

# Passage time and fluctuation calculations for subexponential Lévy processes

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We consider the passage time problem for Lévy processes, emphasising heavy tailed cases. Results are obtained under quite mild assumptions, namely, drift to  $-\infty$  a.s. of the process, possibly at a linear rate (the finite mean case), but possibly much faster (the infinite mean case), together with subexponential growth on the positive side. Local and functional versions of limit distributions are derived for the passage time itself, as well as for the position of the process just prior to passage, and the overshoot of a high level. A significant connection is made with extreme value theory via regular variation or maximum domain of attraction conditions imposed on the positive tail of the canonical measure, which are shown to be necessary for the kind of convergence behaviour we are interested in.

*Keywords:* fluctuation theory; Lévy process; maximum domain of attraction; overshoot; passage time; regular variation; subexponential growth; undershoot

## 1. Introduction

The exit time of a Lévy process  $X$  above a horizontal boundary has been studied extensively in a variety of situations with a view to relating its distributional behaviour to the tail behaviour of the canonical measure of  $X$ . It is helpful to categorise the latter into three general regimes:

- Light tailed (Cramér case).
- Medium tailed (convolution equivalent case).
- Heavy tailed (subexponential tails).

This classification is not prescriptive – categories may overlap – but it provides a convenient general framework in which to summarise results. Representative papers covering the first two categories are Bertoin and Doney [4] for the Cramér and Klüppelberg, Kyprianou and Maller [18] for the convolution equivalent case. The intention of the present paper is to consider in some detail the passage time problem with special emphasis on the third category – the heavy tailed cases.

We assume subexponential growth together with regular variation or maximum domain of attraction conditions for the positive part of the canonical measure of  $X$ , or of its increasing ladder height process; on the negative side, we assume regular variation of the renewal measure

of the descending ladder process, allowing both finite and infinite mean cases. To these is added the assumption of a drift to  $-\infty$  a.s. of the process, possibly at a linear rate, as is the case when the process has finite mean, but possibly at a much faster rate. We obtain very explicit and detailed descriptions of the asymptotic behaviours of the process, in these situations. In particular, we obtain local, and functional, versions of limit distributions for the passage time itself, as well as for the position of the process just prior to passage, and the overshoot of a high level.

Our results are original in a number of respects. We give a very general treatment for Lévy processes, imposing no overt moment conditions, though it will transpire that our conditions imply the positive tail of the canonical measure is integrable (a finite mean for the positive jump process). Extreme value theory enters via the regular variation or maximum domain of attraction conditions we impose on the positive tail of the canonical measure. These are shown to be *necessary* as well as sufficient for convergence of the type we investigate. Subsidiary results in Proposition 4.1 (concerning the convergence of the overshoot for a general subordinator) and Proposition 4.2 (concerning connections between the regular variation or maximum domain of attraction behaviour of the upward ladder height measure as compared with the Lévy measure of the underlying process), are also new, and extend the domain of applicability of the paper.

In the next section, we introduce the setup. The main results are stated in Section 3, and proofs are in Sections 4–6. The final Section 7 discusses similar results for random walks and compound Poisson processes.

## 2. Preliminary setting up

Let  $(X_t)_{t \geq 0}$ ,  $X_0 = 0$ , be a real-valued Lévy process on a probability space  $\{\Omega, \mathcal{F}, \mathcal{P}\}$  with triplet  $(\gamma_X, \sigma_X^2, \Pi_X)$ , where  $\gamma_X \in \mathbb{R}$ ,  $\sigma_X^2 \geq 0$  and  $\Pi_X$  is a Lévy measure on  $\mathbb{R}$ . Throughout,  $X$  is assumed to satisfy

$$\lim_{t \rightarrow \infty} X_t = -\infty \quad \text{a.s.} \tag{2.1}$$

We refer to Bertoin [3] and Doney [9] for this notation and the ensuing notions of fluctuation theory. Denote by  $(H_t)_{t \geq 0}$  the ascending ladder height subordinator generated by  $X$ . In view of (2.1), it is defective, obtained from a non-defective subordinator  $\mathcal{H}$  by independent exponential killing with a rate  $q > 0$  given by  $e^{-q} = P(H_1 < \infty)$ . By this, we mean there is a non-defective subordinator  $\mathcal{H}$  and an independent exponential variable  $e_q$  with expectation  $1/q$  such that  $(H_t)_{0 \leq t < L_\infty}$  has the distribution of  $(\mathcal{H}_t)_{0 \leq t < e_q}$ , where  $L_t$ ,  $t > 0$ , is a local time of  $X$  (cf. Bertoin [3] Lemma VI.2, page 157). It follows that

$$P(H_t \leq x) = P(H_t \leq x, t < L_\infty) = e^{-qt} P(\mathcal{H}_t \leq x), \quad t, x > 0. \tag{2.2}$$

The descending ladder height subordinator, denoted by  $(H_t^*)_{t \geq 0}$ , is the ascending ladder height subordinator corresponding to the dual process  $(X_t^*)_{t \geq 0} := (-X_t)_{t \geq 0}$ . Under (2.1), the process  $(H_t^*)_{t \geq 0}$  is proper, and the corresponding  $q^* = 0$ .

Let  $\Pi_{\mathcal{H}}(\cdot)$  be the Lévy measure of  $\mathcal{H}$ , with tail  $\bar{\Pi}_{\mathcal{H}}(x) = \Pi_{\mathcal{H}}\{(x, \infty)\}$ ,  $x > 0$ , assumed positive for all  $x > 0$ . Similarly,  $\Pi_{H^*}(\cdot)$  is the Lévy measure of  $H^*$ , with tail  $\bar{\Pi}_{H^*}$ , and we write  $d_{\mathcal{H}}$  and  $d_{H^*}$  for the drift coefficients of  $\mathcal{H}$  and  $H^*$ . We have  $d_{\mathcal{H}} = d_H$  and  $\Pi_{\mathcal{H}} = \Pi_H$ . Let  $\bar{\Pi}_X^+$

and  $\overline{\Pi}_X^-$  be the positive and negative Lévy tails of  $X$ , equal to  $\Pi_X\{(x, \infty)\}$  and  $\Pi_X\{(-\infty, -x]\}$ ,  $x > 0$ . Write  $\Pi_X^{(+)}$  and  $\Pi_X^{(-)}$  for  $\Pi_X$  restricted to  $(0, \infty)$  and  $(-\infty, 0)$ , respectively. Assume throughout that  $\overline{\Pi}_X^+(x) > 0$  for all  $x > 0$ .

Our results will be phrased in terms of  $\Pi_X$ ,  $\Pi_{\mathcal{H}}$ , and  $\Pi_{H^*}$ , or, more specifically, in terms of the behaviour of their tails for large values. After normalisation, we can regard these as being the tails of probability distributions. Then a condition applied to the tail of a probability measure can equally be applied to the tails of the probability measures defined, for example, by

$$\frac{\Pi_X(dx)\mathbf{1}_{\{x>1\}}}{\overline{\Pi}_X^+(1)} \quad \text{and} \quad \frac{\Pi_{\mathcal{H}}(dx)\mathbf{1}_{\{x>1\}}}{\overline{\Pi}_{\mathcal{H}}(1)}. \tag{2.3}$$

We will need certain functionals of these tails, in particular,

$$A_X^+(x) := \int_1^x \overline{\Pi}_X^+(y) dy \quad \text{and} \quad A_X^*(x) := \int_1^x \overline{\Pi}_X^-(y) dy, \quad x > 1 \tag{2.4}$$

and

$$A_{\mathcal{H}}(x) := \int_0^x \overline{\Pi}_{\mathcal{H}}(y) dy \quad \text{and} \quad A_{H^*}(x) := \int_0^x \overline{\Pi}_{H^*}(y) dy, \quad x > 0, \tag{2.5}$$

which are kinds of truncated or Winsorised means.

Particular classes of tail functions we are interested in are the regularly varying ones and the class of probability distributions in the maximum domain of attraction of the Gumbel distribution. Write  $\text{RV}(\alpha)$  for the class of real valued functions regularly varying at  $\infty$  with index  $\alpha \in \mathbb{R}$ , so that  $\text{RV}(0)$  are the slowly varying functions. We refer to Bingham, Goldie and Teugels [5] for definitions and properties of regularly varying functions.

Denote the tail of a distribution function  $F$  on  $[0, \infty)$  by  $\overline{F} = 1 - F$ , and assume  $\overline{F}(u) > 0$  for all  $u > 0$ .  $\overline{F} \in \text{RV}(-\beta)$  for some  $\beta \in (0, \infty)$  is equivalent to  $F$  being in the *maximum domain of attraction of a Fréchet distribution with parameter  $\beta > 0$* , denoted  $F \in \text{MDA}(\Phi_\beta)$ . A positive random variable having distribution tail  $\overline{F}$  is said to be in the *maximum domain of attraction of the Gumbel distribution*, which we denote as  $\text{MDA}(\Lambda)$ , with *auxiliary function  $a(u) > 0$* , if

$$\frac{\overline{F}(u + a(u)x)}{\overline{F}(u)} \rightarrow e^{-x}, \quad x \geq 0. \tag{2.6}$$

(Here and throughout, all limits are as  $u \rightarrow \infty$  unless otherwise stated.) Useful properties of such distributions can be found in Bingham, Goldie and Teugels [5], page 410, Resnick [21], Chapters 0 and 1, Embrechts, Klüppelberg and Mikosch [14], Chapter 3 and de Haan and Ferreira [7], Chapter 1. In particular, when (2.6) holds,  $F$  has finite moments of all orders, and the auxiliary function  $a(u)$  satisfies  $a(u) = o(u)$  and is *self-neglecting*, that is,  $a(u + Ka(u)) \sim a(u)$  for any fixed  $K$ . Typical distributions in  $\text{MDA}(\Phi_\beta)$  are the Pareto distributions, while  $\text{MDA}(\Lambda)$  includes the Weibull and lognormal.

Further, it is well known from extreme value theory [cf. Theorems 1.1.2, 1.1.3 and 1.1.6 in de Haan and Ferreira [7]] that (2.6) can be extended to give that there is a function  $0 < a(u) \rightarrow \infty$

and a positive random variable  $C$  such that

$$\frac{\overline{F}(u + a(u)x)}{\overline{F}(u)} \rightarrow P(C > x), \quad x > 0, \tag{2.7}$$

if and only if (for distributions with unbounded support to the right, as we have)  $F \in \text{MDA}(\Phi_\beta)$  for some  $\beta \in (0, \infty)$ , or  $\overline{F} \in \text{MDA}(\Lambda)$ . Furthermore,  $a(u)$  can be chosen as  $a(u) = u$  in the first case, and as  $a(u) = \int_u^\infty \overline{F}(y) dy / \overline{F}(u)$  (finite) in the second case, and  $C$  has a  $\text{Par}(\beta)$  distribution (i.e., a Pareto distribution with parameter  $\beta > 0$ ) having density  $\beta(1+x)^{-\beta-1}$ ,  $x > 0$ , in the first case, and an exponential distribution with unit parameter ( $\text{Exp}(1)$ ) in the second case.

We introduce also the class of *long-tailed distributions*,  $\mathcal{L}$ , and the subexponential class,  $\mathcal{S}$ . The distribution  $F$  (or its tail  $\overline{F} = 1 - F$ ) is said to be in class  $\mathcal{L}$  if

$$\frac{\overline{F}(u+x)}{\overline{F}(u)} \rightarrow 1 \quad \text{for } x \in (-\infty, \infty), \tag{2.8}$$

while  $F$  (or its tail  $\overline{F}$ ) is said to be in the class  $\mathcal{S}$  of *subexponential* distributions if  $F \in \mathcal{L}$  and

$$\frac{\overline{F^{2*}}(u)}{\overline{F}(u)} \rightarrow 2, \tag{2.9}$$

where  $F^{2*} = F * F$ . For background, see Foss, Korshunov and Zachary [15]. We have  $\text{RV}(\alpha) \subset \mathcal{S} \subset \mathcal{L}$  but  $\text{MDA}(\Lambda)$  is not contained in  $\mathcal{S}$  [Goldie and Resnick [16]].

