

Passive Constrained Rational Approximation Algorithm using Nevanlinna-Pick Interpolation

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Abstract

As system integration evolves and tighter design constraints must be met, it becomes necessary to account for the non-ideal behavior of all the elements in a system. For high-speed digital, and microwave systems, it is increasingly important to model previously neglected frequency domain effects.

In this paper, results from Nevanlinna-Pick interpolation theory are used to develop a bounded real matrix rational approximation algorithm. A method is presented that allows for the generation of guaranteed passive rational function models of passive systems by approximating their scattering parameter matrices. Since the order of the models may in some cases be high, an incremental fitting strategy is also proposed that allows for the generation of smaller models while still meeting the required passivity and accuracy requirements. Results of the application of the proposed method to several real-world examples are also shown.

1 Introduction

Especially in high-frequency applications, certain devices are usually described and studied in the frequency domain. Devices such as coil inductors, SAW filters, non-ideal transmission lines and high-frequency transistors are commonly described by manufacturers and designers by their frequency dependent scattering parameter or admittance matrices. Frequency domain data is either obtained through measurement or through physical simulation. In either case, the available data is sampled, incomplete, noisy, and covers only a finite range of the spectrum.

It is not trivial to generate accurate circuit-level models for all the devices in a system. Such models are however necessary for the time domain simulation of larger designs and to account for the non-ideal characteristics of the devices. Although harmonic balance simulators can eas-

ily handle devices described by their frequency response, they cannot adequately treat highly non-linear designs such as oscillators and mixers. On the other hand, time-domain simulators, using state-space model integration, that can deal with high-order non-linearity, require time-domain models. It is necessary that these models have frequency responses that match the available data but, they must also possess stability and passivity properties similar to those of the physical system that they represent.

Several algorithms for stable rational approximation exist and can be used for frequency domain identification of stable systems, see for example [15, 16, 18, 9, 5] and the references therein. Several other algorithms can be used for frequency domain identification of passive systems. See [14, 10] and [8] for a convex programming approach to positive real constrained rational approximation that can be used to generate multivariable models for the impedance or admittance matrices of a passive system. In [1, 12, 2] several rational interpolation and approximation algorithms that allow the generation of multivariable models with a prescribed frequency response and norm bounds are presented. Also in [13, 7, 6] norm-bounded and positive real models are generated using rational interpolation and approximation algorithms and convex programming.

In this paper we propose a simple rational approximation algorithm based on multivariable Nevanlinna-Pick interpolation. We also consider interpolant parameterization and its state-space representation and present results of the application of the proposed method to several real-world examples.

This paper is structured as follows. In section 2, the scattering, impedance and admittance matrices of passive electric networks are characterized and background information is provided. In section 3, a version of the matrix Nevanlinna-Pick interpolation problem is presented. Since the Nevanlinna-Pick algorithm assumes that the data points are in the interior of the analyticity domain, a boundary interpolation, or Löwner, problem is also considered. In section 4, an incremental rational approximation algorithm us-

ing boundary Nevanlinna-Pick interpolation is proposed. In section 5, a state-space representation for the Nevanlinna-Pick interpolant is presented. In section 6, models are generated for several real-world data-sets. Effects of model parameterization on the interpolants frequency response are also illustrated. Finally, section 7 conclusions and suggestions for further work are drawn.

2 Background

Passivity is an important property of certain physical systems. Networks composed by resistors, capacitors and inductors are passive, they do not generate energy. Systems that always consume energy are called strictly passive. Interconnected (strictly) passive systems are (strictly) passive. Stable systems do not possess this closure property. Stable systems loaded by stable, and even passive systems, may not constitute an overall stable system. Therefore, it is important that passive systems be represented by passive models.

The properties of the transfer function of a model representing a passive system depend on the physical interpretation of the model's inputs and outputs. There exist several standard system representations such as the impedance parameter matrix \mathbf{Z} , the admittance parameter matrix \mathbf{Y} , the hybrid parameter matrix \mathbf{H} , the chain parameter matrix \mathbf{ABCD} , the scattering parameter matrix \mathbf{S} , and the transmission parameter matrix \mathbf{T} .

For an n-port network \mathbf{Y} and \mathbf{Z} relate the port currents and voltages. For higher frequency systems, impedances and admittances cannot be accurately measured because the required short-circuit and open-circuit tests are difficult to achieve over a broad range of frequencies. The scattering matrix, s-parameter, representation is used for higher-frequency by characterizing the n-ports through relations between the incident traveling waves \mathbf{a} and the reflected traveling waves \mathbf{b} , satisfying $\mathbf{b} = \mathbf{S}\mathbf{a}$. The definition of the scattering parameter matrix, \mathbf{S} and its relation to the other system representations depends on the reference impedance used in the measurement process, Z_0 . It is assumed that Z_0 is real. In practice most measurement equipment uses the standard 50Ω . The reference impedance at each port is represented by $Z_{0,i}$. Let \mathbf{Z}_0 be a diagonal matrix such that $\mathbf{Z}_0(i,i) = Z_{0,i}$. The incident and reflected waves are linearly related to the circuit's input voltage and currents according to

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{Z}_0^{-1/2} & \mathbf{Z}_0^{1/2} \\ \mathbf{Z}_0^{-1/2} & -\mathbf{Z}_0^{1/2} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{i} \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} \mathbf{v} \\ \mathbf{i} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_0^{1/2} & \mathbf{Z}_0^{1/2} \\ \mathbf{Z}_0^{-1/2} & -\mathbf{Z}_0^{-1/2} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \quad (2)$$

