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Partition functions of 3d $\hat{D}$-quivers and their mirror duals from 1d free fermions

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Abstract: We study the matrix models calculating the sphere partition functions of 3d gauge theories with $\mathcal{N} = 4$ supersymmetry and a quiver structure of a $\hat{D}$ Dynkin diagram (where each node is a unitary gauge group). As in the case of necklace ($\hat{A}$) quivers, we can map the problem to that of free fermion quantum mechanics whose complicated Hamiltonian we find explicitly. Many of these theories are conjectured to be dual under mirror symmetry to certain unitary linear quivers with extra Sp nodes or antisymmetric hypermultiplets. We show that the free fermion formulation of such mirror pairs are related by a linear symplectic transformation.

We then study the large N expansion of the partition function, which as in the case of the $\hat{A}$ quivers is given to all orders in $1/N$ by an Airy function. We simplify the algorithm to calculate the numerical coefficients appearing in the Airy function and evaluate them for a wide class of $\hat{D}$-quiver theories.

Keywords: Matrix Models, Supersymmetry and Duality, Field Theories in Lower Dimensions, Duality in Gauge Field Theories

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Contents

1 Introduction

Three-dimensional gauge theories with $\mathcal{N} = 4$ supersymmetry have an $\mathrm{SU}(2)_C \times \mathrm{SU}(2)_H$ R-symmetry group, with $\mathrm{SU}(2)_C$ acting on the Coulomb branch moduli and $\mathrm{SU}(2)_H$ acting on the Higgs branch moduli. A large class of three-dimensional $\mathcal{N} = 4$ gauge theories flow to interacting fixed points in the infrared limit. These theories possess a remarkable duality known as mirror symmetry [1], which is the statement that pairs of UV gauge theories flow to the same infrared fixed point with the action of $\mathrm{SU}(2)_C$ and $\mathrm{SU}(2)_H$ exchanged. In particular the classical (protected) Higgs branch of one theory matches the quantum-corrected

Coulomb branch of the other theory [2, 3]. The exact results obtained from the technique of supersymmetric localization provide a privileged testing ground for mirror symmetry. In particular the full partition function of 3d $\mathcal{N} = 4$ gauge theories defined on a three-sphere reduces to a relatively simple matrix model [4]. This matrix model is independent of running coupling constants and therefore computes the three-sphere partition function of the infrared fixed point. Mirror symmetry was successfully tested by matching the matrix models of pairs of mirror-dual theories in [5–7]. More tests of mirror symmetry by matching the Coulomb branch and Higgs branch Hilbert series of dual theories were achieved in [8–11].

The most studied $\mathcal{N} = 4$ gauge theories subject to mirror symmetry are infrared fixed points of quiver theories of type $\hat{A}, \hat{D}$ or $\hat{E}$, referring to the shape of the quiver as an extended Dynkin diagram and likewise to the orbifold singularity in their M-theory realization [12, 13]. It was found in [14] that the mapping between matrix models of mirror-dual theories of type $\hat{A}$ can be expressed as a very simple canonical transformation, exchanging position and momentum, in the 1d free fermion formalism developed in [15].

The free fermion formalism arises from the observation that the matrix model computing the three-sphere partition function of $\mathcal{N} \geqslant 3$ Chern-Simons-Matter quiver theories can be re-expressed as the partition function of a gas of non-interacting fermions in one dimension with a non-trivial Hamiltonian. This formalism allowed the use of powerful techniques from quantum and statistical mechanics to solve the matrix models as an $\hbar$ expansion, $\hbar$ being related to the Chern-Simons levels of the 3d theory. It was found in [15] that the perturbative part of the partition function at large N takes the form of an Airy function.

This Airy function behavior was first found for the particular case of ABJM theory [16], in [17], based on its original large N perturbative solution [18]. Non perturbative corrections have been studied intensively for ABJM [19–27], ABJ [28, 29], and some other examples of $\hat{A}$-quiver theories [30, 31]. Recently, the grand partition function of ABJ(M) was determined exactly [32, 33], in perfect agreement with numerical results [34].

In this paper we find the free fermion formalism associated to infrared fixed points of $\mathcal{N} = 4$ Yang-Mills quiver theories of type $\hat{D}$ and similar Chern-Simons-matter theories.[1] A $\hat{D}_{L+2}$-quiver is described by a linear chain of $(L-1)$ $\mathrm{U}(2N)$ gauge nodes, connected by bifundamental hypermultiplets, with pairs of $\mathrm{U}(N)$ nodes attached at both ends of the $\mathrm{U}(2N)$ chain, as shown in figure 1. In addition each node may couple to an arbitrary number of fundamental hypermultiplets, indicated by the boxes in the diagram. Contrarily to type $\hat{A}$ theories, mirror symmetry does not relate pairs of type $\hat{D}$ theories. Instead, $\hat{D}$-quivers are mapped under mirror symmetry to linear quivers with $\mathrm{U}(2N)$ gauge nodes and extra symplectic gauge nodes $\mathrm{Sp}(2N)$ or antisymmetric hypermultiplets at both ends of the quiver chain [13, 35–37], that we call generically *linear quivers*.[2]

Roughly speaking, the length of the linear quiver mirror depends on the number of fundamental hypermultiplets coupling to the $\mathrm{U}(2N)$ nodes and the structure of the end of the linear quiver depends on the number of fundamental hypermultiplets on the $\mathrm{U}(N)$

[1] Chern-Simons terms generically lead to only $\mathcal{N} = 3$ supersymmetry, and we mostly set them to zero after section 2.

[2] There are many other types of linear quivers, which we do not study in this paper. We use this name to refer to the quivers as in figure 2 and hope that this will not cause confusion.

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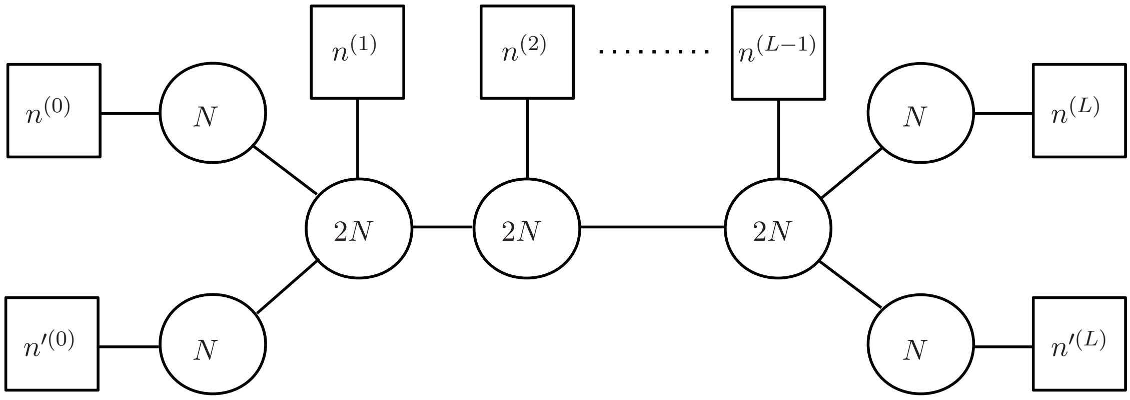

Figure 1: The general $\hat{D}_{L+2}$-quiver with arbitrary fundamental matter.

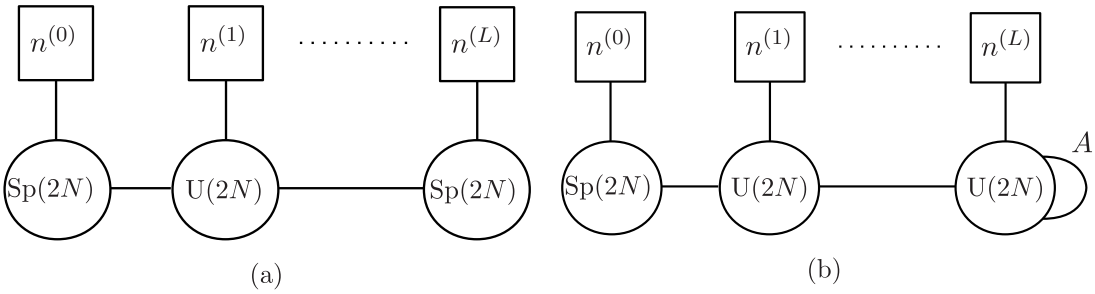

Figure 2: Examples of linear quivers that are mirror dual to $\hat{D}$-quivers.

nodes. When the number of fundamental hypermultiplets on the two $\mathrm{U}(N)$ nodes on one side of the $\hat{D}$-quiver are equal, the mirror theory has a terminating $\mathrm{Sp}(2N)$ node. When the numbers of fundamental hypermultiplets differ by one, the mirror theory has a terminating antisymmetric hypermultiplet. For instance if the numbers of $\mathrm{U}(N)$ fundamental hypermultiplets are given by $n^{(0)} = n'^{(0)} = 0$ and $n^{(L)} = n'^{(L)} = 1$ (in the notations of figure 1), then the mirror theory has $\mathrm{Sp}(2N)$ nodes at each end as in figure 2a, while $n^{(0)} = n'^{(0)} = n^{(L)} = 0$ and $n'^{(L)} = 1$ leads to a mirror linear quiver with a terminating $\mathrm{Sp}(2N)$ node at one end and an antisymmetric hypermultiplet at the other end as in figure 2b.[3]

When the numbers of fundamental hypermultipets on the two $\mathrm{U}(N)$ nodes differ by two or more, we do not know what is the precise mirror dual theory. Naive considerations on the Hanany-Witten IIB brane setup realizing these quivers [36, 38], lead to 'bad' mirror dual linear quivers [11]. 'Bad' quivers are quivers whose gauge group cannot be completely higgsed and whose three-sphere partition function is divergent. The formal manipulations leading to a Fermi gas description presented in section 2 can be done for any quiver, however when we study mirror symmetry, we focus only on duality between 'good' quivers.

Our starting point is the matrix model computing the three-sphere partition function of the quiver theories. This matrix model is expressed as an integral over the Cartan subalge-

[3]Note that the parameters $n^{(a)}$ in figures 1 and 2a, 2b do not get mapped to each other under mirror symmetry. The actual mirror map is more involved, see section 3.

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bra of the gauge group and the integrand is a product of classical contributions and one-loop contributions in the (exact) saddle point analysis of supersymmetric localization [4]

$$Z = \frac{1}{|\mathcal{W}|}\int_{\text{Cartan}} d\lambda\, Z_{\text{class}} \cdot Z_{\text{vec}} \cdot Z_{\text{hyp}}\,, \tag{1.1}$$

where $|\mathcal{W}|$ is the order of the Weyl group. The classical contribution depends on Fayet-Iliopoulos (FI) and Chern-Simons (CS) parameters. These are given by

$$Z^{\text{FI}}_{\text{class}} = e^{2\pi i\,\zeta\,\text{Tr}\,\lambda}\,, \qquad Z^{\text{CS}}_{\text{class}} = e^{\pi i\,k\,\text{Tr}\,\lambda^2}\,, \tag{1.2}$$

with ζ the FI parameter, k the Chern-Simons level and Tr the trace in the fundamental representation. The one-loop contributions of the $\mathcal{N}=4$ vector multiplet and hypermultiplet in a representation $\mathcal{R}$ are

$$Z_{\text{vec}} = \prod_{\alpha>0} 4\sinh^2(\pi\,\alpha\cdot\lambda)\,, \qquad Z_{\text{hyp}} = \prod_{w\in\mathcal{R}} \frac{1}{2\cosh(\pi(w\cdot\lambda+m))}\,, \tag{1.3}$$

where α runs over the positive roots of the Lie algebra and w over the weights of $\mathcal{R}$. m is a real mass parameter for the hypermultiplet. We provide in appendix A the explicit matrix factors relevant to the $\hat{D}$-quivers and their mirror linear quivers.

By manipulating the matrix models of $\hat{D}$-quivers we are able to re-express it in the form

$$Z_{\hat{D}} = \frac{1}{N!}\sum_{\sigma\in S_N}\frac{(-1)^\sigma}{2^{n_\sigma}}\int\prod_{i=1}^N d\lambda_i \prod_{i=1}^N \rho(\lambda_i,\lambda_{\sigma(i)})\,, \tag{1.4}$$

where ρ depends on the content of the theory and n_σ is the number of cycles in the permutation σ. This expression differs from the analogue expression for $\hat{A}$ quivers by the presence of the factor $1/2^{n_\sigma}$. Defining the density operator $\hat{\rho}$ by $\langle\lambda|\,\hat{\rho}\,|\lambda'\rangle = \rho(\lambda,\lambda')$, we are able to recast the partition function of theories with vanishing mass and FI parameters into

$$Z_{\hat{D}} = \frac{1}{N!}\sum_{\sigma\in S_N}(-1)^\sigma\int\prod_{i=1}^N d\lambda_i \prod_{i=1}^N \langle\lambda_i|\,\hat{\rho}\left(\frac{1+\hat{R}}{2}\right)|\lambda_{\sigma(i)}\rangle\,, \tag{1.5}$$

where $\hat{R}$ is the reflection operator $\hat{R}\,|\lambda\rangle = |-\lambda\rangle$. This can be interpreted as the partition function of N non-interacting fermions on a *half line* with Neumann boundary condition at the origin, with a Hamiltonian $\hat{H} = -\log\hat{\rho}$. We also find that $Z_{\hat{D}}$ can alternatively be recast into the partition function of N non-interacting fermions on a half line with Dirichlet boundary condition at the origin, which translates into having the projection $\frac{1-\hat{R}}{2}$ instead of $\frac{1+\hat{R}}{2}$ in (1.5). This freedom exists because, as we show in appendix C, the density operators associated with $\hat{D}$ quivers have the remarkable property that their spectrum is pairwise degenerate between even and odd states on the line.

Similarly we find that the matrix models associated to linear quivers can be set in the same form (1.4) or (1.5), with the same property that the density operator has pairwise degenerate spectrum. The derivation of the density operators associated to the $\hat{D}$-quivers and their mirror-dual linear quivers are presented in section 2.

Having found the density operators for $\hat{D}$-quivers and linear quivers, we observe in section 3 that density operators of mirror theories are the mapped by the transformation on position and momentum operators

$$p \to q, \qquad q \to -p\,, \tag{1.6}$$

as was first found in [14] for type $\hat{A}$ quivers.[4]

With a free fermion formalism, we are in a position to evaluate the *perturbative part* of the large N expansion of the S^3 partition function of $\hat{D}$-quivers, as was done for $\hat{A}$-quiver theories in [39]. We present the analysis in section 4 and derive explicit expressions for theories with vanishing masses and FI parameters. This is conveniently described from the grand potential $J(\mu)$, which is the logarithm of the grand canonical partition function

$$\Xi(z) = 1 + \sum_{N=1}^{\infty} Z(N) z^N = e^{J(\mu)}, \qquad z = e^{\mu}\,. \tag{1.7}$$

Using *phase space* techniques, the grand potential evaluates to

$$J(\mu) = \frac{C}{3}\mu^3 + B\mu + A + \mathcal{O}(e^{-\alpha\mu})\,, \qquad \alpha > 0\,, \tag{1.8}$$

which after inverting the relation (1.7) leads to

$$Z(N) = C^{-\frac{1}{3}} e^A \,\mathrm{Ai}\left[C^{-\frac{1}{3}}(N-B)\right] + Z_{\mathrm{np}}(N)\,, \tag{1.9}$$

where Ai is the Airy function and $Z_{\mathrm{np}}(N)$ denotes non-perturbative, exponentially suppressed contributions at large N. For quiver theories without CS terms and without mass or FI deformations, we are able to find the exact values of the coefficients C and B, but not A, nor the non-perturbative corrections. The method we develop is simplified compared to earlier literature.

The fact that the result for $\hat{D}$-quivers is similar to that of $\hat{A}$-quivers is consistent with the localization calculation of supergravity in AdS_4 [40]. That suggests a universal answer for all conformal 3d theories with enough supersymmetry and an M-theory dual, which includes both the $\hat{A}$ and $\hat{D}$-quivers.

The free fermion formalism for 3d $\mathcal{N}=4$ theories with no U$(2N)$ and a single Sp$(2N)$ or SO$(2N)$ node was studied previously in [41]. We compare our formalism to theirs in appendix D.[5]

We conclude with a discussion of some open questions and possible extensions of our work in section 5. We also include a simple holographic test of our results.

During the course of the work we learned that similar questions were studied by Moriyama and Nosaka, whose results are published concurrently [42].

[4]Mirror symmetry for type $\hat{A}$-quivers extends to an SL$(2,\mathbb{Z})$ group of dualities [7, 36]. We are not aware of a similar extension for $\hat{D}$-quivers and consequently our discussion concerns only the mirror transformation corresponding to the so-called S-duality.

[5]We point out a mistake in their next-to-leading contribution to the grand potential.

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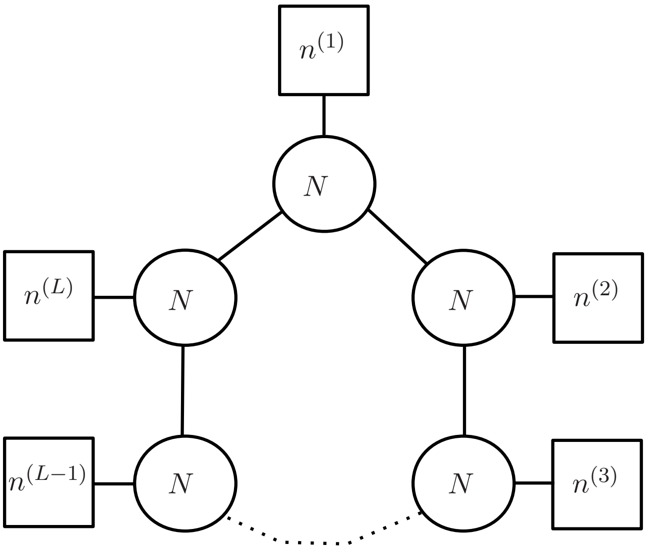

Figure 3: An $\hat{A}_{L-1}$-quiver. $n^{(a)}$ denotes the number of fundamental hypermultiplets at the corresponding U(N) node.

Notations. We consider quivers with nodes of rank N or $2N$ and we use the indices i, j when the label runs over $1, \cdots, N$, and I, J when the label runs over $1, \cdots, 2N$. Moreover we define

$$\mathrm{sh}\, x \equiv 2 \sinh \pi x \,, \qquad \mathrm{ch}\, x \equiv 2 \cosh \pi x \,, \qquad \mathrm{th}\, x \equiv \frac{\mathrm{sh}\, x}{\mathrm{ch}\, x} = \tanh \pi x \,. \tag{1.10}$$

2 $\hat{D}$-quivers and linear quivers as free 1d fermions

In this section we show how to re-express the S^3 partition function of $\hat{D}$-quivers and linear quivers as the partition function of a gas of free fermions on a half line.