Consistent with the convention noted in (2.3), abbreviate  $\Pi_X^{(+)}(dx) \mathbf{1}_{\{x>1\}} / \overline{\Pi}_X^+(1) \in \text{MDA}(\Lambda)$  to  $\Pi_X^{(+)} \in \text{MDA}(\Lambda)$  and  $\Pi_{\mathcal{H}}(dx) \mathbf{1}_{\{x>1\}} / \overline{\Pi}_{\mathcal{H}}(1) \in \mathcal{S}$  to  $\Pi_{\mathcal{H}} \in \mathcal{S}$ , etc. With this notation, our second basic assumption is

$$\Pi_{\mathcal{H}} \in \mathcal{S}. \tag{2.10}$$

Equation (2.10) is equivalent to  $P(\mathcal{H}_1 \in \cdot) \in \mathcal{S}$ , and then  $P(\mathcal{H}_1 > u) \sim \overline{\Pi}_{\mathcal{H}}(u)$  as  $u \rightarrow \infty$  [Embrechts, Goldie and Veraverbeke [13], Pakes [19,20]]. Together with (2.1), (2.10) implies that

$$P\left(\sup_{t \geq 0} X_t > u\right) \sim q^{-1} \overline{\Pi}_{\mathcal{H}}(u) \quad \text{as } u \rightarrow \infty \tag{2.11}$$

[from Lemma 3.5 of Klüppelberg, Kyprianou and Maller [18]].

For  $u > 0$  let

$$\tau_u := \inf\{t > 0: X_t > u\}, \quad Z^{(u)} = -X_{\tau_u-}, \quad O^{(u)} = X_{\tau_u} - u \tag{2.12}$$

denote the passage time above level  $u > 0$ , the negative of the position reached just prior to passage, and the overshoot above the level. (The reason for taking  $-X$  in the definition of  $Z$  will become apparent later.) Note that  $P(\tau_u < \infty) = P(H_\infty > u) < 1$  for all  $u > 0$  by (2.1), while  $P(\tau_u < \infty) > 0$  for all  $u > 0$  because of our assumption that  $\overline{\Pi}_X^+(x) > 0$  for all  $x > 0$  and  $\lim_{u \rightarrow \infty} P(\tau_u < \infty) = 0$  by (2.11). We use  $P^{(u)}(\cdot) = P(\cdot | \tau_u < \infty)$ ,  $u > 0$ , defined in an

elementary way, for the probability measure conditional on passage above  $u$ . We also use the notation  $\bar{X}_t = \sup_{0 \leq s \leq t} X_s, t \geq 0$ .

Recall the definition of  $A_{H^*}(\cdot)$  in (2.5). Our third main assumption is of the form:

$$A_{H^*}(\cdot) \in \text{RV}(\gamma), \tag{2.13}$$

where the precise value of the index  $\gamma \in [0, 1)$  will be specified later. By, for example, Bingham, Goldie and Teugels [5], page 364, (2.13) is equivalent to  $G^*(\cdot) \in \text{RV}(1 - \gamma)$ , where  $G^*$  is the renewal measure for the strict decreasing ladder height process, and then we have, as  $x \rightarrow \infty$ ,

$$A_{H^*}(x) \sim \frac{k_\gamma x}{G^*(x)} \in \text{RV}(\gamma) \quad \text{where } k_\gamma = \frac{1}{\Gamma(1 + \gamma)\Gamma(2 - \gamma)}. \tag{2.14}$$

Equation (2.13) is also equivalent to

$$\lim_{x \rightarrow \infty} \frac{x \bar{\Pi}_{H^*}(x)}{A_{H^*}(x)} = \gamma, \quad 0 \leq \gamma < 1 \tag{2.15}$$

(Bingham, Goldie and Teugels [5], Theorem 1.5.11, page 18, Theorem 1.6.1, page 30).

### 3. Main results

We now state our two main results. Both assume (2.1) and (2.10), and the first assumes in addition that  $A_{H^*} \in \text{RV}(0)$ , that is, that  $A_{H^*}$  is slowly varying as  $x \rightarrow \infty$ . This implies that  $X_t^*$  is positively relatively stable as  $t \rightarrow \infty$ , so there is a continuous, strictly increasing function  $c(\cdot) \in \text{RV}(1)$  such that  $X_t^*/c(t) \xrightarrow{P} 1$  as  $t \rightarrow \infty$ . This in turn implies that the process  $(X_{st}^*/c(t))_{0 \leq s \leq 1}$  converges weakly in  $\mathbb{D}_0[0, 1]$  (i.e., in the sense of weak convergence of càdlàg functions on  $[0, 1]$  with the Skorokhod topology) as  $t \rightarrow \infty$  to the process  $\mathbf{D}^{(0)}$ , where  $\mathbf{D}^{(0)}(s) \equiv s$ . This situation includes the possibility of a finite, positive mean for  $X_1^*$ . Write  $b(\cdot)$  for the inverse function of  $c(\cdot)$ . We sometimes write  $X^*(t)$  for  $X_t^*$ .

**Theorem 3.1.** *Assume  $\lim_{t \rightarrow \infty} X_t = -\infty$  a.s.,  $\Pi_{\mathcal{H}} \in \mathcal{S}$ , and  $A_{H^*} \in \text{RV}(\gamma)$  with  $\gamma = 0$ .*

(1) *Then the following are equivalent;*

(a) *there exists  $a(u) > 0$  with  $\lim_{u \rightarrow \infty} a(u) = \infty$  such that  $P^{(u)}(O^{(u)} \in a(u) dx), x > 0$ , has a non-degenerate limit as  $u \rightarrow \infty$ ;*

(b) *either  $\bar{\Pi}_{\mathcal{H}} \in \text{RV}(1 - \gamma - \beta)$  for some  $\beta > 1 - \gamma$  and then (a) holds with  $a(u) = u$  [case (i)] or else  $\Pi_{\mathcal{H}} \in \text{MDA}(\Lambda)$ , and then (a) holds with  $a(u) = \int_u^\infty \bar{\Pi}_{\mathcal{H}}(y) dy / \bar{\Pi}_{\mathcal{H}}(u)$  [case (ii)];*

(c) *either  $\bar{\Pi}_X^+ \in \text{RV}(-\beta)$  for some  $\beta > 1$  (case (i)) or else  $\Pi_X^{(+)} \in \text{MDA}(\Lambda)$  [case (ii)], and  $a(\cdot)$  may then be chosen as  $a(u) = u$  in the first case or as  $a(u) = \int_u^\infty \bar{\Pi}_X^+(y) dy / \bar{\Pi}_X^+(u)$  in the second case.*

(2) When (a)–(c) hold, the  $P^{(u)}$ -distribution of  $\tau_u$ , restricted to the event  $X_{\tau_u-} < u$ , has a density  $g^{(u)}(\cdot)$  which satisfies

$$\lim_{u \rightarrow \infty} b(a(u))g^{(u)}(tb(a(u))) = \begin{cases} \frac{\beta - 1}{(1 + t)^\beta}, & \text{in case (i),} \\ e^{-t}, & \text{in case (ii),} \end{cases} \tag{3.1}$$

uniformly on compact subintervals of  $(0, \infty)$ . Moreover, conditioned on  $\tau_u = tb(a(u))$ , the  $P^{(u)}$ -finite-dimensional distributions of the process

$$\left\{ \frac{X^*(s\tau_u)}{c(\tau_u)}, 0 \leq s \leq 1 \right\}$$

converge to those of  $\mathbf{D}^{(0)}$  as  $u \rightarrow \infty$ .

(3) Further: when (a)–(c) hold, under  $P^{(u)}$  the process

$$\mathbf{Y}^{(u)} := \left( \frac{Z^{(u)}}{a(u)}, \frac{O^{(u)}}{a(u)}, \frac{\tau_u}{b(a(u))}, \left( \frac{X^*(s\tau_u)}{a(u)} \right)_{0 \leq s \leq 1} \right) \tag{3.2}$$

converges weakly as  $u \rightarrow \infty$  in  $\mathbb{R}^3 \times \mathbb{D}_0[0, 1]$  to  $(V, U, V, (V\mathbf{D}^{(0)}(s))_{0 \leq s \leq 1})$ , where in case (i)

$$P(V \in dz, U \in dx) = \frac{\beta(\beta - 1) dz dx}{(1 + z + x)^{\beta+1}}, \quad x, z > 0, \tag{3.3}$$

and in case (ii)

$$P(V \in dz, U \in dx) = e^{-z-x} dz dx, \quad x, z > 0. \tag{3.4}$$

**Remark 3.1.** (i) The redundant parameter  $\gamma = 0$  is introduced in Theorem 3.1 for conformity with Theorem 3.2, below.

(ii) The event  $\{X_{\tau_u-} < u\}$  in Theorem 3.1 has  $P^{(u)}$ -probability approaching 1 as  $u \rightarrow \infty$ ; see Remark 5.1 in Section 5.

(iii) In general, we cannot replace condition (2.10) with simpler equivalent conditions on  $\Pi_X$  directly, but easily checked sufficient conditions are available; see Remark 6.1 in Section 6.

(iv) The assumption  $A_{H^*} \in \text{RV}(0)$  in Theorem 3.1 is true in particular when  $0 < A_{H^*}(\infty) < \infty$ , or, equivalently, when  $0 < EX_1^* < \infty$ , so the case of a finite mean for  $EX_1^*$  is included in the theorem. Note that part 1(c) implies  $EX_1^+ = E(X_1 \vee 0) < \infty$  in any case. A related result for random walks and compound Poisson processes with finite mean is in Asmussen and Klüppelberg [2].

In our next result, we replace the assumption  $A_{H^*} \in \text{RV}(0)$  by the condition that  $A_{H^*} \in \text{RV}(\gamma)$  for some  $\gamma \in (0, 1)$ . This can only happen when  $E|X_1| = \infty$ , and we will show that it is in fact equivalent, under our basic assumptions, to  $\overline{\Pi}_X \in \text{RV}(\gamma - 1)$  [see Proposition 4.3, where  $A_{H^*}$  is

shown to be asymptotically equivalent to  $q^{-1}A_X^*$ , and note (4.31)]. It then follows that  $X^*$  is in the domain of attraction of  $\mathbf{D}$ , a standard stable subordinator of parameter  $\bar{\gamma} := 1 - \gamma \in (0, 1)$ . Let  $c(\cdot)$  be such that  $(X_{st}^*/c(t))_{0 \leq s \leq 1} \xrightarrow{D} \mathbf{D}$  as  $t \rightarrow \infty$ , and let  $b(\cdot)$  denote the inverse function of  $c(\cdot)$ , so that  $b(\cdot) \in \text{RV}(\bar{\gamma})$ , and let  $\widehat{\mathbf{D}}_{t,z}$  denote an associated “stable subordinator bridge”, which is a rescaled version of  $\mathbf{D}$  conditioned to be at  $z > 0$  at time  $t$ ; namely,

$$P(\widehat{\mathbf{D}}_{t,z} \in \mathcal{B}) = P((D(ts))_{0 \leq s \leq 1} \in \mathcal{B} | D_t = z),$$

for any Borel set  $\mathcal{B}$ . Thus, with

$$h_t(x) dx = P(D_t \in dx) \tag{3.5}$$

as the density of  $D$ , we have for  $0 = s_0 < s_1 < s_2 < \dots < s_k < 1$ ,  $y_0 = 0$ , and  $y_1 < y_2 < \dots < y_k < z$ ,

$$P\left(\bigcap_{r=1}^k \{\widehat{D}_{t,z}(s_r) \in dy_r\}\right) = \frac{h_{t(1-s_k)}(z - y_k)}{h_t(z)} \prod_{r=1}^k h_{t(s_r - s_{r-1})}(y_r - y_{r-1}) dy_r. \tag{3.6}$$

We will use  $\widehat{\mathbf{D}}_{W,V}$  in the obvious sense, where  $(W, V)$  are positive random variables independent of the family  $\widehat{\mathbf{D}}_{t,z}$ .

**Theorem 3.2.** Assume  $\lim_{t \rightarrow \infty} X_t = -\infty$  a.s.,  $\Pi_{\mathcal{H}} \in \mathcal{S}$ , and  $A_{H^*} \in \text{RV}(\gamma)$  with  $\gamma \in (0, 1)$ .

(1) Then conditions (a)–(c) of Theorem 3.1 remain equivalent as stated for the current value of  $\gamma \in (0, 1)$ .

