Partitioning a regular n-gon into $n+1$ congruent pieces is impossible for sufficiently large n

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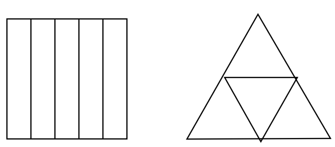
1 Introduction

The interest in polygon decomposition emanates from the theoretical importance of the problem on one hand and the many applications that it has on the other. The decomposition problem has been extensively studied in the literature and yet variations of the problem remain open [3]. The existence of a huge literature on this problem can be informally explained by the fact that there are numerous ways in which we can decompose a polygon and there are many types of polygons to decompose. Decomposition can be defined as partitioning a polygon into components according to a set of rules. In other words, each kind of decomposition has a set of constraints either on the type of the pieces, the number of pieces, the length of the cuts, the areas of the partitions. In this paper, we discuss the following problem posed by Joseph O'Rourke at the Fall Workshop on Computational Geometry in 2004:

Is it possible to partition a regular n-gon into $n+1$ congruent pieces? It is obviously possible for the equilateral triangle and the square as shown below. However, is it ever possible for $n \geq 5$?



The problem can be categorized as a special tiling problem consisting of one single tile, generally known as *monohedral* tiling [2], and where the area of the tiling is finite and restricted to the boundary of a regular n-gon. We begin by proving that no regular n-gon can be partitioned into $n+1$ congruent convex pieces each with e_c boundary edges for all $e_c \geq 7$, nor into hexagonal pieces when $n \leq 6$. The same result applies for congruent concave pieces where no two interior angles sum up to 360°. We then prove with a long case analysis that no regular n-gon can be partitioned into $n+1$ convex congruent pieces for $e_c = 3, 4, 5, 6$, for sufficiently large n. The proofs presented are necessarily sketches due to space limitations. We conclude with open problems.

2 Notation and definitions

We refer to [1] for different meanings of congruence. Two polygons are *properly congruent* if they are equivalent up to rotations and translations and *mirror congruent* if they are equivalent up to mirror reflection but not *properly congruent* and simply *congruent* if they are equivalent up to rotations, translations and mirror reflections. In this paper we use *congruent* unqualified to mean properly congruent.

A partition of a polygon defines an embedded planar graph. We call a planar graph *region regular* if all the faces have the same number of edges except for the outer face. In the partition, a *T-node* is an interior node of degree 3 with a 180° angle. A partition of a regular n-gon is an *edge-to-edge* partition if it is *T-node free*. Two angles are called *circular* if they sum up to 360°. A concave polygon is called *circularity-free* if no two of its angles are *circular*.

For convenience, we call the regular n-gon the (or more generally the polygon to be partitioned) the *large polygon* and the congruent pieces that partition it the *small polygons*. We stipulate without loss of generality that the large polygon is unit-sided.

In a partition, some of the congruent small polygons are embedded on the boundary of the large polygon. They can be embedded to contain zero or several consecutive unit-distant angles from the

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boundary of the large polygon. By consecutive unit-distant angles (CUDA), we mean consecutive angles congruent to the angles of the large polygon ($\frac{(n-2)180^\circ}{n}$) and that are unit distance apart by our definition of the large polygon.

For the case analysis, we characterize a small polygon of the partition according to the number of consecutive unit-distant angles it contains from the boundary of the large polygon. Therefore a small polygon is said to be (a_c, n)-*CUDA* if it contains a_c unit-distant angles consecutive on the boundary of the large n-gon. Figure 1 shows examples.

A partition is said to be a_c-CUDA if a_c is the CUDA number of its pieces and if there exists at least one small polygon embedded with a_c angles on the exterior of the polygon. It is sufficient in the case analysis to consider all CUDA values because every partition is a_c-CUDA for some value of a_c. Two small polygons are said to be *perfectly fitting* on the boundary of the large polygon when the small polygons are adjacent to the boundary and to each other with two supplementary angles.

