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Partition function of the eight-vertex model with domain wall boundary condition

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We derive the recursive relations of the partition function for the eight-vertex model on an $N\times N$ square lattice with domain wall boundary condition. Solving the recursive relations, we obtain the explicit expression of the domain wall partition function of the model. In the trigonometric/rational limit, our results recover the corresponding ones for the six-vertex model. © *2009 American Institute of Physics.* [DOI: 10.1063/1.3205448]

I. INTRODUCTION

The domain wall (DW) boundary condition for the six-vertex model on a finite square lattice was introduced by Korepin in Ref. 1, where some recursion relations of the partition function which fully determine the partition function were also derived. It was then found in Refs. 2 and 3 that the partition function can be represented as a determinant. Such an explicit expression of the partition function has played an important role in constructing norms of Bethe states, correlation functions,[4–6] and thermodynamical properties of the six-vertex model,[7,8] and also in the Toda theories.[9] Moreover, it has been proven to be very useful in solving some pure mathematical problems, such as the problem of alternating sign matrices.[10] Recently, the partition functions with DW boundary condition have been obtained for the high-spin models[11] and the fermionic models.[12,13]

Among solvable models, elliptic ones stand out as a particularly important class due to the fact that most trigonometric and rational models can be obtained from them by certain limits. In this paper, we focus on the most fundamental elliptic model—the eight-vertex model[14,15] whose trigonometric limit gives the six-vertex model. By means of the algebraic Bethe ansatz method we derive an explicit expression of the partition function for the eight-vertex model on an $N\times N$ square lattice with the DW boundary condition. In the trigonometric limit, our results recover those obtained in Refs. 1–3 for the six-vertex model.

The paper is organized as follows. In Sec. II, we introduce our notation and some basic ingredients. In Sec. III, after briefly reviewing the vertex-face correspondence, we introduce the four boundary states which specify the DW boundary condition of the eight-vertex model. In Sec. IV, some properties of the partition function of the eight-vertex model with the DW boundary condition are obtained by using the algebraic Bethe ansatz method. With the help of these properties, we derive in Sec. V the recursive relations of the partition function and obtain the explicit expression of the DW partition function by resolving the recursive relations. In Sec. VI, we summarize our results and give some discussions. Some detailed technical proofs are given in Appendixes A and B.

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II. THE EIGHT-VERTEX MODEL

In this section, we define the DW boundary condition for the eight-vertex model on an $N \times N$ square lattice.[15]

A. The eight-vertex *R*-matrix

Let us fix τ such that $\mathrm{Im}(\tau)>0$ and a generic complex number η. Introduce the following elliptic functions:

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(u,\tau) = \sum_{n=-\infty}^{\infty} \exp\{i\pi[(n+a)^2\tau + 2(n+a)(u+b)]\}, \tag{2.1}$$

$$\theta^{(j)}(u) = \theta\begin{bmatrix} \frac{1}{2}-\frac{j}{2} \\ \frac{1}{2} \end{bmatrix}(u,2\tau), \quad j=1,2, \tag{2.2}$$

$$\sigma(u) = \theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(u,\tau), \quad \sigma'(u) = \frac{\partial}{\partial u}\{\sigma(u)\}. \tag{2.3}$$

The σ-function satisfies the so-called Riemann identity:

$$\sigma(u+x)\sigma(u-x)\sigma(v+y)\sigma(v-y) - \sigma(u+y)\sigma(u-y)\sigma(v+x)\sigma(v-x) = \sigma(u+v)\sigma(u-v)\sigma(x+y)\sigma(x-y), \tag{2.4}$$

which will be useful in the following. [Our σ-function is the ϑ-function $\vartheta_1(u)$.[16] It has the following relation with the *Weierstrassian* σ-function $\sigma_w(u)$: $\sigma_w(u) \propto e^{\eta_1 u^2}\sigma(u)$ with $\eta_1=\pi^2(1/6 - 4\Sigma_{n=1}^{\infty}(nq^{2n}/(1-q^{2n})))$ and $q=e^{i\tau}$.]

Let V be a two-dimensional vector space $\mathbb{C}^2$ and $\{\epsilon_i | i=1,2\}$ be the orthonormal basis of V such that $\langle \epsilon_i, \epsilon_j\rangle = \delta_{ij}$. The well-known eight-vertex model R-matrix $R(u) \in \mathrm{End}(V\otimes V)$ is given by

$$R(u) = \begin{pmatrix} a(u) & & & d(u) \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ d(u) & & & a(u) \end{pmatrix}. \tag{2.5}$$

The nonvanishing matrix elements are[15]

$$a(u) = \frac{\theta^{(1)}(u)\theta^{(0)}(u+\eta)\sigma(\eta)}{\theta^{(1)}(0)\theta^{(0)}(\eta)\sigma(u+\eta)}, \quad b(u) = \frac{\theta^{(0)}(u)\theta^{(1)}(u+\eta)\sigma(\eta)}{\theta^{(1)}(0)\theta^{(0)}(\eta)\sigma(u+\eta)},$$

$$c(u) = \frac{\theta^{(1)}(u)\theta^{(1)}(u+\eta)\sigma(\eta)}{\theta^{(1)}(0)\theta^{(1)}(\eta)\sigma(u+\eta)}, \quad d(u) = \frac{\theta^{(0)}(u)\theta^{(0)}(u+\eta)\sigma(\eta)}{\theta^{(1)}(0)\theta^{(1)}(\eta)\sigma(u+\eta)}. \tag{2.6}$$

Here u is the spectral parameter and η is the so-called crossing parameter. The R-matrix satisfies the quantum Yang–Baxter equation (QYBE)

$$R_{1,2}(u_1-u_2)R_{1,3}(u_1-u_3)R_{2,3}(u_2-u_3) = R_{2,3}(u_2-u_3)R_{1,3}(u_1-u_3)R_{1,2}(u_1-u_2), \tag{2.7}$$

and the following properties:

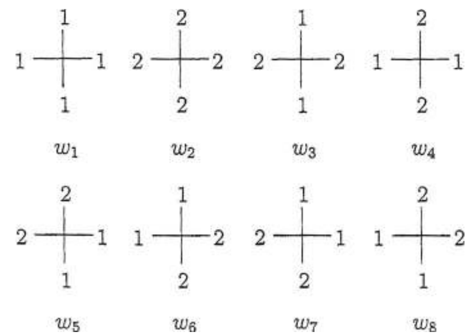

FIG. 1. Vertex configurations and their associated Boltzmann weights.

$$Z_2\text{-symmetry:}\ \ \sigma_1^i\sigma_2^i R_{1,2}(u) = R_{1,2}(u)\sigma_1^i\sigma_2^i \quad \text{for}\ \ i = x,y,z, \tag{2.8}$$

initial condition: $R_{1,2}(0) = P_{12}$, the Boltzmann weights and elements of the eight-vertex R -matrix. (2.9)

Here σ^x, σ^y, and σ^z are the Pauli matrices and P_{12} is the usual permutation operator. Throughout this paper we adopt the standard notations: for any matrix $A \in \mathrm{End}(V)$, A_j is an embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as A on the jth space and as identity on the other factor spaces; $R_{i,j}(u)$ is an embedding operator of the R-matrix in the tensor space, which acts as identity on the factor spaces except for the ith and jth ones.