Assuming no singularities such as ideal short-circuits or ideal open-circuits occur, each parameter set may be transformed onto another. From (1), since \mathbf{Z} and \mathbf{Y} are related

by matrix inversion $\mathbf{Z} = \mathbf{Y}^{-1}$, the scattering matrix and the impedance matrix are related by

$$\mathbf{Z}(s) = \mathbf{Z}_0^{1/2}(\mathbf{I} + \mathbf{S}(s))(\mathbf{I} - \mathbf{S}(s))^{-1}\mathbf{Z}_0^{1/2}. \quad (3)$$

Furthermore, the scattering matrix is related to the admittance matrix by

$$\mathbf{Y}(s) = \mathbf{Z}_0^{-1/2}(\mathbf{I} - \mathbf{S}(s))(\mathbf{I} + \mathbf{S}(s))^{-1}\mathbf{Z}_0^{-1/2}. \quad (4)$$

2.1 State-space Representation Transformations

Given a state space model of a device's admittance or impedance parameter matrix and \mathbf{Z}_0 , it is possible to obtain a state space model of the device's scattering parameter matrix. The converse is also true. These results are useful because certain time domain simulators still require an admittance or impedance representation of the model.

Assume a possible state space representation of a system with a frequency response $\mathbf{S}(s)$, $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is known. The problem at hand is to determine a state space representation, $(\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}})$, of a system whose transfer function is given by (4).

Noticing that $\mathbf{Y}(s) = \mathbf{Z}_0^{-1/2}(2(\mathbf{I} + \mathbf{S}(s))^{-1} - \mathbf{I})\mathbf{Z}_0^{-1/2}$, and that a state space representation for a system with a transfer function $(\mathbf{I} + \mathbf{S}(s))^{-1}$ is $(\mathbf{A} - \mathbf{B}(\mathbf{I} + \mathbf{D})^{-1}\mathbf{C}, \mathbf{B}(\mathbf{I} + \mathbf{D})^{-1}, -(\mathbf{I} + \mathbf{D})^{-1}\mathbf{C}, (\mathbf{I} + \mathbf{D})^{-1})$, assuming $(\mathbf{I} + \mathbf{D})^{-1}$ exists [11], a possible representation is

$$\begin{aligned} \hat{\mathbf{A}} &= \mathbf{A} - \mathbf{B}(\mathbf{I} + \mathbf{D})^{-1}\mathbf{C} \\ \hat{\mathbf{B}} &= \mathbf{B}(\mathbf{I} + \mathbf{D})^{-1}\mathbf{Z}_0^{-1/2} \\ \hat{\mathbf{C}} &= -2\mathbf{Z}_0^{-1/2}(\mathbf{I} + \mathbf{D})^{-1}\mathbf{C} \\ \hat{\mathbf{D}} &= \mathbf{Z}_0^{-1/2}(\mathbf{I} - \mathbf{D})(\mathbf{I} + \mathbf{D})^{-1}\mathbf{Z}_0^{-1/2} \end{aligned} \quad (5)$$

Conversely, assume that a state space representation $(\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}})$ for a system whose transfer function is $\mathbf{Y}(s)$ is known. A state space representation of a system whose transfer function is given by

$$\mathbf{S}(s) = (\mathbf{I} - \mathbf{Z}_0^{1/2}\mathbf{Y}(s)\mathbf{Z}_0^{1/2})(\mathbf{I} + \mathbf{Z}_0^{1/2}\mathbf{Y}(s)\mathbf{Z}_0^{1/2})^{-1}, \quad (6)$$

can be obtained by solving (5) for $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$. If $(\mathbf{I} + \mathbf{Z}_0^{1/2}\hat{\mathbf{D}}\mathbf{Z}_0^{1/2})^{-1}$ exists, a possible representation is

$$\begin{aligned} \mathbf{A} &= \hat{\mathbf{A}} - \hat{\mathbf{B}}(\hat{\mathbf{D}} + \mathbf{Z}_0^{-1})^{-1}\hat{\mathbf{C}} \\ \mathbf{B} &= 2\hat{\mathbf{B}}\mathbf{Z}_0^{1/2}(\mathbf{I} + \mathbf{Z}_0^{1/2}\hat{\mathbf{D}}\mathbf{Z}_0^{1/2})^{-1} \\ \mathbf{C} &= -(\mathbf{I} + \mathbf{Z}_0^{1/2}\hat{\mathbf{D}}\mathbf{Z}_0^{1/2})^{-1}\mathbf{Z}_0^{1/2}\hat{\mathbf{C}} \\ \mathbf{D} &= (\mathbf{I} - \mathbf{Z}_0^{1/2}\hat{\mathbf{D}}\mathbf{Z}_0^{1/2})(\mathbf{I} + \mathbf{Z}_0^{1/2}\hat{\mathbf{D}}\mathbf{Z}_0^{1/2})^{-1} \end{aligned} \quad (7)$$

The state space representation transformations (5) and (7) allow the model generation process to take place by approximating either admittance, impedance or scattering parameters. Not only can the data be transformed through (3), (4) and their inverses, but also the models themselves can be transformed to a more convenient representation.

2.2 Model Properties

The admittance, impedance and scattering parameter matrices of passive systems satisfy a set of physically important conditions.