(2) Assume conditions (a)–(c) as stated in Theorem 3.1 hold for the current value of  $\gamma \in (0, 1)$ , and further assume that  $X_t$  has a non-lattice distribution for each fixed  $t > 0$ . Then, uniformly for  $z \in [\Delta_0, \Delta_1]$ , for any fixed  $0 < \Delta_0 < \Delta_1 < \infty$ , and  $t \in [T_0, T_1]$  for any fixed  $0 < T_0 < T_1 < \infty$ ,

$$\lim_{u \rightarrow \infty} a(u)b(a(u))P^{(u)}(Z^{(u)} \in (a(u)z, a(u)z + \Delta], \tau_u \in b(a(u)) dt) = h_t(z)f(z)\Delta dt, \tag{3.7}$$

where, in case (i),

$$f(z) = \frac{\Gamma(\beta)}{\Gamma(\beta + \gamma - 1)(1 + z)^\beta}, \tag{3.8}$$

and in case (ii)

$$f(z) = e^{-z}, \quad z > 0.$$

Moreover, for  $k = 2, 3, \dots$ , take  $z_i > 0$  and  $I_i = (a(u)z_i, a(u)z_i + \Delta_i]$ ,  $i = 1, 2, \dots, k - 1$ , and write, for  $0 < s_1 < \dots < s_{k-1} < s_k = 1$ ,

$$A_k = \{X^*(s_i t b(a(u))) \in I_i, i = 1, 2, \dots, k - 1\}.$$

Then, uniformly for  $z_i \in [\Delta_0, \Delta_1]$ ,  $i = 1, 2, \dots, k$ , for any fixed  $0 < \Delta_0 < \Delta_1 < \infty$ , and  $t \in [T_0, T_1]$  for any fixed  $0 < T_0 < T_1 < \infty$ , we have

$$\begin{aligned} & \lim_{u \rightarrow \infty} (a(u))^k b(a(u)) P^{(u)}(A_k, Z^{(u)} \in (z_k a(u), z_k a(u) + \Delta_k], \tau_u \in b(a(u)) dt) \\ & = \theta(z_1, z_2, \dots, z_k, t) \prod_{i=1}^k \Delta_i dt, \quad k = 1, 2, \dots \end{aligned} \tag{3.9}$$

Here, with  $s_0 = z_0 = 0$ ,

$$\theta(z_1, z_2, \dots, z_k, t) = \prod_{i=1}^k h_{t(s_i - s_{i-1})}(z_i - z_{i-1}) f(z_k).$$

(3) Further: assume conditions (a)–(c) as stated in Theorem 3.1 hold for the current value of  $\gamma \in (0, 1)$ , and that  $X_t$  has a non-lattice distribution for each  $t > 0$ . Then, under  $P^{(u)}$ , the process  $\mathbf{Y}^{(u)}$  defined in (3.2) converges weakly in  $\mathbb{R}^3 \times \mathbb{D}_0[0, 1]$  as  $u \rightarrow \infty$  to the process  $(V, U, W, (\widehat{\mathbf{D}}_{W,V}(s))_{0 \leq s \leq 1})$ , where in case (i)

$$\begin{aligned} & P(V \in dz, U \in dx, W \in dt) \\ & = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \gamma - 1)(1 + z + x)^{\beta+1}} h_t(z) dz dx dt, \quad t, x, z > 0, \end{aligned} \tag{3.10}$$

and in case (ii)

$$P(V \in dz, U \in dx, W \in dt) = e^{-z-x} h_t(z) dz dx dt, \quad t, x, z > 0. \tag{3.11}$$

**Remark 3.2.** (i) The further assumption in part 2 of Theorem 3.2, that for each  $t > 0$ ,  $X_t$  has a non-lattice distribution, is equivalent to assuming that  $X$  is not a compound Poisson process whose step distribution takes values on a lattice. We can cover the lattice case also with only minor adjustments. Thus, if the lattice has span 1, we need only restrict  $\Delta$  to take integer values and replace  $(a(u)z, a(u)z + \Delta]$  in (3.7) by  $(\lfloor a(u)z \rfloor, \lfloor a(u)z \rfloor + \Delta]$ , and similarly in (3.9), for a valid conclusion. The only difference in the proof is which version of a local limit theorem is used.

(ii) The right-hand sides of (3.3) and (3.4) and (3.10) and (3.11) are probability densities on  $x, z > 0$  and  $t, x, z > 0$ , so, under the conditions of Theorems 3.1 and 3.2, the limiting distributions of  $Z^{(u)}/a(u)$  (and of course those of  $O^{(u)}/a(u)$  and  $\tau_u/b(a(u))$ ) are concentrated on  $[0, \infty)$ . Thus,  $\lim_{u \rightarrow \infty} P(Z^{(u)}/a(u) \leq -z) = 0$  for all  $z > 0$ . So it is convenient to define  $Z^{(u)} = -X_{\tau_u-}$  as we did in (2.12).

(iii) In connection with Theorem 3.2, we mention the paper by Klüppelberg and Kyprianou [17], which deals with the infinite mean case under special assumptions.

(iv) The marginal limiting distributions of the fluctuation quantities are easily computed from (3.3) and (3.4) and (3.10) and (3.11). The identities  $t^{1/\bar{\gamma}} h_t(z) = h_1(z/t^{1/\bar{\gamma}})$  and  $\int_0^\infty h_t(z) dt = z^{-\gamma} / \Gamma(\bar{\gamma})$ , where  $\bar{\gamma} = 1 - \gamma$  [see Sato [22], pages 87, 261], are useful. Thus, for example, under



the conditions of case (i) of Theorem 3.2, the limiting densities of  $(Z^{(u)}, O^{(u)})$  and  $\tau_u$ , suitably normalised, are derived from (3.10) as

$$P(V \in dz, U \in dx) = \frac{\Gamma(\beta + 1)z^{-\gamma}}{\Gamma(1 - \gamma)\Gamma(\beta + \gamma - 1)(1 + z + x)^{\beta+1}} dz dx, \quad y, x > 0 \quad (3.12)$$

and

$$P(W \in dt) = \frac{\Gamma(\beta)}{\Gamma(\beta + \gamma - 1)} \int_0^\infty \frac{h_1(z) dz}{(1 + t^{1/\gamma}z)^\beta} dt, \quad t > 0. \quad (3.13)$$

It can be checked that no pair of  $(V, U, W)$  are independent, in case (i). For case (ii),

$$P(V \in dz, U \in dx) = \frac{z^{-\gamma}e^{-z-x}}{\Gamma(1 - \gamma)} dz dx, \quad x, z > 0 \quad (3.14)$$

and

$$P(W \in dt) = \int_0^\infty e^{-zt^{1/\gamma}} h_1(z) dz dt, \quad t > 0. \quad (3.15)$$

In this case,  $V$  is independent of  $U$ ,  $U$  is independent of  $W$ , but  $V$  is not independent of  $W$ .

### 4. Preliminaries to the proofs

Our first proposition applies to any defective subordinator, so we change notation slightly just for this result.

**Proposition 4.1.** *Let  $Y$  be any defective subordinator, obtained from a non-defective subordinator  $\mathcal{Y}$  with killing rate  $q$ , whose Lévy measure is  $\Pi_Y$ , with tail  $\bar{\Pi}_Y$ . Assume  $\Pi_Y \in \mathcal{S}$ . Write  $P_Y^{(u)}$  for  $P(\cdot | T_u^Y < \infty)$ , where  $T_u^Y = \inf\{t: Y_t > u\}$ ,  $u > 0$ , and put  $O_Y^{(u)} = Y_{T_u^Y} - u$  on the event  $\{T_u^Y < \infty\}$ .*

*Then  $P_Y^{(u)}(O_Y^{(u)} \in a(u) dx)$  has a non-degenerate limit  $P(O \in dx)$  for some  $a(u) > 0$ ,  $a(u) \rightarrow \infty$ , if and only if either  $\bar{\Pi}_Y \in \text{RV}(-\alpha)$  for some  $\alpha > 0$ , or  $\Pi_Y \in \text{MDA}(\Lambda)$ .*

*Moreover, in the first case we can take  $a(u) = u$  and  $O$  to have density  $\alpha(1 + x)^{-1-\alpha}$ , and in the second case we can take  $a(u) = \int_u^\infty \bar{\Pi}_Y(y) dy / \bar{\Pi}_Y(u) = o(u)$  and  $O$  to have density  $e^{-x}$ .*

**Proof.** For the distribution of  $O_Y^{(u)}$ , use of the compensation formula for Poisson point processes as in Bertoin [3], Proposition 2, page 76, or Klüppelberg, Kyprianou and Maller [18], Theorem 2.4, gives

$$\begin{aligned} P(O_Y^{(u)} > xa(u), T_u^Y < \infty) &= P(Y_{T_u^Y} > u + xa(u), T_u^Y < \infty) \\ &= E \sum_{0 < t < L_\infty} \mathbf{1}_{\{Y_t > u + xa(u), T_u^Y = t\}} \\ &= \int_0^\infty e^{-qt} \int_{(0,u]} \bar{\Pi}_Y(u + xa(u) - y) P(\mathcal{Y}_t \in dy) dt. \end{aligned}$$

From this, writing  $e(q)$  for an independent  $\text{Exp}(q)$  random variable, we have for any  $C_0 > 0$

$$\begin{aligned} P(\mathcal{O}_Y^{(u)} > xa(u), T_u^Y < \infty) &= q^{-1} \int_{(0,u]} P(\mathcal{Y}_{e(q)} \in dy) \bar{\Pi}_Y(u + xa(u) - y) \\ &= q^{-1} \left( \int_{(0,C_0]} + \int_{(C_0,u]} \right) P(\mathcal{Y}_{e(q)} \in dy) \bar{\Pi}_Y(u + xa(u) - y). \end{aligned} \tag{4.1}$$

Assume at this stage that  $\Pi_Y \in \mathcal{S}$ . Then  $\Pi_Y \in \mathcal{L}$ , so we have

$$\bar{\Pi}_Y(u - y + xa(u)) \sim \bar{\Pi}_Y(u + xa(u)) \quad \text{uniformly for } y \in (0, C_0] \text{ and } x \geq 0. \tag{4.2}$$

Thus,

$$\int_{(0,C_0]} P(\mathcal{Y}_{e(q)} \in dy) \bar{\Pi}_Y(u + xa(u) - y) \sim P(\mathcal{Y}_{e(q)} \leq C_0) \bar{\Pi}_Y(u + xa(u)). \tag{4.3}$$

Since  $\Pi_Y \in \mathcal{S}$ , we know from Lemma 3.5 of Klüppelberg, Kyprianou and Maller [18] (with  $\alpha = 0$ ) that  $\bar{\Pi}_Y(u) \sim q P(T_u^Y < \infty)$ . Given arbitrary  $\varepsilon \in (0, 1)$ , we can choose  $C_0 > 0$  such that  $P(\mathcal{Y}_{e(q)} > C_0) \leq \varepsilon$ . Then for  $u$  large enough, again using (4.2),

$$\begin{aligned} (1 + \varepsilon) \bar{\Pi}_Y(u) &\geq q P(T_u^Y < \infty) \\ &= \left( \int_{(0,C_0]} + \int_{(C_0,\infty)} \right) P(\mathcal{Y}_{e(q)} \in dy) \bar{\Pi}_Y(u - y) \\ &\geq (1 - \varepsilon) P(\mathcal{Y}_{e(q)} \leq C_0) \bar{\Pi}_Y(u) \\ &\quad + \int_{(C_0,u]} P(\mathcal{Y}_{e(q)} \in dy) \bar{\Pi}_Y(u - y), \end{aligned}$$

giving

$$\int_{(C_0,u]} P(\mathcal{Y}_{e(q)} \in dy) \bar{\Pi}_Y(u - y) \leq ((1 + \varepsilon) - (1 - \varepsilon)^2) \bar{\Pi}_Y(u) \leq 3\varepsilon \bar{\Pi}_Y(u).$$

From this, and (4.1) and (4.3), and since  $\bar{\Pi}_Y(u) \sim q P(T_u^Y < \infty)$ , we have

$$\begin{aligned} P^{(u)}(\mathcal{O}_Y^{(u)} > xa(u)) &= \frac{P(\mathcal{O}_Y^{(u)} > xa(u), T_u^Y < \infty)}{P(T_u^Y < \infty)} \\ &= (1 + o(1)) P(\mathcal{Y}_{e(q)} \leq C_0) \frac{\bar{\Pi}_Y(u + xa(u))}{\bar{\Pi}_Y(u)} + o(1). \end{aligned} \tag{4.4}$$

As discussed in (2.7), the condition  $\overline{\Pi}_Y \in \text{RV}(-\alpha)$  for some  $\alpha > 0$ , or  $\Pi_Y \in \text{MDA}(\Lambda)$ , is equivalent to the existence of  $a(u) \rightarrow \infty$  such that

$$\frac{\overline{\Pi}_Y(u + xa(u))}{\overline{\Pi}_Y(u)} \rightarrow P(O > x), \tag{4.5}$$

and when it holds  $a(u)$  and  $O$  have the stated properties. The conclusions of the proposition then follow from this and (4.4).  $\square$

We will make use of the “équations amicales” of Vigon [23], which are

$$\overline{\Pi}_X^+(u) = \int_{(0,\infty)} \overline{\Pi}_{H^*}(y)\Pi_{\mathcal{H}}(u + dy) + d_{H^*}n(u), \quad u > 0 \tag{4.6}$$

and

$$\overline{\Pi}_X^-(u) = \int_{(0,\infty)} \overline{\Pi}_{\mathcal{H}}(y)\Pi_{H^*}(u + dy) + d_{\mathcal{H}}n^*(u) + q\overline{\Pi}_{H^*}(u), \quad u > 0, \tag{4.7}$$

where  $n(\cdot), n^*(\cdot)$  denote càdlàg versions of the densities of  $\Pi_{\mathcal{H}}, \Pi_{H^*}$ , defined if  $d_{\mathcal{H}} > 0, d_{H^*} > 0$ , respectively. Recall that  $q$  is the killing rate in (2.2).