2.1 Free fermions formalism for $\hat{A}$ quivers

We start by reviewing how the free fermion formalism arises for $\mathcal{N} = 3$ $\hat{A}$ quiver theories with U(N) gauge groups [15]. The $\hat{A}_{L-1}$-quiver diagram is presented in figure 3. The matrix model for these theories is given generically by

$$Z(N) = \frac{1}{N!^L} \int \prod_{a=1}^{L} d^N \lambda^{(a)} \prod_i F^{(a)}(\lambda_i^{(a)}) \frac{\prod_{i<j} \mathrm{sh}^2(\lambda_i^{(a)} - \lambda_j^{(a)})}{\prod_{i,j} \mathrm{ch}(\lambda_i^{(a)} - \lambda_j^{(a+1)} + m^{(a)})} \,, \tag{2.1}$$

where $\lambda_i^{(a)}$, $i = 1, \cdots, N$ are the eigenvalues of the a^{th} node (with the identification $\lambda^{(L+1)} \equiv \lambda^{(1)}$) and $m^{(a)}$ are the bifundamental masses. The factor $\prod_i F^{(a)}(\lambda_i^{(a)})$ represents all terms which depend only on single eigenvalues and includes contributions from the CS term with level $k^{(a)}$, the FI term with parameter $\zeta^{(a)}$ and $n^{(a)}$ fundamental hypermultiplets with individual masses $\mu_\alpha^{(a)}$, $\alpha = 1, \cdots, n^{(a)}$

$$F^{(a)}(\lambda) = e^{\pi i k^{(a)} \lambda^2} \, e^{2\pi i \zeta^{(a)} \lambda} \prod_{\alpha=1}^{n^{(a)}} \frac{1}{\mathrm{ch}(\lambda + \mu_\alpha^{(a)})} \,. \tag{2.2}$$

Using the Cauchy determinant identity

$$\frac{\prod_{i<j}\mathrm{sh}(\lambda_i-\lambda_j)\,\mathrm{sh}(\tilde{\lambda}_i-\tilde{\lambda}_j)}{\prod_{i,j}\mathrm{ch}(\lambda_i-\tilde{\lambda}_j)}=\sum_{\sigma\in S_N}(-1)^{\sigma}\prod_{i=1}^{N}\frac{1}{\mathrm{ch}(\lambda_i-\tilde{\lambda}_{\sigma(i)})}\,, \tag{2.3}$$

the partition function can be re-expressed as a sum over L permutations

$$Z(N)=\frac{1}{N!^L}\sum_{\sigma^{(a)}\in S_N}(-1)^{\sum_{a=1}^{L}\sigma^{(a)}}\int\prod_{a=1}^{L}d^N\lambda^{(a)}\prod_i\frac{F^{(a)}(\lambda_i^{(a)})}{\mathrm{ch}(\lambda_i^{(a)}-\lambda_{\sigma^{(a)}(i)}^{(a+1)}+m^{(a)})}\,. \tag{2.4}$$

By relabelling the eigenvalues one can factor out all but one of the permutations, picking up an overall factor of $N!^{L-1}$. This gives

$$\begin{aligned}Z(N)=\frac{1}{N!}\sum_{\sigma\in S_N}(-1)^{\sigma}\int\prod_{a=1}^{L}d^N\lambda^{(a)}\prod_{i}^{N}&\left(\prod_{a=1}^{L-1}\frac{F^{(a)}(\lambda_i^{(a)})}{\mathrm{ch}(\lambda_i^{(a)}-\lambda_i^{(a+1)}+m^{(a)})}\right)\\&\times\frac{F^{(L)}(\lambda_i^{(L)})}{\mathrm{ch}(\lambda_i^{(L)}-\lambda_{\sigma(i)}^{(1)}+m^{(L)})}\,.\end{aligned} \tag{2.5}$$

This integrand is a series of kernels of pairs of specific eigenvalues of successive nodes ultimately coupling each $\lambda_i^{(1)}$ with $\lambda_{\sigma(i)}^{(1)}$. This can be encoded graphically by the following diagram

$$\left\{\lambda^{(1)}\right\}\to\left\{\lambda^{(2)}\right\}\to\cdots\to\left\{\lambda^{(L)}\right\}\xrightarrow{\sigma}\left\{\lambda^{(1)}\right\}\,. \tag{2.6}$$

One can express the kernels in terms of canonical position and momentum operators $\hat{q}$, $\hat{p}$, which satisfy $[\hat{q},\hat{p}]=i\hbar$. Taking λ to be position eigenvalues, we have

$$F(\lambda)\delta(\lambda'-\lambda)=\langle\lambda'|\,F(\hat{q})\,|\lambda\rangle\,,\qquad\frac{1}{\mathrm{ch}(\lambda-\lambda')}=\langle\lambda|\,\frac{1}{\mathrm{ch}\,\hat{p}}\,|\lambda'\rangle\,,\qquad e^{2\pi im\hat{p}}\,|\lambda\rangle=|\lambda-m\rangle\,. \tag{2.7}$$

Here we used the standard relation between the position and momentum bases $|p\rangle=\int\frac{d\lambda}{\sqrt{2\pi\hbar}}\,e^{\frac{ip\lambda}{\hbar}}\,|\lambda\rangle$ and we have $\langle\lambda_1|\,\lambda_2\rangle=\delta(\lambda_1-\lambda_2)$ and $\langle p_1|\,p_2\rangle=\delta(p_1-p_2)$. We choose to normalize $\hat{p}$ such that $\hbar=\frac{1}{2\pi}$. This allows one to write the integrand of (2.5) as

$$\langle\lambda_i^{(1)}|\,F^{(1)}(\hat{q})\frac{e^{2\pi im^{(1)}\hat{p}}}{\mathrm{ch}\,\hat{p}}\,|\lambda_i^{(2)}\rangle\langle\lambda_i^{(2)}|\,F^{(2)}(\hat{q})\frac{e^{2\pi im^{(2)}\hat{p}}}{\mathrm{ch}\,\hat{p}}\,|\lambda_i^{(3)}\rangle\cdots\langle\lambda_i^{(L)}|\,F^{(L)}(\hat{q})\frac{e^{2\pi im^{(L)}\hat{p}}}{\mathrm{ch}\,\hat{p}}\,|\lambda_{\sigma(i)}^{(1)}\rangle\,. \tag{2.8}$$

We obtain the final expression for $Z(N)$,

$$Z(N)=\frac{1}{N!}\sum_{\sigma\in S_N}(-1)^{\sigma}\int d^N\lambda\prod_{i=1}^{N}\langle\lambda_i|\,\hat{\rho}\,|\lambda_{\sigma(i)}\rangle \tag{2.9}$$

with

$$\hat{\rho}=F^{(1)}(\hat{q})\frac{e^{2\pi im^{(1)}\hat{p}}}{\mathrm{ch}\,\hat{p}}F^{(2)}(\hat{q})\frac{e^{2\pi im^{(2)}\hat{p}}}{\mathrm{ch}\,\hat{p}}\cdots F^{(L)}(\hat{q})\frac{e^{2\pi im^{(L)}\hat{p}}}{\mathrm{ch}\,\hat{p}}\,. \tag{2.10}$$

This expression coincides with the partition function of N non-interacting fermions[6] living on a line, with a Hamiltonian $\hat{H}$ given by $\hat{\rho}=e^{-\hat{H}}$. In this language, λ_i is the position of the i^{th} fermion on the line.

[6]The fermionic statistics is understood from the anti-symmetrization over permutations of the positions λ_i.

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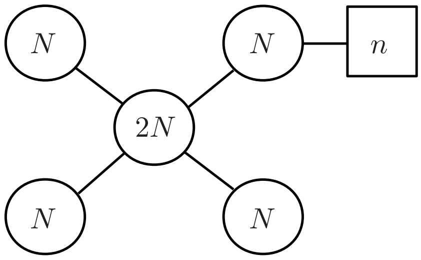

Figure 4: A $\hat{D}_4$-quiver with n fundamental hypermultiplets on one of the U(N) nodes.

Clearly (2.9) is fully determined by the spectrum of $\hat{\rho}$. In this formulation there is thus a natural splitting of the computation of the partition function into two distinct steps. The first is to find a suitable density operator that encodes the desired quiver theory, and the second is to solve the resulting quantum mechanics problem. In this and the next section we concern ourselves only with finding the density operators and studying relations between them, we deal with the explicit computation of the partition function in section 4

For the remainder of this section we suppress the hats above operators.

2.2 $\hat{D}$-quiver analysis

We now implement the Fermi-gas formulation to the matrix models computing the S^3 partition functions of $\hat{D}_{L+2}$-quiver theories with unitary gauge groups. These theories are characterised by gauge group $\mathrm{U}(2N)^{L-1} \times \mathrm{U}(N)^4$ and a quiver structure as in figure 1. We work out the full details in a simple example, a $\hat{D}_4$-quiver theory with n fundamental hypermultiplets attached to a single U(N) node. We do not turn on mass or FI deformations. The quiver diagram for this theory is shown in figure 4. The generalization to other $\hat{D}_L$-quivers is outlined in section 2.4.

Our aim is now to find a suitable density operator that repackages the matrix model for this theory into an expression like (2.9). In order to do this it is useful to collect the eigenvalues of pairs of terminal U(N) into a single set of $2N$ eigenvalues. Anticipating this, we label the eigenvalues of one pair of U(N) nodes respectively by[7] $\lambda_i^{(0)}$ and $\lambda_{N+i}^{(0)}$, $i = 1, \cdots, N$, and the other pair by $\lambda_i^{(2)}$, $\lambda_{N+i}^{(2)}$. The eigenvalues of the U($2N$) node are labelled $\lambda_I^{(1)}$, $I = 1, \cdots, 2N$. Following the rules from appendix A, the matrix model for this theory is given by

$$Z(N) = \frac{1}{N!^4(2N)!} \int \prod_{a=0}^{2} d^{2N}\lambda^{(a)} \prod_{I<J} \mathrm{sh}^2\left(\lambda_I^{(1)} - \lambda_J^{(1)}\right) \tag{2.11}$$
$$\frac{\prod_{i<j} \mathrm{sh}^2\left(\lambda_i^{(0)} - \lambda_j^{(0)}\right) \mathrm{sh}^2\left(\lambda_{N+i}^{(0)} - \lambda_{N+j}^{(0)}\right) \mathrm{sh}^2\left(\lambda_i^{(2)} - \lambda_j^{(2)}\right) \mathrm{sh}^2\left(\lambda_{N+i}^{(2)} - \lambda_{N+j}^{(2)}\right)}{\prod_{i,J} \mathrm{ch}\left(\lambda_i^{(0)} - \lambda_J^{(1)}\right) \mathrm{ch}\left(\lambda_{N+i}^{(0)} - \lambda_J^{(1)}\right) \mathrm{ch}\left(\lambda_i^{(2)} - \lambda_J^{(1)}\right) \mathrm{ch}\left(\lambda_{N+i}^{(2)} - \lambda_J^{(1)}\right) \prod_i \mathrm{ch}^n \lambda_i^{(2)}}.$$

[7]This is a slight abuse of our earlier notation where superscripts distinguished gauge group factors.

Including artificially the factors $\prod_{i,j} \mathrm{sh}\left(\lambda_i^{(0)} - \lambda_{N+j}^{(0)}\right) \prod_{i,j} \mathrm{sh}\left(\lambda_i^{(2)} - \lambda_{N+j}^{(2)}\right)$ in the numerator and denominator, one can use the Cauchy identity (2.3), as well as a modified version

$$\frac{\prod_{i<j} \mathrm{sh}(\lambda_i - \lambda_j)\, \mathrm{sh}(\tilde{\lambda}_i - \tilde{\lambda}_j)}{\prod_{i,j} \mathrm{sh}(\lambda_i - \tilde{\lambda}_j)} = (-1)^{\frac{N(N-1)}{2}} \sum_{\sigma \in S_N} (-1)^\sigma \prod_{i=1}^{N} \frac{1}{\mathrm{sh}(\lambda_i - \tilde{\lambda}_{\sigma(i)})}\,, \tag{2.12}$$

to re-expressed $Z(N)$ as

$$Z(N) = \frac{1}{N!^4 (2N)!} \sum_{\substack{\sigma^{(a)} \in S_N \\ \tau^{(a)} \in S_{2N}}} (-1)^{\sigma^{(0)}+\sigma^{(2)}+\tau^{(0)}+\tau^{(2)}} \int \prod_{a=0}^{2} d^{2N}\lambda^{(a)} \frac{1}{\prod_i \mathrm{ch}^n \lambda_i^{(2)}} \tag{2.13}$$
$$\prod_{i=1}^{N} \frac{1}{\mathrm{sh}\left(\lambda_i^{(0)} - \lambda_{N+\sigma^{(0)}(i)}^{(0)}\right)} \frac{1}{\mathrm{sh}\left(\lambda_i^{(2)} - \lambda_{N+\sigma^{(2)}(i)}^{(2)}\right)} \prod_{I=1}^{2N} \frac{1}{\mathrm{ch}\left(\lambda_I^{(0)} - \lambda_{\tau^{(0)}(I)}^{(1)}\right)} \frac{1}{\mathrm{ch}\left(\lambda_I^{(1)} - \lambda_{\tau^{(2)}(I)}^{(2)}\right)}\,.$$

Successive relabelings of the indices allow us to remove the sum over $\sigma^{(0)}$, $\sigma^{(2)}$ and $\tau^{(0)}$ and compensate for it by an overall factor of $N!^2(2N)!$

$$Z(N) = \frac{1}{N!^2} \sum_{\tau \in S_{2N}} (-1)^\tau \int \prod_{a=0}^{2} d^{2N}\lambda^{(a)} \prod_{i=1}^{N} \frac{1}{\mathrm{ch}^n \lambda_i^{(2)}} \tag{2.14}$$
$$\prod_{i=1}^{N} \frac{1}{\mathrm{sh}\left(\lambda_i^{(0)} - \lambda_{N+i}^{(0)}\right)} \frac{1}{\mathrm{sh}\left(\lambda_i^{(2)} - \lambda_{N+i}^{(2)}\right)} \prod_{I=1}^{2N} \frac{1}{\mathrm{ch}\left(\lambda_I^{(0)} - \lambda_I^{(1)}\right)} \frac{1}{\mathrm{ch}\left(\lambda_I^{(1)} - \lambda_{\tau(I)}^{(2)}\right)}\,.$$

As in the case of the $\hat{A}$-quivers (2.10), we would like to write this as the successive interaction between pairs of eigenvalues. Defining the reflection permutation R by

$$R(i) = N + i\,, \qquad R(N+i) = i\,, \tag{2.15}$$

the integrand of the matrix model can be viewed as a series of kernels pairing eigenvalues of adjacent nodes in a chain that goes back and forth along the quiver, according to the diagram (cf., (2.6))

$$R \circlearrowright \left\{\lambda^{(0)}\right\} \rightleftarrows \left\{\lambda^{(1)}\right\} \underset{\tau^{-1}}{\overset{\tau}{\rightleftarrows}} \left\{\lambda^{(2)}\right\} \circlearrowleft R\,. \tag{2.16}$$

Traversing the quiver back and forth we end up with the composite permutation

$$R\tau^{-1}R\tau\,, \tag{2.17}$$

so we can write the partition function in terms of a kernel relating $\lambda_I^{(1)}$ and $\lambda_{R\tau^{-1}R\tau(I)}^{(1)}$. Note however that another eigenvalue of the central node $\lambda_{\tau^{-1}R\tau(I)}^{(1)}$ is integrated over to get this kernel. So for each permutation τ we need to choose half the eigenvalues of $\lambda^{(1)}$ on which the kernel acts. Let us call the set of N indices of those eigenvalues $\mathcal{K}(\tau)$. It is chosen to be closed under the composite permutation $R\tau^{-1}R\tau$ and such that R takes this set to its complement $R(\mathcal{K}(\tau)) = \overline{\mathcal{K}(\tau)}$. The partition function can be expressed in the following way

$$Z(N) = \frac{1}{N!^2} \sum_{\tau\in S_{2N}} (-1)^\tau \int \prod_{a=0}^{2} d^{2N}\lambda^{(a)} \prod_{i=1}^{N} \frac{1}{\operatorname{ch}^n \lambda_i^{(2)}} \prod_{k\in\mathcal{K}(\tau)} \frac{1}{\operatorname{ch}\left(\lambda_k^{(1)}-\lambda_{\tau(k)}^{(2)}\right)}$$
$$\times \frac{(-1)^{s(\tau(k))}}{\operatorname{sh}\left(\lambda_{\tau(k)}^{(2)}-\lambda_{R\tau(k)}^{(2)}\right)} \frac{1}{\operatorname{ch}\left(\lambda_{R\tau(k)}^{(2)}-\lambda_{\tau^{-1}R\tau(k)}^{(1)}\right)} \frac{1}{\operatorname{ch}\left(\lambda_{\tau^{-1}R\tau(k)}^{(1)}-\lambda_{\tau^{-1}R\tau(k)}^{(0)}\right)} \tag{2.18}$$
$$\times \frac{(-1)^{s(\tau^{-1}R\tau(k))}}{\operatorname{sh}\left(\lambda_{\tau^{-1}R\tau(k)}^{(0)}-\lambda_{R\tau^{-1}R\tau(k)}^{(0)}\right)} \frac{1}{\operatorname{ch}\left(\lambda_{R\tau^{-1}R\tau(k)}^{(0)}-\lambda_{R\tau^{-1}R\tau(k)}^{(1)}\right)},$$

where

$$s(k) = \begin{cases} 0\,, & k=1,\cdots,N\,,\\ 1\,, & k=N+1,\cdots,2N\,. \end{cases} \tag{2.19}$$

To be able to write the partition function in terms of a density operator we need to include the contribution from the fundamental hypermultiplets into the product over k in (2.18). However, the fundamental hypermultiplets couple only to the eigenvalues $\lambda_i^{(2)}$ with $i=1,\cdots N$, which depending on τ is either $\lambda_{\tau(k)}^{(2)}$ *or* $\lambda_{R\tau(k)}^{(2)}$, but not both. These two options happen with equal probability for each combined permutation $R\tau^{-1}R\tau$, so we can write it as the sum (normalized by $1/2^N$)[8]

$$Z(N) = \frac{1}{2^N N!^2} \sum_{\tau\in S_{2N}} (-1)^\tau \int \prod_{a=0}^{2} d^{2N}\lambda^{(a)} \prod_{k\in\mathcal{K}(\tau)} (-1)^{s(k)+s(\tau(k))+1} \tag{2.20}$$
$$\prod_{k\in\mathcal{K}(\tau)} \left[\frac{1}{\operatorname{ch}\left(\lambda_k^{(1)}-\lambda_{\tau(k)}^{(2)}\right)} \frac{1}{\operatorname{sh}\left(\lambda_{\tau(k)}^{(2)}-\lambda_{R\tau(k)}^{(2)}\right)} \frac{1}{\operatorname{ch}\left(\lambda_{R\tau(k)}^{(2)}-\lambda_{\tau^{-1}R\tau(k)}^{(1)}\right)}\right.$$
$$\left. + \frac{1}{\operatorname{ch}\left(\lambda_k^{(1)}-\lambda_{R\tau(k)}^{(2)}\right)} \frac{1}{\operatorname{sh}\left(\lambda_{R\tau(k)}^{(2)}-\lambda_{\tau(k)}^{(2)}\right)} \frac{1}{\operatorname{ch}\left(\lambda_{\tau(k)}^{(2)}-\lambda_{\tau^{-1}R\tau(k)}^{(1)}\right)}\right] \frac{1}{\operatorname{ch}^n \lambda_{\tau(k)}^{(2)}}$$
$$\times \frac{1}{\operatorname{ch}\left(\lambda_{\tau^{-1}R\tau(k)}^{(1)}-\lambda_{\tau^{-1}R\tau(k)}^{(0)}\right)} \frac{1}{\operatorname{sh}\left(\lambda_{\tau^{-1}R\tau(k)}^{(0)}-\lambda_{R\tau^{-1}R\tau(k)}^{(0)}\right)} \frac{1}{\operatorname{ch}\left(\lambda_{R\tau^{-1}R\tau(k)}^{(0)}-\lambda_{R\tau^{-1}R\tau(k)}^{(1)}\right)},$$

where we have used $\prod_{k\in\mathcal{K}(\tau)}(-1)^{s(\tau^{-1}R\tau(k))} = \prod_{k\in\mathcal{K}(\tau)}(-1)^{s(k)+1}$. This expression can now be recast as

$$Z(N) = \frac{1}{2^{2N}N!^2} \sum_{\tau\in S_{2N}} (-1)^\tau \int d^N\lambda \prod_{k\in\mathcal{K}(\tau)} (-1)^{s(k)+s(\tau(k))} \rho(\lambda_k, \lambda_{R\tau^{-1}R\tau(k)})\,, \tag{2.21}$$

with

$$\rho(\lambda,\lambda') = 2\int \prod_{a=1}^{5} d\lambda_a \frac{-1}{\operatorname{ch}(\lambda-\lambda_1)} \left(\frac{1}{\operatorname{ch}^n\lambda_1}\frac{1}{\operatorname{sh}(\lambda_1-\lambda_2)} + \frac{1}{\operatorname{sh}(\lambda_1-\lambda_2)}\frac{1}{\operatorname{ch}^n\lambda_2}\right)$$
$$\frac{1}{\operatorname{ch}(\lambda_2-\lambda_3)} \frac{1}{\operatorname{ch}(\lambda_3-\lambda_4)} \frac{1}{\operatorname{sh}(\lambda_4-\lambda_5)} \frac{1}{\operatorname{ch}(\lambda_5-\lambda')}\,, \tag{2.22}$$

[8]In more detail, note that $R\tau^{-1}R\tau$ remains the same if one multiplies τ on the left by any combinations of two-cycles appearing in R. For a given τ this generates a set of 2^N terms, half with $\tau(k)$ in $\{1,\cdots,N\}$ and half in the compliment.

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where we chose the normalization factor for convenience.

