ , so that

$$\alpha \approx \frac{\eta |(\dot{x}_0/u) - 3|}{3} \quad (\text{A26})$$

The number 3 in (A26) comes from the 3 in (A23). This would be decreased to 3/2 by the second approximation mentioned above. The following results (A40) imply that the 3 should be replaced by 2.

### APPENDIX C

A Lorentz transformation can be made using the fields given by (2) and (7) to a system where the electric field is zero. This is possible for the present model, because of the form of the fields and because  $B_x$  (equation 7) is a constant. If the transformation velocity is chosen to be

$$\mathbf{v} = (-a/\eta b) \hat{e}_y \quad (\text{A27})$$

then the fields become, in the transformed system,

$$\mathbf{E}' = 0$$

$$\mathbf{B}' = b[\eta \sqrt{1 - v^2/c^2} \hat{e}_x - (x'/d) \hat{e}_y] \quad (\text{A28})$$

where the primes refer to the transformed system.

The equations of motion corresponding to (8), (9), and (10) are

$$\dot{x}' = C_1 x' \dot{z}' \quad (\text{A29})$$

$$\dot{y}' = C_2 \eta' \dot{z}' \quad (\text{A30})$$

$$\dot{z}' = -C_1 x' \dot{x}' - C_2 \eta' \dot{y}' \quad (\text{A31})$$

where

$$\eta' = \eta \sqrt{1 - v^2/c^2}$$

If we borrow the previous result that the  $x$  coordinate oscillates until  $t = \tau$ , then the term  $-C_1 x' \dot{x}'$  in (A31) will be assumed to be small initially. Equations A30 and A31 then become

$$\dot{y}' = \omega \dot{z}' \quad (\text{A32})$$

$$\dot{z}' = -\omega \dot{y}' \quad (\text{A33})$$

where  $\omega \equiv C_2 \eta'$ . These are just the coupled equations for simple harmonic motion, and the solutions are:

$$\dot{y}' = \dot{y}'_0 \cos \omega t' + \dot{z}'_0 \sin \omega t' \quad (\text{A34})$$

$$\dot{z}' = \dot{z}'_0 \cos \omega t' - \dot{y}'_0 \sin \omega t' \quad (\text{A35})$$

For transformation velocities small compared with the speed of light, the previous boundary conditions (appendix B) become, in the transformed system,

$$\dot{y}'_0 = +\frac{a}{\eta b} \quad \dot{z}'_0 = -C_1 x_0^2/2 \quad (\text{A36})$$

For small  $\eta$ ,  $\dot{y}'_0$  can be large compared to  $\dot{z}'_0$ , and so (A34) and (A35) are approximately

$$\dot{y}' = \dot{y}'_0 \cos \omega t' \quad (\text{A37})$$

$$\dot{z}' = -\dot{y}'_0 \sin \omega t' \quad (\text{A38})$$

implying a circular drift in these components. Thus  $\dot{z}'$  starts at zero and grows negatively until  $\omega t' = \pi/2$  and then decreases in absolute value, going to zero at  $\omega t' = \pi$ .  $\dot{z}'$  becomes positive thereafter, and so by the same arguments in the preceding section and by (A29) we see that this time is the ejection time.

$$\tau = \frac{\pi}{\omega} = \frac{\pi}{(q/m)b\eta} \quad (\text{A39})$$

For velocities small compared to the speed of light, this time is about the same in the two systems. The second approximation mentioned in appendix B replaced  $6^{1/2}$  in (A19) by  $10^{1/2}$ , which is very close to  $\pi$  from (A39). From (A37) at the time of ejection ( $\omega t' = \pi$ ),  $\dot{y}'(\tau) = -\dot{y}'_0$ . Transforming back to the unprimed system gives

$$\dot{y}(\tau) = -2a/\eta b \quad (\text{A40})$$

rather than the factor of 3 (first approximation, appendix B) or  $3/2$  (second approximation, appendix B).

More information can now be obtained than in appendix B by approximating  $\dot{z}'$  from (A38) as

$$\dot{z}' \approx -\dot{y}_0'/2 \quad (\text{A41})$$

Using (A41) with (A29), we see that the  $x'$  coordinate will oscillate with frequency

$$\omega_1 = \left( \frac{C_1 \dot{y}_0'}{2} \right)^{1/2} = \left( \frac{(q/m)a}{2\eta d} \right)^{1/2} \quad (\text{A42})$$

and the number of oscillations about the neutral sheet before the particle is ejected is

$$n = \frac{1}{2b} \left( \frac{a}{2(q/m)\eta^3 d} \right)^{1/2} \quad (\text{A43})$$

The above results can be extended to larger velocities (larger  $a/\eta b$ ) by not making the approximation  $v \ll c$ , and keeping all of the terms from the Lorentz transformation.

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