We are looking for limit theorems which will always include the convergence of the normed overshoot, and Proposition 4.1 suggests the relevance of conditions like

$$\overline{\Pi}_{\mathcal{H}} \in \text{RV}(-\alpha) \quad \text{for some } \alpha > 0 \text{ [case (i)]} \quad \text{or} \quad \Pi_{\mathcal{H}} \in \text{MDA}(\Lambda) \quad \text{[case (ii)]}. \tag{4.8}$$

The next proposition shows that these imply similarly stated conditions on  $\overline{\Pi}_X^{(+)}$ . At this stage, we are not assuming  $\Pi_{\mathcal{H}} \in \mathcal{S}$ .

**Proposition 4.2.** *Assume  $\lim_{t \rightarrow \infty} X_t = -\infty$  a.s. and  $A_{H^*} \in \text{RV}(\gamma)$  with  $\gamma \in [0, 1)$ . Suppose (4.8) holds with  $\alpha = \beta + \gamma - 1 > 0$ , where  $\beta > 0$ , in case (i).*

*Then  $\overline{\Pi}_X^+ \in \text{RV}(-\beta)$  (case (i)), or  $\overline{\Pi}_X^{(+)} \in \text{MDA}(\Lambda)$  [case (ii)], or, equivalently,*

$$\frac{\overline{\Pi}_X^+(u + xa(u))}{\overline{\Pi}_X^+(u)} \rightarrow P(C > x), \quad x > 0, \tag{4.9}$$

where  $a(u) = u$  and  $P(C > x) = (1 + x)^{-\beta}$  (case (i)), or  $a(u) = \int_u^\infty \overline{\Pi}_{\mathcal{H}}(y) dy / \overline{\Pi}_{\mathcal{H}}(u)$  and  $P(C > x) = e^{-x}$  [case (ii)]. Further, in both cases we have, for some constants  $c_{\gamma,\beta} \in (0, \infty)$  (whose values are made explicit in the proof),

$$\overline{\Pi}_X^+(u) \sim \frac{c_{\gamma,\beta} \overline{\Pi}_{\mathcal{H}}(u) A_{H^*}(a(u))}{a(u)}. \tag{4.10}$$

Moreover, in case (ii) we can alternatively take  $a(u) = \int_u^\infty \overline{\Pi}_X^+(y) dy / \overline{\Pi}_X^+(u), u > 0$ .

**Proof.** Assume (2.1), and that (2.13) holds with  $\gamma \in [0, 1)$ .

The starting point is Vigon’s *équation amicale*, (4.6), which we write as  $\overline{\Pi}_X^+(u) = I(u) + d_{H^*n}(u)$ , with

$$\begin{aligned}
 I(u) &= \int_{(0,\infty)} \Pi_{\mathcal{H}}(u + dy) \int_{(y,\infty)} \Pi_{H^*}(dz) = \int_{(0,\infty)} \Pi_{H^*}(dz) \int_{(0,z)} \Pi_{\mathcal{H}}(u + dy) \\
 &= \int_{(0,\infty)} \Pi_{H^*}(a(u) dz) \Pi_{\mathcal{H}}\{(u, u + a(u)z]\} \\
 &= \left( \int_{(0,K]} + \int_{(K,\infty)} \right) \Pi_{H^*}(a(u) dz) \Pi_{\mathcal{H}}\{(u, u + a(u)z]\} \\
 &=: I_1(u) + I_2(u), \quad \text{say,}
 \end{aligned}
 \tag{4.11}$$

where  $K > 0$ . Recall the definition of  $A_{H^*}$  in (2.5), and note that

$$u \overline{\Pi}_{H^*}(u) \leq \int_0^u \overline{\Pi}_{H^*}(y) dy = A_{H^*}(u), \quad u > 0,$$

so we have by the regular variation of  $A_{H^*}$

$$\frac{a(u)I_2(u)}{A_{H^*}(a(u))\overline{\Pi}_{\mathcal{H}}(u)} \leq \frac{a(u)\overline{\Pi}_{H^*}(Ka(u))}{A_{H^*}(a(u))} \leq \frac{A_{H^*}(Ka(u))}{KA_{H^*}(a(u))} \sim \frac{1}{K^{1-\gamma}}.$$

Since  $0 \leq \gamma < 1$  it follows that

$$\lim_{K \rightarrow \infty} \limsup_{u \rightarrow \infty} \frac{a(u)I_2(u)}{A_{H^*}(a(u))\overline{\Pi}_{\mathcal{H}}(u)} = 0. \tag{4.12}$$

Now assume (4.8), in which we set  $\alpha = \beta + \gamma - 1 > 0$ . By (2.7) with  $\overline{F}$  replaced by  $\overline{\Pi}_{\mathcal{H}}$ , this implies

$$\frac{\Pi_{\mathcal{H}}\{(u, u + a(u)z]\}}{\overline{\Pi}_{\mathcal{H}}(u)} \rightarrow \int_0^z p(y) dy \tag{4.13}$$

uniformly for  $z \in [0, K]$ , where  $p(\cdot)$  is the limiting density associated with  $\Pi_{\mathcal{H}}$ , that is,  $\text{Par}(\beta + \gamma - 1)$  in case (i), or  $\text{Exp}(1)$  in case (ii). So the component  $I_1(u)$  in (4.11) satisfies

$$\begin{aligned}
 I_1(u) &\sim \overline{\Pi}_{\mathcal{H}}(u) \int_0^K \Pi_{H^*}(a(u) dz) \int_0^z p(y) dy \\
 &= \overline{\Pi}_{\mathcal{H}}(u) \int_0^K p(y) dy \int_y^K \Pi_{H^*}(a(u) dz) \\
 &= \overline{\Pi}_{\mathcal{H}}(u) \int_0^K p(y)\overline{\Pi}_{H^*}(a(u)y) dy - \overline{\Pi}_{\mathcal{H}}(u)\overline{\Pi}_{H^*}(a(u)K) \int_0^K p(y) dy.
 \end{aligned}
 \tag{4.14}$$

(a) When  $\gamma \in (0, 1)$ ,  $A_{H^*} \in \text{RV}(\gamma)$  is equivalent, by the monotone density theorem (Bingham, Goldie and Teugels [5], Theorem 1.7.2, page 39), to  $\overline{\Pi}_{\mathcal{H}^*} \in \text{RV}(\gamma - 1)$ , and then  $\overline{\Pi}_{\mathcal{H}^*}(x) \sim \gamma x^{-1} A_{H^*}(x)$ . So

$$\int_0^K p(y) \overline{\Pi}_{H^*}(a(u)y) dy \sim \frac{\gamma A_{H^*}(a(u))}{a(u)} \int_0^K p(y) y^{\gamma-1} dy, \tag{4.15}$$

and by taking  $u \rightarrow \infty$  then  $K \rightarrow \infty$  in (4.14) we conclude

$$\lim_{K \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{a(u) I_1(u)}{A_{H^*}(a(u)) \overline{\Pi}_{\mathcal{H}}(u)} = \gamma \int_0^\infty p(y) y^{\gamma-1} dy = \gamma E(C^{\gamma-1}). \tag{4.16}$$

(b) When  $\gamma = 0$ , so that  $A_{H^*}$  is slowly varying, we use the feature that  $\lim_{x \downarrow 0} p(x) = p(0) > 0$  in either case,  $\text{Par}(\beta - 1 + \gamma)$  or  $\text{Exp}(1)$ , to argue, given arbitrary  $\varepsilon > 0$ , the existence of a  $\delta_\varepsilon > 0$  such that for all large enough  $u$

$$a(u) \int_0^{\delta_\varepsilon} p(y) \overline{\Pi}_{\mathcal{H}}(a(u)y) dy \leq p(0)(1 + \varepsilon) A_{H^*}(\delta_\varepsilon a(u)) \sim p(0)(1 + \varepsilon) A_{H^*}(a(u))$$

and

$$a(u) \int_0^{\delta_\varepsilon} p(y) \overline{\Pi}_{H^*}(a(u)y) dy \geq p(0)(1 - \varepsilon) A_{H^*}(\delta_\varepsilon a(u)) \sim p(0)(1 - \varepsilon) A_{H^*}(a(u)).$$

$A_{H^*}$  slowly varying implies  $x \overline{\Pi}_{H^*}(x) = o(A_{H^*}(x))$  as  $x \rightarrow \infty$ , so with  $\delta_\varepsilon$  fixed we can argue

$$\begin{aligned} \int_{\delta_\varepsilon}^K p(y) \overline{\Pi}_{H^*}(a(u)y) dy &= o\left(\frac{1}{a(u)} \int_{\delta_\varepsilon}^K p(y) A_{H^*}(a(u)y) \frac{dy}{y}\right) \\ &= o\left(\frac{A_{H^*}(a(u))}{a(u)}\right), \end{aligned}$$

and we deduce for  $\gamma = 0$  that

$$\lim_{K \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{a(u) I_1(u)}{A_{H^*}(a(u)) \overline{\Pi}_{\mathcal{H}}(u)} = p(0). \tag{4.17}$$

Thus, in all cases we have

$$I(u) \sim \frac{c(\gamma, \beta) A_{H^*}(a(u)) \overline{\Pi}_{\mathcal{H}}(u)}{a(u)} \tag{4.18}$$

for a constant  $c(\gamma, \beta) \in (0, \infty)$ , which we can evaluate as follows.

(a) When  $\gamma \in (0, 1)$ , in case (i)

$$c(\gamma, \beta) = \gamma E(C^{\gamma-1}) = \gamma(\beta + \gamma - 1) \int_0^\infty \frac{x^{\gamma-1} dx}{(1+x)^{\beta+\gamma}} = \frac{\Gamma(\gamma+1)\Gamma(\beta)}{\Gamma(\beta+\gamma-1)}. \tag{4.19}$$

[Note that the density  $p(\cdot)$  here is the one associated with  $\mathcal{H}$ , not  $X^+$ , that is, it is Pareto with parameter  $\alpha = \beta + \gamma - 1$ ; see (4.13).]

In case (ii)

$$c(\gamma, \beta) = \gamma E(C^{\gamma-1}) = \Gamma(\gamma + 1). \tag{4.20}$$

(b) When  $\gamma = 0$ ,  $p(0) = 1 - \beta$  in case (i), and in case (ii),  $p(0) = 1$ , so we set  $c(0, \beta) = 1 - \beta$  in case (i), and  $c(0, \beta) = 1$  in case (ii).