In [4], Brune proved that the admittance and impedance parameter matrices of passive electrical networks are positive real matrix rational functions. A matrix function $\mathbf{H}(s)$ is positive real if,

$$\mathbf{H}(s) = \overline{\mathbf{H}(\bar{s})} \quad (8)$$

$$\mathbf{H}(s) \text{ is analytic in } Re[s] > 0 \quad (9)$$

$$\mathbf{H}(s) + \mathbf{H}(s)^H \geq 0 \text{ in } Re[s] > 0. \quad (10)$$

A matrix rational function is positive real if and only if (8) and (9) hold and

$$\mathbf{H}(jw) + \mathbf{H}(jw)^H \geq 0 \text{ for } w \in \mathbb{R}. \quad (11)$$

Any pole on the imaginary axis must be simple and its residue matrix must be Hermitian and nonnegative definite. If $\mathbf{H}(s)$ has no poles on the closed right-half plane, it is positive real if and only if (11) holds.

Since a passive system is necessarily stable and has a power gain at most one, the scattering matrix of a passive system is bounded real. This property may also be inferred through (3) or (4) and the definition of positive real transfer matrices. A matrix function $\mathbf{H}(s)$ is bounded real if it satisfies (8), (9) and

$$\|\mathbf{H}(s)\| \leq 1 \text{ for } Re[s] > 0. \quad (12)$$

A matrix rational function is bounded real if it satisfies, (8) and (9) and

$$\|\mathbf{H}(jw)\| \leq 1 \text{ for } w \in \mathbb{R}. \quad (13)$$

It is not necessary to actually calculate the infinity norm model's frequency response to determine that it is less than one. If $\mathbf{\Gamma} = (\mathbf{I} - \mathbf{D}^T \mathbf{D})^{-1}$ exists, the system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is bounded real if and only if the Hamiltonian matrix,

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} + \mathbf{B}\mathbf{\Gamma}\mathbf{C} & \mathbf{B}\mathbf{\Gamma}\mathbf{B}^T \\ -\mathbf{C}^T \mathbf{\Gamma} \mathbf{C} & -\mathbf{A}^T - \mathbf{C}^T \mathbf{\Gamma} \mathbf{B}^T \end{bmatrix} \quad (14)$$

has no purely imaginary eigenvalues [3]. If $\mathbf{I} - \mathbf{D}^T \mathbf{D}$ is ill-conditioned, the alternative proposed in [17] may be used. Note that the infinity norm is usually calculated through bisection and that, at each step, a similar eigenvalue problem is solved [3, 17].

3 Matrix Nevanlinna-Pick Interpolation

Let Δ represent the open right-half plane. Given a set of N data matrices, $\mathbf{s} \in \Delta^N$ and $\mathbf{H} \in \mathbb{C}^{N(n_o \times n_i)}$. The $n_o \times n_i$ matrix rational function \mathbf{F} interpolates the data set (\mathbf{s}, \mathbf{H}) if

$$\mathbf{F}(s_k) = \mathbf{H}_k, \text{ for } k = 1, \dots, N. \quad (15)$$

The Nevanlinna-Pick problem is to describe all interpolating functions \mathbf{F} , that are analytic in Δ and that satisfy

$$\sup_{s \in \Delta} \|\mathbf{F}(s)\| < 1. \quad (16)$$

Since the scattering parameter matrix of a passive system also satisfies (16), using the Nevanlinna-Pick interpolation algorithm for passive multivariable system identification seems to be a natural choice.

The matrix Nevanlinna-Pick problem may be formulated as a special case of the two-sided Nudelman problem. Let

$$\mathbf{C}_- = [\mathbf{I}_{n_i} \quad \dots \quad \mathbf{I}_{n_i}], \quad (17)$$

a $n_i \times Nn_i$ matrix, and

$$\mathbf{C}_+ = [\mathbf{H}_1 \quad \dots \quad \mathbf{H}_N], \quad (18)$$

also a $n_o \times Nn_i$ matrix, and

$$\mathbf{A}_\pi = \begin{bmatrix} s_1 \mathbf{I}_{n_i} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & s_N \mathbf{I}_{n_i} \end{bmatrix}, \quad (19)$$

a $Nn_i \times Nn_i$ block diagonal matrix.

Theorem 3.1 (Matrix Nevanlinna-Pick) *There exists a rational function $\mathbf{F}(s)$ that interpolates a given data set (\mathbf{s}, \mathbf{H}) , that is analytic in Δ and satisfies $\sup_{s \in \Delta} \|\mathbf{F}(s)\| < 1$, if and only if the Pick matrix,*

$$\mathbf{\Lambda}_{p,q} = \left[\frac{\mathbf{I} - \mathbf{H}_p^H \mathbf{H}_q}{\bar{s}_p + s_q} \right]_{1 \leq p,q \leq N} \quad (20)$$

is positive definite. In this case, there is a 2×2 block matrix function

$$\mathbf{\Theta}(s) = \mathbf{I}_{n_o+n_i} + \begin{bmatrix} \mathbf{C}_+ \\ \mathbf{C}_- \end{bmatrix} (s \mathbf{I}_{Nn_i} - \mathbf{A}_\pi)^{-1} \mathbf{\Lambda}^{-1} \begin{bmatrix} -\mathbf{C}_+^H & \mathbf{C}_-^H \end{bmatrix} \quad (21)$$

such that the solutions of the Nevanlinna-Pick interpolation problem become,

$$\mathbf{F}(s) = [\mathbf{\Theta}_{1,1}(s) \mathbf{G}(s) + \mathbf{\Theta}_{1,2}(s)] [\mathbf{\Theta}_{2,1}(s) \mathbf{G}(s) + \mathbf{\Theta}_{2,2}(s)]^{-1}, \quad (22)$$

where $\mathbf{G}(s)$ is an arbitrary $n_o \times n_i$ rational function that is analytic on Δ and that satisfies $\sup_{s \in \Delta} \|\mathbf{G}(s)\| < 1$.