The kernel ρ defines a density operator through the relation $\rho(\lambda_1, \lambda_2) = \langle\lambda_1|\,\rho\,|\lambda_2\rangle$, which has a representation in terms of canonical position and momentum operators (2.7)

$$\rho = \frac{1}{2}\frac{1}{\operatorname{ch} p}\left(\frac{1}{\operatorname{ch}^n q}\frac{\operatorname{sh} p}{\operatorname{ch} p} + \frac{\operatorname{sh} p}{\operatorname{ch} p}\frac{1}{\operatorname{ch}^n q}\right)\frac{\operatorname{sh} p}{\operatorname{ch}^4 p}\,, \tag{2.23}$$

where in addition to (2.7) we have used

$$\frac{1}{\operatorname{sh}(\lambda - \lambda')} = -\frac{i}{2}\,\langle\lambda|\,\frac{\operatorname{sh} p}{\operatorname{ch} p}\,|\lambda'\rangle\,. \tag{2.24}$$

To make further progress, we need to study the combinatorics of the composite permutations $R\tau^{-1}R\tau$. We relegate these additional technical calculations to appendix B and provide the final simplified result, which involves only a sum over permutations of S_N

$$Z(N) = \frac{1}{N!}\sum_{\sigma\in S_N}\frac{(-1)^\sigma}{2^{n_\sigma}}\int\prod_{i=1}^{N}d\lambda_i\,\prod_{i=1}^{N}\rho(\lambda_i, \lambda_{\sigma(i)})\,, \tag{2.25}$$

where n_σ is the number of cycles in σ.

Because of the factor $1/2^{n_\sigma}$, (2.25) cannot be interpreted directly as the partition function of N non-interacting fermions. When all FI and mass parameters are turned off, we find that it can be understood as resulting from a projection onto half of the states of a fermionic system. We show in appendix C that the density operator ρ commutes with the reflection operator $\hat{R}$, defined by

$$\hat{R}\,|\lambda\rangle = |-\lambda\rangle\,. \tag{2.26}$$

Consequently, the Hilbert space can be split into even and odd eigenstates. Furthermore, the spectra of even and odd eigenstates are identical, allowing us to rewrite $Z(N)$, using the projector $\frac{1+\hat{R}}{2}$, as

$$Z(N) = \frac{1}{N!}\sum_{\sigma\in S_N}(-1)^\sigma\int\prod_{i=1}^{N}d\lambda_i\,\prod_{i=1}^{N}\langle\lambda_i|\,\rho\left(\frac{1+\hat{R}}{2}\right)|\lambda_{\sigma(i)}\rangle\,. \tag{2.27}$$

This can be readily interpreted as the partition function of N non-interacting fermions at positions $|\lambda_i|$ on a half-line with a Hamiltonian $H = -\log\rho$ where the operator $\frac{1+\hat{R}}{2}$ is responsible for the projection onto particle states with even wavefunction on the line, or equivalently particle states on a half-line with Neumann boundary conditions.

Likewise we can use the projector $\frac{1-\hat{R}}{2}$ to express the partition function in terms of the odd states

$$Z(N) = \frac{1}{N!}\sum_{\sigma\in S_N}(-1)^\sigma\int\prod_{i=1}^{N}d\lambda_i\,\prod_{i=1}^{N}\langle\lambda_i|\,\rho\left(\frac{1-\hat{R}}{2}\right)|\lambda_{\sigma(i)}\rangle\,. \tag{2.28}$$

In this case we would interpret $Z(N)$ as the partition function of N non-interacting fermions on a half-line with Dirichlet boundary condition at the origin.

We did not find a free fermion interpretation of the partition function (2.25) for the cases with non-vanishing masses and FI parameters.

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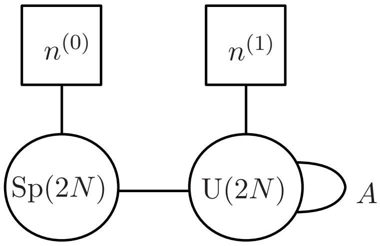

Figure 5: A linear quiver with an $\mathrm{Sp}(2N)$ node at one end and an antisymmetric hypermultiplet at the other.

2.3 Linear quiver analysis

We turn now to studying the matrix models of $\mathrm{U}(2N)$ linear quivers terminating at each end with either an $\mathrm{Sp}(2N)$ node or an antisymmetric hypermultiplet. In many cases these quivers are known to be the mirror duals of $\hat{D}$-quivers [2].

We perform the analysis in the case of the quiver with a single $\mathrm{U}(2N)$ node which is coupled to a single $\mathrm{Sp}(2N)$ node and has also an antisymmetric hypermultiplet. This example contains all the ingredients to treat any other quiver of this class, as we elaborate on in section 2.4.

The $\mathrm{Sp}(2N)$ node in our example has eigenvalues labelled $\lambda_i^{(0)}$, $i = 1, \cdots, N$ and $n^{(0)}$ fundamental hypermultiplets. The $\mathrm{U}(2N)$ node has eigenvalues $\lambda_I^{(1)}$, $I = 1, \cdots, 2N$ an antisymmetric hypermultiplet as well as $n^{(1)}$ fundamental hypermultiplets. To simplify the expressions we do not include any FI terms or masses. Following the rules in appendix A, the matrix model is given by

$$\begin{aligned} Z(N) = {} & \frac{1}{2^N N!(2N)!} \int d^N\lambda^{(0)} d^{2N}\lambda^{(1)} \prod_{i=1}^{N} \frac{\mathrm{sh}^2\, 2\lambda_i^{(0)}}{\mathrm{ch}^{2n^{(0)}} \lambda_i^{(0)}} \prod_{I=1}^{2N} \frac{1}{\mathrm{ch}^{n^{(1)}} \lambda_I^{(1)}} \\ & \times \frac{\prod_{i<j} \mathrm{sh}^2\left(\lambda_i^{(0)} - \lambda_j^{(0)}\right) \mathrm{sh}^2\left(\lambda_i^{(0)} + \lambda_j^{(0)}\right) \prod_{I<J} \mathrm{sh}^2\left(\lambda_I^{(1)} - \lambda_J^{(1)}\right)}{\prod_{i,J} \mathrm{ch}\left(\lambda_i^{(0)} - \lambda_J^{(1)}\right) \mathrm{ch}\left(\lambda_i^{(0)} + \lambda_J^{(1)}\right) \prod_{I<J} \mathrm{ch}\left(\lambda_I^{(1)} + \lambda_J^{(1)}\right)}\,. \end{aligned} \tag{2.29}$$

A first step is to write the contribution of the Sp node in terms of $2N$ eigenvalues satisfying

$$\lambda_{N+i}^{(0)} = -\lambda_i^{(0)}\,. \tag{2.30}$$

The interaction between the $\mathrm{Sp}(2N)$ and $\mathrm{U}(2N)$ nodes combine to a single Cauchy determinant (2.3)

$$\begin{aligned} & \frac{\prod_{i<j} \mathrm{sh}^2\left(\lambda_i^{(0)} - \lambda_j^{(0)}\right) \mathrm{sh}^2\left(\lambda_i^{(0)} + \lambda_j^{(0)}\right) \prod_{i=1}^{N} \mathrm{sh}\, 2\lambda_i^{(0)} \prod_{I<J} \mathrm{sh}\left(\lambda_I^{(1)} - \lambda_J^{(1)}\right)}{\prod_{i,J} \mathrm{ch}\left(\lambda_i^{(0)} - \lambda_J^{(1)}\right) \mathrm{ch}\left(\lambda_i^{(0)} + \lambda_J^{(1)}\right)} \\ & \quad = \frac{\prod_{I<J} \mathrm{sh}\left(\lambda_I^{(0)} - \lambda_J^{(0)}\right) \mathrm{sh}\left(\lambda_I^{(1)} - \lambda_J^{(1)}\right)}{\prod_{I,J} \mathrm{ch}\left(\lambda_I^{(0)} - \lambda_J^{(1)}\right)} = \sum_{\tau^{(0)} \in S_{2N}} (-1)^{\tau^{(0)}} \prod_{I=1}^{2N} \frac{1}{\mathrm{ch}\left(\lambda_I^{(0)} - \lambda_{\tau^{(0)}(I)}^{(1)}\right)}\,. \end{aligned} \tag{2.31}$$

The remaining terms involving the eigenvalues of the U($2N$) node can be interpreted as a Pfaffian, rather than a determinant. We can use the identity [6, 43]

$$\prod_{I<J\leqslant 2N}\frac{x_I-x_J}{1+x_Ix_J}=\mathrm{Pf}\left(\frac{x_I-x_J}{1+x_Ix_J}\right)=\frac{1}{2^N N!}\sum_{\tau\in S_{2N}}(-1)^\tau\prod_{i=1}^{N}\frac{x_{\tau(i)}-x_{\tau R(i)}}{1+x_{\tau(i)}x_{\tau R(i)}}\,, \tag{2.32}$$

where R is again the permutation $R(i)=N+i$ modulo $2N$. Plugging $x=e^{2\pi\lambda^{(1)}}$, we obtain

$$\prod_{I<J}\frac{\mathrm{sh}\left(\lambda_I^{(1)}-\lambda_J^{(1)}\right)}{\mathrm{ch}\left(\lambda_I^{(1)}+\lambda_J^{(1)}\right)}=\frac{1}{2^N N!}\sum_{\tau\in S_{2N}}(-1)^\tau\prod_{i=1}^{N}\frac{\mathrm{sh}\left(\lambda_{\tau(i)}^{(1)}-\lambda_{\tau R(i)}^{(1)}\right)}{\mathrm{ch}\left(\lambda_{\tau(i)}^{(1)}+\lambda_{\tau R(i)}^{(1)}\right)}\,. \tag{2.33}$$

As before, we can remove one of the permutations coming from (2.31) and (2.33) by a relabelling of eigenvalues, picking up an overall factor of $(2N)!$. This gives

$$\begin{aligned}Z(N)=\frac{1}{2^{2N}N!^2}\sum_{\tau\in S_{2N}}(-1)^\tau\int d^N\lambda^{(0)}d^{2N}\lambda^{(1)}\prod_{i=1}^{N}\frac{\mathrm{sh}\,2\lambda_i^{(0)}}{\mathrm{ch}^{2n^{(0)}}\lambda_i^{(0)}}\prod_{I=1}^{2N}\frac{1}{\mathrm{ch}^{n^{(1)}}\lambda_I^{(1)}}\\ \prod_{i=1}^{N}\frac{1}{\mathrm{ch}\left(\lambda_{\tau(i)}^{(0)}-\lambda_{\tau(i)}^{(1)}\right)}\frac{\mathrm{sh}\left(\lambda_{\tau(i)}^{(1)}-\lambda_{\tau R(i)}^{(1)}\right)}{\mathrm{ch}\left(\lambda_{\tau(i)}^{(1)}+\lambda_{\tau R(i)}^{(1)}\right)}\frac{1}{\mathrm{ch}\left(\lambda_{\tau R(i)}^{(1)}-\lambda_{\tau R(i)}^{(0)}\right)}\,.\end{aligned} \tag{2.34}$$

replacing for convenience $\tau\to\tau^{-1}$, we can again rewrite the expression as a product over the set $\mathcal{K}(\tau)$, consisting of N indices closed under the permutation $R\tau^{-1}R\tau$

$$\begin{aligned}Z(N)\;=\;&\frac{1}{2^{2N}N!^2}\sum_{\tau\in S_{2N}}(-1)^\tau\int d^N\lambda^{(0)}d^{2N}\lambda^{(1)}\prod_{k\in\mathcal{K}(\tau)}\frac{(-1)^{s(k)}\,\mathrm{sh}\,2\lambda_k^{(0)}}{\mathrm{ch}^{2n^{(0)}}\lambda_k^{(0)}}\frac{1}{\mathrm{ch}^{n^{(1)}}\lambda_k^{(1)}\,\mathrm{ch}^{n^{(1)}}\lambda_{R(k)}^{(1)}}\\ &\frac{1}{\mathrm{ch}\left(\lambda_k^{(0)}-\lambda_k^{(1)}\right)}\frac{(-1)^{s(\tau(k))}\,\mathrm{sh}\left(\lambda_k^{(1)}-\lambda_{\tau^{-1}R\tau(k)}^{(1)}\right)}{\mathrm{ch}\left(\lambda_k^{(1)}+\lambda_{\tau^{-1}R\tau(k)}^{(1)}\right)}\frac{1}{\mathrm{ch}\left(\lambda_{\tau^{-1}R\tau(k)}^{(1)}+\lambda_{R\tau^{-1}R\tau(k)}^{(0)}\right)}\,.\end{aligned} \tag{2.35}$$

The $(-1)^{s(\tau(k))}$ signs comes from re-expressing $\mathrm{sh}\left(\lambda_{\tau(i)}^{(1)}-\lambda_{\tau R(i)}^{(1)}\right)$ in terms of k, while (2.30) is responsible for the $(-1)^{s(k)}$ signs as well as allowing the replacement in the last denominator

$$\lambda_{\tau^{-1}R\tau(k)}^{(0)}=-\lambda_{R\tau^{-1}R\tau(k)}^{(0)}\,. \tag{2.36}$$

As in the case of the $\hat{D}$-quivers (2.21), we obtain a density operator between two $\lambda^{(0)}$ eigenvalues related by the permutation $R\tau^{-1}R\tau$

$$\rho(\lambda,\lambda')=\int d\lambda_1 d\lambda_2\,\frac{\mathrm{sh}\,2\lambda}{\mathrm{ch}^{2n^{(0)}}\lambda}\frac{1}{\mathrm{ch}(\lambda-\lambda_1)}\frac{1}{\mathrm{ch}^{n^{(1)}}\lambda_1}\frac{\mathrm{sh}(\lambda_1-\lambda_2)}{\mathrm{ch}(\lambda_1+\lambda_2)}\frac{1}{\mathrm{ch}^{n^{(1)}}\lambda_2}\frac{1}{\mathrm{ch}(\lambda_2+\lambda')}\,, \tag{2.37}$$

in terms of which the partition function is given exactly as in (2.21). Expanding $\mathrm{sh}(\lambda_1-\lambda_2)$ and reversing the sign of λ_2 allows us again to represent the operator in terms of canonical position and momentum operators

$$\begin{aligned}\rho&=\frac{1}{2}\frac{\mathrm{sh}\,2q}{\mathrm{ch}^{2n^{(0)}}q}\frac{1}{\mathrm{ch}\,p}\frac{1}{\mathrm{ch}^{n^{(1)}}q}\left(\mathrm{sh}\,q\frac{1}{\mathrm{ch}\,p}\mathrm{ch}\,q+\mathrm{ch}\,q\frac{1}{\mathrm{ch}\,p}\mathrm{sh}\,q\right)\frac{1}{\mathrm{ch}^{n^{(1)}}q}\frac{1}{\mathrm{ch}\,p}\\ &=\frac{1}{2}\frac{\mathrm{sh}\,2q}{\mathrm{ch}^{2n^{(0)}}q}\frac{1}{\mathrm{ch}\,p}\frac{1}{\mathrm{ch}^{n^{(1)}}q}\left(e^{\pi q}\frac{1}{\mathrm{ch}\,p}e^{\pi q}+e^{-\pi q}\frac{1}{\mathrm{ch}\,p}e^{-\pi q}\right)\frac{1}{\mathrm{ch}^{n^{(1)}}q}\frac{1}{\mathrm{ch}\,p}\,.\end{aligned} \tag{2.38}$$

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The same arguments as for the $\hat{D}$-quiver allow us to express $Z(N)$ as the partition function of N non-interacting fermions on a half line with Neumann (2.27) or Dirichlet (2.28) boundary conditions. Indeed, as we show in section 3, the density operators of many linear quivers are related to those of $\hat{D}$-quivers by a linear canonical transformation.

2.4 Generalization to longer quivers

It is straight-forward to generalize the analysis to quivers with an arbitrary number of $\mathrm{U}(2N)$ nodes and arbitrary number of fundamental hypermultiplets on each node. Just as for the $\hat{A}$-quiver theories, the matrix model contributions from hypermultiplets transforming in the bifundamental representation of pairs of $\mathrm{U}(2N)$ gauge nodes combine with the vector multiplet contributions to form one Cauchy determinant between each pair of adjacent nodes. This translates into $\mathrm{ch}^{-1} p$ terms in the density operator. We represent the contribution from fundamental hypermultiplets and the FI and CS terms of the a^{th} node again by $F^{(a)}(q)$ (2.2). This leads to a piece in the density operator of the form

$$\frac{1}{\mathrm{ch}\, p} F^{(1)}(q) \frac{1}{\mathrm{ch}\, p} F^{(2)}(q) \frac{1}{\mathrm{ch}\, p} F^{(3)}(q) \frac{1}{\mathrm{ch}\, p} \cdots \tag{2.39}$$

In all of our examples the density operator combines kernels going back and forth along the quiver. The contribution from the ends of the quivers ($\mathrm{U}(N)$ nodes for $\hat{D}$-quivers and $\mathrm{Sp}(2N)$/antisymmetric hypermultiplet for linear quivers) are the same as in the previous sections.

For the $\hat{D}$-quivers the contribution from going back along the quiver as in (2.16) gives,

$$\cdots \frac{1}{\mathrm{ch}\, p} F^{(3)}(q) \frac{1}{\mathrm{ch}\, p} F^{(2)}(q) \frac{1}{\mathrm{ch}\, p} F^{(1)}(q) \frac{1}{\mathrm{ch}\, p}\,. \tag{2.40}$$

For the linear quivers, the antisymmetric hypermultiplet or Sp node introduce a minus sign, like the replacement $\lambda_2 \to -\lambda_2$ that gave (2.38) from (2.37). Therefore the second part of the density operator includes

$$\cdots \frac{1}{\mathrm{ch}\, p} F^{(3)}(-q) \frac{1}{\mathrm{ch}\, p} F^{(2)}(-q) \frac{1}{\mathrm{ch}\, p} F^{(1)}(-q) \frac{1}{\mathrm{ch}\, p}\,. \tag{2.41}$$

2.4.1 Generalized $\hat{D}$-quivers

Let us consider a $\hat{D}_{L+2}$ quiver with arbitrary number of gauge nodes and fundamental hypermultiplets on each node,[9] as shown in figure 1. We label the $\mathrm{U}(2N)$ nodes by $1, \cdots, L-1$, and as in figure 1, we distinguish parameters from pairs of terminal $\mathrm{U}(N)$ nodes by primes: $F^{(0)}$, $F'^{(0)}$,$F^{(L)}$, $F'^{(L)}$. We note that all bifundamental hypermultiplet masses can be set

[9]Note that in order for the matrix model to be convergent, such a theory must have at least one fundamental hypermultiplet. With this condition violated the formal manipulations still go through, but the divergence of the matrix model translates to a density operator which is not of trace class.

to zero by shifting eigenvalues.[10] The above rules lead to the density operator

$$\rho = \frac{1}{4}\frac{1}{\operatorname{ch} p}\left(F^{(0)}(q)\frac{\operatorname{sh} p}{\operatorname{ch} p}F'^{(0)}(q) + F'^{(0)}(q)\frac{\operatorname{sh} p}{\operatorname{ch} p}F^{(0)}(q)\right)\frac{1}{\operatorname{ch} p}\left(\prod_{a=1}^{L-1} F^{(a)}(q)\frac{1}{\operatorname{ch} p}\right) \\ \left(F^{(L)}(q)\frac{\operatorname{sh} p}{\operatorname{ch} p}F'^{(L)}(q) + F'^{(L)}(q)\frac{\operatorname{sh} p}{\operatorname{ch} p}F^{(L)}(q)\right)\frac{1}{\operatorname{ch} p}\left(\prod_{a=1}^{L-1} F^{(L-a)}(q)\frac{1}{\operatorname{ch} p}\right). \tag{2.42}$$

We can easily recover (2.23) by setting $L = 2$, $F^{(2)}(q) = \operatorname{ch}^{-n} q$ and all other $F^{(a)} = F'^{(a)} = 1$.

2.4.2 Generalized linear quivers

We can proceed in a similar fashion to write down the density operators for longer linear quivers, where each end of the $\mathrm{U}(2N)$ linear chain has either an $\mathrm{Sp}(2N)$ node or an antisymmetric hypermultiplet and any number of fundamental hypermultiplets on all the nodes.[11] Again we note that the masses for all bifundamental hypermultiplets between $\mathrm{U}(2N)$ nodes can be set to zero by shifts of the eigenvalues. We cannot always do the same for the mass of antisymmetric hypermultiplets, or for those charged under $\mathrm{Sp}(2N)$, so we keep these masses as well as the fundamental hypermultiplet masses. We consider a quiver with L $\mathrm{U}(2N)$ nodes and again package the FI, CS and fundamental hypermultiplet contributions into $F^{(a)}$. Two instances (out of three) of such general linear quivers are pictured in figure 2. The density operator is given by

$$\rho = B^{(0)}(p,q)\left(\prod_{a=1}^{L-1} F^{(a)}(q)\frac{1}{\operatorname{ch} p}\right)F^{(L)}(q)B^{(L+1)}(p,q)\left(\prod_{a=1}^{L-1} F^{(L+1-a)}(-q)\frac{1}{\operatorname{ch} p}\right)F^{(1)}(-q)\,, \tag{2.43}$$

where the functions $B^{(a)}(p,q)$ account for whether the ends of the quiver terminate with an $\mathrm{Sp}(2N)$ node or antisymmetric hypermultiplet.