B. The model

The partition function of a statistical model on a two-dimensional lattice is defined by the following:

$$Z = \sum \exp\left\{-\frac{E}{kT}\right\},$$

where E is the energy of the system, k is the Boltzmann constant, T is the temperature of the system, and the summation is taken over all possible configurations under the particular boundary condition such as the DW boundary condition. The model we consider here has eight allowed local vertex configurations (see Fig. 1, where 1 and 2 respectively denote the spin up and down states). Each of these eight configurations is assigned a statistical weight (or Boltzmann weight) w_i. Then the partition function can be rewritten as

$$Z = \sum w_1{}^{n_1} w_2{}^{n_2} w_3{}^{n_3} w_4{}^{n_4} w_5{}^{n_5} w_6{}^{n_6} w_7{}^{n_7} w_8{}^{n_8},$$

where the summation is over all possible vertex configurations with n_i being the number of the vertices of type i. If the local Boltzmann weights have Z_2-symmetry, i.e.,

$$a \equiv w_1 = w_2, \quad b \equiv w_3 = w_4, \quad c \equiv w_5 = w_6, \quad d \equiv w_7 = w_8, \tag{2.10}$$

and the variables a, b, c, and d satisfy a function relation, or equivalently, the local Boltzmann weights $\{w_i\}$ can be parametrized by the matrix elements of the eight-vertex R-matrix R (2.5) and (2.6) as in Fig. 2, then the corresponding model is called the eight-vertex model which can be exactly solved.[15] Therefore the partition function of the eight-vertex model is given by

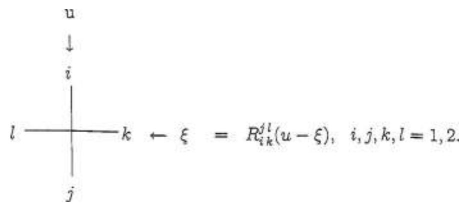

FIG. 2. The Bolzmann weights and elements of the eight-vertex R-matrix.

$$Z=\sum a^{n_1+n_2}b^{n_3+n_4}c^{n_5+n_6}d^{n_7+n_8}.$$

In order to parametrize the local Boltzmann weights in terms of the elements of the R-matrix, one needs to assign spectral parameters u and ξ, respectively, to the vertical line and horizontal line of each vertex of the lattice, as shown in Fig. 2. In an inhomogeneous model, the statistical weights are site dependent. Hence two sets of spectral parameters $\{u_\alpha\}$ and $\{\xi_i\}$ are needed, see Fig. 3. The horizontal lines are enumerated by indices $1,\ldots,N$ with spectral parameters $\{\xi_i\}$, while the vertical lines are enumerated by indices $\bar{1},\ldots,\bar{N}$ with spectral parameters $\{u_\alpha\}$. The DW boundary condition is specified by four boundary states $|\Omega^{(1)}(\lambda)\rangle$, $|\bar{\Omega}^{(1)}(\lambda)\rangle$, $\langle\Omega^{(2)}(\lambda+\eta N\hat{2})|$, and $\langle\bar{\Omega}^{(2)}(\lambda+\eta N\hat{2})|$ [$\hat{2}$ and the definitions of the boundary states will be given later in Sec. III, see (3.1) and (3.12)–(3.15) below]. These four states correspond to the particular choices of spin states on the four boundaries of the lattice. In contrast to the six-vertex case,[4] our boundary states depend not only on the spectral parameters ($|\Omega^{(1)}(\lambda)\rangle$ and $\langle\Omega^{(2)}(\lambda+\eta N\hat{2})|$ depend on $\{\xi_i\}$, while $|\bar{\Omega}^{(1)}(\lambda)\rangle$ and $\langle\bar{\Omega}^{(2)}(\lambda+\eta N\hat{2})|$ depend on $\{u_\alpha\}$) but also on two continuous parameters λ_1 and λ_2 (it is convenient to introduce a vector $\lambda\in V$ associated with these two parameters $\{\lambda_i\}$: $\lambda=\sum_{i=1}^{2}\lambda_i\epsilon_i$). However, in the trigonometric limit (i.e., setting $\lambda_2=\tau/2$ and then taking $\tau\to+i\infty$), the corresponding boundary states $|\Omega^{(1)}(\lambda)\rangle$ and $\langle\bar{\Omega}^{(1)}(\lambda)|$ (or $|\bar{\Omega}^{(2)}(\lambda+\eta N\hat{2})\rangle$ and $\langle\Omega^{(2)}(\lambda+\eta N\hat{2})|$) become the state of all spin up and its dual (or the state of all spin down and its dual) up to some overall scalar factors. Therefore the partition function in the limit reduces to that of the six-vertex model.[1–3] In this sense, we call the partition function corresponding to the boundary condition given in Fig. 3 the DW partition function of the eight-vertex model.

Now the partition function of the eight-vertex model with DW boundary condition is a function of $2N+2$ variables $\{u_\alpha\}$, $\{\xi_i\}$, λ_1, and λ_2, which will be denoted by $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$. Due to the fact that the local Boltzmann weights of each vertex of the lattice are given by the matrix elements of the eight-vertex R-matrix (see Fig. 2), the partition function can be expressed in terms of the product of the R-matrices and the four boundary states,

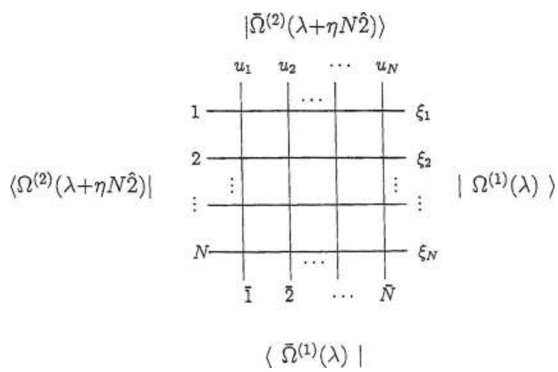

FIG. 3. The eight-vertex model with DW boundary condition.

$$Z_N(\{u_\alpha\};\{\xi_i\};\lambda) = \langle\Omega^{(2)}(\lambda+\eta N\hat{2})|\langle\bar{\Omega}^{(1)}(\lambda)|R_{\bar{1},N}(u_1-\xi_N)\cdots R_{\bar{1},1}(u_1-\xi_1)\cdots R_{\bar{N},N}(u_N - \xi_N)\cdots R_{\bar{N},1}(u_N-\xi_1)|\bar{\Omega}^{(2)}(\lambda+\eta N\hat{2})\rangle|\Omega^{(1)}(\lambda)\rangle. \tag{2.11}$$

The aim of this paper is to obtain an explicit expression for $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$.

One can rearrange the product of the R-matrices in (2.11) in terms of a product of the row-to-row monodromy matrices, namely,

$$Z_N(\{u_\alpha\};\{\xi_i\};\lambda) = \langle\Omega^{(2)}(\lambda+\eta N\hat{2})|\langle\bar{\Omega}^{(1)}(\lambda)|T_{\bar{1}}(u_1)\cdots T_{\bar{N}}(u_N)|\bar{\Omega}^{(2)}(\lambda+\eta N\hat{2})\rangle|\Omega^{(1)}(\lambda)\rangle, \tag{2.12}$$

where the monodromy matrix $T_{\bar{i}}(u)$ is given by

$$T_{\bar{i}}(u) \equiv T_{\bar{i}}(u;\xi_1,\ldots,\xi_N) = R_{\bar{i},N}(u-\xi_N)\cdots R_{\bar{i},1}(u-\xi_1). \tag{2.13}$$

The QYBE (2.7) of the R-matrix gives rise to the so-called RLL relation satisfied by the monodromy matrix $T_{\bar{i}}(u)$,

$$R_{\bar{i},\bar{j}}(u_i-u_j)T_{\bar{i}}(u_i)T_{\bar{j}}(u_j) = T_{\bar{j}}(u_j)T_{\bar{i}}(u_i)R_{\bar{i},\bar{j}}(u_i-u_j). \tag{2.14}$$