3.1 Boundary Interpolation

In Theorem 3.1, the data points are considered to lie in the interior of Δ . Since we intend to use Nevanlinna-Pick for frequency domain identification, the available data points will usually be placed on the imaginary axis. This is the boundary interpolation problem, also called a Loewner interpolation problem.

Definition 3.1 (Loewner Interpolation Problem) Given a set of $n_o \times n_i$ data matrices, $\mathbf{H}_1, \dots, \mathbf{H}_N$, corresponding to s_1, \dots, s_N on $\delta\Delta$, find an interpolating function F , analytic in $\overline{\Delta}$, such that

$$\sup_{s \in \overline{\Delta}} \|\mathbf{F}(s)\| \leq 1 \quad (23)$$

and

$$\mathbf{F}(s_k) = \mathbf{H}_k, \text{ for } k = 1, \dots, N. \quad (24)$$

It is clear that the data must satisfy $\|\mathbf{H}_k\| \leq 1$, for all k .

The boundary interpolation problem may be solved by a shifting procedure. Let \mathbf{F}_σ be the Nevanlinna-Pick interpolant of the data set $(\mathbf{s} + \sigma, \mathbf{H})$, with σ real satisfying $\sigma > 0$. The interpolating function $\mathbf{F}(s) = \mathbf{F}_\sigma(s + \sigma)$ satisfies conditions (23) and (24) and is analytic for $\text{Re}[s] > -\sigma$ which contains Δ . The Nevanlinna-Pick interpolant exists if the Pick matrix

$$\Lambda_{p,q} = \left[\frac{\mathbf{I} - \mathbf{H}_p^H \mathbf{H}_q}{\overline{s}_p + s_q + 2\sigma} \right]_{1 \leq p, q \leq N} \quad (25)$$

is positive definite. If $\|\mathbf{H}_k\| < 1$ for all k , it is always possible to choose a small enough σ so that (25) is positive definite.

Although it is always possible to generate an interpolant by choosing a small enough σ the behavior between data points is severely affected by these parameters. As σ tends to zero, the poles of the rational function are allowed to approach the imaginary axis or the unit circle resulting in a highly oscillatory frequency response.

4 An Incremental Fitting Strategy

From equation (22), the Nevanlinna-Pick interpolant will have at least Nn_i poles. The minimum order is achieved if $\mathbf{G}(s)$ is chosen to be a constant matrix. Unfortunately, even for scalar data sets of moderate size, the generated models become too large for practical use. In this section an iterative algorithm, based on the boundary Nevanlinna-Pick interpolation algorithm, is presented. The basic idea is to choose, either randomly, manually, or through some heuristic procedure, an initial set of points. Then perform the Cholesky factorization of the Pick matrix associated with the initial set and determine the $\Theta(s)$ matrix. Choose \mathbf{G} , a $n_i \times n_i$ constant matrix, so that the weighted quadratic error between the unused data points and the model's frequency response is minimized. Then, iteratively, choose the set of data points with the largest error norm, update the Cholesky factorization of the Pick matrix, determine the new $\Theta(s)$ and a new \mathbf{G} matrix. The iterative procedure should be stopped when the error reaches a certain threshold or when the order exceeds a predetermined limit.

The choice of the constant matrix \mathbf{G} may use a different criteria such as the square error between the models frequency response and a spline-interpolated version of the

Algorithm 1 Iterative Interpolation Algorithm

input $N, n_i, n_o, s, H, \sigma, n_0, \varepsilon, \text{maxord}$.

output F

Set $it = 0$,

Choose n_0 data points and form the set of the interpolation points \mathcal{A} .

Form the Cholesky factor \mathbf{R}_0 of Λ_0 .

if The Cholesky factorization fails **then**

 Choose a new set of points until \mathbf{R}_0 exists.

 Move the points that lead to failure to the set of the rejected points \mathcal{B} .

end if

Choose \mathbf{G} such that $E_0^2 = \sum_{k=1}^N \|\mathbf{F}(s_k) - \mathbf{H}_k\|_2^2$ is minimized.

while $E_{it} > \varepsilon$ and $|\mathcal{A}| < \text{maxord}$ **do**

 Choose a set of data points not in \mathcal{A} or \mathcal{B} .

 Update the Cholesky factor of the Pick matrix

$$\begin{bmatrix} \Lambda_{it-1} & \delta\Lambda_A \\ \delta\Lambda_A^H & \delta\Lambda_B \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{it-1}^H & \mathbf{0} \\ \delta\mathbf{R}_A^H & \delta\mathbf{R}_B^H \end{bmatrix} \begin{bmatrix} \mathbf{R}_{it-1} & \delta\mathbf{R}_A \\ \mathbf{0} & \delta\mathbf{R}_B \end{bmatrix} \quad (26)$$

 where $\delta\mathbf{R}_A = \mathbf{R}_{it-1}^{-1} \delta\Lambda_A$ and $\delta\mathbf{R}_B$ is the Cholesky factor of $\Lambda_B - \delta\mathbf{R}_A^H \mathbf{R}_A$ if it exists.

if The Cholesky factorization of $\Lambda_B - \delta\mathbf{R}_A^H \delta\mathbf{R}_A$ fails **then**

 Move the subset of the new points that lead to failure to the set \mathcal{B} .