In what follows, n denotes the number of sides of the large polygon. In any assumed partitioning, let n_b be the Steiner points to be added on the border of the large polygon, n_m the Steiner points to be added in the interior of the large polygon and e_c the number of edges of the small polygons. Trivially, $n \geq 3$ and $e_c \geq 3$.

3 Main Result

Euler part. Assume that the partitioning we are seeking is an edge-to-edge convex or concave circularity-free.

Our problem can then be stated as partitioning the regular n-gon into $n+1$ pieces where each piece is e_c-sided. The partition will have the structure of a region-regular planar graph with $n + n_b$ edges on the outer face and $\frac{e_c(n+1)-n-n_b}{2}$ edges in the interior.

Lemma 1. *The number of Steiner points to add to the large polygon in order to form the region-regular graph is given by:* $2n_m + n_b = e_c(n+1) - 3n$

Proof: Proved by application of Euler's formula for planar graphs □

Lemma 2. $n_m + n_b \leq 2n$

Proof: Proved by application of Euler's formula for planar graphs. □

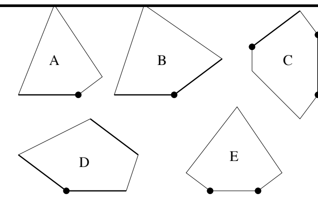

Figure 1: Here we illustrate the CUDA values of several examples of small polygons that are being considered to tile a regular decagon. On each of these polygons, a bold edge represents a unit distance edge, and a marked angle is an angle of a regular decagon. Polygon E is $(0, 10)$-CUDA while A, B, and D are $(1, 10)$-CUDA, and polygon C is $(2, 10)$-CUDA

Lemma 3. *No n-gon can be partitioned into $n+1$ circularity-free e_c-gons for $e_c \geq n$ and $n \geq 6$.*

Proof: By Lemma 1, for $e_c \geq n$, if the large polygon can be partitioned into $n+1$ convex or concave $2n_m + n_b \geq n^2 - 2n$. By Lemma 2 and the previous inequality, $n^2 - 2n - n_m \leq n^2 - 4n \leq n_m + n_b \leq 2n$. However, $n^2 - 4n \geq 2n$ for all $n \geq 6$ which contradicts the existence of such a partition. □

Lemma 4. *No n-gon can be partitioned into $n+1$ circularity-free e_c-gons for $e_c \geq 7$ nor into 6-gons for $n < 6$.*

Proof: By Lemmas 1 and 2, in such a partition, , $e_c(n+1) - 3n - n_m \leq 2n$ and therefore $e_c(n+1) - 3n - 2n \leq 2n$. The following inequality holds: $e_c(n+1) \leq 7n$ However, the previous inequality is not satisfied when $e_c \geq 7$ nor when $n < \frac{e_c}{7-e_c}$ for $e_c < 7$. By hypothesis, $e_c \geq 3$. Hence the only allowable values remaining are $e_c = 3, 4, 5, 6$ for which $\frac{e_c}{7-e_c} = \frac{3}{4}, \frac{4}{3}, \frac{5}{2}, 6$. By hypothesis, $n \geq 3$. The only value for which $n \leq \frac{e_c}{7-e_c}$ is when: $\frac{e_c}{7-e_c} = 6$. Proof of the lemma follows. □

If we remove the assumption that the partition is edge-to-edge and let n_c be the number of T-node vertices, then the same results hold with slightly different arguments. It is important to note that from all the previous lemmas, it is implied that the partitioning for the mentioned values of n or e_c is not possible for convex n-gons and for regular n-gons since convex and regular n-gons are subclasses of circularity-free n-gons.

Triangles through hexagons. By Lemmas 3 and 4, it remains to disprove that the partitioning is not possible when $e_c = 3, 4, 5$ for all values n and the case where $e_c = 6$ for $n \geq 7$. We proceed by an case analysis based on the values of e_c and n, and also the CUDA numbers which vary between 0 and $e_c - 2$ and which depend on n. In what follows, many arguments depend on n exceeding some constant and this requirement is implicit in most of the subsequent claims.