If the quiver terminates with an $\mathrm{Sp}(2N)$ node we have

$$B^{(a)}_{Sp}(p,q) = \frac{e^{2\pi i m^{(a)} p}}{\operatorname{ch} p}\operatorname{sh}(2q)\widetilde{F}^{(a)}(q)\frac{e^{2\pi i m^{(a)} p}}{\operatorname{ch} p}\,, \tag{2.44}$$

where $m^{(a)}$ is the bifundamental mass and $\widetilde{F}^{(a)}(q)$ captures the contributions from the CS term with level $k^{(a)}$ and $n^{(a)}$ fundamental hypermultiplets of $\mathrm{Sp}(2N)$ with masses $\mu_\alpha^{(a)}$

$$\widetilde{F}^{(a)}(q) = \frac{e^{2\pi i k^{(a)} q^2}}{\prod_{\alpha=1}^{n^{(a)}}\operatorname{ch}(q-\mu_\alpha^{(a)})\operatorname{ch}(q+\mu_\alpha^{(a)})}\,. \tag{2.45}$$

[10]Shifting eigenvalues to remove masses can introduce an overall phase in the matrix model. Such phases are unphysical, in the sense that they arise from background (mixed) Chern-Simons terms that can be added to the regularization scheme when computing the partition function of the 3d theories [44, 45].

[11]While the formal manipulations again go through in all cases (see also footnote 9), convergence of the matrix model requires a total of at least three fundamental hypermultipets, with at least one coupling to the terminating nodes at each end of the quiver.

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If it terminates with an antisymmetric hypermultiplet we have

$$B_A^{(a)}(p,q) = \frac{1}{2}\left(e^{\pi q}\frac{e^{2\pi i M^{(a)}p}}{\operatorname{ch} p}e^{\pi q} + e^{-\pi q}\frac{e^{2\pi i M^{(a)}p}}{\operatorname{ch} p}e^{-\pi q}\right), \tag{2.46}$$

where $M^{(a)}$ is the antisymmetric mass.

Note that the expression (2.43) assumes there is at least one $\mathrm{U}(2N)$ node. There are two relevant cases without $\mathrm{U}(2N)$ nodes: for the single node $\mathrm{Sp}(2N)$ theory with an antisymmetric hypermultiplet the density operator is given by[12]

$$\rho = \frac{1}{2}\operatorname{sh}(2q)\widetilde{F}^{(0)}(q)\left(\operatorname{sh} q\frac{e^{2\pi i M p}}{\operatorname{ch} p}\operatorname{ch} q + \operatorname{ch} q\frac{e^{2\pi i M p}}{\operatorname{ch} p}\operatorname{sh} q\right). \tag{2.47}$$

For the $\mathrm{Sp}(2N)\times\mathrm{Sp}(2N)$ theory the density operator is given by

$$\rho = \frac{e^{2\pi i m^{(0)}p}}{\operatorname{ch} p}\operatorname{sh}(2q)\widetilde{F}^{(0)}(q)\frac{e^{2\pi i m^{(0)}p}}{\operatorname{ch} p}\operatorname{sh}(2q)\widetilde{F}^{(1)}(q)\,. \tag{2.48}$$

3 Mirror symmetry

In [14] it was found that pairs of mirror dual $\hat{A}$-quiver theories give rise to Fermi gas formulations which are related to each other by simple linear canonical transformations acting on their density operators. In this way, once one understands how to go from a quiver theory to the Fermi gas density operator, one can with little effort find the mirror map between dual theories. In this section we show that this continues to hold true for the mirror pairs involving $\hat{D}$-quivers and linear quivers. In this section we consider theories with mass and FI term deformations, since those get mapped to each other under mirror symmetry. For simplicity we restrict our attention in the remainder of the paper to theories without Chern-Simons terms

We proceed with a number of examples, starting in each case with a $\hat{D}$-quiver, and demonstrating how the canonical transformation

$$p \to q, \qquad q \to -p\,, \tag{3.1}$$

maps the density operator onto that of the mirror linear quiver theory.

3.1 $\hat{D}_4$-quiver with two fundamentals

The first example we consider is the $\hat{D}_4$-quiver with a fundamental hypermultiplet on two of the terminal $\mathrm{U}(N)$ nodes. This example is somewhat special as there are three inequivalent ways of pairing up the terminal $\mathrm{U}(N)$ nodes, which leads to different density operators. The canonical transformations of these three descriptions are related to three different mirror theories. The existence of several mirror dual theories was already noted in section 4.4.3 of [11].

[12] This theory was previously studied from the Fermi gas perspective in [41], where a rather different density operator was obtained. We compare the different formalisms in appendix D.

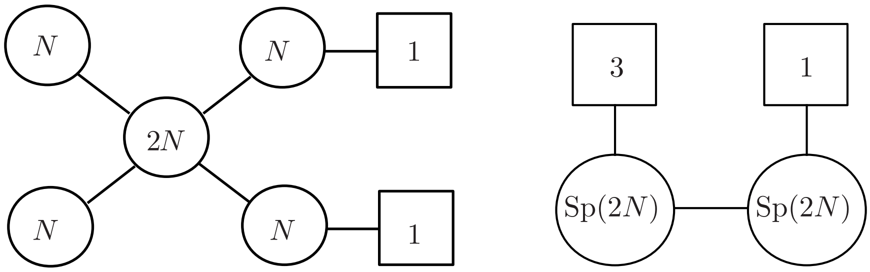

Figure 6: $\hat{D}_4$-quiver and its mirror dual theory with $\mathrm{Sp}(2N)\times\mathrm{Sp}(2N)$ gauge group.

The first possibility is to pair the two terminal nodes without fundamental matter and the two with fundamental matter, as shown in figure 6. Within each pairing we distinguish the FI parameters of the two $\mathrm{U}(N)$ nodes by giving one of them a prime. We do not turn on masses for the two $\mathrm{U}(N)$ fundamental hypermultiplets.[13] The density operator of the $\hat{D}_4$ theory can be read from (2.42)

$$\begin{aligned}\rho = &\frac{1}{4}\frac{1}{\mathrm{ch}\, p}\left(e^{2\pi i\zeta^{(0)}q}\frac{\mathrm{sh}\, p}{\mathrm{ch}\, p}e^{2\pi i\zeta'^{(0)}q}+e^{2\pi i\zeta'^{(0)}q}\frac{\mathrm{sh}\, p}{\mathrm{ch}\, p}e^{2\pi i\zeta^{(0)}q}\right)\frac{1}{\mathrm{ch}\, p}e^{2\pi i\zeta^{(1)}q}\\ &\times\frac{1}{\mathrm{ch}\, p}\frac{1}{\mathrm{ch}\, q}\left(e^{2\pi i\zeta^{(2)}q}\frac{\mathrm{sh}\, p}{\mathrm{ch}\, p}e^{2\pi i\zeta'^{(2)}q}+e^{2\pi i\zeta'^{(2)}q}\frac{\mathrm{sh}\, p}{\mathrm{ch}\, p}e^{2\pi i\zeta^{(2)}q}\right)\frac{1}{\mathrm{ch}\, q}\frac{1}{\mathrm{ch}\, p}e^{2\pi i\zeta^{(1)}q}\,.\end{aligned}\tag{3.2}$$

To map this to the density operator of the mirror dual theory, we first use the relation

$$e^{2\pi i\zeta q}f(p)e^{-2\pi i\zeta q}=f(p-\zeta)\,,\tag{3.3}$$

to simplify the terms in parenthesis

$$\begin{aligned}&e^{\pi i(\zeta-\zeta')q}\frac{\mathrm{sh}\, p}{\mathrm{ch}\, p}e^{-\pi i(\zeta-\zeta')q}+e^{-\pi i(\zeta-\zeta')q}\frac{\mathrm{sh}\, p}{\mathrm{ch}\, p}e^{\pi i(\zeta-\zeta')q}\\ &=\frac{\mathrm{sh}(p+\frac{1}{2}\zeta'-\frac{1}{2}\zeta)}{\mathrm{ch}(p+\frac{1}{2}\zeta'-\frac{1}{2}\zeta)}+\frac{\mathrm{sh}(p-\frac{1}{2}\zeta'+\frac{1}{2}\zeta)}{\mathrm{ch}(p-\frac{1}{2}\zeta'+\frac{1}{2}\zeta)}=\frac{2\,\mathrm{sh}\, 2p}{\mathrm{ch}(p+\frac{1}{2}\zeta-\frac{1}{2}\zeta')\,\mathrm{ch}(p-\frac{1}{2}\zeta+\frac{1}{2}\zeta')}\,.\end{aligned}\tag{3.4}$$

By further commuting exponential factors we get

$$\begin{aligned}\rho = &\, e^{\pi i(\zeta^{(0)}+\zeta'^{(0)})q}\frac{1}{\mathrm{ch}\left(p+\tilde{\mu}_1^{(1)}\right)}\frac{\mathrm{sh}\, 2p}{\mathrm{ch}\left(p+\tilde{\mu}_2^{(1)}\right)\mathrm{ch}\left(p-\tilde{\mu}_2^{(1)}\right)}\frac{1}{\mathrm{ch}\left(p-\tilde{\mu}_1^{(1)}\right)}\frac{1}{\mathrm{ch}\left(p+\tilde{\mu}_3^{(1)}\right)}\\ &\times\frac{e^{-\pi i\widetilde{m}q}}{\mathrm{ch}\, q}\frac{\mathrm{sh}\, 2p}{\mathrm{ch}\left(p+\tilde{\mu}^{(0)}\right)\mathrm{ch}\left(p-\tilde{\mu}^{(0)}\right)}\frac{e^{-\pi i\widetilde{m}q}}{\mathrm{ch}\, q}\frac{1}{\mathrm{ch}\left(p-\tilde{\mu}_3^{(1)}\right)}e^{-\pi i(\zeta^{(0)}+\zeta'^{(0)})q}\,.\end{aligned}\tag{3.5}$$

with

$$\begin{aligned}&\tilde{\mu}_1^{(1)}=\frac{1}{2}(\zeta^{(0)}+\zeta'^{(0)})\,,\quad\tilde{\mu}_2^{(1)}=\frac{1}{2}(\zeta^{(0)}-\zeta'^{(0)})\,,\quad\tilde{\mu}_3^{(1)}=-\zeta^{(1)}-\frac{1}{2}(\zeta^{(0)}+\zeta'^{(0)})\,,\\ &\tilde{\mu}^{(0)}=\frac{1}{2}(\zeta^{(2)}-\zeta'^{(2)})\,,\\ &\widetilde{m}=-\zeta^{(1)}-\frac{1}{2}(\zeta^{(0)}+\zeta'^{(0)}+\zeta^{(2)}+\zeta'^{(2)})\,.\end{aligned}\tag{3.6}$$

[13] One of these mass parameters can be removed by shifts of the matrix model eigenvalues. The other is mapped under mirror symmetry to a "hidden" FI parameter [46, 47], which does not have a clear interpretation in the mirror gauge theory.

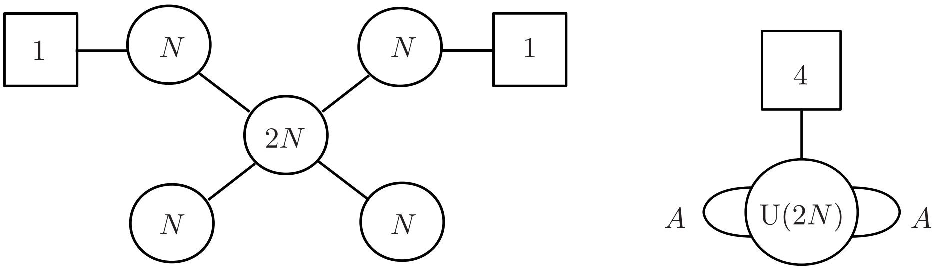

Figure 7: $\hat{D}_4$-quiver and its mirror dual theory with $\mathrm{U}(2N)$ gauge group and antisymmetric hypermultiplets.

Now we can act on the density operator by canonical transformation (3.1). In addition we conjugate the operator to remove the exponential factors at the beginning and end, which clearly does not alter the spectrum. This gives

$$\tilde{\rho} = \frac{e^{2\pi i\tilde{m}p}}{\operatorname{ch} p}\frac{\operatorname{sh} 2q}{\operatorname{ch}\left(q+\tilde{\mu}^{(0)}\right)\operatorname{ch}\left(q-\tilde{\mu}^{(0)}\right)}\frac{e^{2\pi i\tilde{m}p}}{\operatorname{ch} p}\frac{\operatorname{sh} 2q}{\prod_{\alpha=1}^{3}\operatorname{ch}\left(q+\tilde{\mu}_\alpha^{(1)}\right)\operatorname{ch}\left(q-\tilde{\mu}_\alpha^{(1)}\right)}\,. \tag{3.7}$$

As advertized we recover the density operator for a linear quiver with two $\mathrm{Sp}(2N)$ nodes. One with a single fundamental hypermultiplet and the other with three (2.48). The relations between the FI and mass deformation parameters of the mirror dual theories are expressed in (3.6), where $\tilde{m}$ is the bifundamental hypermultiplet mass and $\tilde{\mu}^{(0)}, \tilde{\mu}_\alpha^{(1)}$ are the fundamental hypermultiplet masses of the dual theory. This mirror map generalises slightly the one found already in [6], allowing for $\zeta^{(0)} \neq \zeta'^{(0)}$ and $\zeta^{(2)} \neq \zeta'^{(2)}$, which translate to additional mass deformations in the dual linear quiver theory.

A second way of obtaining a density operator for the $\hat{D}_4$-quiver comes from pairing nodes with and without fundamental hypermultiplet, as in figure 7. After shifting eigenvalues to remove masses we are left with only one nonzero mass μ for one of the fundamental hypermultiplets. From (2.42) we can write down the density operator

$$\begin{aligned}\rho = {}& \frac{1}{4}\frac{1}{\operatorname{ch} p}\left(\frac{e^{2\pi i\zeta^{(0)}q}}{\operatorname{ch} q}\frac{\operatorname{sh} p}{\operatorname{ch} p}e^{2\pi i\zeta'^{(0)}q} + e^{2\pi i\zeta'^{(0)}q}\frac{\operatorname{sh} p}{\operatorname{ch} p}\frac{e^{2\pi i\zeta^{(0)}q}}{\operatorname{ch} q}\right)\frac{1}{\operatorname{ch} p}e^{2\pi i\zeta^{(1)}q} \\ & \times\frac{1}{\operatorname{ch} p}\left(\frac{e^{2\pi i\zeta^{(2)}q}}{\operatorname{ch}(q+\mu)}\frac{\operatorname{sh} p}{\operatorname{ch} p}e^{2\pi i\zeta'^{(2)}q} + e^{2\pi i\zeta'^{(2)}q}\frac{\operatorname{sh} p}{\operatorname{ch} p}\frac{e^{2\pi i\zeta^{(2)}q}}{\operatorname{ch}(q+\mu)}\right)\frac{1}{\operatorname{ch} p}e^{2\pi i\zeta^{(1)}q}\,.\end{aligned} \tag{3.8}$$

Once again we start by manipulating the expression of the density operator, using the relation (3.3)

$$\begin{aligned}\rho = {}& \frac{1}{4}e^{-i\pi\zeta^{(1)}q}\frac{1}{\operatorname{ch}(p+\frac{\zeta^{(1)}}{2})}\frac{1}{\operatorname{ch}(p-\frac{\zeta^{(1)}}{2})} \\ & \times\left(\frac{e^{2\pi i(\zeta^{(0)}+\zeta'^{(0)}+\zeta^{(1)})q}}{\operatorname{ch} q}\frac{\operatorname{sh}(p+\zeta'^{(0)}+\frac{\zeta^{(1)}}{2})}{\operatorname{ch}(p+\zeta'^{(0)}+\frac{\zeta^{(1)}}{2})} + \frac{\operatorname{sh}(p-\zeta'^{(0)}-\frac{\zeta^{(1)}}{2})}{\operatorname{ch}(p-\zeta'^{(0)}-\frac{\zeta^{(1)}}{2})}\frac{e^{2\pi i(\zeta^{(0)}+\zeta'^{(0)}+\zeta^{(1)})q}}{\operatorname{ch} q}\right) \\ & \times\frac{1}{\operatorname{ch}(p+\frac{\zeta^{(1)}}{2})}\frac{1}{\operatorname{ch}(p-\frac{\zeta^{(1)}}{2})}\end{aligned}$$

$$\times \left(\frac{e^{2\pi i(\zeta^{(2)}+\zeta'^{(2)}+\zeta^{(1)})q}}{\mathrm{ch}(q+\mu)} \frac{\mathrm{sh}(p+\zeta'^{(2)}+\frac{\zeta^{(1)}}{2})}{\mathrm{ch}(p+\zeta'^{(2)}+\frac{\zeta^{(1)}}{2})} + \frac{\mathrm{sh}(p-\zeta'^{(2)}-\frac{\zeta^{(1)}}{2})}{\mathrm{ch}(p-\zeta'^{(2)}-\frac{\zeta^{(1)}}{2})} \frac{e^{2\pi i(\zeta^{(2)}+\zeta'^{(2)}+\zeta^{(1)})q}}{\mathrm{ch}(q+\mu)} \right)$$
$$\times\, e^{i\pi\zeta^{(1)}q}. \tag{3.9}$$

We now use the identity

$$\begin{aligned} &\frac{e^{2\pi i\zeta q}}{\mathrm{ch}(q+\mu)} \frac{\mathrm{sh}\, p+\zeta'}{\mathrm{ch}\, p+\zeta'} + \frac{\mathrm{sh}\, p-\zeta'}{\mathrm{ch}\, p-\zeta'} \frac{e^{2\pi i\zeta q}}{\mathrm{ch}(q+\mu)} \\ &\qquad = e^{-2\pi i\zeta\mu} \frac{e^{2\pi i\mu p}}{\mathrm{ch}(p-\zeta')} \left(e^{\pi p} \frac{e^{2\pi i\zeta q}}{\mathrm{ch}\, q} e^{\pi p} - e^{-\pi p} \frac{e^{2\pi i\zeta q}}{\mathrm{ch}\, q} e^{-\pi p} \right) \frac{e^{-2\pi i\mu p}}{\mathrm{ch}(p+\zeta')}\,, \end{aligned} \tag{3.10}$$

to bring the density operator into the form

$$\begin{aligned} \rho = &\frac{1}{4} e^{-2\pi i(\zeta^{(1)}+\zeta^{(2)}+\zeta'^{(2)})\mu} e^{-\pi i\zeta^{(1)}q} e^{2\pi i\mu p} \\ &\times \frac{e^{-2\pi i\mu p}}{\mathrm{ch}\left(p+\frac{1}{2}\zeta^{(1)}+\zeta'^{(2)}\right)} \frac{1}{\mathrm{ch}\left(p+\frac{1}{2}\zeta^{(1)}\right)} \frac{1}{\mathrm{ch}\left(p-\frac{1}{2}\zeta^{(1)}\right)} \frac{1}{\mathrm{ch}\left(p-\zeta'^{(0)}-\frac{1}{2}\zeta^{(1)}\right)} \\ &\times \left(e^{\pi p} \frac{e^{2\pi i(\zeta^{(0)}+\zeta'^{(0)}+\zeta^{(1)})q}}{\mathrm{ch}\, q} e^{\pi p} + e^{-\pi p} \frac{e^{2\pi i(\zeta^{(0)}+\zeta'^{(0)}+\zeta^{(1)})q}}{\mathrm{ch}\, q} e^{-\pi p} \right) \\ &\times \frac{e^{2\pi i\mu p}}{\mathrm{ch}\left(p+\zeta'^{(0)}+\frac{1}{2}\zeta^{(1)}\right)} \frac{1}{\mathrm{ch}\left(p+\frac{1}{2}\zeta^{(1)}\right)} \frac{1}{\mathrm{ch}\left(p-\frac{1}{2}\zeta^{(1)}\right)} \frac{1}{\mathrm{ch}\left(p-\frac{1}{2}\zeta^{(1)}-\zeta'^{(2)}\right)} \\ &\times \left(e^{\pi p} \frac{e^{2\pi i(\zeta^{(2)}+\zeta'^{(2)}+\zeta^{(1)})q}}{\mathrm{ch}\, q} e^{\pi p} + e^{-\pi p} \frac{e^{2\pi i(\zeta^{(2)}+\zeta'^{(2)}+\zeta^{(1)})q}}{\mathrm{ch}\, q} e^{-\pi p} \right) e^{-2\pi i\mu p} e^{\pi i\zeta^{(1)}q}\,. \end{aligned} \tag{3.11}$$

Applying the canonical transformation (3.1) and removing the exponential factors at the beginning and end by conjugation, we obtain (up to an overall phase)[14] the density operator of the (second) mirror dual theory, which is a $\mathrm{U}(2N)$ theory with two antisymmetric hypermultiplets of masses $\widetilde{M}_1, \widetilde{M}_2$, four fundamental hypermultiplets of masses $\tilde{\mu}_\alpha$, $\alpha = 1, \cdots, 4$ and an FI parameter $\tilde{\zeta}$. The explicit mirror map between parameters is given by

$$\begin{aligned} &\widetilde{M}_1 = -\zeta^{(0)} - \zeta'^{(0)} - \zeta^{(1)}, \qquad \widetilde{M}_2 = -\zeta^{(2)} - \zeta'^{(2)} - \zeta^{(1)}, \\ &\tilde{\mu}_1 = \frac{1}{2}\zeta^{(1)} + \zeta'^{(0)}, \quad \tilde{\mu}_2 = \frac{1}{2}\zeta^{(1)}, \quad \tilde{\mu}_3 = -\frac{1}{2}\zeta^{(1)}, \quad \tilde{\mu}_4 = -\frac{1}{2}\zeta^{(1)} - \zeta'^{(2)}, \\ &\tilde{\zeta} = \mu\,. \end{aligned} \tag{3.12}$$

Finally we may think of other ways to pair the $\mathrm{U}(N)$ nodes of the $\hat{D}_4$-quiver which are similar to the two cases above. For instance we can consider the exchange $\zeta'^{(0)} \leftrightarrow \zeta'^{(2)}$ in (3.8). This symmetry, which is completely trivial on the $\hat{D}$-quiver side, manifests as a relation between two mirror $\mathrm{U}(2N)$ theories which differ only by the values of their mass parameters

$$\begin{aligned} &\widetilde{M}_1 \to \widetilde{M}_1 + \tilde{\mu}_1 + \tilde{\mu}_4\,, \qquad \widetilde{M}_2 \to \widetilde{M}_2 - \tilde{\mu}_1 - \tilde{\mu}_4\,, \\ &\tilde{\mu}_1 \to -\tilde{\mu}_4, \quad \tilde{\mu}_2 \to \tilde{\mu}_2, \quad \tilde{\mu}_3 \to \tilde{\mu}_3, \quad \tilde{\mu}_4 \to -\tilde{\mu}_1\,, \\ &\tilde{\zeta} \to \tilde{\zeta}\,. \end{aligned} \tag{3.13}$$

[14] Such phases are unphysical, see footnote 10.