 Choose a new set of points until either $\delta\mathbf{R}_B$ exists or all the remaining points are in \mathcal{B} .

end if

 Choose \mathbf{G} such that $E_0^2 = \sum_{k \notin \mathcal{A}} \|\mathbf{F}(s_k) - \mathbf{H}_k\|_2^2$ is minimized.

end while

original data. This may be used to regularize the behavior of the interpolant between data points.

Given the boundary interpolation shift parameter σ , the size of the initial set of data points n_0 and the data set itself, Algorithm 1 uses Nevanlinna-Pick boundary interpolation to generate a bounded real rational approximant.

If a fixed number of points is added at each iteration, the computational cost of updating the Cholesky factorization in (26), is asymptotically proportional to $|\mathcal{A}|^2$, where $|\mathcal{A}|$ is the number of elements in the set \mathcal{A} . The cost of recalculating the Cholesky factorization at each step would be asymptotically proportional to $|\mathcal{A}|^3$.

In order to get a model with a real impulse response, \mathbf{G} must be a real matrix and any complex interpolation point (s_k, \mathbf{H}_k) must be accompanied by its complex conjugate $(\overline{s}_k, \overline{\mathbf{H}}_k)$.

The choice of the initial data set and the shift determines most of the outcome. If the points are chosen randomly, a fixed number of approximants should be generated and the best one chosen. If a small shift is used, most combinations of data points will yield a positive definite Pick matrix but the model poles will have a small real part and the fre-

quency response will be oscillatory. If a large shift is used, not many points will be accepted because it will not be possible to generate an interpolation function whose poles have such a large real part. However, in the later case, the frequency response tends to be smoother and, with an appropriate choice of \mathbf{G} , a better approximation for the non-interpolated points is usually more likely to be achieved than with a small shift. However, if the shift is too large, most of the points will be rejected. By starting with a large shift and reducing it until enough points are accepted, a heuristic algorithm for the choice of σ is obtained. If the interpolation points are somehow arbitrated, the smallest eigenvalue of $\mathbf{\Lambda}(\sigma)$ may be imposed to any desired accuracy by using a bisection method in σ . Since all the eigenvalues of $\mathbf{\Lambda}$ are real, if the smallest eigenvalue is positive, all the others will also be positive and $\mathbf{\Lambda}$ will be positive definite. By choosing a small enough value, the largest σ that still yields a positive definite $\mathbf{\Lambda}$ may be approximated.

At each step, the choice of \mathbf{G} is determined by the solution of a non-linear least squares problem involving the matrix linear fraction (22). Assuming that \mathbf{G} is a constant matrix and knowing that the error at the interpolation points is zero, the problem may be stated as

$$\begin{aligned} & \text{minimize} \quad \sum_{k \notin \mathcal{A}} \left\| \left[\begin{array}{c} \Theta_{1,1}(s_k) \mathbf{G} + \Theta_{1,2}(s_k) \\ \Theta_{2,1}(s_k) \mathbf{G} + \Theta_{2,2}(s_k) \end{array} \right]^{-1} - \mathbf{H}_k \right\|_2^2 \\ & \text{subject to} \quad \|\mathbf{G}\| < 1. \end{aligned} \quad (27)$$

In our implementation this problem is solved by using `constr` from Matlab's optimization toolbox. For greater control on the behavior between interpolation points a finer grid, containing spline interpolated values of $\mathbf{H}(s)$ can be used.

For certain applications it is desirable to change the unitary upper bound. A constant non-unitary upper bound for the norm, L , may be obtained by first scaling the data by $1/L$, determining the Nevanlinna-Pick interpolant, and scaling the interpolant by L .

Using the incremental fitting strategy described, the size of models can be reduced without noticeable deterioration in accuracy and while guaranteeing passivity of the resulting model.

5 State-space Representations

For time domain simulation, a state space model is usually required. In this section a state space representation of a system with a transfer function (22) is presented. It is assumed that $\mathbf{G}(s)$ is a constant $n_o \times n_i$ matrix with norm less than one and that there are \tilde{N} interpolation points.

Recalling equations (22) and (21), and defining $\mathbf{K} = \mathbf{\Lambda}^{-1}(\mathbf{C}_-^H - \mathbf{C}_+^H \mathbf{G})$, we get

$$\Theta_{1,1}(s) = \mathbf{I}_{n_o} - \mathbf{C}_+(s\mathbf{I}_{\tilde{N}n_i} - \mathbf{A}_\pi)^{-1} \mathbf{\Lambda}^{-1} \mathbf{C}_+^H \quad (28)$$

$$\Theta_{1,2}(s) = \mathbf{C}_+(s\mathbf{I}_{\tilde{N}n_i} - \mathbf{A}_\pi)^{-1} \mathbf{\Lambda}^{-1} \mathbf{C}_-^H \quad (29)$$

$$\Theta_{2,1}(s) = -\mathbf{C}_-(s\mathbf{I}_{\tilde{N}n_i} - \mathbf{A}_\pi)^{-1} \mathbf{\Lambda}^{-1} \mathbf{C}_+^H \quad (30)$$

$$\Theta_{2,2}(s) = \mathbf{I}_{n_i} + \mathbf{C}_-(s\mathbf{I}_{\tilde{N}n_i} - \mathbf{A}_\pi)^{-1} \mathbf{\Lambda}^{-1} \mathbf{C}_-^H \quad (31)$$

and hence $\mathbf{F}(s) = \mathbf{N}(s)\mathbf{D}(s)^{-1}$ is given by

$$\mathbf{F}(s) = (\mathbf{G} + \mathbf{C}_+(s\mathbf{I} - \mathbf{A}_\pi)^{-1} \mathbf{K}) (\mathbf{I}_{n_i} + \mathbf{C}_-(s\mathbf{I} - \mathbf{A}_\pi)^{-1} \mathbf{K})^{-1}. \quad (32)$$