III. THE BOUNDARY STATES

A. The vertex-face correspondence

Let us briefly review the face-type R-matrix associated with the eight-vertex model. From the orthonormal basis $\{\epsilon_i\}$ of V, we define

$$\hat{i} = \epsilon_i - \bar{\epsilon}, \quad \bar{\epsilon} = \frac{1}{2}\sum_{k=1}^{2}\epsilon_k, \quad i=1,2, \quad \text{then} \quad \sum_{k=1}^{2}\hat{k}=0. \tag{3.1}$$

For a generic $m \in V$, define

$$m_i = \langle m,\epsilon_i\rangle, \quad m_{ij} = m_i - m_j = \langle m,\epsilon_i-\epsilon_j\rangle, \quad i,j=1,2. \tag{3.2}$$

Let $\bar{R}(u;m) \in \mathrm{End}(V\otimes V)$ be the R-matrix of the eight-vertex SOS model[15] given by

$$\bar{R}(u;m) = \sum_{i=1}^{2}\bar{R}_{ii}^{ii}(u;m)E_{ii}\otimes E_{ii} + \sum_{i\neq j}^{2}\{\bar{R}_{ij}^{ij}(u;m)E_{ii}\otimes E_{jj} + \bar{R}_{ij}^{ji}(u;m)E_{ji}\otimes E_{ij}\}, \tag{3.3}$$

where E_{ij} is the matrix with elements $(E_{ij})_k^l = \delta_{jk}\delta_{il}$. The coefficient functions are

$$\bar{R}_{ii}^{ii}(u;m) = 1, \quad \bar{R}_{ij}^{ij}(u;m) = \frac{\sigma(u)\sigma(m_{ij}-\eta)}{\sigma(u+\eta)\sigma(m_{ij})}, \quad i\neq j, \tag{3.4}$$

$$\bar{R}_{ij}^{ji}(u;m) = \frac{\sigma(\eta)\sigma(u+m_{ij})}{\sigma(u+\eta)\sigma(m_{ij})}, \quad i\neq j, \tag{3.5}$$

and m_{ij} is defined in (3.2). The R-matrix $\bar{R}$ satisfies the dynamical (modified) QYBE.

Let us introduce two intertwiners which are two-component column vectors $\phi_{m,m-\eta\hat{j}}(u)$ labeled by $\hat{j}=\hat{1},\hat{2}$. The kth element of $\phi_{m,m-\eta\hat{j}}(u)$ is given by

$$\phi_{m,m-\eta\hat{j}}^{(k)}(u) = \theta^{(k)}(u+2m_j). \tag{3.6}$$

Explicitly,

$$\phi_{m,m-\eta\hat{1}}(u)=\begin{pmatrix}\theta^{(1)}(u+2m_1)\\ \theta^{(2)}(u+2m_1)\end{pmatrix},\quad \phi_{m,m-\eta\hat{2}}(u)=\begin{pmatrix}\theta^{(1)}(u+2m_2)\\ \theta^{(2)}(u+2m_2)\end{pmatrix}.$$

It is easy to check that these two intertwiner vectors $\phi_{m,m-\eta\hat{i}}(u)$ are linearly *independent* for a generic $m\in V$.

Using the intertwiner vectors, one can derive the following vertex-face correspondence relation:[15,17]

$$R_{1,2}(u_1-u_2)\phi^1_{m,m-\eta\hat{i}}(u_1)\phi^2_{m-\eta\hat{i},m-\eta(\hat{i}+\hat{j})}(u_2)=\sum_{k,l}\bar{R}(u_1-u_2;m)^{kl}_{ij}\phi^1_{m-\eta\hat{l},m-\eta(\hat{l}+\hat{k})}(u_1)\phi^2_{m,m-\eta\hat{l}}(u_2). \tag{3.7}$$

Hereafter we adopt the convention $\phi^1=\phi\otimes\mathrm{id}\otimes\cdots,\phi^2=\mathrm{id}\otimes\phi\otimes\mathrm{id}\otimes\cdots,\ldots$. The QYBE (2.7) of the vertex-type R-matrix $R(u)$ is equivalent to the dynamical Yang–Baxter equation of the solid-on-solid (SOS) R-matrix $\bar{R}(u;m)$. For a generic m, we can introduce two row-vector intertwiners $\tilde{\phi}$ satisfying the conditions

$$\tilde{\phi}_{m+\eta\hat{\mu},m}(u)\phi_{m+\eta\hat{\nu},m}(u)=\delta_{\mu\nu},\quad \mu,\nu=1,2, \tag{3.8}$$

from which one derives the relation

$$\sum_{\mu=1}^{2}\phi_{m+\eta\hat{\mu},m}(u)\tilde{\phi}_{m+\eta\hat{\mu},m}(u)=\mathrm{id}. \tag{3.9}$$

With the help of (3.6)–(3.9), we obtain the following relations from the vertex-face correspondence relation (3.7):

$$\tilde{\phi}^1_{m+\eta\hat{k},m}(u_1)R_{1,2}(u_1-u_2)\phi^2_{m+\eta\hat{j},m}(u_2)=\sum_{i,l}\bar{R}(u_1-u_2;m)^{kl}_{ij}\tilde{\phi}^1_{m+\eta(\hat{i}+\hat{j}),m+\eta\hat{j}}(u_1)\phi^2_{m+\eta(\hat{k}+\hat{l}),m+\eta\hat{k}}(u_2), \tag{3.10}$$

$$\tilde{\phi}^1_{m+\eta\hat{k},m}(u_1)\tilde{\phi}^2_{m+\eta(\hat{k}+\hat{l}),m+\eta\hat{k}}(u_2)R_{1,2}(u_1-u_2)=\sum_{i,j}\bar{R}(u_1-u_2;m)^{kl}_{ij}\tilde{\phi}^1_{m+\eta(\hat{i}+\hat{j}),m+\eta\hat{j}}(u_1)\tilde{\phi}^2_{m+\eta\hat{j},m}(u_2). \tag{3.11}$$

The intertwiners ϕ and $\tilde{\phi}$ and the associated vertex-face correspondence relations will play an important role in determining the very properties of the partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ that enable us in Sec. IV to fully determine its explicit expression.

B. The boundary states

Now we are in the position to construct the boundary states which have been used in Sec. II to specify the DW boundary condition of the eight-vertex model, see Fig. 3.