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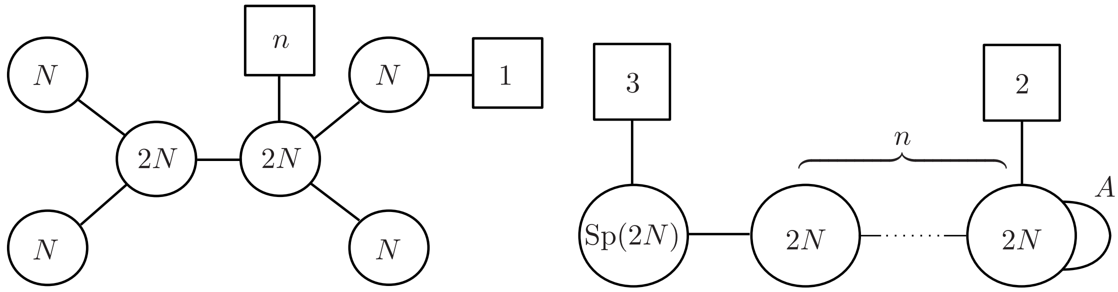

Figure 8: $\hat{D}_5$-quiver and its mirror dual theory.

3.2 $\hat{D}_5$-quiver

The next example of mirror map involves a $\hat{D}_5$-quiver with n fundamental hypermultiplets on one $\mathrm{U}(2N)$ node and a single fundamental hypermultiplet on one $\mathrm{U}(N)$ node. The quiver is shown in figure 8. Shifting eigenvalues to set all bifundamental masses, and one of the $\mathrm{U}(2N)$ fundamental masses to zero, we find the density operator (with obvious notations for the mass parameters)

$$
\begin{aligned}
\rho = {} & \frac{1}{4}\frac{1}{\operatorname{ch} p}\left(e^{2\pi i\zeta^{(0)}q}\frac{\operatorname{sh} p}{\operatorname{ch} p}e^{2\pi i\zeta'^{(0)}q}+e^{2\pi i\zeta'^{(0)}q}\frac{\operatorname{sh} p}{\operatorname{ch} p}e^{2\pi i\zeta^{(0)}q}\right)\frac{1}{\operatorname{ch} p}e^{2\pi i\zeta^{(1)}q} \\
& \times \frac{1}{\operatorname{ch} p}\frac{e^{2\pi i\zeta^{(2)}q}}{\operatorname{ch} q\prod_{\alpha=1}^{n-1}\operatorname{ch}\left(q+\mu_\alpha^{(2)}\right)}\frac{1}{\operatorname{ch} p}\left(\frac{e^{2\pi i\zeta^{(3)}q}}{\operatorname{ch}\left(q+\mu^{(3)}\right)}\frac{\operatorname{sh} p}{\operatorname{ch} p}e^{2\pi i\zeta'^{(3)}q}\right. \\
& \left.+\,e^{2\pi i\zeta'^{(3)}q}\frac{\operatorname{sh} p}{\operatorname{ch} p}\frac{e^{2\pi i\zeta^{(3)}q}}{\operatorname{ch}\left(q+\mu^{(3)}\right)}\right)\frac{1}{\operatorname{ch} p}\frac{e^{2\pi i\zeta^{(2)}q}}{\operatorname{ch} q\prod_{\alpha=1}^{n-1}\operatorname{ch}\left(q+\mu_\alpha^{(2)}\right)}\frac{1}{\operatorname{ch} p}e^{2\pi i\zeta^{(1)}q}\,.
\end{aligned}
\tag{3.14}
$$

Using the identities (3.3), (3.4) and (3.10), we can write the density operator as

$$
\begin{aligned}
\rho = {} & \frac{1}{2}e^{-2\pi i(\zeta^{(3)}+\zeta'^{(3)})\mu^{(3)}}\frac{1}{\operatorname{ch} p}e^{\pi i(\zeta^{(0)}+\zeta'^{(0)})q}\frac{\operatorname{sh} 2p}{\operatorname{ch}(p+\frac{1}{2}\zeta^{(0)}-\frac{1}{2}\zeta'^{(0)})\operatorname{ch}(p-\frac{1}{2}\zeta^{(0)}+\frac{1}{2}\zeta'^{(0)})} \\
& \times e^{\pi i(\zeta^{(0)}+\zeta'^{(0)})q}\frac{1}{\operatorname{ch} p}e^{2\pi i\zeta^{(1)}q}\frac{1}{\operatorname{ch} p}\frac{e^{2\pi i\zeta^{(2)}q}}{\operatorname{ch} q\prod_{\alpha=1}^{n-1}\operatorname{ch}\left(q+\mu_\alpha^{(2)}\right)}\frac{1}{\operatorname{ch} p} \\
& \times \frac{e^{2\pi i\mu^{(3)}p}}{\operatorname{ch}(p-\zeta'^{(3)})}\left(e^{\pi p}\frac{e^{2\pi i(\zeta^{(3)}+\zeta'^{(3)})q}}{\operatorname{ch} q}e^{\pi p}+e^{-\pi p}\frac{e^{2\pi i(\zeta^{(3)}+\zeta'^{(3)})q}}{\operatorname{ch} q}e^{-\pi p}\right)\frac{e^{-2\pi i\mu^{(3)}p}}{\operatorname{ch}(p+\zeta'^{(3)})} \\
& \times \frac{1}{\operatorname{ch} p}\frac{e^{2\pi i\zeta^{(2)}q}}{\operatorname{ch} q\prod_{\alpha=1}^{n-1}\operatorname{ch}\left(q+\mu_\alpha^{(2)}\right)}\frac{1}{\operatorname{ch} p}e^{2\pi i\zeta^{(1)}q}\,.
\end{aligned}
\tag{3.15}
$$

Further commuting exponential terms using (3.3) we obtain

$$\begin{aligned}\rho = &\frac{1}{2} e^{\pi i(\zeta^{(0)}+\zeta'^{(0)})q} \operatorname{ch}\left(p+\tilde{\mu}_3^{(0)}\right) \operatorname{ch} q\, e^{2\pi i \tilde{m} q}\, e^{-2\pi i(\zeta^{(3)}+\zeta'^{(3)})\mu^{(3)}} \\ &\times \frac{e^{-2\pi i \tilde{m} q}}{\operatorname{ch} q} \frac{\operatorname{sh} 2p}{\prod_{\beta=1}^{3} \operatorname{ch}\left(p+\tilde{\mu}_\beta^{(0)}\right) \operatorname{ch}\left(p-\tilde{\mu}_\beta^{(0)}\right)} \frac{e^{-2\pi i \tilde{m} q}}{\operatorname{ch} q} \left(\prod_{a=1}^{n-1} e^{2\pi i \tilde{\zeta}^{(a)} p} \frac{1}{\operatorname{ch} q} \right) \\ &\times \frac{e^{2\pi i \tilde{\zeta}^{(n)} p}}{\operatorname{ch} p \operatorname{ch}\left(p+\tilde{\mu}^{(n)}\right)} \left(e^{\pi p} \frac{e^{-2\pi i \widetilde{M} q}}{\operatorname{ch} q} e^{\pi p} + e^{-\pi p} \frac{e^{-2\pi i \widetilde{M} q}}{\operatorname{ch} q} e^{-\pi p} \right) \frac{e^{-2\pi i \tilde{\zeta}^{(n)} p}}{\operatorname{ch} p \operatorname{ch}\left(p-\tilde{\mu}^{(n)}\right)} \\ &\times \left(\prod_{a=1}^{n-1} \frac{1}{\operatorname{ch} q} e^{-2\pi i \tilde{\zeta}^{(n-a)} p} \right) \frac{e^{-2\pi i \tilde{m} q}}{\operatorname{ch} q} \frac{1}{\operatorname{ch}\left(p+\tilde{\mu}_3^{(0)}\right)} e^{-\pi i(\zeta^{(0)}+\zeta'^{(0)})q},\end{aligned} \tag{3.16}$$

with

$$\begin{aligned} &\tilde{m} = -\frac{1}{2}\zeta^{(0)} - \frac{1}{2}\zeta'^{(0)} - \zeta^{(1)} - \zeta^{(2)}, \qquad \widetilde{M} = -\zeta^{(3)} - \zeta'^{(3)} \\ &\tilde{\mu}_1^{(0)} = \frac{1}{2}\zeta^{(0)} - \frac{1}{2}\zeta'^{(0)}, \qquad \tilde{\mu}_2^{(0)} = \frac{1}{2}\zeta^{(0)} + \frac{1}{2}\zeta'^{(0)}, \qquad \tilde{\mu}_3^{(0)} = \frac{1}{2}\zeta^{(0)} + \frac{1}{2}\zeta'^{(0)} + \zeta^{(1)}, \\ &\tilde{\mu}^{(n)} = -\zeta'^{(3)}, \\ &\tilde{\zeta}^{(1)} = \mu_1^{(2)}, \qquad \tilde{\zeta}^{(n)} = \mu^{(3)} - \mu_{n-1}^{(2)}, \qquad \tilde{\zeta}^{(a)} = \mu_a^{(2)} - \mu_{a-1}^{(2)}, \quad a = 2, \cdots, n-1\,. \end{aligned} \tag{3.17}$$

Applying the canonical transformation (3.1) and removing terms at the beginning and end of (3.16) by conjugation, we obtain up to an overall phase (see footnote 10) the density operator of the mirror theory. This is a linear quiver with one $\mathrm{Sp}(2N)$ node with three fundamental hypermultiplets, connected to n $\mathrm{U}(2N)$ nodes, where the last $\mathrm{U}(2N)$ node has one antisymmetric and two fundamental hypermultiplets. By shifting eigenvalues the masses of hypermultiplets transforming in the bifundamental representation of two $\mathrm{U}(2N)$ nodes and the mass of one of the $\mathrm{U}(2N)$ fundamental hypermultiplets can be set to zero. This leaves us with the masses: $\tilde{m}$ for the $\mathrm{Sp}(2N) \times \mathrm{U}(2N)$ bifundamental hypermultiplet, $\widetilde{M}$ for the antisymmetric hypermultiplet, $\tilde{\mu}_{1,2,3}^{(0)}$ for the three $\mathrm{Sp}(2N)$ fundamental hypermultiplets, $\tilde{\mu}^{(n)}$ for one $\mathrm{U}(2N)$ fundamental hypermultiplet. Moreover the theory has FI parameters $\tilde{\zeta}^{(a)}$, for $a = 1, \cdots, n$. The mirror map between parameters is given by (3.17).

This general approach to finding mirror maps can be easily extended to more general $\hat{D}$-quivers. For mirrors involving a 'bad' linear quiver, proposed in [11] the matrix model of the linear quiver is divergent,[15] while the matrix model of the $\hat{D}$ quiver is still finite. As we mention in the discussion section, the density operators of the 'bad' linear quivers can be matched to those of the 'good' $\hat{D}$-quivers with a simple replacement, alas, this replacement is rather ad hoc and we do not know whether it represents a true regularization of the non trace class density operator of the 'bad' quiver.

[15]It was proposed in [5] that the divergence of 'bad' quiver matrix model is related to a mismatch between the R-symmetry group in the UV localization computation, and the R-symmetry group at the infrared fixed point.

4 Computing the partition function

Having re-expressed the S^3 partition function of $\hat{D}$-quiver theories as a free fermion partition function, we proceed now with its evaluation following the technique developed in [15]. More precisely we compute the perturbative part in the large μ expansion of the grand potential J and extract from it the perturbative part in the large N expansion of the partition function Z, which turns out to be the Airy function (1.9). Our strategy to compute J has new ingredients compared to [15], in particular we use a simplifying recursion method to compute the perturbative expansion of $\mathrm{Tr}\,\rho^l$, for arbitrary $l \geqslant 1$. This method can be used for any density operator ρ. We also rely on the integral representation of J, following [48], to evaluate its perturbative part. Masses and FI parameters introduce extra difficulties in the computation and we set all of them to zero in this section.

We proceed to present this strategy and provide the exact evaluations for all the $\hat{D}$-quivers considered in this paper.

4.1 General strategy

The starting point for our computation of the perturbative part of the $\hat{D}$-quiver (and linear mirror) partition functions (2.25) is the Fermi gas reformulation

$$Z(N) = \frac{1}{N!} \sum_{\sigma \in S_N} \frac{(-1)^\sigma}{2^{n_\sigma}} \int \prod_{i=1}^{N} d\lambda_i \prod_{i=1}^{N} \rho(\lambda_i, \lambda_{\sigma(i)}) \,, \tag{4.1}$$

where we recall that n_σ counts the number of cycles in σ. The standard analysis [49] is to factor the integral into its closed loops

$$\int d\lambda_1 \cdots d\lambda_l\, \rho(\lambda_1, \lambda_2)\rho(\lambda_2, \lambda_3) \cdots \rho(\lambda_l, \lambda_1) = \mathrm{Tr}\,\rho^l \equiv Z_l \,. \tag{4.2}$$

These loops of course correspond to the cycles of the permutation σ, and so the summand of (4.1) depends only on Z_l and the conjugacy class of σ. Conjugacy classes of S_N can be labelled a set of integers $\{m_l\}$, where m_l is the number of cycles of length l. In terms of this labelling we have

$$\frac{1}{2^{n_\sigma}} = \prod_l \frac{1}{2^{m_l}} \,, \tag{4.3}$$

and the number of permutations in a given conjugacy class is given by

$$\frac{N!}{\prod_l m_l!\, l^{m_l}} \,. \tag{4.4}$$

With these combinatorics (4.1) becomes

$$Z(N) = {\sum_{\{m_l\}}}' \prod_l \frac{(\frac{1}{2} Z_l)^{m_l}\, (-1)^{(l-1)m_l}}{m_l!\, l^{m_l}} \,, \tag{4.5}$$

where the primed sum denotes a sum over sets that satisfy $\sum_l l m_l = N$.

The computation of $Z(N)$ thus boils down to the evaluation of Z_l. To compute results for large N, the standard approach [49] is consider the grand canonical partition function

$$\Xi(z) = 1 + \sum_{N=1}^{\infty} Z(N) z^N = e^{J(\mu)}, \qquad z = e^{\mu}\,, \tag{4.6}$$

where μ is the chemical potential, and $J(\mu)$ the grand canonical potential, given by

$$J(\mu) = -\sum_{l=1}^{\infty} \frac{(-1)^l Z_l e^{\mu l}}{2l}\,. \tag{4.7}$$

The strategy is to first find an expression for Z_l, then to resum the expression and obtain $J(\mu)$ using (4.7), and finally to recover $Z(N)$ by computing

$$Z(N) = \frac{1}{2\pi i}\int_{\mu_0-\pi i}^{\mu_0+\pi i} d\mu\, e^{J(\mu)-\mu N}\,, \tag{4.8}$$

where μ_0 can be chosen arbitrarily without affecting the result.

In practice, computing Z_l exactly for arbitrary l is highly non trivial. To make the problem tractable it is useful to reformulate it within Wigner's phase space [15]. For a general review of Wigner's phase space see [50]; here we simply summarise the properties that we require.

The Wigner transform of an operator[16] $\hat{A}$ is given (with $\hbar = \frac{1}{2\pi}$) by

$$A_W(q,p) = \int dq' \left\langle q - \frac{q'}{2}\right| \hat{A} \left| q + \frac{q'}{2}\right\rangle e^{2\pi i p q'}\,. \tag{4.9}$$

Some important identities are

$$(\hat{A}\hat{B})_W = A_W \star B_W, \qquad \star = \exp\left[\frac{i}{4\pi}\left(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overrightarrow{\partial}_q \overleftarrow{\partial}_p\right)\right], \qquad \mathrm{Tr}(\hat{A}) = \int dp dq A_W\,. \tag{4.10}$$

In the language of phase space Z_l becomes

$$Z_l = \int dp dq\, \overbrace{\rho_W \star \rho_W \cdots \star \rho_W}^{l}\,. \tag{4.11}$$

We generate an expansion for the integrand of (4.11) by performing a derivative expansion of the star products (4.10). To this end, we introduce into the star product an expansion

[16] To avoid confusion between phase space variables p, q, and the canonical position and momentum operators, we give all operators hats for the remainder of this section.

parameter ϵ which will be set at the end to 1.[17]

$$\begin{aligned}\star &= \exp\left[\frac{i\epsilon}{4\pi}\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overrightarrow{\partial}_q\overleftarrow{\partial}_p\right)\right] \\ &= 1 + \frac{i\epsilon}{4\pi}\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overrightarrow{\partial}_q\overleftarrow{\partial}_p\right) - \frac{\epsilon^2}{32\pi^2}\left(\overleftarrow{\partial}^2_q\overrightarrow{\partial}^2_p + \overrightarrow{\partial}^2_q\overleftarrow{\partial}^2_p - 2\overleftarrow{\partial}_{q,p}\overrightarrow{\partial}_{q,p}\right) + \mathcal{O}(\epsilon^3)\,.\end{aligned} \tag{4.13}$$

In [15], the role of the expansion parameter ϵ was played by the Planck constant $\hbar$, which was proportional to the Chern-Simons level of the ABJM theory. We do not have a tunable $\hbar$ here, however it still proves useful to consider the derivative expansion associated to ϵ, as we now explain.