Let $\mathbf{C}_+^{(k)}$ represent the k th $n_o \times n_i$ matrix entry in \mathbf{C}_+ , $\mathbf{C}_-^{(k)}$ the k th $n_i \times n_i$ matrix entry in \mathbf{C}_- , and $\mathbf{K}_-^{(k)}$ the k th $n_i \times n_i$ matrix entry in \mathbf{K} . By (19) and (32),

$$\mathbf{F}(s) = \left(\mathbf{G} + \sum_{k=1}^{\tilde{N}} \frac{\mathbf{C}_+^{(k)} \mathbf{K}^{(k)}}{s - s_k} \right) \left(\mathbf{I}_{n_i} + \sum_{k=1}^{\tilde{N}} \frac{\mathbf{C}_-^{(k)} \mathbf{K}^{(k)}}{s - s_k} \right)^{-1}. \quad (33)$$

After some algebraic manipulation

$$\mathbf{F}(s) = \left(\mathbf{G} \prod_{k=1}^{\tilde{N}} (s - s_k) + \sum_{k=1}^{\tilde{N}} \mathbf{C}_+^{(k)} \mathbf{K}^{(k)} \prod_{p \neq k} (s - s_p) \right) \left(\mathbf{I}_{n_i} \prod_{k=1}^{\tilde{N}} (s - s_k) + \sum_{k=1}^{\tilde{N}} \mathbf{C}_-^{(k)} \mathbf{K}^{(k)} \prod_{p \neq k} (s - s_p) \right)^{-1} \quad (34)$$

is obtained. It can be seen that if $s = s_k$ then $\mathbf{F}(s_k) = \mathbf{C}_+^{(k)}$ which, by definition, is \mathbf{H}_k .

A transfer function establishes a frequency domain input-output relationship $\mathbf{Y}(s) = \mathbf{F}(s)\mathbf{U}(s)$, where $\mathbf{Y}(s)$ is the output and $\mathbf{U}(s)$ is the input. Define $\mathbf{V}(s)$ and $\mathbf{V}_k(s)$ for $k = 0, \dots, \tilde{N}$, such that $\mathbf{V}(s) = \mathbf{D}(s)^{-1}\mathbf{U}(s)$, $\mathbf{Y}(s) = \mathbf{N}(s)\mathbf{V}(s)$ and

$$\begin{aligned} \mathbf{U}(s) &= \prod_{k=1}^{\tilde{N}} (s - s_k) \mathbf{V}(s) + \sum_{k=1}^{\tilde{N}} \mathbf{C}_-^{(k)} \mathbf{K}^{(k)} \prod_{p \neq k} (s - s_p) \mathbf{V}(s) \\ &= \mathbf{V}_0(s) + \sum_{k=1}^{\tilde{N}} \mathbf{C}_-^{(k)} \mathbf{K}^{(k)} \mathbf{V}_k(s). \end{aligned} \quad (35)$$

The output can be obtained as a combination of $\mathbf{V}_k(s)$ and $\mathbf{U}(s)$ as indicated below

$$\begin{aligned} \mathbf{Y}(s) &= \mathbf{G} \prod_{k=1}^{\tilde{N}} (s - s_k) \mathbf{V}(s) + \sum_{k=1}^{\tilde{N}} \mathbf{C}_+^{(k)} \mathbf{K}^{(k)} \prod_{p \neq k} (s - s_p) \mathbf{V}(s) \\ &= \mathbf{G}\mathbf{U}(s) + \sum_{k=1}^{\tilde{N}} \left(\mathbf{C}_+^{(k)} - \mathbf{G}\mathbf{C}_-^{(k)} \right) \mathbf{K}^{(k)} \mathbf{V}_k(s). \end{aligned} \quad (36)$$

Noticing that $(s - s_k)\mathbf{V}_k(s) = \mathbf{V}_0(s)$ and using the input equation to solve $\mathbf{V}_0(s)$ for $\mathbf{V}_k(s)$ and $\mathbf{U}(s)$, an expression for $s\mathbf{V}_k(s)$ is obtained

$$s\mathbf{V}_k(s) = \mathbf{U}(s) + s_k \mathbf{V}_k(s) - \sum_{p=1}^{\tilde{N}} \mathbf{C}_-^{(p)} \mathbf{K}^{(p)} \mathbf{V}_p(s). \quad (37)$$

The inverse Laplace transform of $\mathbf{V}_k(s)$ may be used as a state space variable and the state space representation

$$\begin{aligned} \mathbf{A}_{NP} &= \mathbf{A}_\pi - \begin{bmatrix} \mathbf{I}_{n_i} \\ \vdots \\ \mathbf{I}_{n_i} \end{bmatrix} \begin{bmatrix} \mathbf{K}^{(1)T} \\ \vdots \\ \mathbf{K}^{(\tilde{N})T} \end{bmatrix}^T \\ \mathbf{B}_{NP} &= \left[\mathbf{I}_{n_i} \quad \cdots \quad \mathbf{I}_{n_i} \right]_{n_i \times \tilde{N}n_i}^T \\ \mathbf{C}_{NP} &= \left[\left(\mathbf{C}_+^{(1)} - \mathbf{G} \right) \mathbf{K}^{(1)} \quad \cdots \quad \left(\mathbf{C}_+^{(\tilde{N})} - \mathbf{G} \right) \mathbf{K}^{(\tilde{N})} \right] \\ \mathbf{D}_{NP} &= \mathbf{G}, \end{aligned} \quad (38)$$

may be formed.