For any vector $m\in V$, we introduce four states which live in the two N-tensor spaces of V (one is indexed by $1,\ldots,N$ and the other is indexed by $\bar{1},\ldots,\bar{N}$) or their dual spaces as follows:

$$|\Omega^{(i)}(m)\rangle=\phi^1_{m,m-\eta\hat{i}}(\xi_1)\phi^2_{m-\eta\hat{i},m-2\eta\hat{i}}(\xi_2)\cdots\phi^N_{m-\eta(N-1)\hat{i},m-\eta N\hat{i}}(\xi_N),\quad i=1,2, \tag{3.12}$$

$$|\bar{\Omega}^{(i)}(m)\rangle=\phi^{\bar{1}}_{m,m-\eta\hat{i}}(u_1)\phi^{\bar{2}}_{m-\eta\hat{i},m-2\eta\hat{i}}(u_2)\cdots\phi^{\bar{N}}_{m-\eta(N-1)\hat{i},m-\eta N\hat{i}}(u_N),\quad i=1,2, \tag{3.13}$$

$$\langle\Omega^{(i)}(m)|=\tilde{\phi}^1_{m,m-\eta\hat{i}}(\xi_1)\tilde{\phi}^2_{m-\eta\hat{i},m-2\eta\hat{i}}(\xi_2)\cdots\tilde{\phi}^N_{m-\eta(N-1)\hat{i},m-\eta N\hat{i}}(\xi_N),\quad i=1,2, \tag{3.14}$$

$$\langle\bar{\Omega}^{(i)}(m)| = \tilde{\bar{\phi}}^{\bar{1}}_{m,m-\eta\hat{i}}(u_1)\,\tilde{\bar{\phi}}^{\bar{2}}_{m-\eta\hat{i},m-2\eta\hat{i}}(u_2)\cdots\tilde{\bar{\phi}}^{\bar{N}}_{m-\eta(N-1)\hat{i},m-\eta N\hat{i}}(u_N), \quad i=1,2. \tag{3.15}$$

(Among them, $|\Omega^{(i)}(m)\rangle$, with special choices of m, are the complete reference states of the open XYZ spin chain[18] and have played an important role in constructing the extra center elements of the elliptic algebra at roots of unity.[19,20]) The boundary states which have been used to define the DW boundary condition in Sec. II can be obtained through the above states by special choices of m and i (for example, m is specified to λ or $\lambda+\eta N\hat{2}$). Then the DW partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ given by (2.11) becomes

$$\begin{aligned}
Z_N(\{u_\alpha\};\{\xi_i\};\lambda) = {}& \tilde{\phi}^{1}_{\lambda+\eta N\hat{2},\lambda+\eta(N-1)\hat{2}}(\xi_1)\cdots\tilde{\phi}^{N}_{\lambda+\eta\hat{2},\lambda}(\xi_N)\,\tilde{\bar{\phi}}^{\bar{1}}_{\lambda,\lambda-\eta\hat{1}}(u_1)\cdots\tilde{\bar{\phi}}^{\bar{N}}_{\lambda-\eta(N-1)\hat{1},\lambda-\eta N\hat{1}}(u_N)\\
&\times R_{\bar{1},N}(u_1-\xi_N)\cdots R_{\bar{1},1}(u_1-\xi_1)\cdots R_{\bar{N},N}(u_N-\xi_N)\cdots R_{\bar{N},1}(u_N\\
&-\xi_1)\,\phi^{\bar{1}}_{\lambda+\eta N\hat{2},\lambda+\eta(N-1)\hat{2}}(u_1)\cdots\phi^{\bar{N}}_{\lambda+\eta\hat{2},\lambda}(u_N)\,\phi^{1}_{\lambda,\lambda-\eta\hat{1}}(\xi_1)\cdots\phi^{N}_{\lambda-\eta(N-1)\hat{1},\lambda-\eta N\hat{1}}(\xi_N).
\end{aligned} \tag{3.16}$$

IV. THE PROPERTIES OF THE PARTITION FUNCTION

In this section we will derive certain properties of the DW partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ which enable us to determine its explicit expression. For the case of $N=1$, the corresponding partition function (3.16) becomes

$$\begin{aligned}
Z_1(u_1;\xi_1;\lambda) &= \tilde{\phi}^{1}_{\lambda+\eta\hat{2},\lambda}(\xi_1)\,\tilde{\bar{\phi}}^{\bar{1}}_{\lambda,\lambda-\eta\hat{1}}(u_1)R_{\bar{1},1}(u_1-\xi_1)\,\phi^{\bar{1}}_{\lambda+\eta\hat{2},\lambda}(u_1)\,\phi^{1}_{\lambda,\lambda-\eta\hat{1}}(\xi_1)\\
&\overset{(3.7)}{=} \sum_{k,l=1}^{2}\bar{R}^{kl}_{2\ 1}(u_1-\xi_1;\lambda+\eta\hat{2})\big(\tilde{\phi}^{1}_{\lambda+\eta\hat{2},\lambda}(\xi_1)\,\phi^{1}_{\lambda+\eta\hat{2},\lambda+\eta(\hat{2}-\hat{l})}(\xi_1)\big)\\
&\quad\times\big(\tilde{\bar{\phi}}^{\bar{1}}_{\lambda,\lambda-\eta\hat{1}}(u_1)\,\phi^{\bar{1}}_{\lambda+\eta(\hat{k}-\hat{1}),\lambda-\eta\hat{1}}(u_1)\big)\\
&\overset{(3.8)}{=} \bar{R}^{1\ 2}_{2\ 1}(u_1-\xi_1;\lambda+\eta\hat{2}).
\end{aligned}$$

Thus, we have the first property of the partition function:

$$Z_1(u_1;\xi_1;\lambda) = \frac{\sigma(\eta)\sigma(u_1-\xi_1+\lambda_{21}+\eta)}{\sigma(u_1-\xi_1+\eta)\sigma(\lambda_{21}+\eta)}, \tag{4.1}$$

where λ_{21} is defined by (3.2). Using the fundamental exchange relation (2.14) and the vertex-face relations (3.7), (3.10), and (3.11), we derive the second property of the partition function:

$$Z_N(\{u_\alpha\};\{\xi_i\};\lambda) \text{ is a symmetric function of } \{u_\alpha\} \text{ and } \{\xi_i\} \text{ separatively}. \tag{4.2}$$

The proof of the above property is relegated to Appendix A.

In addition to the Riemann identity (2.4), the σ-function enjoys the following quasiperiodic properties:

$$\sigma(u+1) = -\sigma(u), \quad \sigma(u+\tau) = -e^{-2i\pi(u+\tau/2)}\sigma(u), \tag{4.3}$$

which are useful in deriving the quasiperiodicity of the partition function. The expansions of the boundary states in terms of the intertwiner vectors (3.12)–(3.15) and the partition function in terms of the monodromy matrices (2.12) allow us to rewrite $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ as

$$Z_N(\{u_\alpha\};\{\xi_i\};\lambda) = \tilde{\phi}^{1}_{\lambda+\eta N\hat{2},\lambda+\eta(N-1)\hat{2}}(\xi_1)\cdots\tilde{\phi}^{N}_{\lambda+\eta\hat{2},\lambda}(\xi_N)\tilde{\bar{\phi}}^{\bar{1}}_{\lambda,\lambda-\eta\hat{1}}(u_1)T_{\bar{1}}(u_1)\phi^{\bar{1}}_{\lambda+\eta N\hat{2},\lambda+\eta(N-1)\hat{2}}(u_1)$$

$$\times\cdots\tilde{\phi}^{\bar{N}}_{\lambda-\eta(N-1)\hat{1},\lambda-\eta N\hat{1}}(u_N)T_{\bar{N}}(u_N)\phi^{\bar{N}}_{\lambda+\eta\hat{2},\lambda}(u_N)\phi^{1}_{\lambda,\lambda-\eta\hat{1}}(\xi_1)\cdots\phi^{N}_{\lambda-\eta(N-1)\hat{1},\lambda-\eta N\hat{1}}(\xi_N).$$