The ϵ expansion of the integrand of (4.11) takes the form

$$(\hat{\rho}^l)_W(p,q) = \sum_{n\geqslant 0} \epsilon^n\, \rho_{l(n)}(p,q). \tag{4.14}$$

Note that the ϵ factors come from the expansion of the star products present in (4.11), as well as those arising when replacing the density operator (2.42) by its Wigner transform. Z_l can then be evaluated order by order in ϵ

$$Z_l = \sum_{n\geqslant 0} \epsilon^n Z_{l(n)}, \qquad Z_{l(n)} = \int dpdq\, \rho_{l(n)}\,. \tag{4.15}$$

Resumming each term using (4.7) then generates an ϵ expansion for $J(\mu)$

$$J(\mu) = \sum_{n\geqslant 0} \epsilon^n J_{(n)}(\mu), \qquad J_{(n)}(\mu) = -\sum_{l=1}^{\infty} \frac{(-1)^l Z_{l(n)} e^{\mu l}}{2l}\,. \tag{4.16}$$

As for the $\hat{A}$-quivers [15], we anticipate that for $\hat{D}$-quivers (and their linear mirrors), $J(\mu)$ also admits an asymptotic expansion of the form

$$J(\mu) = \frac{C(\epsilon)}{3}\mu^3 + B(\epsilon)\mu + A(\epsilon) + \mathcal{O}(e^{-\alpha\mu})\,, \qquad \alpha > 0\,, \tag{4.17}$$

where each of the coefficients A, B and C are given by power expansions in ϵ. In principle, to obtain a meaningful result for A, B and C we should now compute and then resum an infinite series of corrections in powers of ϵ, which are really all of the same order since $\epsilon = 1$. From the study of $\hat{A}$-quivers, it is expected that the expansions of $C(\epsilon)$ and $B(\epsilon)$ truncate at orders ϵ^0 and ϵ^2 respectively, so that the first few orders of the ϵ expansion are sufficient to compute them exactly. We give a proof of this truncation for $\hat{D}$-quivers with

[17] In addition to the differential expression for the star product (4.13), it also has an equivalent integral representation

$$(f\star g)(p,q) = \frac{4}{\epsilon^2}\int dq'dp'dq''dp''\, f(p+p',q+q')g(p+p'',q+q'')e^{4\pi i/\epsilon(q'p''-q''p')}\,. \tag{4.12}$$

In some cases (for instance when f or g involve delta functions) (4.12) produces an exact result that is non perturbative in ϵ [51]. In these cases extra care should be taken, because the perturbative star product (4.10) is not valid.

equal number of fundamental hypermultiplets on each pair of terminating $U(N)$ nodes in appendix E. We assume that it holds for the other quivers as well.[18]

It remains to plug the result (4.17) into (4.8) to extract the perturbative part of Z at large N. In practice the evaluation is done by setting the contour integral parameter μ_0 to the saddle point μ^* of the integrand and by extending the contour along all the imaginary axis to $(\mu^* - i\infty, \mu^* + i\infty)$. As explained in [15, 22], this change of contour does not affect the perturbative part of the result. The integration leads to the Airy function behaviour of the partition function at large N, which is our main result

$$Z(N) = C^{-\frac{1}{3}} e^A \,\mathrm{Ai}\Big[C^{-\frac{1}{3}}(N - B)\Big] + Z_{\mathrm{np}}(N)\,, \tag{4.18}$$

where $Z_{\mathrm{np}}(N)$ denotes non-perturbative, exponentially suppressed contributions, and we note that the undetermined coefficient A only affects the overall prefactor.

4.2 Recursive formula for $(\hat{\rho}^l)_W$

In this subsection we present a simple recursive approach for evaluating the coefficients in the ϵ expansion of $(\hat{\rho}^l)_W$ (4.14). This comes from the ϵ expansion of

$$(\hat{\rho}^{l+1})_W = (\hat{\rho}^l)_W \star \rho_W\,. \tag{4.19}$$

One first needs to evaluate the ϵ expansion of ρ_W (which is due to replacing all operator products by star products)

$$\rho_W(p,q) = \sum_{n\geqslant 0} \epsilon^n \,\rho_{(n)}(p,q)\,, \tag{4.20}$$

which also serves as the initial conditions for the recursion $\rho_{1(n)} = \rho_{(n)}$.

At order ϵ^0, equation (4.19) gives

$$\rho_{l+1(0)} = \rho_{l(0)}\rho_{(0)}\,, \tag{4.21}$$

which is solved by

$$\rho_{l(0)} = \rho_{(0)}^l\,. \tag{4.22}$$

At order ϵ^1 we get

$$\rho_{l+1(1)} = \rho_{(0)}^l \rho_{(1)} + \rho_{(0)}\rho_{l(1)} \qquad \Rightarrow \qquad \rho_{l(1)} = l\rho_{(0)}^{l-1}\rho_{(1)}\,. \tag{4.23}$$

At order ϵ^2 we then get

$$\begin{aligned}\rho_{l+1(2)} = \rho_{(0)}\rho_{l(2)} + \rho_{(0)}^l\rho_{(2)} + l\rho_{(0)}^{l-1}\rho_{(1)}^2 - \frac{1}{32\pi^2}\rho_{(0)}^{l-2} l\Big[2\big(\rho_{(0)}\rho''_{(0)}\ddot{\rho}_{(0)} - \rho_{(0)}\dot{\rho}'^2_{(0)}\big) \\ + (l-1)\big(\ddot{\rho}_{(0)}\rho'^2_{(0)} + \rho''_{(0)}\dot{\rho}^2_{(0)} - 2\rho'_{(0)}\dot{\rho}_{(0)}\dot{\rho}'_{(0)}\big)\Big],\end{aligned} \tag{4.24}$$

where

$$\dot{f}(p,q) \equiv \partial_p\, f(p,q)\,, \qquad f'(p,q) \equiv \partial_q\, f(p,q)\,. \tag{4.25}$$

[18] We checked that C and B do not receive contributions at order ϵ^4 for all $\hat{D}$-quivers.

Solving the recurrence relation, with initial condition $\rho_{1(2)} = \rho_{(2)}$ yields

$$\begin{aligned}\rho_{l(2)} = l\rho_{(0)}^{l-1}\rho_{(2)} + \frac{1}{2}l(l-1)\rho_{(0)}^{l-2}\rho_{(1)}^2 - \frac{1}{96\pi^2}\rho_{(0)}^{l-3}l(l-1)\Big[3\big(\rho_{(0)}\rho''_{(0)}\ddot{\rho}_{(0)} - \rho_{(0)}\dot{\rho}'^2_{(0)}\big)\\ + (l-2)\big(\ddot{\rho}_{(0)}\rho'^2_{(0)} + \rho''_{(0)}\dot{\rho}^2_{(0)} - 2\rho'_{(0)}\dot{\rho}_{(0)}\dot{\rho}'_{(0)}\big)\Big].\end{aligned} \tag{4.26}$$

This procedure can be straightforwardly continued to higher order in ϵ. Finally we plug the expansion coefficients into (4.15) to obtain the ϵ expansion of Z_l. In particular this gives

$$\begin{aligned}Z_{l(0)} &= \int dpdq\, \rho_{(0)}^l\,,\\ Z_{l(1)} &= \int dpdq\, l\rho_{(0)}^{l-1}\rho_{(1)}\,,\\ Z_{l(2)} &= \int dpdq\, \left(l\rho_{(0)}^{l-1}\rho_{(2)} + \frac{1}{2}l(l-1)\rho_{(0)}^{l-2}\rho_{(1)}^2 - \frac{1}{96\pi^2}l^2(l-1)(l-2)\rho_{(0)}^{l-4}\dot{\rho}^2_{(0)}\rho'^2_{(0)}\right).\end{aligned} \tag{4.27}$$

Where the last line follows from integrating (4.26) by parts, and using the fact that for $\hat{A}$ or $\hat{D}$ quivers we always have the decomposition $\rho_{(0)} = t(p)u(q)$.

We stress that this algorithm is very general and can be applied to any $\hat{A}$ or $\hat{D}$-quiver. All one needs in order to compute Z_l is to plug into (4.27) the ϵ expansion of the density operator itself.

4.3 Computing $Z(N)$ for $\hat{D}$-quivers

We now show how to apply the approach outlined in the previous sections and compute $Z(N)$ for a generic $\hat{D}$-quiver with arbitrary number of nodes and arbitrary number of fundamental hypermultiplets on each node. We set here all masses and FI parameters to zero, as this simplifies the explicit evaluation of the phase space integrals. The quiver diagram is shown in figure 1, and the density operator is given by (2.42) with

$$F^{(a)}(q) = \frac{1}{\mathrm{ch}^{n^{(a)}}q}, \qquad F'^{(a)}(q) = \frac{1}{\mathrm{ch}^{n'^{(a)}}q}. \tag{4.28}$$

We first work out the ϵ-expansion of ρ_W itself, which is then plugged in to the result of the recursive formula (4.27).

Using manipulations similar to those used in section 3

$$\begin{aligned}&\frac{1}{\mathrm{ch}^{n^{(0)}}q}\star\frac{\mathrm{sh}\,p}{\mathrm{ch}\,p}\star\frac{1}{\mathrm{ch}^{n'^{(0)}}q} + \frac{1}{\mathrm{ch}^{n'^{(0)}}q}\star\frac{\mathrm{sh}\,p}{\mathrm{ch}\,p}\star\frac{1}{\mathrm{ch}^{n^{(0)}}q}\\ &= \frac{1}{\mathrm{ch}^{\min(n^{(0)},n'^{(0)})}q}\star\frac{1}{\mathrm{ch}\,p}\star\left(\mathrm{sh}\,p\star\frac{1}{\mathrm{ch}^{|n^{(0)}-n'^{(0)}|}q}\star\mathrm{ch}\,p + \mathrm{ch}\,p\star\frac{1}{\mathrm{ch}^{|n^{(0)}-n'^{(0)}|}q}\star\mathrm{sh}\,p\right)\\ &\quad\star\frac{1}{\mathrm{ch}\,p}\star\frac{1}{\mathrm{ch}^{\min(n^{(0)},n'^{(0)})}q}\\ &= \frac{1}{\mathrm{ch}^{\min(n^{(0)},n'^{(0)})}q}\star\frac{1}{\mathrm{ch}\,p}\star\frac{2\,\mathrm{sh}\,2p}{\mathrm{ch}^{|n^{(0)}-n'^{(0)}|}q}\star\frac{1}{\mathrm{ch}\,p}\star\frac{1}{\mathrm{ch}^{\min(n^{(0)},n'^{(0)})}q}\,,\end{aligned} \tag{4.29}$$

where in the last step we evaluated the expression inside parentheses using the exact star product (4.12). This allows us to write the Wigner transform of the density operator (2.42) as[19]

$$
\begin{aligned}
\rho_W = {} & e^{T(p)} \star e^{U_1(q)} \star e^{T(p)} \star e^{S(p)+2U_0(q)} \star e^{T(p)} \star \left(\prod_{k=1}^{\lambda-1}{}_{\star} e^{U_k(q)} \star e^{T(p)} \right) \star e^{S(p)+2U_\lambda(q)} \\
& \star \left(\prod_{k=1}^{\lambda-2}{}_{\star} e^{T(p)} \star e^{U_{\lambda-k}(q)} \right),
\end{aligned}
\tag{4.30}
$$

where

$$
S(p) = \log \operatorname{sh} 2p, \qquad T(p) = \log \frac{1}{\operatorname{ch} p}, \qquad U_i(q) = \log \frac{1}{\operatorname{ch}^{\eta_i} q}, \tag{4.31}
$$

and

$$
\begin{aligned}
\eta_0 &= \frac{1}{2}|n^{(0)} - n'^{(0)}|, \qquad \eta_1 = \min(n^{(0)}, n'^{(0)}), \qquad \eta_i = n^{(i-1)}, \quad i = 2, \cdots, L \\
\eta_\lambda &= \frac{1}{2}|n^{(L)} - n'^{(L)}|, \qquad \eta_{\lambda-1} = \min(n^{(L)}, n'^{(L)}), \qquad \lambda = L+2.
\end{aligned}
\tag{4.32}
$$

A first useful manipulation is to conjugate[20] the density operator into a more symmetric form that resembles a palindrome

$$
\begin{aligned}
\rho_W \approx {} & \sqrt[\star]{e^{S(p)+2U_\lambda(q)}} \star e^{T(p)} \star \left(\prod_{k=1}^{\lambda-1}{}_{\star} e^{U_{\lambda-k}(q)} \star e^{T(p)} \right) \star e^{S(p)+2U_0(q)} \\
& \star \left(\prod_{k=1}^{\lambda-1}{}_{\star} e^{T(p)} \star e^{U_k(q)} \right) \star e^{T(p)} \star \sqrt[\star]{e^{S(p)+2U_\lambda(q)}},
\end{aligned}
\tag{4.33}
$$

where $\sqrt[\star]{\ }$ is the star square root, and its expansion gives

$$
\sqrt[\star]{e^{S(p)+2U_\lambda(q)}} = e^{\frac{1}{2}S+U_\lambda} \left(1 + \frac{\epsilon^2}{128\pi^2} \left(2\ddot{S}U''_\lambda + 2\ddot{S}U'^2_\lambda + \dot{S}^2 U''_\lambda \right) + \mathcal{O}(\epsilon^4) \right). \tag{4.34}
$$

The reason for making this conjugation to palindromic form is that the ϵ expansion of (4.33) is purely in even powers of ϵ. The easiest way then to compute the ϵ expansion is to build it up from the central $e^{S(p)+2U_0(q)}$ piece. We first compute

$$
e^T \star e^{S+2U_0} \star e^T = e^{2T+S+2U_0} \left(1 - \frac{\epsilon^2}{8\pi^2} \ddot{T} \left(U''_0 + 2U'^2_0 \right) + \mathcal{O}(\epsilon^4) \right). \tag{4.35}
$$

By plugging this result in, we can then easily compute

$$
\begin{aligned}
& e^{U_1} \star \left(e^T \star e^{S+2U_0} \star e^T \right) \star e^{U_1} \\
& \quad = e^{2T+S+2U_0+2U_1} \left(1 - \frac{\epsilon^2}{8\pi^2} \ddot{T}(U''_0 + 2U'^2_0) - \frac{\epsilon^2}{16\pi^2} U''_1 \left(2\ddot{T} + \ddot{S} + (2\dot{T} + \dot{S})^2 \right) + \mathcal{O}(\epsilon^4) \right).
\end{aligned}
\tag{4.36}
$$

[19] $\prod_\star$ is defined by ordered star multiplication.

[20] Since we are ultimately computing $Z_l = \operatorname{Tr} \hat{\rho}^l$ we can use cyclicity of the trace to make conjugations $\hat{\rho} \to \hat{V}^{-1}\hat{\rho}\hat{V}$, which in the language of phase space becomes $\rho_W \to (\hat{V}^{-1})_W \star \rho_W \star V_W$.

Continuing this procedure we find the full expansion of (4.33) up to $\mathcal{O}(\epsilon^2)$

$$\begin{aligned}\rho_W = e^{2\lambda T+2S+2\sum_{k=0}^{\lambda}U_k}\Bigg[&1-\frac{\epsilon^2}{8\pi^2}\ddot{T}\sum_{k=0}^{\lambda-1}\left(\sum_{j=0}^{k}U_j''+2\Big(\sum_{j=0}^{k}U_j'\Big)^2\right)\\ &-\frac{\epsilon^2}{16\pi^2}\sum_{k=1}^{\lambda}U_k''\left(\ddot{S}+2k\ddot{T}+\big(\dot{S}+2k\dot{T}\big)^2\right)\\ &-\frac{\epsilon^2}{16\pi^2}\left(\ddot{S}\Big(\sum_{k=0}^{\lambda-1}U_k''+2\sum_{k=0}^{\lambda-1}U_k'\sum_{j=0}^{\lambda}U_j'\Big)+U_\lambda''\dot{S}\big(\dot{S}+2\lambda\dot{T}\big)\right)+\mathcal{O}(\epsilon^4)\Bigg].\end{aligned} \tag{4.37}$$

The coefficients of this expansion are $\rho_{(0)}$ and $\rho_{(2)}$ (4.20), which serve as the seed for the recursion and can be plugged directly into (4.27). We find that $Z_{l(0)}$ is given by

$$Z_{l(0)}=\int dpdq\,\rho_{(0)}^l=\int dpdq\,\frac{\mathrm{th}^{2l}\,p}{\mathrm{ch}^{2l(\lambda-2)}\,p\,\mathrm{ch}^{2l\nu}\,q},\qquad \nu=\sum_{k=0}^{\lambda}\eta_k\,. \tag{4.38}$$

The expression for $Z_{l(2)}$ is considerably more involved, but it can be simplified by integrating by parts to remove all double derivatives. Integrating by parts the first term in $Z_{l(2)}$ (4.27) gives

$$\begin{aligned}&\int dpdq\,l\rho_{(0)}^{l-1}\rho_{(2)}\\ &=\frac{1}{4\pi^2}\int dpdq\,l^2e^{2l(\lambda T+S+\sum_{k=0}^{\lambda}U_k)}\Bigg[2\dot{T}\big(\dot{S}+\lambda\dot{T}\big)\sum_{k=0}^{\lambda-1}\sum_{j=0}^{k}U_j'\Big(\sum_{i=0}^{k}U_i'-l\sum_{i=0}^{\lambda}U_i'\Big)\\ &+\frac{1}{2}\sum_{k=1}^{\lambda}U_k'\big(\dot{S}+2k\dot{T}\big)\big(\dot{S}+2k\dot{T}-2l\big(\dot{S}+\lambda\dot{T}\big)\big)\sum_{j=0}^{\lambda}U_j'\\ &-\left(\dot{S}\big(\dot{S}+\lambda\dot{T}\big)(l-1)\sum_{k=0}^{\lambda-1}U_k'-\frac{1}{2}\dot{S}\big(\dot{S}+2\lambda\dot{T}\big)U_\lambda'\right)\sum_{j=0}^{\lambda}U_j'\Bigg].\end{aligned} \tag{4.39}$$

The second (non zero) term in $Z_{l(2)}$ gives

$$\begin{aligned}&\int dpdq\,\frac{-1}{96\pi^2}l^2(l-1)(l-2)\rho_{(0)}^{l-4}\dot{\rho}_{(0)}^2\rho_{(0)}'^2\\ &\quad=\int dpdq\,\frac{-1}{6\pi^2}l^2(l-1)(l-2)e^{2l(\lambda T+S+\sum_{k=0}^{\lambda}U_k)}\left(\sum_{j=0}^{k}\dot{U}_j\right)^2(\dot{S}+\lambda\dot{T})^2\,.\end{aligned} \tag{4.40}$$

Combining these expressions, substituting (4.31) and simplifying leads finally to

$$\begin{aligned}Z_{l(2)}=\int dpdq\,l^2\frac{\pi^2}{24}\frac{\mathrm{th}^{2l-2}\,p\,\mathrm{th}^2\,q}{\mathrm{ch}^{2l(\lambda-2)}\,p\,\mathrm{ch}^{2l\nu}\,q}\Big[&3(2l-1)\Delta\nu-(4l^2-1)\nu^2\\ &+2\,\mathrm{th}^2\,p\Big((4l^2-1)(\lambda-1)\nu^2-3\Delta\nu(l(\lambda-2)+1)-6\Sigma_1\Big)\\ &-\mathrm{th}^4\,p\Big((4l^2-1)(\lambda-1)^2\nu^2-3\lambda^2\nu^2+3\Delta\nu(2l(\lambda-1)+1)-12(\Sigma_2-\Sigma_1)\Big)\Big],\end{aligned} \tag{4.41}$$

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where

$$\Delta = \eta_0 + \eta_\lambda, \qquad \Sigma_1 = \sum_{i>j} \eta_i \eta_j (i-j),$$
$$\Sigma_2 = \sum_{i>j} \eta_i \eta_j \left((\lambda - i)^2 + j^2 + \lambda^2\right) - \sum_{i=0}^{\lambda} \eta_i^2 i(\lambda - i). \tag{4.42}$$

In order to evaluate the integrals appearing in $Z_{l(0)}$ and $Z_{l(2)}$, we require only the identity

$$\int dx \frac{\operatorname{th}^a x}{\operatorname{ch}^b x} = \frac{(1+e^{i\pi a})\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b}{2}\right)}{2^{b+1}\pi\Gamma\left(\frac{a+b+1}{2}\right)}, \qquad \operatorname{Re}(a) > -1, \quad \operatorname{Re}(b) > 0 \tag{4.43}$$

Integrating (4.38) and (4.41) using (4.43), simplifying by shifting the arguments of Gamma functions and choosing to express the result in terms of $L = \lambda - 2$ we find

$$Z_{l(0)} = \frac{1}{2^{2l(\nu+L)}\pi^{3/2}} \frac{\Gamma(l+\frac{1}{2})\Gamma(\nu l)\Gamma(Ll)}{\Gamma(\nu l + \frac{1}{2})\Gamma((L+1)l+\frac{1}{2})},$$
$$Z_{l(2)} = \frac{\pi^{1/2}}{3 \cdot 2^{2l(\nu+L)+3}} l^2 F(l) \frac{\Gamma(l+\frac{1}{2})\Gamma(\nu l)\Gamma(Ll)}{\Gamma(\nu l + \frac{3}{2})\Gamma((L+1)l+\frac{3}{2})}, \tag{4.44}$$

where

$$F(l) = \nu^2(1+(L+1)(L+2)) - 3\Delta\nu - 3(\Sigma_1+\Sigma_2)$$
$$+ l\big(\nu^2(L+2)(L+3) - 6\Delta\nu(L+1) - 6(L+1)\Sigma_1 + 6\Sigma_2\big) - 2l^2\nu^2 L(L+1). \tag{4.45}$$

Having computed the ϵ expansion of Z_l, we can now compute the ϵ expansion for $J(\mu)$ by resuming each term using (4.16). As for the $\hat{A}$ quivers, this gives some complicated hypergeometric functions [15, 39], from which we can then extract the large μ asymptotic expansion (4.17). Ultimately, we are interested only in the perturbative part of this expansion. A recent paper [48] pointed out a very elegant and simple way to extract this perturbative piece of $J(\mu)$, just by evaluating a single residue involving Z_l.