For the boundary interpolation problem, the desired function is $\mathbf{F}(s) = \mathbf{F}_\sigma(s + \sigma)$. In this case, $\mathbf{A}_{NP_\sigma} = \mathbf{A}_{NP} - \mathbf{I}\sigma$ which shifts the poles of the system by $-\sigma$. However, since

$$\mathbf{A}_\pi = \mathbf{I}_{\tilde{N}n_i}\sigma + i\text{diag}(\mathbf{w}) \otimes \mathbf{I}_{n_i}, \quad (39)$$

where $\text{diag}(\mathbf{w})$ represents a diagonal matrix whose diagonal elements are those of the vector $\mathbf{w} = [w_1, \dots, w_{\tilde{N}}]^T$, and \otimes represents the Kronecker product, the $\mathbf{I}_{\tilde{N}n_i}\sigma$ terms cancel out leading to

$$\mathbf{A}_{NP_\sigma} = \begin{bmatrix} iw_1\mathbf{I}_{n_i} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & iw_{\tilde{N}}\mathbf{I}_{n_i} \end{bmatrix} - \begin{bmatrix} \mathbf{I}_{n_i} \\ \vdots \\ \mathbf{I}_{n_i} \end{bmatrix} \begin{bmatrix} \mathbf{K}^{(1)T} \\ \vdots \\ \mathbf{K}^{(\tilde{N})T} \end{bmatrix}^T \quad (40)$$

It is often more convenient to have a real state space representation. Assuming that the interpolation points are all complex conjugates, and that they are ordered so that $s_{2k-1} = \bar{s}_{2k}$ and $\mathbf{K}^{(2k-1)} = \overline{\mathbf{K}^{(2k)}}$ for $k = 1, \dots, \tilde{N}/2$, a real state space representation $(\mathbf{A}_{RNP}, \mathbf{B}_{RNP}, \mathbf{C}_{RNP}, \mathbf{D}_{RNP})$ is obtained through the similarity transform $(\mathbf{T}^{-1}\mathbf{A}_{NP}\mathbf{T}, \mathbf{T}^{-1}\mathbf{B}_{NP}, \mathbf{C}_{NP}\mathbf{T}, \mathbf{D}_{NP})$ where

$$\mathbf{T} = \frac{1}{\sqrt{2}}\mathbf{I}_{\tilde{N}/2} \otimes \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \otimes \mathbf{I}_{n_i}, \quad (41)$$

and

$$\mathbf{T}^{-1} = \frac{1}{\sqrt{2}}\mathbf{I}_{\tilde{N}/2} \otimes \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \otimes \mathbf{I}_{n_i}. \quad (42)$$

More explicitly,

$$\mathbf{A}_{RNP} = \begin{bmatrix} \mathbf{I}_{n_i}\text{Re}[s_1] & \mathbf{I}_{n_i}\text{Im}[s_1] & \cdots & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_{n_i}\text{Im}[s_1] & \mathbf{I}_{n_i}\text{Re}[s_1] & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_{n_i}\text{Re}[s_{\tilde{N}}] & \mathbf{I}_{n_i}\text{Im}[s_{\tilde{N}}] \\ \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{I}_{n_i}\text{Im}[s_{\tilde{N}}] & \mathbf{I}_{n_i}\text{Re}[s_{\tilde{N}}] \end{bmatrix} - \begin{bmatrix} \mathbf{0}_{n_i} \\ \mathbf{I}_{n_i} \\ \vdots \\ \mathbf{0}_{n_i} \\ \mathbf{I}_{n_i} \end{bmatrix} \begin{bmatrix} 2\text{Im}[\mathbf{K}^{(1)}]^T \\ -2\text{Re}[\mathbf{K}^{(1)}]^T \\ \vdots \\ 2\text{Im}[\mathbf{K}^{(\tilde{N})}]^T \\ -2\text{Re}[\mathbf{K}^{(\tilde{N})}]^T \end{bmatrix}^T$$

$$\mathbf{B}_{RNP} = \sqrt{2} \begin{bmatrix} \mathbf{0}_{n_i} & \mathbf{I}_{n_i} & \cdots & \mathbf{0}_{n_i} & \mathbf{I}_{n_i} \end{bmatrix}_{\tilde{N}n_i \times n_i}^T$$

$$\mathbf{C}_{RNP} = \sqrt{2} \begin{bmatrix} -\text{Im}[\mathbf{C}_{NP}^{(1)}] & \text{Re}[\mathbf{C}_{NP}^{(1)}] & \cdots & -\text{Im}[\mathbf{C}_{NP}^{(\tilde{N})}] & \text{Re}[\mathbf{C}_{NP}^{(\tilde{N})}] \end{bmatrix}$$

$$\mathbf{D}_{RNP} = \mathbf{D}_{NP} \quad (43)$$

The resulting model has order $\tilde{N}n_i$. In time domain simulation and certain model order reduction algorithms, matrix-vector products involving \mathbf{A}_{NP} and \mathbf{A}_{RNP} are very important. This operation should take advantage of the fact that \mathbf{A}_{NP} and \mathbf{A}_{RNP} are the result of rank n_i perturbations to block diagonal matrices. The cost of the optimized operation is proportional to $\tilde{N}n_i$ instead of $\tilde{N}^2n_i^2$.