The dependence of $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ on the argument u_N only comes from the last term corresponding to the second line of the above equation. Let us denote the term by $A(u_N)$, namely,

$$\begin{aligned}
A(u_N) &= \tilde{\phi}^{\bar{N}}_{\lambda-\eta(N-1)\hat{1},\lambda-\eta N\hat{1}}(u_N)T_{\bar{N}}(u_N)\phi^{\bar{N}}_{\lambda+\eta\hat{2},\lambda}(u_N)\phi^{1}_{\lambda,\lambda-\eta\hat{1}}(\xi_1)\cdots\phi^{N}_{\lambda-\eta(N-1)\hat{1},\lambda-\eta N\hat{1}}(\xi_N)\\
&= \tilde{\phi}^{\bar{N}}_{\lambda-\eta(N-1)\hat{1},\lambda-\eta N\hat{1}}(u_N)R_{\bar{N},N}(u_N-\xi_N)\phi^{N}_{\lambda-\eta(N-1)\hat{1},\lambda-\eta N\hat{1}}(\xi_N)\cdots\\
&\overset{(3.10)}{=} \phi^{N}_{\lambda-\eta(N-2)\hat{1},\lambda-\eta(N-1)\hat{1}}(\xi_N)\tilde{\phi}^{\bar{N}}_{\lambda-\eta(N-2)\hat{1},\lambda-\eta(N-1)\hat{1}}(u_N)R_{\bar{N},N-1}(u_N-\xi_{N-1})\cdots+\bar{R}(u_N-\xi_N;\lambda\\
&\quad-\eta N\hat{1})^{12}_{21}\phi^{N}_{\lambda-\eta(N-2)\hat{1}+\eta\hat{2},\lambda-\eta(N-1)\hat{1}}(\xi_N)\tilde{\phi}^{\bar{N}}_{\lambda-\eta(N-2)\hat{1}+\eta\hat{2},\lambda-\eta(N-1)\hat{1}}(u_N)\cdots=\cdots\\
&= \phi^{N}_{\lambda-\eta(N-2)\hat{1},\lambda-\eta(N-1)\hat{1}}(\xi_N)\tilde{\phi}^{\bar{N}}_{\lambda-\eta(N-2)\hat{1},\lambda-\eta(N-1)\hat{1}}(u_N)R_{\bar{N},N-1}(u_N-\xi_{N-1})\cdots+\bar{R}(u_N-\xi_N;\lambda\\
&\quad-\eta N\hat{1})^{12}_{21}\prod_{l=1}^{N-1}\bar{R}(u_N-\xi_l;\lambda-\eta l\hat{1})^{21}_{21}\phi^{1}_{\lambda+\eta\hat{2},\lambda+\eta\hat{2}-\eta\hat{1}}(\xi_1)\cdots.
\end{aligned}$$

It can be shown by induction that $A(u_N)$ satisfies the following quasiperiodicity:

$$A(u_N+1)=A(u_N),\quad A(u_N+\tau)=e^{-2i\pi(\lambda_{21})}A(u_N).$$

Since $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ is a symmetric function of $\{u_\alpha\}$, we conclude that $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ has the following quasiperiodic properties:

$$Z_N(u_1,\ldots,u_l+1,u_{l+1},\ldots;\{\xi_i\};\lambda)=Z_N(\{u_\alpha\};\{\xi_i\};\lambda),\quad l=1,\ldots,N, \tag{4.4}$$

$$Z_N(u_1,\ldots,u_l+\tau,u_{l+1},\ldots;\{\xi_i\};\lambda)=e^{-2i\pi(\lambda_{21})}Z_N(\{u_\alpha\};\{\xi_i\};\lambda),\ \ l=1,\ldots,N. \tag{4.5}$$

Using the expressions (3.4) and (3.5) of the matrix elements of the R-matrix $\bar{R}$ and the vertex-face correspondences (3.7) and (3.10), we find the analytic property of the partition function:

$Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ is an analytic function of u_l with simple poles $\{\xi_i-\eta|i=1,\ldots,N\}$ inside the fundamental (upright) rectangle (Ref. 15) generated by 1 and τ. (4.6)

Direct calculation (for details see Appendix B) shows that at each simple pole $\xi_i-\eta$ the corresponding residue is

$$\mathrm{Res}_{u_l=\xi_i-\eta}(Z_N(\{u_\alpha\};\{\xi_j\};\lambda))=\frac{\sigma(\eta)\sigma(\lambda_{21})}{\sigma'(0)\sigma(\lambda_{21}+\eta)}\prod_{\alpha\neq l}\frac{\sigma(u_\alpha-\xi_i)}{\sigma(u_\alpha-\xi_i+\eta)}\prod_{j\neq i}\frac{\sigma(\xi_j-\xi_i+\eta)}{\sigma(\xi_j-\xi_i)}$$
$$\times Z_{N-1}(\{u_\alpha\}_{\alpha\neq l};\{\xi_j\}_{j\neq i};\lambda+\eta\hat{2}),\quad l,i=1,\ldots,N. \tag{4.7}$$

Similarly, using the initial condition (2.9) of the R-matrix R and the vertex-face correspondences (3.7) and (3.10), one can also show that the partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ satisfies the following relations:

$$Z_N(\{u_\alpha\};\{\xi_j\};\lambda)|_{u_l=\xi_i} = Z_{N-1}(\{u_\alpha\}_{\alpha\neq l};\{\xi_j\}_{j\neq i};\lambda), \quad l,i=1,\ldots,N. \tag{4.8}$$

So remarks are in order. The properties (4.1), (4.2), and (4.4)–(4.7) uniquely determine the partition function. On the other hand, the properties (4.1), (4.2), (4.4)–(4.6), and (4.8) also fully fix the partition function. They yield the recursive relations (5.4) and (5.5) (see below), respectively.

V. THE PARTITION FUNCTION

In this section, we will derive two recursive relations from the properties of the partition function obtained in Sec. IV. Each of the recursive relations together with (4.1) uniquely determines the partition function.

A. The recursive relation

We now concentrate on the u_N-dependence of the partition function $Z_N(\{u_\alpha\};\{\xi_j\};\lambda)$. From (4.4) and (4.5), we have

$$Z_N(u_1,\ldots,u_{N-1},u_N+1;\{\xi_j\};\lambda) = Z_N(\{u_\alpha\};\{\xi_j\};\lambda),$$

$$Z_N(u_1,\ldots,u_{N-1},u_N+\tau;\{\xi_j\};\lambda) = e^{-2i\pi(\lambda_{21})}Z_N(\{u_\alpha\};\{\xi_j\};\lambda). \tag{5.1}$$

The analytic properties (4.6) and (4.7) imply that

$$Z_N(\{u_\alpha\};\{\xi_j\};\lambda) = \sum_{i=1}^{N}\left\{\frac{\sigma(\eta)\sigma(u_N-\xi_i+a_i)}{\sigma(u_N-\xi_i+\eta)\sigma(b_i)}\prod_{j\neq i}\frac{\sigma(\xi_j-\xi_i+\eta)}{\sigma(\xi_j-\xi_i)}\prod_{l\neq N}\frac{\sigma(u_l-\xi_i)}{\sigma(u_l-\xi_i+\eta)}\right.$$
$$\left.\times Z_{N-1}(\{u_\alpha\}_{\alpha\neq N};\{\xi_j\}_{j\neq i};\lambda+\eta\hat{2})\right\}+\Delta,$$

where $\{a_i\}$, $\{b_i\}$, and Δ are some constants with respect to u_N, and a_i and b_i satisfy the constraints

$$\frac{\sigma(a_i-\eta)}{\sigma(b_i)} = \frac{\sigma(\lambda_{21})}{\sigma(\lambda_{21}+\eta)}, \quad i=1,\ldots,N. \tag{5.2}$$