Now integrate (4.6) and use the estimate (4.18) to get

$$\begin{aligned} \int_u^\infty \bar{\Pi}_X^+(y) dy &= \int_u^\infty I(v) dv + d_{H^*} \bar{\Pi}_{\mathcal{H}}(u) \\ &\sim c(\gamma, \beta) \int_u^\infty \frac{A_{H^*}(a(v)) \bar{\Pi}_{\mathcal{H}}(v)}{a(v)} dv + d_{H^*} \bar{\Pi}_{\mathcal{H}}(u). \end{aligned} \tag{4.21}$$

Assume in addition that  $\bar{\Pi}_{\mathcal{H}} \in \text{RV}(1 - \gamma - \beta)$ . This together with  $A_{H^*} \in \text{RV}(\gamma)$  means that the product  $\bar{\Pi}_{\mathcal{H}} A_{H^*} \in \text{RV}(1 - \beta)$ . Then, taking  $a(u) = u$  in this case, (4.21) gives

$$\frac{1}{\bar{\Pi}_{\mathcal{H}}(u) A_{H^*}(u)} \int_u^\infty \bar{\Pi}_X^+(y) dy \sim c(\gamma, \beta) \int_1^\infty v^{-\beta} dv + \frac{d_{H^*}}{A_{H^*}(u)}. \tag{4.22}$$

In either case,  $A_{H^*}(\infty) = \infty$  or  $A_{H^*}(\infty) < \infty$ , we can use the monotone density theorem again to deduce from this that  $\bar{\Pi}_X^+ \in \text{RV}(-\beta)$ , and hence that (4.9) holds with  $a(u) = u$ .

Alternatively, suppose  $\bar{\Pi}_{\mathcal{H}} \in \text{MDA}(\Lambda)$ . In this case, (4.21) gives

$$\int_{u+xa(u)}^\infty \bar{\Pi}_X^+(y) dy \sim c(\gamma, \beta) \int_{u+xa(u)}^\infty \frac{A_{H^*}(a(v)) \bar{\Pi}_{\mathcal{H}}(v)}{a(v)} dv + d_{H^*} \bar{\Pi}_{\mathcal{H}}(u + xa(u)), \quad x \geq 0.$$

Change variable by setting  $v = u + v'a(u)$  on the RHS. Since  $a(\cdot)$  is self-neglecting, we have  $a(v) = a(u + v'a(u)) \sim a(u)$ , so by the regular variation of  $A_{H^*}$ ,

$$\frac{A_{H^*}(a(v))}{a(v)} \sim \frac{A_{H^*}(a(u))}{a(u)},$$

and since  $\bar{\Pi}_{\mathcal{H}} \in \text{MDA}(\Lambda)$ ,

$$\bar{\Pi}_{\mathcal{H}}(v) = \bar{\Pi}_{\mathcal{H}}(u + v'a(u)) \sim e^{-v'} \bar{\Pi}_{\mathcal{H}}(u).$$

Thus, for  $x \geq 0$

$$\begin{aligned} &\frac{1}{\bar{\Pi}_{\mathcal{H}}(u)} \int_{u+xa(u)}^\infty \bar{\Pi}_X^+(y) dy \\ &\sim c(\gamma, \beta) a(u) \int_x^\infty \frac{A_{H^*}(a(v')) \bar{\Pi}_{\mathcal{H}}(v')}{a(v') \bar{\Pi}_{\mathcal{H}}(u)} dv' + d_{H^*} \frac{\bar{\Pi}_{\mathcal{H}}(u + xa(u))}{\bar{\Pi}_{\mathcal{H}}(u)} \\ &\sim c(\gamma, \beta) A_{H^*}(a(u)) \int_x^\infty e^{-v'} dv' + e^{-x} d_{H^*}, \end{aligned} \tag{4.23}$$

which, applied with  $x = 0$ , also gives

$$\frac{\int_{u+xa(u)}^{\infty} \bar{\Pi}_X^+(y) dy}{\int_u^{\infty} \bar{\Pi}_X^+(y) dy} \rightarrow e^{-x}, \quad x \geq 0.$$

Applying Theorem 1.2.2(3) of de Haan and Ferreira [7], we get

$$\frac{\bar{\Pi}_X^+(u + xa(u))}{\bar{\Pi}_X^+(u)} \rightarrow e^{-x}, \quad x \geq 0,$$

which is (4.9) in this case, and this implies

$$\frac{\int_{u+xa(u)}^{\infty} \bar{\Pi}_X^+(y) dy}{a(u)\bar{\Pi}_X^+(u)} \rightarrow e^{-x}, \quad x \geq 0, \tag{4.24}$$

hence

$$a(u) \sim \frac{\int_u^{\infty} \bar{\Pi}_X^+(y) dy}{\bar{\Pi}_X^+(u)}, \tag{4.25}$$

as claimed for this case.

It remains to prove (4.10). In case (i), when  $\bar{\Pi}_{\mathcal{H}} \in \text{RV}(1 - \gamma - \beta)$  and  $\bar{\Pi}_X \in \text{RV}(-\beta)$ , the relation (4.22) gives

$$\begin{aligned} \bar{\Pi}_X^+(u) &\sim \frac{\beta - 1}{u} \int_u^{\infty} \bar{\Pi}_X^+(y) dy \\ &\sim \left( c(\gamma, \beta) + \frac{(\beta - 1)d_{H^*}}{A_{H^*}(u)} \right) \frac{\bar{\Pi}_{\mathcal{H}}(u)A_{H^*}(u)}{u}. \end{aligned} \tag{4.26}$$

- (a) When  $\gamma \in (0, 1)$ , this implies (4.10) with  $c_{\gamma, \beta} = c(\gamma, \beta) + (\beta - 1)d_{H^*}/EH_1^*$ , for  $EH_1^* \leq \infty$ .
- (b) When  $\gamma = 0$ ,  $c_{0, \beta} = c(0, \beta)$  for  $EH_1^* = \infty$  and, for  $EH_1^* < \infty$ .

$$\begin{aligned} c_{0, \beta} &= c(0, \beta) + \frac{(\beta - 1)d_{H^*}}{EH_1^* - d_{H^*}} = \beta + \frac{(\beta - 1)d_{H^*}}{EH_1^* - d_{H^*}} \\ &= \frac{\beta EH_1^* - d_{H^*}}{EH_1^* - d_{H^*}} = \frac{\beta EH_1^* - d_{H^*}}{A_{H^*}(\infty)}. \end{aligned}$$

In case (ii), when  $\Pi_X \in \text{MDA}(\Lambda)$ , (4.23) and (4.24) give, instead of (4.26),

$$\bar{\Pi}_X^+(u) \sim \frac{1}{a(u)} \int_u^{\infty} \bar{\Pi}_X^+(y) dy \sim \left( c(\gamma, \beta) + \frac{d_{H^*}}{A_{H^*}(a(u))} \right) \frac{\bar{\Pi}_{\mathcal{H}}(u)A_{H^*}(a(u))}{a(u)}. \tag{4.27}$$

- (a) When  $\gamma \in (0, 1)$  this implies (4.10) with  $c_{\gamma,\beta} = c(\gamma, \beta) + d_{H^*}/EH_1^*$ , for  $EH_1^* \leq \infty$ .
- (b) When  $\gamma = 0$ ,  $c_{0,\beta} = 1$  for  $EH_1^* = \infty$  and, for  $EH_1^* < \infty$ ,

$$\begin{aligned} c_{0,\beta} &= c(0, \beta) + \frac{d_{H^*}}{EH_1^* - d_{H^*}} = 1 + \frac{d_{H^*}}{EH_1^* - d_{H^*}} \\ &= \frac{EH_1^*}{EH_1^* - d_{H^*}} = \frac{EH_1^*}{A_{H^*}(\infty)}. \end{aligned}$$

This completes the proof of Proposition 4.2. □

Doney [9], Corollary 4, page 31 (interchange  $+/-$  in his result), shows that, when  $\lim_{t \rightarrow \infty} X_t = -\infty$  a.s.,  $E|X_1| < \infty$  if and only if  $EH_1^* < \infty$ , and then  $E|X_1| = qEH_1^*$ . The following proposition generalises this, allowing for  $EH_1^* = \infty$ .

**Proposition 4.3.** *Assume  $\lim_{t \rightarrow \infty} X_t = -\infty$  a.s. and  $A_X^*(\infty) = \infty$ , or, equivalently,  $EH_1^* = \infty$ . Then*

$$\lim_{x \rightarrow \infty} \frac{A_X^*(x)}{A_{H^*}(x)} = q. \tag{4.28}$$

**Proof.** Assume  $\lim_{t \rightarrow \infty} X_t = -\infty$  a.s. and  $A_{H^*}(\infty) = \infty$ . The integral term in (4.7) can be written as

$$\begin{aligned} &\int_{(0,\infty)} (\overline{\Pi}_{H^*}(u) - \overline{\Pi}_{H^*}(y+u)) \Pi_{\mathcal{H}}(dy) \\ &= \int_{(0,\infty)} \overline{\Pi}_{\mathcal{H}}(y) dy (\overline{\Pi}_{H^*}(u) - \overline{\Pi}_{H^*}(y+u)) \end{aligned}$$

after integrating by parts. So, by integrating (4.7) over  $1 \leq u \leq x$ , we have

$$A_X^*(x) - q \int_1^x \overline{\Pi}_{H^*}(u) du = d_{\mathcal{H}}(\overline{\Pi}_{H^*}(1) - \overline{\Pi}_{H^*}(x)) + I(x), \tag{4.29}$$

where

$$I(x) = \int_{(0,\infty)} \Pi_{\mathcal{H}}(dy) \int_1^x (\overline{\Pi}_{H^*}(u) - \overline{\Pi}_{H^*}(y+u)) du.$$

We can bound the inner integral by

$$\begin{aligned} \left( \int_1^x - \int_{1+y}^{x+y} \right) \overline{\Pi}_{H^*}(u) du &= \left( \int_1^{1+y} - \int_x^{x+y} \right) \overline{\Pi}_{H^*}(u) du \\ &\leq \int_1^{1+y} \overline{\Pi}_{H^*}(u) du \leq y \overline{\Pi}_{H^*}(1). \end{aligned}$$



Then, for any  $K > 0$ ,

$$\begin{aligned}
 I(x) &\leq \bar{\Pi}_{H^*}(1) \int_0^K y \Pi_{\mathcal{H}}(dy) + \int_K^\infty \Pi_{\mathcal{H}}(dy) \int_1^x (\bar{\Pi}_{H^*}(u) - \bar{\Pi}_{H^*}(y+u)) du \\
 &\leq \bar{\Pi}_{H^*}(1) \int_0^K y \Pi_{\mathcal{H}}(dy) + \int_K^\infty \Pi_{\mathcal{H}}(dy) \int_1^x \bar{\Pi}_{H^*}(u) du \\
 &\leq \bar{\Pi}_{H^*}(1) \int_0^K y \Pi_{\mathcal{H}}(dy) + \bar{\Pi}_{\mathcal{H}}(K) A_{H^*}(x).
 \end{aligned}
 \tag{4.30}$$

Since  $A_{H^*}(\infty) = \infty$ , when we divide by  $A_{H^*}(x)$  and let  $x \rightarrow \infty$  and then  $K \rightarrow \infty$  in (4.30), we get  $\lim_{x \rightarrow \infty} I(x)/A_{H^*}(x) = 0$ . Then (4.28) follows from (4.29).  $\square$

**Remark 4.1.** (i) We mention that a random walk version of Proposition 4.3 is (in a different notation) in Lemma 1 of Denisov, Foss, and Korshunov [8].

(ii) When (2.13) holds, that is,  $A_{H^*} \in \text{RV}(\gamma)$  with  $\gamma \in [0, 1)$ , and  $A_{H^*}(\infty) = \infty$ , then  $A_X^*(\infty) = \infty$  and, by (4.28),  $A_X^* \in \text{RV}(\gamma)$ . The latter is equivalent to

$$\lim_{x \rightarrow \infty} \frac{x \bar{\Pi}_X^-(x)}{A_X^*(x)} = \gamma.
 \tag{4.31}$$

This is also true when  $A_{H^*}(\infty) < \infty$ , equivalently,  $A_X^*(\infty) < \infty$ . [Compare with (2.15).]