The following table shows how the CUDA-values relate to our lemmas: each lemma proves that the partitioning is not possible for the indicated CUDA value.

e_c	CUDA		e_c	CUDA	
3	All	Omitted	5	3+	Lemma 5
4	0	Lemma 11	6	0	Lemma 11
4	1	Omitted	6	1	Omitted
4	2+	Lemma 5	6	2	Omitted
5	0	Lemma 11	6	3	Lemma 13
5	1	Omitted	6	4+	Lemma 5
5	2	Lemma 13			

$(e_c - 2)$-CUDA partitions.

Lemma 5. *For sufficiently large n, at least two sides of the small polygons need to be longer than the side of the large polygon.*

Proof: For $(e_c - 2)$-CUDA, the small polygon for $e_c = 4, 5, 6$ have one interior edge e to the large polygon. The length of e is bounded by $e_c - 1$ and hence as n grows the area of the small polygon is bounded by a constant. The proof of 5 implies that no $(e_c - 2)$-CUDA partitioning is possible. □

0-CUDA partitions. Before the proof itself that no 0-CUDA partition is possible, we need a few preliminary lemmas to establish the properties of possible 0-CUDA partitions. We begin by showing that small polygons can not be embedded with two or more disjoint regions of incidence with the large polygon in a 0-CUDA partition. We then rule out 0-CUDA partitions for various different types of embeddings, before presenting our main result in Lemma 11 by arguing that we have ruled out all possibilities.

Lemma 6. *No small polygon can be incident to two non-adjacent parts of the large polygon in a 0-CUDA partition.*

Proof: Proof omitted. □

Lemma 7. *Given an outerplanar subgraph of G, H, if there are at most k edges in G and not in H from every vertex v in H, then there is a vertex in H with degree at most $k + 2$ in G.*

Proof: Follows immediately from the fact that all outerplanar graphs have a degree two vertex. □

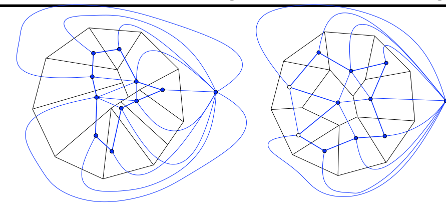

Figure 2: The outerplanar subgraphs chosen in Lemmas 8 (left) and 9 (right) are shown. The illustrated partitions are not congruent, as this is not possible. Left is the case when all small polygons are incident to the large polygon. Here, the shaded lines in the dual graph of the partition define an outerplanar subgraph, where every vertex corresponding to a small polygon has degree 1 outside of this subgraph. Right, the case where exactly one small polygon in not incident to the large polygon is illustrated. In this case, the cycle that defines the outerplanar subgraph detours at the white vertices in the dual to include the center small polygon vertex. These white vertices are degree two outside of the outerplanar subgraph, while the remaining small polygons vertices have degree one.

This result through the appropriate choice of outerplanar subgraphs gives the following lemmas:

Lemma 8. *If all small polygons are incident to the boundary of the large polygon, there can only be a 0-CUDA partition if $e_c = 3$.*

Proof: Since Lemma 6 prohibits small polygons from being incident to non-consecutive parts of the boundary of the large polygon, in the dual graph of any 0-CUDA partition where all small polygons are incident to the boundary there is a cycle among vertices representing the small polygons in the dual graph. This circuit and the edges inside form an outerplanar subgraph of the dual graph. Thus, Lemma 7 can be applied. Noting that the only edges exterior to the circuit in this graph are incident to the exterior vertex, we can conclude that all vertices on the circuit have degree at most 1 outside of the circuit. Thus, according to Lemma 7, there is a vertex on the circuit of degree 3. This corresponds to requiring that at least one small polygon must have at most 3 neighboring regions, which implies the small polygon is a triangle. □

Lemma 9. *If all small polygons are incident to the boundary of the large polygon except one, and all the vertices of the large polygon have at least degree 3, then $e_c \leq 4$*