The quasiperiodic condition (5.1) leads to

$$a_i = \lambda_{21}+\eta, \quad i=1,\ldots,N,$$

$$\Delta = 0. \tag{5.3}$$

The constraint (5.2) then yields that $b_i=a_i=\lambda_{21}+\eta$. Thus the partition function $Z_N(\{u_\alpha\};\{\xi_j\};\lambda)$ satisfies the following recursive relation:

$$Z_N(\{u_\alpha\};\{\xi_j\};\lambda) = \sum_{i=1}^{N}\left\{\frac{\sigma(\eta)\sigma(u_N-\xi_i+\lambda_{21}+\eta)}{\sigma(u_N-\xi_i+\eta)\sigma(\lambda_{21}+\eta)}\prod_{j\neq i}\frac{\sigma(\xi_j-\xi_i+\eta)}{\sigma(\xi_j-\xi_i)}\prod_{l\neq N}\frac{\sigma(u_l-\xi_i)}{\sigma(u_l-\xi_i+\eta)}\right.$$
$$\left.\times Z_{N-1}(\{u_\alpha\}_{\alpha\neq N};\{\xi_j\}_{j\neq i};\lambda+\eta\hat{2})\right\}. \tag{5.4}$$

On the other hand, the quasiperiodicity (5.1) of the partition function, the fact that the partition function only has simple poles at $\{\xi_i-\eta\}$, and the relation (4.8) imply that the partition function has the following expansion:

$$Z_N(\{u_\alpha\};\{\xi_j\};\lambda)=\sum_{i=1}^{N}\left\{\frac{\sigma(\eta)\sigma(u_N-\xi_i+a_i')}{\sigma(u_N-\xi_i+\eta)\sigma(a_i')}\prod_{j\neq i}\frac{\sigma(u_N-\xi_j)\sigma(\xi_i-\xi_j+\eta)}{\sigma(u_N-\xi_j+\eta)\sigma(\xi_i-\xi_j)}Z_{N-1}(\{u_\alpha\}_{\alpha\neq N};\{\xi_j\}_{j\neq i};\lambda)\right\},$$

where $\{a_i\}$ are some constants with respect to u_N. The quasiperiodicity (5.1) further requires $a_i'=\lambda_{21}+N\eta$. Namely, the partition function $Z_N(\{u_\alpha\};\{\xi_j\};\lambda)$ satisfies the following recursive relation:

$$Z_N(\{u_\alpha\};\{\xi_j\};\lambda)=\sum_{i=1}^{N}\left\{\frac{\sigma(\eta)\sigma(u_N-\xi_i+\lambda_{21}+N\eta)}{\sigma(u_N-\xi_i+\eta)\sigma(\lambda_{21}+N\eta)}\prod_{j\neq i}\frac{\sigma(u_N-\xi_j)\sigma(\xi_i-\xi_j+\eta)}{\sigma(u_N-\xi_j+\eta)\sigma(\xi_i-\xi_j)}\right.$$
$$\left.\times Z_{N-1}(\{u_\alpha\}_{\alpha\neq N};\{\xi_j\}_{j\neq i};\lambda)\right\}. \tag{5.5}$$

In the trigonometric limit, the recursive relation (5.5) recovers that of Ref. 6 for the six-vertex model.

B. The DW partition function

The recursive relation (5.4) and the property (4.1) uniquely determine $Z_N(\{u_\alpha\};\{\xi_j\};\lambda)$; on the other hand the recursive relation (5.5) and the property (4.1) also fully fix the partition function. Using the Riemann identity (2.4) of the σ-function, one can check that the solution to each of recursive relations (5.4) and (5.5) gives rise to a symmetric function of $\{u_\alpha\}$ as required. As a consequence, the two expressions of the partition function obtained by solving the recursive relations (5.4) and (5.5), respectively, are equal since they are related to each other by some permutation of $\{u_\alpha\}$. Here, we present the result by resolving the recursive relation (5.5).

Using the property (4.1) and the recursive relation (5.5), we obtain the explicit expression of the partition function $Z_N(\{u_\alpha\};\{\xi_j\};\lambda)$,

$$Z_N(\{u_\alpha\};\{\xi_j\};\lambda)=\sum_{s\in S_N}\prod_{l=1}^{N}\left\{\frac{\sigma(\eta)\sigma(u_l-\xi_{s(l)}+\lambda_{21}+l\eta)}{\sigma(u_l-\xi_{s(l)}+\eta)\sigma(\lambda_{21}+l\eta)}\prod_{k=1}^{l-1}\frac{\sigma(u_l-\xi_{s(k)})\sigma(\xi_{s(l)}-\xi_{s(k)}+\eta)}{\sigma(u_l-\xi_{s(k)}+\eta)\sigma(\xi_{s(l)}-\xi_{s(k)})}\right\}, \tag{5.6}$$

where S_N is the permutation group of N indices. It is easy to see from the above explicit expression that $Z_N(\{u_\alpha\};\{\xi_j\};\lambda)$ is indeed a symmetric function of $\{\xi_i\}$.

VI. CONCLUSIONS AND DISCUSSIONS

We have introduced the DW boundary condition specified by the four boundary states (3.12)–(3.15) for the eight-vertex model on an $N\times N$ square lattice. The boundary states are the two-parameter generalization of the all-spin-up and all-spin-down states and their dual states. With the DW boundary condition, we have obtained the properties (4.1), (4.2), (4.4), (4.5), (4.7), and (4.8) of the partition function $Z_N(\{u_\alpha\};\{\xi_j\};\lambda)$. These properties enable us to derive the two recursive relations (5.4) and (5.5) of $Z_N(\{u_\alpha\};\{\xi_j\};\lambda)$, whose trigonometric limits recover those corresponding to the six-vertex model. The recursive relation (5.4) or (5.5) together with (4.1) uniquely determines the partition function $Z_N(\{u_\alpha\};\{\xi_j\};\lambda)$. Solving the recursive relations, we obtain the explicit expression (5.6) of the partition function.

Note added in proof

After submitting our paper to the arXiv, we became aware that the partition function for the elliptic SOS model with the DW boundary condition has been obtained in Refs. 21–23 by different methods. Our approach is based entirely on the algebraic Bethe ansatz framework and can be used to obtain partition functions of the eight-vertex model with open boundary conditions.[24] Moreover, we have obtained the recursive relations (5.4) and (5.5) of $Z_N(\{u_\alpha\};\{\xi_j\};\lambda)$. We would like to thank V. Mangazeev for kindly drawing our attention to these references.

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APPENDIX A: THE PROOF OF (4.2)

In this appendix, we show that $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ is a symmetric function of $\{u_\alpha\}$ and $\{\xi_i\}$ separatively. Regarding the tensor space indexed by $\bar{1},\ldots,\bar{N}$ as the auxiliary space, (2.12) can be rewritten as

$$Z_N(\{u_\alpha\};\{\xi_i\};\lambda) = \langle\Omega^{(2)}(\lambda+\eta N\hat{2})|\tilde{\phi}^{\bar{1}}_{\lambda,\lambda-\eta\hat{1}}(u_1)T_{\bar{1}}(u_1)\phi^{\bar{1}}_{\lambda+\eta N\hat{2},\lambda+\eta(N-1)\hat{2}}(u_1) \times\cdots\tilde{\phi}^{\bar{N}}_{\lambda-\eta(N-1)\hat{1},\lambda-\eta N\hat{1}}(u_N)T_{\bar{N}}(u_N)\phi^{\bar{N}}_{\lambda+\eta\hat{2},\lambda}(u_N)|\Omega^{(1)}(\lambda)\rangle. \tag{A1}$$

Following Refs. 25 and 26, let us introduce the face-type monodromy matrix with elements given by

$$T(m;m_0|u)^j_\mu = \tilde{\phi}_{m+\eta\hat{j},m}(u)T(u)\phi_{m_0+\eta\hat{\mu},m_0}(u), \quad j,\mu=1,2. \tag{A2}$$