### 5. The case $\gamma = 0$ (including finite mean)

Assume (2.1) and (2.13) with  $\gamma = 0$ , so  $A_{H^*} \in \text{RV}(0)$ , or, equivalently,  $x \bar{\Pi}_{H^*}(x) = o(A_{H^*}(x))$  as  $x \rightarrow \infty$ . Now (e.g., use Theorem 4.4 of Doney and Maller [11] with  $+/-$  interchanged) (2.1) implies

$$\frac{x \bar{\Pi}_X^+(x)}{A_X^*(x)} \leq \frac{A_X^+(x)}{A_X^*(x)} \rightarrow 0 \quad \text{as } x \rightarrow \infty
 \tag{5.1}$$

if  $A_X^*(\infty) = \infty$ , otherwise  $A_X^*(\infty) < \infty$  and then  $A_X^+(\infty) < \infty$  and  $\lim_{x \rightarrow \infty} x \bar{\Pi}_X^+(x) = 0$ . Thus, since also  $x \bar{\Pi}_X^-(x) = o(A_X^*(x))$  by (4.31),

$$A(x) := \gamma + \bar{\Pi}^+(1) - \bar{\Pi}^-(1) + A_X^+(x) - A_X^*(x) \sim -A_X^*(x) \quad \text{as } x \rightarrow \infty,$$

and we see that  $x \bar{\Pi}(x) = o(-A(x))$  as  $x \rightarrow \infty$ . This means that  $X_t$  is negatively relatively stable (Doney and Maller [11]), or, equivalently,  $X_t^*$  is positively relatively stable, as  $t \rightarrow \infty$ . Consequently, we can employ a version of the weak law of large numbers even if the mean is infinite; specifically there is a continuous, increasing function  $c(\cdot) \in \text{RV}(1)$  such that  $X_t^*/c(t) \xrightarrow{P} 1$  as  $t \rightarrow \infty$ . The function  $c(\cdot)$  can be chosen to be strictly increasing and to satisfy

$$c(x) = x A_X^*(c(x)), \quad x > 0,$$

and its inverse function  $b(\cdot) := c^{-1}(\cdot)$  is given by

$$b(y) = \frac{y}{A_X^*(y)}, \quad y > 0.$$

Employing Proposition 4.3, we see that

$$b(y) = \frac{y}{A_X^*(y)} \sim \frac{y}{qA_{H^*}(y)} \quad \text{as } y \rightarrow \infty, \tag{5.2}$$

when  $A_{H^*}(\infty) = \infty$ . When  $A_{H^*}(\infty) < \infty$ , and so  $EX_1 \in (-\infty, 0)$ , we simply take  $c(x) = |EX_1|x$  and  $b(x) = x/|EX_1|$ ,  $x > 0$ .

We define another norming function by  $r(u) = b(a(u))$ , and note that  $c(r(u)) = a(u)$  and

$$r(u) \sim \frac{a(u)}{qA_{H^*}(a(u))} \tag{5.3}$$

when  $A_{H^*}(\infty) = \infty$ , and

$$r(u) = \frac{a(u)}{|EX_1|} = \frac{a(u)}{qEH_1^*} \tag{5.4}$$

when  $A_{H^*}(\infty) < \infty$ . The function  $r(u)$  turns out to be the right norming for  $\tau_u$  in the present situation.

**Proof of Theorem 3.1.** Assume (2.1) and (2.10), and that (2.13) holds with  $\gamma = 0$ . Then parts 1(a) and (b) of the theorem are equivalent by Proposition 4.1 applied to the subordinator  $\mathcal{Y} := \mathcal{H}$ , and part 1(c) follows from part 1(b) by Proposition 4.2. We now show that part 1(c) implies part 2. □

**Proposition 5.1.** Assume (2.1) and (2.10), and additionally that  $A_{H^*} \in \text{RV}(0)$ , and either (i)  $\overline{\Pi}_X^+ \in \text{RV}(-\beta)$ , where  $\beta > 1$ , or (ii)  $\Pi_X \in \text{MDA}(\Lambda)$ . Then the conclusions of part 2 of Theorem 3.1 hold.

**Proof.** A slight extension of a result proved in Doney and Rivero [12] states that, on the event  $X_{\tau_u-} < u$ , the joint distribution of  $(\tau_u, X_{\tau_u-})$  is given by

$$\begin{aligned} &P(\tau_u \in dt, X_{\tau_u-} \in dy) \\ &= P(X_t \in dy, \overline{X}_t \leq u) \overline{\Pi}_X^+(u - y) dt, \quad t > 0, u > 0, y \in \mathbb{R}. \end{aligned} \tag{5.5}$$

Thus,  $\tau_u$  has a density, and for  $\varepsilon > 0$  we can write (recall that  $Z^{(u)} = -X_{\tau_u-} = X_{\tau_u-}^*$ )

$$\begin{aligned} &P(\tau_u \in r(u) dt, Z^{(u)} \in [(1 - \varepsilon)c(\tau(u)), (1 + \varepsilon)c(\tau_u)]) \\ &= \int_{[(1-\varepsilon)c(tr(u)), (1+\varepsilon)c(tr(u))]} \overline{\Pi}_X^+(u + y) P(X_{tr(u)}^* \in dy, \overline{X}_{tr(u)} \leq u) dt. \end{aligned}$$

Under the assumptions of the proposition, the limit relation (4.9) holds, and also  $c(\cdot) \in \text{RV}(1)$  implies  $c(tr(u)) \sim t(c(r(u))) = ta(u)$ . So the last integral is asymptotically equivalent to

$$\begin{aligned} & \int_{[(1-\varepsilon)t, (1+\varepsilon)t]} \overline{\Pi}_X^+(u + ya(u)) P(X_{tr(u)}^* \in a(u) dy, \overline{X}_{tr(u)} \leq u) \\ & \sim r(u) \overline{\Pi}_X^+(u) \int_{[(1-\varepsilon)t, (1+\varepsilon)t]} P(C > y) P(X_{tr(u)}^* \in a(u) dy, \overline{X}_{tr(u)} \leq u) \\ & = r(u) \overline{\Pi}_X^+(u) \left\{ \int_{[1-\varepsilon, 1+\varepsilon]} P(C > ty) P\left(\frac{X_{tr(u)}^*}{ta(u)} \in dy\right) + o(1) \right\}, \end{aligned}$$

where we use the fact that  $P(\overline{X}_{tr(u)} > u) \leq P(\overline{X}_\infty > u) \rightarrow 0$  as  $u \rightarrow \infty$ . This follows because  $\overline{X}_\infty = \sup_{t \geq 0} X_t$  is a finite r.v. a.s. under (2.1).

Next, since

$$\frac{X_{tr(u)}^*}{ta(u)} \sim \frac{X_{tr(u)}^*}{tc(r(u))} \xrightarrow{P} 1,$$

for all  $t > 0$ , we deduce that

$$\int_{[1-\varepsilon, 1+\varepsilon]} P(C > ty) P(X_{tr(u)}^* \in ta(u) dy) = P(C > t) + o(1),$$

so that

$$\begin{aligned} & P^{(u)}(\tau_u \in r(u) dt, Z^{(u)} \in [(1-\varepsilon)c(\tau_u), (1+\varepsilon)c(\tau_u)]) \\ & \sim \frac{r(u) \overline{\Pi}_X^+(u) P(C > t) dt}{P(\tau_u < \infty)} \\ & \sim \frac{a(u) \overline{\Pi}_X^+(u) P(C > t) dt}{\overline{\Pi}_{\mathcal{H}}(u) A_{H^*}(u)} \quad (\text{by (2.11) and (5.3)}) \\ & \rightarrow c_{0,\beta} P(C > t) dt \quad (\text{by (4.10)}). \end{aligned} \tag{5.6}$$

The evaluation of  $c_{0,\beta}$  from (4.10) (and see the end of the proof of Proposition 4.2) shows that the limit here is a probability density function, and since it does not depend on  $\varepsilon$ , we deduce that (3.1) holds, and also that, conditioned on  $\tau_u = tr(u)$ , the  $P^{(u)}$ -distribution of  $X^*(\tau_u-)/c(\tau_u)$  converges to the distribution concentrated on 1.  $\square$

**Remark 5.1.** The event  $\{X_{\tau_u-} < u\}$  to which (5.5) is restricted has  $P^{(u)}$ -probability approaching 1 as  $u \rightarrow \infty$ . This follows since  $\lim_{u \rightarrow \infty} P(Z^{(u)}/a(u) \leq 0) = 0$  in conditions (a)–(c) of Theorem 3.1 (and similarly in Theorem 3.2), so we have

$$P^{(u)}(X_{\tau(u)-} = u) = P^{(u)}(Z^{(u)} = -u) \leq P^{(u)}(Z^{(u)} \leq 0) \rightarrow 0 \quad \text{as } u \rightarrow \infty.$$

To extend (3.1) to the  $k$ -dimensional distributions, we take  $0 < s_1 < s_2 < \dots < s_k < 1$ , set

$$A_k := \left\{ 1 - \varepsilon \leq \frac{X^*(s_i \tau_u)}{s_i c(\tau_u)} \leq 1 + \varepsilon \text{ for } i = 1, 2, \dots, k \right\},$$

and apply the previous argument to

$$P(A_k, \tau_u \in r(u) dt, Z^{(u)} \in [(1 - \varepsilon)c(\tau_u), (1 + \varepsilon)c(\tau_u)]).$$

We find that

$$P^{(u)}(A_k, \tau_u \in r(u) dt, Z^{(u)} \in [(1 - \varepsilon)c(\tau_u), (1 + \varepsilon)c(\tau_u)]) \rightarrow c_{0,\beta} P(C > t) dt,$$

and the convergence of the  $k$ -dimensional distributions follows.

To include the behaviour of the overshoot, we need the following result.

**Lemma 5.1.** *For  $u > 0$ ,  $z \geq 0$ , and  $x \geq 0$  we have*

$$P^{(u)}(Z^{(u)} \in dz, O^{(u)} > x) = P^{(u)}(Z^{(u)} \in dz) \frac{\overline{\Pi}_X^+(u + x + z)}{\overline{\Pi}_X^+(u + z)}.$$

**Proof.** Using the quintuple law in Doney and Kyprianou [10] twice gives

$$\begin{aligned} P(Z^{(u)} \in dz, O^{(u)} > x) &= \int_{0 < w \leq u} G(dw) G^*(u - w - dz) \overline{\Pi}_X^+(u + x + z) \\ &= \int_{0 < w \leq u} G(dw) G^*(u - w - dz) \overline{\Pi}_X^+(u + z) \frac{\overline{\Pi}_X^+(u + x + z)}{\overline{\Pi}_X^+(u + z)} \\ &= P^{(u)}(Z^{(u)} \in dz) \frac{\overline{\Pi}_X^+(u + x + z)}{\overline{\Pi}_X^+(u + z)}. \end{aligned}$$

(Note that there is no issue of creeping to take into account since  $O^{(u)} > 0$  implies  $X_{\tau_u} > u$ .)  $\square$

**Corollary 5.1.** *Under the assumptions of Proposition 5.1, the  $P^{(u)}$ -finite-dimensional distributions  $\mathbf{Y}^{(u)}$ , defined in (3.2), converge to those of  $(V, U, V, (\mathbf{VD}^{(0)}(s))_{0 \leq s \leq 1})$ .*

**Proof.** The result for

$$\left( \frac{Z^{(u)}}{a(u)}, \frac{\tau_u}{b(a(u))}, \left( \frac{X^*(s \tau_u)}{a(u)} \right)_{0 \leq s \leq 1} \right)$$

is immediate from Proposition 5.1, and since, given  $Z^{(u)}$ ,  $O^{(u)}$  is independent of the pre- $\tau_u$   $\sigma$ -field, we need only check that

$$P(O^{(u)} > xa(u) | Z^{(u)} = a(u)z) \rightarrow \begin{cases} \left( \frac{1+z}{1+z+x} \right)^\beta, & \text{in case (i),} \\ e^{-x}, & \text{in case (ii).} \end{cases}$$

But this is immediate from Lemma 5.1. □

In particular, when part 1(c) of Theorem 3.1 holds, we have from Corollary 5.1 that the  $P^{(u)}$ -distribution of  $O^{(u)}$  converges to that of  $U$ , so 1(a) holds. Thus, parts 1(a)–(c) are proved equivalent.

Finally, for part 3 of Theorem 3.1, we show that the convergence in this result can be replaced by weak convergence on the Skorokhod space.