The first step is to write a Mellin-Barnes integral representation for the infinite sum (4.16)

$$J_{(n)}(\mu) = -\sum_{l=1}^{\infty} \frac{(-1)^l Z_{l(n)} e^{\mu l}}{2l} = -\int_{c-i\infty}^{c+i\infty} \frac{dl}{4\pi i} \Gamma(l)\Gamma(-l) Z_{l(n)} e^{l\mu}, \tag{4.46}$$

where c can be chosen arbitrarily in $(0,1)$, and $Z_{l(n)}$ should now be regarded as a function of l, analytically continued to the complex plane. To see how this integral representation reproduces the infinite sum (4.46), note that for $\mu < 0$ we can close the contour of integration around the region with $\operatorname{Re}(l) > c$. Since $Z_{l(n)}$ itself has no poles in this region, the only enclosed poles are the simple poles of $\Gamma(-l)$ for $l = n \in \mathbb{N}^+$. Using the fact that

$$\operatorname*{Res}_{l=n} \Gamma(-l) = \frac{(-1)^n}{n!}, \tag{4.47}$$

we recover (4.46).

Since we are interested in the asymptotic region $\mu \gg 0$, we close the contour of integration around the region $\mathrm{Re}(l) < c$. In this region there can be poles due to both $Z_{l(n)}$ and $\Gamma(l)$. The residue at $l = 0$ turns out to be the only one giving a contribution that is not exponentially suppressed at large μ. Therefore we can immediately evaluate

$$J_{(n)}(\mu) = -\frac{1}{2} \operatorname*{Res}_{l=0} \Gamma(l)\Gamma(-l) Z_{l(n)} e^{l\mu} + \mathcal{O}(e^{-\alpha\mu}), \qquad \alpha > 0\,. \tag{4.48}$$

Evaluating in this way the perturbative contributions to $J(\mu)$ from $J_{(0)}(\mu)$ and $J_{(2)}(\mu)$, we find the asymptotic expansion of the form (4.17), with C and B coefficients

$$\begin{aligned} C &= \frac{1}{4\pi^2 L\nu}, \\ B &= \frac{1 - 3\Delta\nu + 2\nu^2 + L(3+L)(\nu^2-1) - 3(\Sigma_1 + \Sigma_2)}{12 L\nu}\,. \end{aligned} \tag{4.49}$$

As explained before, the coefficients C and B do not receive contributions from higher order terms in the ϵ expansion, so that (4.49) provides the exact result. These coefficients characterize the full perturbative part of $Z(N)$ as an Airy function (4.18), up to the overall coefficient A which is undetermined by our analysis.

5 Discussion

One of the main applications of our results concerns holography. We have found the complete large N perturbative result for the partition function of (good) $\hat{D}$-quivers as an Airy function (4.18). The large N expansion of the free energy starts with the terms

$$-\log Z(N) = \frac{2}{3\sqrt{C}} N^{\frac{3}{2}} - \frac{B}{\sqrt{C}} N^{\frac{1}{2}} + \cdots\,. \tag{5.1}$$

The leading term has the famous $N^{3/2}$ behaviour and its coefficient depends only on the parameter C. This coefficient has a simple geometric interpretation when the theory admits an M-theory holographic dual of the form $AdS_4 \times SE_7$, where SE_7 is a tri-Sasaki-Einstein manifold. Holography predicts the relation [52]

$$C = \frac{6}{\pi^6} \mathrm{Vol}(SE_7)\,. \tag{5.2}$$

This coefficient was already computed for $\hat{D}$-quivers with fundamental hypermultiplets and Chern-Simons terms by saddle point techniques in [53]. Our result for C (4.49) agrees with their computation after setting to zero the CS levels. The $\hat{D}$-quivers studied in this paper (with vanishing masses and FI terms) can be engineered as the low-energy limit of a stack of $2N$ M2-branes at an orbifold singularity in $\mathbb{C}^2/\mathbb{Z}_{2\nu} \times \mathbb{C}^2/\mathbb{D}_L$ [12, 13], where $\mathbb{Z}_{2\nu}$ and $\mathbb{D}_L$ are discrete subgroups of SU(2) of type A and D respectively. The additional data describing the distribution of fundamental hypermultiplets in the quiver should be encoded in four-form fluxes on vanishing cycles at the orbifold singularity [13], however the precise dictionary is not known. The mirror-dual linear quivers have the same M-theory holographic backgrounds. These M-theory backgrounds can be reduced to type IIA and

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T-dualized to type IIB in two ways by exchanging the M-theory and T-duality circles, leading to two IIB backgrounds in S-dual frames.

The backreacted geometry in the large N limit takes the form $AdS_4 \times S^7/(\mathbb{Z}_{2\nu} \times \mathbb{D}_L)$, where the two orbifolds act separately on the two $S^3 \simeq \mathrm{SU}(2)$ inside S^7. Recalling that the volume of S^7 is $\pi^4/3$ and the order of the $\mathbb{Z}_{2\nu}$ and $\mathbb{D}_L$ quotients are 2ν and $4L$ respectively, we obtain the holographic prediction

$$C = \frac{6}{\pi^6}\frac{\pi^4}{24\nu L} = \frac{1}{4\pi^2\nu L}\,, \tag{5.3}$$

in perfect agreement with (4.49). It would be interesting to find a similar nice geometrical interpretation in holography for the coefficient of subleading term $B/\sqrt{C}$ in (5.1) but this coefficient is very subtle and not even fully understood in the case of ABJM [54].

A localization calculation in AdS_4 [40] suggests that the result would be universal for any conformal 3d theory with an M-theory dual and enough supersymmetry. The $\hat{D}$-quivers (and their mirrors) studied in this paper fall into this class, and indeed we found the Airy function behavior, like for the $\hat{A}$-quivers. Note that this calculation does not predict the values of the C and B coefficients, rather they serve as arbitrary parameters of the calculation.

In our analysis of mirror symmetry, we left aside the $\hat{D}$-quivers whose naive mirror duals are bad linear quivers [11]. These are $\hat{D}$-quivers with numbers of fundamental hypermultiplets in a pair of terminating $\mathrm{U}(N)$ nodes differing by two or more. The bad linear quivers have a divergent matrix model leading to a non trace class density operator. One example considered in [11] for instance, is the naive duality between a $\hat{D}_4$-quiver with two fundamental hypermultiplets on an external $\mathrm{U}(N)$ node, and an $\mathrm{Sp}(2N)^2$ linear quiver with four fundamental hypermultiplets on one $\mathrm{Sp}(2N)$ node, and zero on the other. Manipulations as in (4.29) give the Wigner transformed density operator of the $\hat{D}_4$-quiver as

$$\frac{1}{\mathrm{ch}^2 p} \star \frac{\mathrm{sh}\, 2p}{\mathrm{ch}^2 q} \star \frac{\mathrm{sh}\, 2p}{\mathrm{ch}^6 p}\,. \tag{5.4}$$

The Wigner transformed density operator of the naive mirror dual linear quiver is

$$\frac{1}{\mathrm{ch}\, p} \star \mathrm{sh}\, 2q \star \frac{1}{\mathrm{ch}\, p} \star \frac{\mathrm{sh}\, 2q}{\mathrm{ch}^8 q}\,. \tag{5.5}$$

these operators would be related by the canonical transformation (3.1) (and conjugation by $\frac{1}{\mathrm{ch}^2 p}$), if we also included the replacement (with $n = 1$ and $m = 0$)

$$\frac{1}{\mathrm{ch}^n p} \star \frac{\mathrm{sh}\, 2q}{\mathrm{ch}^m p} \star \frac{1}{\mathrm{ch}^n p} \to \frac{\mathrm{sh}\, 2q}{\mathrm{ch}^{2n+m} p} \tag{5.6}$$

This is an ad hoc regularization of the partition function, transforming a non trace class density operator to one that is trace class, based on the assumption that the regularized partition function should satisfy the naive mirror symmetry. In fact, the replacement (5.6) (with suitable n, m) easily regularizes and identifies a mirror dual for any bad linear quiver with no fundamental hypermultiplets on one or both terminal nodes, provided the theory has in total at least four fundamental hypermultiplets.

It would be interesting to understand if (5.6) can be derived from a proper regularization of the divergent integrals in the matrix model.

One can think of several extensions to our work. Having found a Fermi gas formalism for the $\hat{A}$ and $\hat{D}$ quivers, it is natural to ask whether a Fermi gas formalism for the quiver theories of type $\hat{E}_{6,7,8}$ exist. To study these theories one needs to find suitable identities to put the corresponding matrix models in the form of a partition function for non-interacting particles in 1d.

Further generalizations involve necklace quivers with alternating Sp and SO groups. Likewise one can consider linear quivers with terminating SO nodes or symmetric hypermultiplets (see [41]). One could also try to find generalizations of the $\hat{D}$-quivers, replacing one of the $\hat{D}$ ends with some other gauge groups or matter fields.

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A Matrix models for quiver theories

We provide in this appendix the factors appearing in the $\hat{D}$-quiver and linear quiver matrix models. Each factor is associated to an $\mathcal{N}=4$ multiplet of the theory and is a function of the eigenvalues of the matrix model.

For U(N) nodes we have the ingredients

$$\begin{aligned}
\text{Eigenvalues:} \quad & \lambda_i \quad i=1,\cdots,N\,, \\
\text{Weyl group order:} \quad & |W| = N!\,, \\
\text{Vector multiplet:} \quad & Z_{\text{vec}} = \prod_{i<j} \text{sh}^2(\lambda_i-\lambda_j)\,, \\
\begin{array}{l}\text{Fundamental}\\ \text{hypermultiplet:}\end{array} \quad & Z^{\text{fund}}_{\text{hyper}} = \prod_i \frac{1}{\text{ch}(\lambda_i+\mu)}\,, \\
\begin{array}{l}\text{Antisymmetric}\\ \text{hypermultiplet:}\end{array} \quad & Z^{\text{asym}}_{\text{hyper}} = \prod_{i<j} \frac{1}{\text{ch}(\lambda_i+\lambda_j+M)}\,.
\end{aligned} \tag{A.1}$$

The expression for U($2N$) is the same with the obvious change $N \to 2N$, and where we use the convension of upper case I, J for the indices.

For Sp$(2N)$ nodes the analogous expressions are

$$
\begin{aligned}
\text{Eigenvalues:}\quad & \lambda_i \quad i=1,\cdots,N\,, \\
\text{Weyl group order:}\quad & |W| = 2^N N!\,, \\
\text{Vector multiplet:}\quad & Z_{\text{vec}} = \prod_{i<j} \text{sh}^2(\lambda_i-\lambda_j)\,\text{sh}^2(\lambda_i+\lambda_j) \prod_i \text{sh}^2(2\lambda_i)\,, \\
\begin{array}{l}\text{Fundamental}\\ \text{hypermultiplet:}\end{array}\quad & Z_{\text{hyper}}^{\text{fund}} = \prod_i \frac{1}{\text{ch}(\lambda_i+\mu)\,\text{ch}(\lambda_i-\mu)}\,, \\
\begin{array}{l}\text{Antisymmetric}\\ \text{hypermultiplet:}\end{array}\quad & Z_{\text{hyper}}^{\text{asym}} = \prod_{i<j} \frac{1}{\text{ch}(\lambda_i+\lambda_j+M)\,\text{ch}(\lambda_i+\lambda_j-M)} \prod_{i,j} \frac{1}{\text{ch}(\lambda_i-\lambda_j+M)}\,.
\end{aligned}
\tag{A.2}
$$

Finally for the bifundamental hypermultiplets we have

$$
\begin{aligned}
\text{U}(N)\times\text{U}(M):\quad & Z_{\text{hyper}}^{\text{bifund}} = \prod_{i=1}^{N}\prod_{j=1}^{M} \frac{1}{\text{ch}(\lambda_i-\tilde{\lambda}_j-m)}\,, \\
\text{Sp}(2N)\times\text{U}(2N):\quad & Z_{\text{hyper}}^{\text{bifund}} = \prod_{i,J} \frac{1}{\text{ch}(\lambda_i-\tilde{\lambda}_J-m)\,\text{ch}(\lambda_i+\tilde{\lambda}_J+m)}\,, \\
\text{Sp}(2N)\times\text{Sp}(2N):\quad & Z_{\text{hyper}}^{\text{bifund}} = \prod_{i,j}\prod_{\pm} \frac{1}{\text{ch}(\lambda_i+\tilde{\lambda}_j\pm m)\,\text{ch}(\lambda_i-\tilde{\lambda}_j\pm m)}\,.
\end{aligned}
\tag{A.3}
$$

B Combinatorics of the permutations $R\tau^{-1}R\tau$

In this appendix we show how we can simplify the partition function (2.21)

$$
Z(N) = \frac{1}{2^{2N}N!^2} \sum_{\tau\in S_{2N}} (-1)^\tau \int d^N\lambda \prod_{k\in\mathcal{K}(\tau)} (-1)^{s(k)+s(\tau(k))} \rho(\lambda_k, \lambda_{R\tau^{-1}R\tau(k)})\,, \tag{B.1}
$$

by studying more closely the composite permutation $R\tau^{-1}R\tau$ and the set $\mathcal{K}(\tau)$ of N integers in $1,\cdots,2N$ such that $R(\mathcal{K}(\tau)) = \overline{\mathcal{K}(\tau)}$ and $R\tau^{-1}R\tau(\mathcal{K}(\tau)) = \mathcal{K}(\tau)$.

Let us label the N integers in $\mathcal{K}(\tau)$ by $k_1,\cdots,k_N$, such that the action of $R\tau^{-1}R\tau$ on $\mathcal{K}(\tau)$ can be represented in terms of a permutation $\sigma_\tau \in S_N$ as

$$
R\tau^{-1}R\tau(k_i) = k_{\sigma_\tau(i)}\,. \tag{B.2}
$$

We can immediately see that $R\tau^{-1}R\tau$ acts on elements $R(k_i)$ in the compliment of $\mathcal{K}(\tau)$ by the inverse permutation σ_τ^{-1}

$$
R\tau^{-1}R\tau(R(k_i)) = R(R\tau^{-1}R\tau)^{-1}(k_i) = R(k_{\sigma_\tau^{-1}(i)})\,. \tag{B.3}
$$

This property means that $R\tau^{-1}R\tau$ is composed of pairs of cycles that take the form[21]

$$
(k_1k_2\cdots k_l)(R(k_l)R(k_{l-1})\cdots R(k_1))\,. \tag{B.4}
$$

[21]Indeed there is a freedom in choosing the set $\mathcal{K}(\tau)$ by including the elements of either one of each pair of cycles.

For a given $\sigma \in S_N$ we can easily find $\tau \in S_{2N}$ such that $\sigma = \sigma_\tau$ by for example taking for each l-cycle in σ the $2l$-cycle in τ

$$(k_l R(k_l) k_{l-1} R(k_{l-1}) \cdots R(k_1)) \,. \tag{B.5}$$

This particular choice of τ is useful because $\tau(k_i) = R(k_i)$, and it is made up of only cycles of even length, so we can easily compute

$$(-1)^\tau \prod_{k \in \mathcal{K}(\tau)} (-1)^{s(k)+s(\tau(k))} = (-1)^{n_\tau}(-1)^N = (-1)^{n_{\sigma_\tau}}(-1)^N = (-1)^{\sigma_\tau} \,, \tag{B.6}$$

where we recall that

$$s(k) = \begin{cases} 0 \,, & k = 1, \cdots, N \,, \\ 1 \,, & k = N+1, \cdots, 2N \,, \end{cases} \tag{B.7}$$

and n_τ counts the number of cycles in τ. The second equality in (B.6) follows because each cycle in τ gives rise to a cycle in σ_τ, and in the third equality we recognised that the expression appearing is nothing but the signature of σ_τ.

Of course, there are many possible choices of τ giving rise to the same $R\tau^{-1}R\tau$ and we should check that all of them reproduce (B.6). From any particular τ we can generate all τ giving rise to the same $R\tau^{-1}R\tau$ by taking

$$\tau \to \pi\tau \,, \qquad \pi^{-1}R\pi = R \tag{B.8}$$

To solve this condition, π can be any permutation of the form $\pi = \pi_1\pi_2$, where π_1 is any combination of the two cycles appearing in R (2^N possibilities), and π_2 acts as two copies of some S_N permutation ($N!$ possibilities)

$$\pi_2(i) = \sigma'(i), \qquad \pi_2(N+i) = N + \sigma'(i), \qquad i = 1, \cdots, N, \quad \sigma' \in S_N \,. \tag{B.9}$$

It is easy to check that any deformation (B.8) leaves the left-hand side of (B.6) invariant, and so the right-hand side indeed holds for any τ. With this simplification we can rewrite the partition function (B.1) as

$$Z(N) = \frac{1}{2^{2N}N!^2} \sum_{\tau \in S_{2N}} (-1)^{\sigma_\tau} \int d^N\lambda \prod_{i=1}^{N} \rho(\lambda_{k_i}, \lambda_{k_{\sigma_\tau(i)}}) \,, \tag{B.10}$$

The summand of (B.10) depends only on the conjugacy class of σ_τ, determined by the number of cycles m_l of length l. This means we can convert the sum over S_{2N} permutations to a sum over conjugacy classes of S_N, if we know the combinatorics of the map from τ into the conjugacy class of σ_τ.

We have already counted how many τ give rise to any given $R\tau^{-1}R\tau$ permutation ($2^N N!$), so we just need to compute how many distinct $R\tau^{-1}R\tau$ permutations are associated with a σ_τ of a given conjugacy class. We know that $R\tau^{-1}R\tau$ generates all possible permutations made up of pairs of cycles as in (B.4), and we should count how many of them have m_l pairs of cycles of length l. Suppose we fix how we assign to $k_1, \cdots, k_N$

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and $R(k_1), \cdots, R(k_N)$ the integers $1, \cdots, 2N$. With this restriction we are just left with counting the number of ways to distribute the k_i among the cycles, which is the same as counting the number of S_N permutations in a given conjugacy class

$$\frac{N!}{\prod_{l=1}^{N} l^{m_l} m_l!} \,. \tag{B.11}$$

We then have the added possibility of generating more distinct $R\tau^{-1}R\tau$ (without altering the conjugacy class) by exchanging $k_i \leftrightarrow R(k_i)$. Note however that simultaneously exchanging all of the k_i within a given cycle leaves $R\tau^{-1}R\tau$ invariant, so this generates just an additional factor of

$$\frac{2^N}{2^{n_{\sigma_\tau}}} \,, \tag{B.12}$$

Putting all this together and relabelling $\lambda_{k_i} \to \lambda_i$, (B.10) becomes

$$Z(N) = \frac{1}{N!} \sum_{\sigma \in S_N} \frac{(-1)^\sigma}{2^{n_\sigma}} \int d^N\lambda \prod_{i=1}^{N} \rho(\lambda_i, \lambda_{\sigma(i)}) \,, \tag{B.13}$$

where we have absorbed the factor (B.11) to promote the sum over S_N conjugacy classes to a sum over S_N permutations.

C Degeneracy of the spectrum

In this appendix we show that for vanishing masses and FI parameters the expressions for the partition functions found in the main text in terms of density operators (2.25) (which is identical to (B.13) above), correspond to the partition function of fermions on a semi-infinite line (2.27). We do this by proving that the spectrum of ρ splits into odd and even states whose spectrum is identical.[22]

The first statement amounts to ρ commuting with the reflection operator $\hat{R}$ which acts on states by $\hat{R}\,|\lambda\rangle = |-\lambda\rangle$. For vanishing masses and FI parameters the density operator ρ in (2.42) is given by a sequence of even or odd functions of p or q, with precisely two odd functions. Since $\hat{R}f(q) = f(-q)\hat{R}$, $\hat{R}f(p) = f(-p)\hat{R}$, for any function f, we see that $\hat{R}$ commutes with ρ.