Then the partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ can be expressed in terms of the product of the matrix elements of the face-type monodromy matrix

$$Z_N(\{u_\alpha\};\{\xi_i\};\lambda) = \langle\Omega^{(2)}(\lambda+\eta N\hat{2})|T(\lambda-\eta\hat{1};\lambda+\eta(N-1)\hat{2}|u_1)^1_2 T(\lambda-2\eta\hat{1};\lambda+\eta(N-2)\hat{2}|u_2)^1_2\cdots T(\lambda-\eta N\hat{1};\lambda|u_N)^1_2|\Omega^{(1)}(\lambda)\rangle. \tag{A3}$$

The exchange relation (2.14) and the vertex-face correspondence relations (3.7) and (3.11) enable us to derive the following exchange relations for the operators (A2):

$$\sum_{i,j=1}^{2}\bar{R}(u_1-u_2;m)^{kl}_{ij}T(m+\eta\hat{\jmath};m_0+\eta\hat{\nu}|u_1)^i_\mu T(m;m_0|u_2)^j_\nu = \sum_{\alpha,\beta=1}^{2}\bar{R}(u_1-u_2;m_0)^{\alpha\beta}_{\mu\nu}T(m+\eta\hat{k};m_0+\eta\hat{\alpha}|u_2)^l_\beta T(m;m_0|u_1)^k_\alpha.$$

For the case of $k=l=1$ and $\mu=\nu=2$, the above relations become

$$T(m+\eta\hat{1};m_0+\eta\hat{2}|u_1)^1_2 T(m;m_0|u_2)^1_2 = T(m+\eta\hat{1};m_0+\eta\hat{2}|u_2)^1_2 T(m;m_0|u_1)^1_2. \tag{A4}$$

This relation with $T(m;m_0|u)^1_2$ given by (A2) implies that the partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ (A3) is a symmetric function of $\{u_\alpha\}$. Similarly, one can check that the partition function is also a symmetric function of $\{\xi_i\}$.

APPENDIX B: THE PROOFS OF (4.7) and (4.8)

Firstly, let us prove the analytic property (4.7) of the partition function. Since the partition function is a symmetric function of $\{u_\alpha\}$ and $\{\xi_i\}$, it is sufficient to prove (4.7) for the case of $u_N=\xi_N-\eta$. For this purpose, we need to rearrange the order of the product of R-matrices in the expression (2.11) of the partition function as follows:

$$\begin{aligned}Z_N(\{u_\alpha\};\{\xi_i\};\lambda) &= \langle\Omega^{(2)}(\lambda+\eta N\hat{2})|\langle\bar{\Omega}^{(1)}(\lambda)|R_{\bar{1},N}(u_1-\xi_N)R_{\bar{2},N}(u_2-\xi_N)\cdots R_{\bar{N},N}(u_N-\xi_N)\cdots R_{\bar{1},1}(u_1-\xi_1)\cdots R_{\bar{N},1}(u_N-\xi_1)|\bar{\Omega}^{(2)}(\lambda+\eta N\hat{2})\rangle|\Omega^{(1)}(\lambda)\rangle = \langle\Omega^{(2)}(\lambda\\&\quad+\eta N\hat{2})|\tilde{\phi}^{\bar{1}}_{\lambda,\lambda-\eta\hat{1}}(u_1)\cdots\tilde{\phi}^{\overline{N-1}}_{\lambda-\eta(N-2)\hat{1},\lambda-\eta(N-1)\hat{1}}(u_{N-1})R_{\bar{1},N}(u_1-\xi_N)\cdots R_{\overline{N-1},N}(u_{N-1}\\&\quad-\xi_N)\tilde{\phi}^{\bar{N}}_{\lambda-\eta(N-1)\hat{1},\lambda-\eta N\hat{1}}(u_N)R_{\bar{N},N}(u_N-\xi_N)\phi^{N}_{\lambda-\eta(N-1)\hat{1},\lambda-\eta N\hat{1}}(\xi_N)R_{\bar{1},N-1}(u_1\end{aligned}$$

$$- \xi_{N-1}) \cdots R_{\bar{N},N-1}(u_N - \xi_{N-1}) \cdots R_{\bar{N},1}(u_N - \xi_1)\phi^1_{\lambda,\lambda-\eta\hat{1}}(\xi_1) \cdots \phi^{N-1}_{\lambda-\eta(N-2)\hat{1},\lambda-\eta(N-1)\hat{1}}(\xi_{N-1})|\bar{\Omega}^{(2)}(\lambda + \eta N\hat{2})\rangle. \tag{B1}$$

The expressions (3.4) and (3.5) of the matrix elements of $\bar{R}$ imply that

$$\mathrm{Res}_{u=-\eta}R(u;m)^{11}_{11} = 0, \quad \mathrm{Res}_{u=-\eta}R(u;m)^{12}_{21} = \frac{\sigma(\eta)\sigma(m_{21}-\eta)}{\sigma'(0)\sigma(m_{21})}. \tag{B2}$$

Keeping the above equations in mind and using the vertex-face correspondence relation (3.10), we find

$$\begin{aligned}
&\mathrm{Res}_{u_N=\xi_N-\eta}Z_N(\{u_\alpha\};\{\xi_i\};\lambda) \\
&= \frac{\sigma(\eta)\sigma(\lambda_{21} + (N-1)\eta)}{\sigma'(0)\sigma(\lambda_{21} + N\eta)}\langle\Omega^{(2)}(\lambda + \eta N\hat{2})| \\
&\times\tilde{\phi}^{\bar{1}}_{\lambda,\lambda-\eta\hat{1}}(u_1) \cdots \tilde{\phi}^{\overline{N-2}}_{\lambda-\eta(N-3)\hat{1},\lambda-\eta(N-2)\hat{1}}(u_{N-2})\tilde{\phi}^{\bar{N}}_{\lambda+\eta\hat{2}-\eta(N-1)\hat{1},\lambda-\eta(N-1)\hat{1}}(u_N) \\
&\times R_{\bar{1},N}(u_1 - \xi_N) \cdots R_{\overline{N-2},N}(u_{N-2} - \xi_N) \\
&\times\tilde{\phi}^{\overline{N-1}}_{\lambda-\eta(N-2)\hat{1},\lambda-\eta(N-1)\hat{1}}(u_{N-1})R_{\overline{N-1},N} \\
&\times(u_{N-1} - \xi_N)\phi^N_{\lambda+\eta\hat{2}-\eta(N-1)\hat{1},\lambda-\eta(N-1)\hat{1}}(\xi_N) \\
&\times R_{\bar{1},N-1}(u_1 - \xi_{N-1}) \cdots R_{\bar{N},N-1}(u_N - \xi_{N-1}) \cdots R_{\bar{N},1}(u_N - \xi_1) \\
&\times\phi^1_{\lambda,\lambda-\eta\hat{1}}(\xi_1) \cdots \phi^{N-1}_{\lambda-\eta(N-2)\hat{1},\lambda-\eta(N-1)\hat{1}}(\xi_{N-1})|\bar{\Omega}^{(2)}(\lambda + \eta N\hat{2})\rangle \\
&= \\
&\vdots \\
&= \frac{\sigma(\eta)\sigma(\lambda_{21} + (N-1)\eta)}{\sigma'(0)\sigma(\lambda_{21} + N\eta)}\prod_{l=1}^{N-1}\bar{R}(u_l - \xi_N;\lambda - \eta l\hat{1})^{12}_{12} \\
&\times\langle\Omega^{(2)}(\lambda + \eta N\hat{2})|\phi^N_{\lambda+\eta\hat{2},\lambda}(\xi_N) \\
&\times\tilde{\phi}^{\bar{1}}_{\lambda+\eta\hat{2},\lambda+\eta\hat{2}-\eta\hat{1}}(u_1) \cdots \tilde{\phi}^{\overline{N-1}}_{\lambda+\eta\hat{2}-\eta(N-2)\hat{1},\lambda+\eta\hat{2}-\eta(N-1)\hat{1}}(u_{N-1}) \\
&\times R_{\bar{1},N-1}(u_1 - \xi_{N-1}) \cdots R_{\overline{N-1},N-1}(u_{N-1} - \xi_{N-1}) \\
&\times\tilde{\phi}^{\bar{N}}_{\lambda+\eta\hat{2}-\eta(N-1)\hat{1},\lambda-\eta(N-1)\hat{1}}(u_N)R_{\bar{N},N-1} \\
&\times(u_N - \xi_{N-1})\phi^{N-1}_{\lambda-\eta(N-2)\hat{1},\lambda-\eta(N-1)\hat{1}}(\xi_{N-1}) \\
&\times R_{\bar{1},N-2}(u_1 - \xi_{N-2}) \cdots R_{\bar{N},N-2}(u_N - \xi_{N-2}) \cdots R_{\bar{N},1}(u_N - \xi_1) \\
&\times\phi^1_{\lambda,\lambda-\eta\hat{1}}(\xi_1) \cdots \phi^{N-2}_{\lambda-\eta(N-3)\hat{1},\lambda-\eta(N-2)\hat{1}}(\xi_{N-2})|\bar{\Omega}^{(2)}(\lambda + \eta N\hat{2})\rangle \\
&= \\
&\vdots \\
&= \frac{\sigma(\eta)\sigma(\lambda_{21} + (N-1)\eta)}{\sigma'(0)\sigma(\lambda_{21} + N\eta)}\prod_{l=1}^{N-1}\bar{R}(u_l - \xi_N;\lambda - \eta l\hat{1})^{12}_{12}\bar{R}(u_N - \xi_l;\lambda - \eta l\hat{1})^{21}_{21}
\end{aligned}$$