**Proposition 5.2.** *Under the assumptions of Proposition 5.1, the  $P^{(u)}$ -distribution of  $\mathbf{Y}^{(u)}$  converges weakly on  $\mathbb{R}^3 \times \mathbb{D}_0[0, 1]$  as  $u \rightarrow \infty$ .*

**Proof.** Put  $\mathbf{Y}^{(u)} = (W^{(u)}, \mathbf{X}^{(u)})$ , where

$$W^{(u)} := \left( \frac{Z^{(u)}}{a(u)}, \frac{O^{(u)}}{a(u)}, \frac{\tau_u}{b(a(u))} \right) \quad \text{and} \quad \mathbf{X}^{(u)} := \left( \frac{X^*(s\tau_u)}{a(u)} \right)_{0 \leq s \leq 1}.$$

We need only prove tightness. This will follow if we can show that for any  $\varepsilon > 0$  there is a compact subset of  $K$  of  $\mathbb{R}^3 \times \mathbb{D}_0[0, 1]$  such that  $\limsup_{u \rightarrow \infty} P^{(u)}(\mathbf{Y}^{(u)} \in K^c) \leq \varepsilon$ . We will do this with  $K = K_1 \times K_2$ , where  $K_1 \subset \mathbb{R}^3$  is of the form  $\{1/D < x_r < D, r = 1, 2, 3\}$ ,  $K_2 \subset \mathbb{D}_0[0, 1]$  will be specified later, and  $D$  is fixed with  $P^{(u)}(W^{(u)} \in K_1^c) \leq \varepsilon/2$  for large  $u$ . So it suffices to show that  $\limsup_{u \rightarrow \infty} P^{(u)}(\mathbf{Y}^{(u)} \in K_1 \times K_2^c) \leq \varepsilon/2$ . This probability is dominated by

$$P^{(u)}(B \cap (\mathbf{X}^{(u)} \in K_2^c))$$

where

$$B = \left\{ \frac{\tau_u}{r(u)} \in (D^{-1}, D), \frac{Z^{(u)}}{a(u)} \in (D^{-1}, D) \right\}.$$

But (recall  $c(r(u)) = a(u)$  and (5.5))

$$\begin{aligned} & P^{(u)}(\mathbf{X}^{(u)} \in K_2^c, B) \\ & \leq \frac{1}{P(\tau_u < \infty)} \int_{r(u)/D}^{r(u)D} \int_{z \in (D^{-1}, D)} dt P(X_t^* \in a(u) dz, \mathbf{X}^{(u)} \in K_2^c) \overline{\Pi}_X^+(u + a(u)z) \\ & \leq \frac{\overline{\Pi}_X^+(u)}{P(\tau_u < \infty)} \int_{r(u)/D}^{r(u)D} dt P\left(\left(\frac{X_{st}^*}{a(u)}, 0 \leq s \leq 1\right) \in K_2^c\right) \\ & = \frac{r(u)\overline{\Pi}_X^+(u)}{P(\tau_u < \infty)} \int_{1/D}^D dt P\left(\left(\frac{X_{r(u)st}^*}{c(r(u))}, 0 \leq s \leq 1\right) \in K_2^c\right). \end{aligned} \tag{5.7}$$

As shown in (5.6), the factor

$$\frac{r(u)\overline{\Pi}_X^+(u)}{P(\tau_u < \infty)} \rightarrow c_{0,\beta} \quad \text{as } u \rightarrow \infty.$$

Also, since  $(X_{ys}^*/c(y))_{0 \leq s \leq 1}$  is tight as  $y \rightarrow \infty$ , we can choose  $K_2$  such that when  $D^{-1}a(u)$  is sufficiently large,

$$P\left(\sup_{t \in (D^{-1}, D)} \left(\frac{X_{r(u)st}^*}{c(r(u))}, 0 \leq s \leq 1\right) \in K_2^c\right) \leq \varepsilon,$$

and the result follows. □

### 6. The case $0 < \gamma < 1$ (infinite mean)

Throughout this section, our standing assumptions (and notations) will be those of Theorem 3.2, namely, (2.1) and (2.10) hold, and (2.13) holds with  $\gamma \in (0, 1)$ . By the monotone density theorem, the latter is equivalent to

$$\overline{\Pi}_X^-(x) \sim \gamma x^{-1} A_X^*(x) \in \text{RV}(\gamma - 1) \quad \text{as } x \rightarrow \infty. \tag{6.1}$$

From (5.1), we then deduce  $\lim_{x \rightarrow \infty} \overline{\Pi}_X^+(x)/\overline{\Pi}_X^-(x) = 0$ . This together with (6.1) means that  $X^*$  is in the domain of attraction of a standard stable subordinator,  $\mathbf{D}$ , of parameter  $\overline{\gamma} := 1 - \gamma \in (0, 1)$ . Thus, we can find a continuous, increasing function  $c(\cdot)$  such that  $(X_{su}^*/c(u))_{s>0} \xrightarrow{D} \mathbf{D}$ , and one can check that

$$u \overline{\Pi}_X^-(c(u)) \rightarrow 1/\Gamma(\gamma).$$

Write  $b(\cdot)$  for the inverse of  $c(\cdot)$ , so that  $b(\cdot) \in \mathcal{R}_{\overline{\gamma}}$ , and

$$b(u) \sim \frac{1}{\Gamma(\gamma) \overline{\Pi}_X^-(u)}. \tag{6.2}$$

Put  $r(u) = b(a(u))$ , so that

$$r(u) \sim \frac{1}{\Gamma(\gamma) \overline{\Pi}_X^-(a(u))} \sim \frac{a(u)}{\Gamma(1 + \gamma) A_X^*(a(u))} \quad (\text{by (6.1)}). \tag{6.3}$$

A version of Stone’s stable local limit theorem (see Proposition 13 of Doney and Rivero [12]) implies that

$$P(X_{tv}^* \in (c(v)z, c(v)z + \Delta]) = \frac{\Delta}{c(v)} (h_t(z) + o(1)) \tag{6.4}$$

as  $v \rightarrow \infty$ , uniformly for  $z \in \mathbb{R}$ ,  $\Delta \in [\Delta_0, \Delta_1]$ , for any fixed  $0 < \Delta_0 < \Delta_1 < \infty$ , and  $t \in [T_0, T_1]$ , for any fixed  $0 < T_0 < T_1 < \infty$ . Here  $h_t(z) dz = P(D_t \in dz)$  [see (3.5)], so that, in particular, the term  $h_t(z)$  is zero for  $z < 0$ . A simple consequence of this is the existence of constants  $v_0$  and  $C$  such that for all  $v \geq v_0$ ,  $\Delta \in [\Delta_0, \Delta_1]$ , and  $t \in [T_0, T_1]$ ,

$$P(X_{tv}^* \in (c(v)z, c(v)z + \Delta]) \leq \frac{C \Delta}{c(v)}. \tag{6.5}$$

Notice that if we put  $v = r(u)$  in (6.4) we have  $c(v) = c(b(a(u))) = a(u)$ , so an equivalent version of (6.4) is

$$P(X_{tr(u)}^* \in (a(u)z, a(u)z + \Delta]) = \frac{\Delta}{a(u)}(h_t(z) + o(1)) \quad \text{as } u \rightarrow \infty. \quad (6.6)$$

We have already proved part 1 of Theorem 3.2, except for the implication from parts 1(c) to (a), and we now show that part 1(c) implies part 2, and then that this implies part 1(a).

**Proposition 6.1.** *Assume (2.1) and (2.10), and that  $A_{H^*} \in \text{RV}(\gamma)$  with  $\gamma \in (0, 1)$ . Suppose either (i)  $\overline{\Pi}_X^+(x) \in \text{RV}(-\beta)$ , where  $\beta > 1 - \gamma$ , or (ii)  $\overline{\Pi}_X^+(x) \in \text{MDA}(\Delta)$  and  $\overline{\Pi}_{\mathcal{H}} \in \mathcal{S}$ . Then part 2 of Theorem 3.2 holds.*

**Proof.** Under the conditions of the proposition, we have from (5.5)

$$\begin{aligned} & P(\tau_u \in r(u) dt, Z^{(u)} \in [za(u), za(u) + \Delta]) \\ &= \int_{y \in [0, \Delta]} \overline{\Pi}_X^+(u + za(u) + y) P(X_{tr(u)}^* \in za(u) + dy, \overline{X}_{tr(u)} \leq u) dt \\ &\sim \overline{\Pi}_X^+(u + za(u)) \int_{y \in [0, \Delta]} P(X_{tr(u)}^* \in za(u) + dy, \overline{X}_{tr(u)} \leq u) dt \\ &\sim \overline{\Pi}_X^+(u) P(C > z) P(X_{tr(u)}^* \in [za(u), za(u) + \Delta], \overline{X}_{tr(u)} \leq u) dt. \end{aligned} \quad (6.7)$$

Write

$$P(X_{tr(u)}^* \in [za(u), za(u) + \Delta], \overline{X}_{tr(u)} \leq u) = P_1(u) - P_2(u),$$

where, by (6.4),

$$P_1(u) := P(X_{tr(u)}^* \in [za(u), za(u) + \Delta]) = \frac{\Delta}{a(u)}(h_t(z) + o(1)), \quad (6.8)$$

and we will show that

$$P_2(u) := P(X_{tr(u)}^* \in [za(u), za(u) + \Delta], \overline{X}_{tr(u)} > u) = o\left(\frac{\Delta}{a(u)}\right), \quad u \rightarrow \infty. \quad (6.9)$$

To do this, observe that  $\{\overline{X}_{tr(u)} > u\} \subseteq \{\tau_u \leq tr(u)\}$ , and decompose  $P_2(u)$  further according as  $\tau_u \leq tr(u)/2$  or  $tr(u)/2 < \tau_u \leq tr(u)$ . Thus, write  $P_2(u) = P_2^{(1)}(u) + P_2^{(2)}(u)$ , recall that  $\mathbf{O}^{(u)}$  is independent of the pre- $\tau_u$   $\sigma$ -field, and argue as follows:

$$\begin{aligned} P_2^{(1)}(u) &:= P(\tau_u \leq tr(u)/2, X_{tr(u)}^* \in [za(u), za(u) + \Delta]) \\ &= \int_{0 \leq s \leq tr(u)/2} \int_{x > 0} P(\tau_u \in ds, \mathbf{O}^{(u)} \in dx) \\ &\quad \times P(X_{tr(u)-s}^* \in [u + x + za(u), u + x + za(u) + \Delta]) \end{aligned} \quad (6.10)$$

$$\begin{aligned} &\leq \int_{0 \leq s \leq tr(u)/2} \int_{x>0} P(\tau_u \in ds, O^{(u)} \in dx) \frac{C \Delta}{c(tr(u) - s)} \quad (\text{by (6.5)}) \\ &\leq \frac{C' \Delta}{c(tr(u))} P(\tau_u < \infty) \\ &= o\left(\frac{\Delta}{c(tr(u))}\right). \end{aligned}$$

Next, introduce  $\tau^*(u) = \inf\{s: X_s^* > u\}$  and  $\sigma_v(u) = \sup\{s \leq v: X_s > u\}$ . Use the duality lemma (Bertoin [3], page 45) to see that for any  $w$  and any  $v > 0$

$$P(\sigma_v(u) \in ds | X_v^* = w) = P(\tau^*(u + w) \in v - ds | X_v^* = w).$$

Applying this with  $v = tr(u)$  and  $w = za(u) + y$  gives

$$\begin{aligned} P_2^{(2)}(u) &= \int_{[0, \Delta]} P(tr(u)/2 < \tau_u \leq tr(u), X_{tr(u)}^* \in za(u) + dy) \\ &\leq \int_{[0, \Delta]} P(tr(u)/2 < \sigma_{tr(u)}(u) \leq tr(u), X_{tr(u)}^* \in za(u) + dy) \\ &= \int_{[0, \Delta]} P(0 < \tau^*(u + za(u) + y) < tr(u)/2, X_{tr(u)}^* \in za(u) + dy) \\ &\leq P(0 < \tau^*(u + za(u)) < tr(u)/2, X_{tr(u)}^* \in (za(u), za(u) + \Delta]) \\ &= \int_{0 \leq v \leq tr(u)/2} \int_{y>0} P(\tau^*(u + za(u)) \in dv, X_v^* \in u + za(u) + dy) \\ &\quad \times P(X_{tr(u)-v} \in (u + y - \Delta, u + y]) \\ &= o(1) \int_{0 \leq v \leq tr(u)/2} P(\tau^*(u + za(u)) \in dv) \frac{\Delta}{c(tr(u) - v)} \\ &= o\left(\frac{\Delta}{c(tr(u))}\right). \end{aligned}$$

In the last few steps, we used the strong Markov property at  $\tau^*(u + za(u))$ , equated  $P(X_{tr(u)-v} \in (u + y - \Delta, u + y])$  with  $P(X_{tr(u)-v}^* \in (-u - y + \Delta, -u - y])$ , and used (6.5). Since  $c(tr(u)) \sim t^{1/\bar{\gamma}} c(r(u)) = t^{1/\bar{\gamma}} a(u)$ , this together with (6.10) gives (6.9).