To show that the spectrum of odd and even states is the same, we prove that $\mathrm{Tr}(\rho^l \hat{R}) = 0$ for all l. Let us focus first on the case $l = 1$. We notice that the density operator takes the specific form

$$\rho = B_D^{(0)}(p,q)\,\tilde{\rho}\,B_D^{(L)}(p,q)\,\tilde{\rho}^\dagger \,, \tag{C.1}$$

with $\tilde{\rho}$ a sequence of even functions of p or q, $\tilde{\rho}^\dagger$ is the Hermitean conjugate and

$$B_D^{(a)}(p,q) = \left(F^{(a)}(q) \frac{\operatorname{sh} p}{\operatorname{ch} p} F'^{(a)}(q) + F'^{(a)}(q) \frac{\operatorname{sh} p}{\operatorname{ch} p} F^{(a)}(q) \right), \qquad a = 0, L \,. \tag{C.2}$$

Using the properties[23]

$$\hat{R}^\dagger = \hat{R} \,, \quad f(q)^\dagger = f(q) \,, \quad f(p)^\dagger = f(-p) \,, \tag{C.3}$$

[22] Apart for a single even zero-mode, which is non-normalizable.

[23] Here $\dagger$ acts as transposition, since we consider only real operators.

we can derive the chain of equalities

$$\begin{aligned}\mathrm{Tr}(\rho\hat{R}) &= \mathrm{Tr}\left((\rho\hat{R})^\dagger\right) = \mathrm{Tr}\left(\hat{R}\rho^\dagger\right) = \mathrm{Tr}\left(\hat{R}\,\tilde{\rho}\,B_D^{(L)}(p,q)\,\tilde{\rho}^\dagger\,B_D^{(0)}(p,q)\right)\\ &= -\,\mathrm{Tr}\left(B_D^{(0)}(p,q)\,\tilde{\rho}\,B_D^{(L)}(p,q)\,\tilde{\rho}^\dagger\,\hat{R}\right) = -\,\mathrm{Tr}(\rho\hat{R})\,,\end{aligned} \tag{C.4}$$

where we have used $(B_D^{(a)}(p,q))^\dagger = -B_D^{(a)}(p,q)$, the cyclicity of the trace and commuted $\hat{R}$ and $B_D^{(0)}(p,q)$, producing a minus sign. This yieds $\mathrm{Tr}(\rho\hat{R}) = 0$. The argument generalizes easily $l \geqslant 2$.

To derive (2.25), notice that the effect of the projection $\frac{1\pm\hat{R}}{2}$ in (2.27), (2.28) is to add a factor of 1/2 to every cycle in a given permutation $\sigma \in S_N$,

$$\begin{aligned}\int\prod_{k=1}^{l} d\lambda_{i_k}\,\langle\lambda_{i_l}|\,\rho\left(\frac{1\pm\hat{R}}{2}\right)|\lambda_{i_1}\rangle\prod_{k=1}^{l-1}\langle\lambda_{i_k}|\,\rho\left(\frac{1\pm\hat{R}}{2}\right)|\lambda_{i_{k+1}}\rangle &= \mathrm{Tr}\left(\left(\rho\,\frac{1\pm\hat{R}}{2}\right)^l\right)\\ &= \mathrm{Tr}\left(\rho^l\,\frac{1\pm\hat{R}}{2}\right) = \frac{1}{2}\,\mathrm{Tr}\left(\rho^l\right).\end{aligned} \tag{C.5}$$

The same results hold for density operators of linear quivers (2.43) at vanishing masses and FI parameters. The arguments are the same except that one must consider the momentum space basis $|p\rangle$, with for instance $\mathrm{Tr}\,\hat{A} = \int dp\,\langle p|\,\hat{A}\,|p\rangle$.

D Relation with previous Fermi gas formulation of Sp($2N$) quivers

In this appendix we compare our results to those in [41], which also developed a Fermi gas approach to Sp quivers. As we show below, the two formulations are equivalent, but in the final expression for the coefficients B in the Airy function, we disagree with their results and explain why.

The overlap between theories discussed in [41] and our work is the single node Sp quiver with an antisymmetric hypermultiplet and n fundamental hypermultiplets. The partition function is given by the matrix model

$$Z(N) = \frac{1}{4^N N!}\int d^N\lambda\prod_{i=1}^{N}\frac{\mathrm{sh}^2\,2\lambda_i\,\mathrm{ch}\,2\lambda_i}{\mathrm{ch}^{2n}\,\lambda_i}\,\frac{\prod_{i<j}\mathrm{sh}^2(\lambda_i-\lambda_j)\,\mathrm{sh}^2(\lambda_i+\lambda_j)}{\prod_{i,j}\mathrm{ch}(\lambda_i-\lambda_j)\,\mathrm{ch}(\lambda_i+\lambda_j)}\,. \tag{D.1}$$

In [41] the matrix model (D.1) was manipulated with a modified Cauchy identity

$$\frac{\prod_{i<j}(x_i-x_j)(y_i-y_j)(x_ix_j-1)(y_iy_j-1)}{\prod_{i,j}(x_i+y_j)(x_iy_j+1)} = \sum_\sigma(-1)^\sigma\prod_i\frac{1}{(x_i+y_{\sigma(i)})(x_iy_{\sigma(i)}+1)}\,. \tag{D.2}$$

This identity, with the usual the usual replacements $x_i \to e^{\lambda_i}$, $y_i \to e^{\lambda_i}$, gives

$$Z(N) = \frac{1}{4^N N!}\sum_{\sigma\in S_N}(-1)^\sigma\int d^N\lambda\prod_{i=1}^{N}\frac{\prod_i\mathrm{sh}^2\,2\lambda_i\,\mathrm{ch}\,2\lambda_i}{\prod_i\mathrm{ch}^{2n}\,\lambda_i}\,\frac{1}{\mathrm{ch}(\lambda_i-\lambda_{\sigma(i)})\,\mathrm{ch}(\lambda_i+\lambda_{\sigma(i)})} \tag{D.3}$$

Using the relations[24]

$$\begin{aligned} \frac{\operatorname{ch}\pi\lambda_1 \operatorname{ch}\lambda_2}{\operatorname{ch}(\lambda_1-\lambda_2)\operatorname{ch}(\lambda_1+\lambda_2)} &= \langle\lambda_1|\,\frac{1+\hat{R}}{2\operatorname{ch}p}\,|\lambda_2\rangle\,, \\ \frac{\operatorname{sh}\lambda_1 \operatorname{sh}\lambda_2}{\operatorname{ch}(\lambda_1-\lambda_2)\operatorname{ch}(\lambda_1+\lambda_2)} &= \langle\lambda_1|\,\frac{1-\hat{R}}{2\operatorname{ch}p}\,|\lambda_2\rangle\,, \end{aligned} \tag{D.4}$$

from which one immediately has the corollary

$$\frac{1-\hat{R}}{2\operatorname{ch}p} = \frac{\operatorname{sh}q}{\operatorname{ch}q}\frac{1+\hat{R}}{2\operatorname{ch}p}\frac{\operatorname{sh}q}{\operatorname{ch}q}\,, \tag{D.5}$$

they rewrite the partition function in two equivalent forms

$$Z(N) = \frac{1}{N!}\sum_{\sigma\in S_N}(-1)^\sigma \int \prod_{i=1}^N d\lambda_i \prod_{i=1}^N \langle\lambda_i|\,\rho_\pm\left(\frac{1\pm\hat{R}}{2}\right)|\lambda_{\sigma(i)}\rangle\,, \tag{D.6}$$

where

$$\begin{aligned} \rho_+ &= \frac{\operatorname{sh}^2 q\operatorname{ch}2q}{\operatorname{ch}^{2n}q}\frac{1}{\operatorname{ch}p}\,, \\ \rho_- &= \frac{\operatorname{ch}2q}{\operatorname{ch}^{2n-2}q}\frac{1}{\operatorname{ch}p}\,. \end{aligned} \tag{D.7}$$

Indeed, this looks very similar to our rewriting of the matrix model as (2.27) or (2.28)

$$Z(N) = \frac{1}{N!}\sum_{\sigma\in S_N}(-1)^\sigma \int \prod_{i=1}^N d\lambda_i \prod_{i=1}^N \langle\lambda_i|\,\rho\left(\frac{1\pm\hat{R}}{2}\right)|\lambda_{\sigma(i)}\rangle\,, \tag{D.8}$$

but with a different density operator (2.47)

$$\rho = \frac{1}{2}\frac{\operatorname{sh}2q}{\operatorname{ch}^{2n}q}\left(\operatorname{sh}q\frac{1}{\operatorname{ch}p}\operatorname{ch}q + \operatorname{ch}q\frac{1}{\operatorname{ch}p}\operatorname{sh}q\right). \tag{D.9}$$

All those expressions are in fact equivalent, since we can show that the spectrum of even states of ρ agrees with that of ρ_+ and the spectrum of odd states to that of ρ_-. Indeed the projected operators are similar to each other by the simple manipulations

$$\begin{aligned} \rho\,\frac{1-\hat{R}}{2} &= \frac{1}{2}\frac{\operatorname{sh}2q}{\operatorname{ch}^{2n}q}\left(\operatorname{sh}q\frac{1-\hat{R}}{2\operatorname{ch}p}\operatorname{ch}q + \operatorname{ch}q\frac{1+\hat{R}}{2\operatorname{ch}p}\operatorname{sh}q\right) \\ &= \frac{1}{2}\frac{\operatorname{sh}2q}{\operatorname{ch}^{2n}q}\left(\operatorname{sh}q\frac{1-\hat{R}}{2\operatorname{ch}p}\operatorname{ch}q + \frac{\operatorname{ch}^2 q}{\operatorname{sh}q}\frac{1-\hat{R}}{2\operatorname{ch}p}\operatorname{ch}q\right) \\ &= \frac{\operatorname{ch}2q\operatorname{ch}q}{\operatorname{ch}^{2n}q}\frac{1-\hat{R}}{2\operatorname{ch}p}\operatorname{ch}q = \frac{1}{\operatorname{ch}q}\left(\rho_-\frac{1-\hat{R}}{2}\right)\operatorname{ch}q\,, \end{aligned} \tag{D.10}$$

where in the first line we used that $\hat{R}$ commutes with even and anticommutes with odd functions of q, and in the second line we used (D.5). Similar manipulations lead to the relations

$$\frac{1}{\operatorname{ch}q}\left(\rho_-\frac{1-\hat{R}}{2}\right)\operatorname{ch}q = \frac{1}{\operatorname{sh}q}\left(\rho_+\frac{1+\hat{R}}{2}\right)\operatorname{sh}q = \frac{\operatorname{ch}q}{\operatorname{ch}2q\operatorname{sh}q}\left(\rho\,\frac{1+\hat{R}}{2}\right)\frac{\operatorname{ch}2q\operatorname{sh}q}{\operatorname{ch}q}\,. \tag{D.11}$$

[24]Note that $\hat{R}$ commutes with $\frac{1}{\operatorname{ch}p}$, and so we can write $\frac{1+\hat{R}}{2\operatorname{ch}p}$ ($= \frac{1}{\operatorname{ch}p}\frac{1+\hat{R}}{2}$) which would otherwise be ill defined.

We can also derive the Airy function expression for these theories based on our analysis in sections 3 and 4. The mirrors of this class of theories are $\hat{D}$ quivers with $(n-3)$ U$(2N)$ nodes and and a single fundamental hypermultiplet on one U(N) node. The asymptotic expansion of the grand potential (4.49) for this theory ($L = n-2$, $\nu = \Delta = \frac{1}{2}$, $\Sigma_1 = \Sigma_2 = 0$) is given by[25]

$$C = \frac{1}{2\pi^2(n-2)}, \qquad B = \frac{1}{8}\left(-n-1+\frac{1}{n-2}\right). \tag{D.12}$$

The coefficient C is the same as found in [41], but B is not. The reason for the discrepancy is that in our formulation, the operator ρ has degenerate odd/even spectrum, as shown in appendix C. While the odd and even spectra agree with ρ_+ and ρ_- of [41], those latter operators do not have degenerate spectra. The saddle point calculation of [41] based on the Fermi surface of ρ_- considered the full spectrum of this operator and divided the end result by 2. Since the operator does not have a degenerate spectrum, this is merely an approximation that is good enough to evaluate the leading order term C, but fails for the subleading coefficient B.

E Truncation of the ϵ expansion

Here we show that the ϵ expansions of C and B coefficients in (4.17) truncates at order ϵ^0 and ϵ^2 respectively, adapting a similar proof from [15] for $\hat{A}$ quiver theories.

We recall that corrections to C, B and A coefficients at order ϵ^n can be computed from a single residue involving $Z_{l(n)}$

$$J(\mu) = \sum_{n\geqslant 0} \epsilon^n J_{(n)}(\mu), \qquad J_{(n)}(\mu) = -\frac{1}{2}\operatorname*{Res}_{l=0}\Gamma(l)\Gamma(-l)Z_{l(n)}e^{l\mu} + \mathcal{O}(e^{-\alpha\mu}), \qquad \alpha > 0\,. \tag{E.1}$$

From this expression it is clear that if $Z_{l(n)}$ vanishes at $l=0$, then $J_{(n)}(\mu)$ can only correct the A coefficient. We should prove then that $Z_{l(n)}$ has at least a simple zero at $l=0$ for all $n>2$.

It is useful to consider the Wigner-Kirkwood expansion of Z_l [15]. The idea is to express Z_l in terms of the Fermi gas Hamiltonian, $H_W = -\log_\star \rho_W$

$$Z_l = \sum_{r\geqslant 0}\frac{(-l)^r}{r!}\int dpdq\, e^{-lH_W}\mathcal{G}_r, \qquad \mathcal{G}_r = \left((\hat{H}-H_W)^r\right)_W. \tag{E.2}$$

We recall that the $\hat{D}$ density operator is given by (4.33)

$$\begin{aligned}\rho_W = \sqrt[\star]{e^{S(p)+2U_\lambda(q)}} \star e^{T(p)} \star \left(\prod_{k=1}^{\lambda-1}{}_\star\, e^{U_{\lambda-k}(q)} \star e^{T(p)}\right) \star e^{S(p)+2U_0(q)} \\ \star \left(\prod_{k=1}^{\lambda-1}{}_\star\, e^{T(p)} \star e^{U_k(q)}\right) \star e^{T(p)} \star \sqrt[\star]{e^{S(p)+2U_\lambda(q)}}\,.\end{aligned} \tag{E.3}$$

[25] We thank Silviu Pufu for correcting a mistake in this formula.

Since (E.3) has an expansion in purely even powers of ϵ, H_W likewise takes the form

$$H_W = H_{(0)} + \epsilon^2 H_{(2)} + \epsilon^4 H_{(4)} + \cdots . \tag{E.4}$$

This gives

$$Z_l = \sum_{r\geqslant 0} \frac{(-l)^r}{r!} \int dpdq\, e^{-lH_{(0)}} \left(1 - l \sum_{n\geqslant 1} \epsilon^{2n} H_{(2n)} + \frac{l^2}{2}\left(\sum_{n\geqslant 1} \epsilon^{2n} H_{(2n)}\right)^2 + \mathcal{O}(l^3)\right) \mathcal{G}_r\,. \tag{E.5}$$

We know that all of the integrals that appear in this expansion are of the form (4.43)

$$\int dx \frac{\mathrm{th}^a\, x}{\mathrm{ch}^b\, x} = \frac{(1+e^{i\pi a})\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b}{2}\right)}{2^{b+1}\pi\Gamma\left(\frac{a+b+1}{2}\right)}\,. \tag{E.6}$$

The coefficients a and b are given by linear functions of l and so the integrals over p and q in (E.5) can each produce at most a simple pole at $l = 0$, from the $\Gamma\left(\frac{b}{2}\right)$. Therefore, we need only concern ourselves with terms in (E.5) with prefactors of at most order l^2. Discarding also (most of) the terms of order ϵ^2 or less, we are left with[26]

$$\begin{aligned} \int dpdq\, e^{-lH_{(0)}} \left(-l \sum_{n\geqslant 2} \epsilon^{2n} H_{(2n)} + \frac{l^2}{2}\left(\sum_{n\geqslant 1} \epsilon^{2n} H_{(2n)}\right)^2\right) \\ + \frac{l^2}{2} \int dpdq e^{-lH_{(0)}} (H_W \star H_W - H_W^2)\,, \end{aligned} \tag{E.7}$$

For simplicity let us now restrict the density operator (E.3) to cases with $U_0(q) = U_\lambda(q) = 0$.[27] With this restriction, the exponentials in (E.3) can be freely exchanged with star exponentials, and we can straightforwardly evaluate the star logarithm by the star product version of the Baker-Campbell-Hausdorff expansion. The leading term gives

$$e^{-lH_{(0)}} = \left(\frac{\mathrm{th}^2\, p}{\mathrm{ch}^{2L}\, p\, \mathrm{ch}^{2\nu}\, q}\right)^l\,. \tag{E.8}$$

All ϵ corrections to H_W are then given by nested star commutators involving $T(p)$, $S(p)$ and $U_k(q)$. An example of such a term contributing to $H_{(4)}$ would be

$$[T(p), [T(p), [T(p), [T(p), U_k(q)]]]]_\star = \frac{\epsilon^4}{16\pi^4} \dot{T}^4 U_k^{(4)} + \mathcal{O}(\epsilon^6)\,, \tag{E.9}$$

where we have used

$$[f, g]_\star = f \star g - g \star f = i\frac{\epsilon}{2\pi}\{f, g\} + \text{higher derivative terms}\,. \tag{E.10}$$

[26]The final term $(H_W \star H_W - H_W^2)$ has still some ϵ^2 piece, but as we shall see this also has at least a simple zero at $l = 0$.

[27]This corresponds to restricting the $\hat{D}$ quiver theories shown in figure 1 to a subclass with $n^{(0)} = n'^{(0)}$ and $n^{(L)} = n'^{(L)}$. We expect that theories outside this subclass also have B and C coefficients truncating at ϵ^2. We have verified that this holds true up to ϵ^4.

As illustrated in (E.9), in order to have a term at order ϵ^n with only single derivatives acting on functions of p (or q), such a term has a single function of q (or p) with an n^{th} derivative.

This is important, because every term in the epsilon expansion has therefore at least one multiple derivative of S, T or U_k. From (4.31) it follows that these derivatives take the form

$$U_k^{(n)}(q) = \frac{1}{\mathrm{ch}^2 q} \sum_{a\in\mathbb{Z}} \sum_{b\geqslant 0} C_{ab} \frac{\mathrm{th}^a q}{\mathrm{ch}^b q}\,, \qquad n \geqslant 2\,, \tag{E.11}$$

and similarly for S, T with $q \to p$.

We now should combine these derivative terms with (E.8) in (E.7), and integrate with (E.6). It is clear that since the derivative terms contribute always a $\frac{1}{\mathrm{ch}^2 q}$ or $\frac{1}{\mathrm{ch}^2 p}$, the resulting Gamma functions can contribute at most a simple pole at $l = 0$. This guarantees the terms in (E.7) with an l^2 outside have at least a simple zero, leaving us with

$$-\,l \int dpdq\, e^{-lH_{(0)}} \sum_{n\geqslant 2} \epsilon^{2n} H_{(2n)}\,. \tag{E.12}$$

By the same reasoning, terms with multiple derivatives on both a function of p and a function of q have an overall $\frac{1}{\mathrm{ch}^2 q}\frac{1}{\mathrm{ch}^2 p}$ which kills both of the poles one could get from integrating. The remaining terms in (E.12) which don't obviously have a simple zero are those where all derivatives on functions of p (or q) are first order, like (E.9). But such terms can be integrated by parts, for instance (E.9) would give

$$-\,l \int dpdq\, e^{-lH_{(0)}} \frac{\epsilon^4}{16\pi^4} \dot{T}^4 U_k^{(4)} = l^2 \int dpdq\, e^{-lH_{(0)}} \frac{\epsilon^4}{16\pi^4} \dot{T}^4 U_k^{(3)} H'_{(0)}\,. \tag{E.13}$$

Integrating by parts pulls down an additional factor of l, which guarantees that there is an overall simple zero at $l = 0$, since the integral on the right hand side still produces just a simple pole. Since we have shown that at order ϵ^4 and higher Z_l has at least a simple zero at $l = 0$, this concludes the proof that C and B do not get contributions beyond order ϵ^2.

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