$$\langle\Omega^{(2)}(\lambda+\eta N\hat{2})|\phi^{N}_{\lambda+\eta\hat{2},\lambda}(\xi_N)$$
$$\tilde{\phi}^{\bar{1}}_{\lambda+\eta\hat{2},\lambda+\eta\hat{2}-\eta\hat{1}}(u_1)$$
$$\times\cdots\tilde{\phi}^{\overline{N-1}}_{\lambda+\eta\hat{2}-\eta(N-2)\hat{1},\lambda+\eta\hat{2}-\eta(N-1)\hat{1}}(u_{N-1})$$
$$R_{\bar{1},N-1}(u_1-\xi_{N-1})\cdots R_{\overline{N-1},N-1}(u_{N-1}-\xi_{N-1})\cdots R_{\bar{1},1}(u_1-\xi_1)\cdots R_{\overline{N-1},1}(u_{N-1}-\xi_1)$$
$$\phi^{1}_{\lambda+\eta\hat{2},\lambda+\eta\hat{2}-\eta\hat{1}}(\xi_1)\cdots\phi^{N-1}_{\lambda+\eta\hat{2}-\eta(N-2)\hat{1},\lambda+\eta\hat{2}-\eta(N-1)\hat{1}}(\xi_{N-1})$$
$$\tilde{\phi}^{\bar{N}}_{\lambda+\eta\hat{2},\lambda}(u_N)|\bar{\Omega}^{(2)}(\lambda+\eta N\hat{2})\rangle. \tag{B3}$$

It is understood that $u_N=\xi_N-\eta$ in the above equations. With help of the definitions (3.13) and (3.14) of the boundary states and the condition (3.8), we finally obtain the residue of $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ at the simple pole $\xi_N-\eta$:

$$\mathrm{Res}_{u_N=\xi_N-\eta}Z_N(\{u_\alpha\};\{\xi_i\};\lambda)=\frac{\sigma(\eta)\sigma(\lambda_{21})}{\sigma'(0)\sigma(\lambda_{21}+\eta)}\prod_{l=1}^{N-1}\frac{\sigma(u_l-\xi_N)}{\sigma(u_l-\xi_N+\eta)}\prod_{j=1}^{N-1}\frac{\sigma(\xi_j-\xi_N+\eta)}{\sigma(\xi_j-\xi_N)}$$
$$\times\tilde{\phi}^{1}_{\lambda+\eta\hat{2}+\eta(N-1)\hat{2},\lambda+\eta\hat{2}+\eta(N-2)\hat{2}}(\xi_1)\cdots\tilde{\phi}^{N-1}_{\lambda+\eta\hat{2}+\eta\hat{2},\lambda+\eta\hat{2}}(\xi_{N-1})$$
$$\times\tilde{\phi}^{\bar{1}}_{\lambda+\eta\hat{2},\lambda+\eta\hat{2}-\eta\hat{1}}(u_1)\cdots\tilde{\phi}^{\overline{N-1}}_{\lambda+\eta\hat{2}-\eta(N-2)\hat{1},\lambda+\eta\hat{2}-\eta(N-1)\hat{1}}(u_{N-1})$$
$$\times R_{\bar{1},N-1}(u_1-\xi_{N-1})\cdots R_{\overline{N-1},N-1}(u_{N-1}-\xi_{N-1})\cdots$$
$$\times R_{\bar{1},1}(u_1-\xi_1)\cdots R_{\overline{N-1},1}(u_{N-1}-\xi_1)$$
$$\times\phi^{\bar{1}}_{\lambda+\eta\hat{2}+\eta(N-1)\hat{2},\lambda+\eta\hat{2}+\eta(N-2)\hat{2}}(u_1)\cdots\phi^{\overline{N-1}}_{\lambda+\eta\hat{2}+\eta\hat{2},\lambda+\eta\hat{2}}(u_{N-1})$$
$$\times\phi^{1}_{\lambda+\eta\hat{2},\lambda+\eta\hat{2}-\eta\hat{1}}(\xi_1)\cdots\phi^{N-1}_{\lambda+\eta\hat{2}-\eta(N-2)\hat{1},\lambda+\eta\hat{2}-\eta(N-1)\hat{1}}(\xi_{N-1})$$
$$=\frac{\sigma(\eta)\sigma(\lambda_{21})}{\sigma'(0)\sigma(\lambda_{21}+\eta)}\prod_{l=1}^{N-1}\frac{\sigma(u_l-\xi_N)}{\sigma(u_l-\xi_N+\eta)}\prod_{j=1}^{N-1}\frac{\sigma(\xi_j-\xi_N+\eta)}{\sigma(\xi_j-\xi_N)}$$
$$\times Z_{N-1}(\{u_\alpha\}_{\alpha\neq N};\{\xi_i\}_{i\neq N};\lambda+\eta\hat{2}). \tag{B4}$$

Therefore, we have completed the proof of (4.7).

Noting $\bar{R}(u;m)^{ii}_{ii}=1$ for any values of u, m, and $i=1,2$ and the initial condition (2.9) of R, by a similar procedure as above, one can show that

$$Z_N(\{u_\alpha\};\{\xi_i\};\lambda)|_{u_N=\xi_1}=Z_{N-1}(\{u_\alpha\}_{\alpha\neq N};\{\xi_i\}_{i\neq 1};\lambda). \tag{B5}$$

(In the proof of (B5), it is convenient to keep the same order of the product of the R-matrices as that of (2.11) [cf. (B1)] in the calculation.) The fact that $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ is a symmetric function of $\{u_\alpha\}$ and $\{\xi_i\}$ then leads to (4.8).

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