Now for case (i), with  $a(u) = u$  and  $P(C > z) = (1 + z)^{-\beta}$ ,

$$\begin{aligned} &P^{(u)}(\tau_u \in r(u) dt, Z^{(u)} \in [za(u), za(u) + \Delta]) \\ &\sim \frac{\bar{\Pi}_X^+(u) P(C > z) P(X_{tr(u)}^* \in [za(u), za(u) + \Delta]) dt}{P(\tau_u < \infty)} \quad (\text{by (4.9) and (6.7)}) \\ &\sim \frac{(1 + z)^{-\beta} \bar{\Pi}_X^+(u) h_t(z) \Delta dt}{q^{-1} \bar{\Pi}_{\mathcal{H}}(u) a(u)} \quad (\text{by (6.8) and (2.11)}) \quad (6.11) \end{aligned}$$



$$\begin{aligned} &\sim \frac{(1+z)^{-\beta} q c_{\gamma, \beta} A_{H^*}(u) h_t(z) \Delta dt}{a^2(u)} && \text{(by (4.26), with } A_{H^*}(\infty) = \infty) \\ &\sim \frac{(1+z)^{-\beta} c(\gamma, \beta) h_t(z) \Delta dt}{\Gamma(1+\gamma) a(u) r(u)} && \text{(by (6.3), and } c_{\gamma, \beta} = c(\gamma, \beta)) \\ &= \frac{(1+z)^{-\beta} \Gamma(\beta) h_t(z) \Delta dt}{\Gamma(\beta + \gamma - 1) a(u) r(u)} && \text{(by (4.19)).} \end{aligned}$$

This gives

$$\lim_{u \rightarrow \infty} a(u) r(u) P^{(u)}(Z^{(u)} \in (za(u), za(u) + \Delta], \tau_u \in r(u) dt) = h_t(z) f(z) \Delta dt,$$

where  $f(\cdot)$  is as defined in (3.8), and proves (3.7) for case (i).

In case (ii), we get from (4.10)

$$P^{(u)}(Z^{(u)} \in (za(u), za(u) + \Delta], \tau_u \in r(u) dt) \sim e^{-z} \frac{h_t(z) \Delta}{r(u) a(u)} dt = \frac{h_t(z) f(z) \Delta}{r(u) a(u)} dt,$$

and (3.7) is established in this case.

Notice also that, since  $h_t(\cdot)$  vanishes on the negative half-line, the previous estimates show that  $P^{(u)}(-Z^{(u)} \in (za(u), za(u) + \Delta], \tau_u \in r(u) dt)/dt$  is uniformly  $o((r(u)a(u))^{-1})$  for  $z \in [\Delta_0, \Delta_1]$  and  $t \in [0, T_0]$ .

We have now proved (3.7). It remains to prove (3.9).

For  $k \geq 2$ , we assume first that  $z_1 < z_2 < \dots < z_k$  and write (3.9) as

$$(a(u))^k r(u) P^{(u)}\left(\bigcap_{i=1}^k C_i \cap B\right) = \theta_k(z_1, z_2, \dots, z_k, t) \left(\prod_{i=1}^k \Delta_i + o(1)\right) dt,$$

where

$$\begin{aligned} C_i &:= \{X^*(s_i t r(u)) \in (z_i a(u), z_i a(u) + \Delta_i]\}, && i = 1, 2, \dots, k \quad \text{and} \\ B &:= \{\tau_u \in r(u) dt\}. \end{aligned} \tag{6.12}$$

As in the lines leading up to (6.8), we have

$$P\left(\bigcap_{i=1}^k C_i \cap B\right) \sim P\left(\bigcap_{i=1}^k C_i \cap \{\bar{X}_{tr(u)} \leq u\}\right) \bar{\Pi}_X^+(u + z_k a(u)) dt. \tag{6.13}$$

The event in brackets on the RHS coincides with  $\bigcap_{i=1}^k \tilde{C}_i$  where

$$\tilde{C}_i := \left\{ X^*(s_i r) \in (z_i a(u), z_i a(u) + \Delta_i], \sup_{rs_{i-1} < v \leq rs_i} X_v \leq u \right\}$$

and we set  $r := tr(u)$ . Note that each  $r(u)(s_i - s_{i-1}) \rightarrow \infty$  uniformly in  $i = 1, 2, \dots, k$  as  $u \rightarrow \infty$ . So by the Markov property and stationarity we find that  $P(\bigcap_{i=1}^k \tilde{C}_i)$  is equal to

$$\begin{aligned} & \int_{a(u)z_{k-1}}^{a(u)z_{k-1} + \Delta_{k-1}} P\left(X^*(rs_k) \in (z_k a(u), z_k a(u) + \Delta_k], \sup_{rs_{k-1} < v \leq rs_k} X_v \leq u \mid X^*(rs_{k-1}) = y\right) \\ & \quad \times P\left(\bigcap_{i=1}^{k-1} \tilde{C}_i, X^*(rs_{k-1}) \in dy\right) \\ & = \int_{a(u)z_{k-1}}^{a(u)z_{k-1} + \Delta_{k-1}} P\left(X^*(r(s_k - s_{k-1})) \in (z_k a(u) - y, z_k a(u) - y + \Delta_k], \right. \\ & \quad \left. \bar{X}_{r(s_k - s_{k-1})} \leq u - y\right) \\ & \quad \times P\left(\bigcap_{i=1}^{k-1} \tilde{C}_i, X^*(rs_{k-1}) \in dy\right) \\ & = \frac{\Delta_k}{a(u)} (h_{t(s_k - s_{k-1})}(z_k - z_{k-1})) + o(1) \times P^{(u)}\left(\bigcap_{i=1}^{k-1} \tilde{C}_i\right), \end{aligned}$$

where the last line uses the result for  $k = 1$  in (3.7). Repeating this argument, a further  $k - 1$  times gives

$$P\left(\bigcap_{i=1}^k \tilde{C}_i\right) = (a(u))^{-k} \prod_{i=1}^k \Delta_i \left(\prod_{i=1}^k h_{t(s_i - s_{i-1})}(z_i - z_{i-1}) + o(1)\right),$$

and the result then follows from (6.13) and the previous calculation. Clearly, if any  $z_i \leq z_{i-1}$  the calculation is still valid, but the above product vanishes.  $\square$

Using this local result and Lemma 5.1, we easily obtain convergence of the finite-dimensional distributions, as claimed in part 3.

Now argue as follows. Equation (3.7) implies that  $Z^{(u)}/a(u)$  has a proper limiting distribution under  $P^{(u)}$ . By Lemma 5.1, this means that  $(Z^{(u)}/a(u), O^{(u)}/a(u))$  has a proper limiting distribution under  $P^{(u)}$ , thus, in particular,  $O^{(u)}/a(u)$  has a proper limiting distribution under  $P^{(u)}$ . From Proposition 4.1, we then deduce Properties 1(a) and 1(b), and the proof of Theorem 3.2 is completed by repeating the tightness argument of the previous section, almost word for word.  $\square$

**Remark 6.1.** Assumption (2.10), that  $\mathcal{H} \in \mathcal{S}$ , is only needed for application of Proposition 4.1, where it is used in effect to deduce that  $\bar{\Pi}_{\mathcal{H}}(u) \sim qP(\tau_u < \infty)$  via (2.11). We could replace assumption (2.10) with the assumption  $\bar{\Pi}_{\mathcal{H}}(u) \sim qP(\tau_u < \infty)$  throughout. But general necessary and sufficient conditions for the latter in terms of more basic quantities are currently not known.

Further note that  $\bar{\Pi}_{\mathcal{H}}(u)$  is not asymptotically equivalent to the more basic quantity  $\bar{\Pi}_X^+(u)$  in our situation. Vigon’s “équation amicale inversée” is

$$\bar{\Pi}_{\mathcal{H}}(u) = \int_{(0,\infty)} \bar{\Pi}_X^+(y+u)G^*(dy) \tag{6.14}$$

(recall that  $G^*$  is the renewal measure in the down-going ladder height process  $H^*$ , see (2.14)). Under the assumption  $\lim_{t \rightarrow \infty} X_t = -\infty$  a.s., we have  $G^*(\infty) = \infty$ , and it is not hard to show from (6.14) that either  $\bar{\Pi}_{\mathcal{H}} \in \mathcal{L}$  (see (2.8), or  $\bar{\Pi}_X^+ \in \mathcal{L}$  implies  $\bar{\Pi}_{\mathcal{H}}(u)/\bar{\Pi}_X^+(u) \rightarrow \infty$ .

In general, a sufficient condition for  $\bar{\Pi}_{\mathcal{H}} \in \mathcal{S}$  is  $\bar{\Pi}_X^+ \in \mathcal{D} \cap \mathcal{L}$ , where  $\mathcal{D}$  is the class of dominatedly varying functions; that is, those for which  $\limsup_{x \rightarrow \infty} \bar{\Pi}_X^+(x/2)/\bar{\Pi}_X^+(x) < \infty$ ; see, for example, Foss, Korshunov and Zachary [15], page 11. So we can replace Assumption (2.10) by  $\bar{\Pi}_X^+ \in \mathcal{D} \cap \mathcal{L}$  throughout. In particular,  $\bar{\Pi}_X^+ \in \mathcal{D}$  if  $\bar{\Pi}_X^+$  is regularly varying with index  $-\alpha$  for  $\alpha \geq 0$ .

Further connections between  $\bar{\Pi}_{\mathcal{H}}$  and  $\bar{\Pi}_X$  are in Proposition 5.4 of Klüppelberg, Kyprianou and Maller [18] and the related discussion.

## 7. Random walks and compound Poisson processes

We can specialize our results to the case that  $X$  is a compound Poisson process of the form  $X_t = S_{N_t}$ , where  $(S_n, n \geq 0)$  is a random walk and  $(N_t, t \geq 0)$  is an independent Poisson counting process of unit rate. Then, writing  $Z_n$  and  $Z_n^*$  for the  $n$ th strict increasing and weak decreasing ladder heights in  $S$ , we have also that  $H_t = Z_{N_t}$  and  $H_t^* = Z_{N_t}^*$  for all  $t \geq 0$ . Then our basic assumptions, (2.1) and (2.10) are equivalent to

$$S_n \xrightarrow{\text{a.s.}} -\infty \quad \text{and} \quad J \in \mathcal{S},$$

where  $J(dx) = P(Z_1 \in dx | Z_1 \in (0, \infty))$ . It is also clear that, with  $\tau^S(u) := \inf\{n: S_n > u\}$ , we have the identity

$$\tau_u = \sum_1^{\tau^S(u)} e_i,$$

where the  $e_i$  are i.i.d.  $\text{Exp}(1)$  random variables. Clearly, the event  $\{\tau_u < \infty\}$  coincides a.s. with the event  $\{\tau^S(u) < \infty\}$ , so  $P^{(u)}(\cdot)$  has an unambiguous meaning and, furthermore, it is straightforward to show that for any  $r(u) \rightarrow \infty$  as  $u \rightarrow \infty$ , the statements

$$r(u)P^{(u)}(\tau^S(u) = \lceil tr(u) \rceil) \rightarrow g(t)$$

and

$$r(u)P^{(u)}(\tau_u \in r(u) dt) \rightarrow g(t) dt$$

are equivalent. Also the spatial quantities  $Z_S^{(u)} := S^*(\tau^S(u))$  and  $O_S^{(u)} := S(\tau^S(u)) - u$  coincide with  $Z^{(u)}$  and  $O^{(u)}$ .

We claim that this allows us to deduce versions of Theorems 3.2 and 3.1 for random walks, with very minor changes. Specifically, if  $F$  is the distribution of  $S_1$  and we replace  $\Pi$  and  $\Pi_{\mathcal{H}}$  in those results by  $F$  and  $J$ , then Theorem 3.1 requires only replacing  $g^{(u)}(tr(u))$  by  $P^{(u)}(\tau^S(u) = [tr(u)])$ , and Theorem 3.2 requires only an analogous change to (3.7).

Alternatively, we can prove the random walk results by repeating the Lévy process proof, with appropriate changes. We refer to Borovkov and Borovkov [6] for general results on heavy-tailed random walks.

**Remark 7.1.** An alternative approach to our proofs, suggested by a referee, based on “the principle of a single large jump” (developed in Asmussen and Foss [1] for a more general setting and then considered in Chapter 5, Section 13 of Foss, Korshunov and Zachary [15] for random walks), may provide a shorter and more intuitive treatment. However, extending these techniques to the Lévy process situation and dealing with the infinite mean case is not straightforward, and it is not clear that this approach would deliver the local results or the if and only if conditions which we establish.

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