Proof: The proof proceeds identically as in the previous lemma, except for the choice of circuit. □

Lemma 10. *There does not exist a 0-CUDA partition where $e_c = 4$ and all but one small polygon is incident to the large polygon.*
Proof: Proof omitted. □

Lemma 11. *If the small polygon is 0-CUDA then in any partition $e_c \neq 4, 5, 6$.*
Proof: Lemma 6 assures us that in any 0-CUDA partition, small polygons can not be incident to the large polygon in two disjoint parts of the boundary. Since small polygons in a 0-CUDA partition can not have any angles superimposed on the angles of the large polygons, that means that small and large polygons can be incident at most at a single unit edge. As there are n such unit edges on the large polygon, and $n+1$ small polygons to embed inside with these restrictions, any valid partition can have at most one small polygon that is not incident to the large polygon. From Lemmas 8 and 9 we can rule out all possibilities of partitioning except for when $e_c = 4$ and there is one inside small polygon. Lemma 10 shows this case can not occur. □

Pentagons 2-CUDA and Hexagons 3-CUDA paritions. Here we prove that 2-CUDA partitions with pentagons and a 3-CUDA partitions with hexagons are not possible.
Lemma 12. *For $c = 2, 3$, any regular $\frac{n}{c}$-gon in a regular unit-sided n-gon has $O(1)$ sized sides.*
Proof: Proved by straightforward trigonometry. □

Lemma 13. *No $n+1$ $(e_c - 3)$-CUDA congruent partitioning of a large polygon exists when $e_c = 5, 6$.*
Proof: For $e_c = 5$, two cases need to be considered for a 2-CUDA partition where the boundary of the large polygon is embedded as shown in Figure 3 (a) or (b): when the pieces perfectly fit on the boundary of the large polygon and when they don't. In the former case, we have two subcases depending on whether n is even or odd. When n is even, $\frac{n}{2}$ pieces are adjacent to the boundary of the large polygon which forms a regular $\frac{n}{2}$-gon in the interior with $\frac{n}{2}+1$ remaining pieces to complete the partition, see Figure 3 (a).
The argument is based on which angles of the congruent pieces can fit in the regular $\frac{n}{2}$-gon angles. Trivially, none of the two consecutive angles congruent to the large polygon (the CUDA angles) can be embedded. This leaves 3 angles: α, β and γ shown in Figure 3 (a). Let one of the angles of

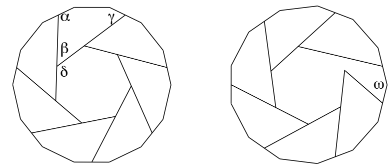

Figure 3: Perfectly fitting pentagons, even and odd

the regular $\frac{n}{2}$-gon be δ. α and γ are supplementary angles and thus any combination of them cannot be embedded in δ. Lemma 5 ensures that the two internal sides of the pentagon get arbitrarily large. A combination of α and γ or β and γ or of γs cannot be embedded in δ because the two internal sides grow arbitrarily with n while the side of the regular $\frac{n}{2}$-gon is $O(1)$ by Lemma 12. For the case where n is odd, the same argument about angles with long sides holds. It is impossible to embed only one of the consecutive large polygon angles of the small polygons in ω because there is no room for its unit-away adjacent angle. Similar arguments show that the partitioning where the small polygons do not fit perfectly on the boundary is not realizable. Similar proofs hold for 3-*CUDA* for $e_c = 6$. □

Summary. All previous lemmas yield:
Theorem 14. *No regular n-gon can be partitioned into $n+1$ congruent convex for sufficiently large n.*

It remains open whether the partitioning is possible with congruent concave pieces with circular angles for $e_c \geq 7$, the partitioning is possible with all classes of concave pieces for $e_c = 4, 5, 6$ and for all types of congruence and if the partitioning is possible for mirror congruence with convex pieces for $e_c = 4, 5, 6$.

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