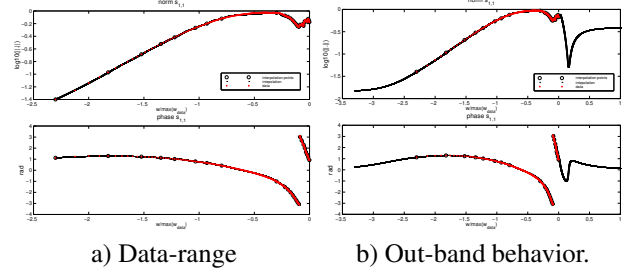


Figure 1. Order 50 approximation of the $s_{1,1}$ parameter of a coil inductor.

6 Experimental Results

In this section several experimental results are presented. The examples were chosen to illustrate certain issues that were found to be relevant and the properties of the generated models. The frequency responses were obtained using both the state-space representation and (22) thus validating the results from section 5.

The first example is the $s_{1,1}$ parameter of a coil inductor. In Figure 1-a), the data is approximated by a 25 point interpolant, the value of σ was manually chosen, $\sigma = 0.0125$ was found to be adequate. The out band behavior is quite smooth as can be seen in Figure 1-b). Since 25 interpolation points were used, the model has order 50 which is clearly excessive. By determining the poles of the system, it can be seen that the approximation is stable (it was verified that, as expected, the real part of the poles is always less than $\sigma = 0.0125$). The infinity norm of the system's frequency response, calculated to an accuracy of $1e-3$, using the method in [3], was determined to be $\|H\| = 0.935 < 1$ which proves the model is passive.

In the next example, the 2 by 2 scattering parameter matrix of the same coil inductor is approximated. The generated model uses 50 interpolation points which means that it has 200 states. The approximation is illustrated in Figure 2. The poles of the system matrix are all in the left half plane and have a real part smaller than $-\sigma$, where $\sigma = 2.5 \times 10^{-3}$. It was interesting to notice that, the condition number of the system matrix is relatively small for its size, $\text{cond}(\mathbf{A}) = 1.766 \times 10^4$. The infinity norm, calculated to a tolerance of 10^{-3} , is $\|H(s)\|_\infty = 0.991$ which proves that the model is passive. The behavior of the model outside the data set is also seen to be smooth.

In the next example the $s_{1,1}$ parameter of a SAW filter is approximated. The frequency band is very narrow and the data is very close to one showing little magnitude variation. Since the values are very close to one, σ must be small. However, a small σ leads to an oscillatory frequency response. In order to reduce the magnitude of the oscillations a large number of interpolation points is required. Figure 3 shows an approximation for $\sigma = 2 \times 10^{-4}$ using 61 interpolation points. The oscillations of the 122 order model are at most 0.4dB but, since the data has almost constant modulus,

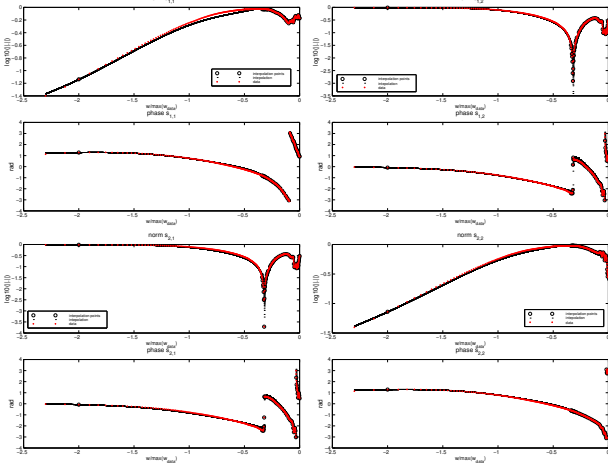


Figure 2. Multivariable approximation of the scattering matrix of a coil inductor using $\sigma = 0.0025$ and 50 points.

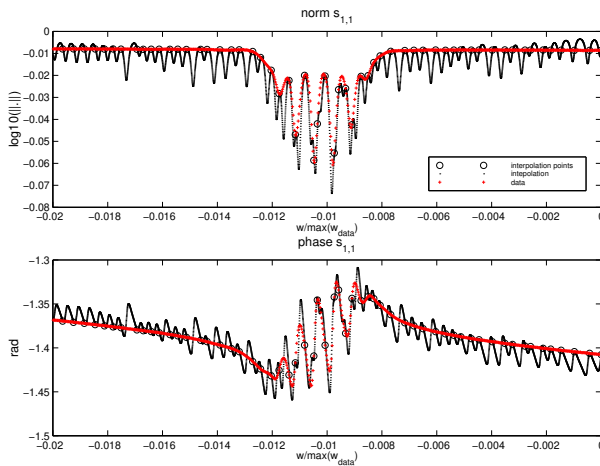


Figure 3. Approximation of a SAW filter's $s_{1,1}$ parameter using $\sigma = 2 \times 10^{-4}$ and 61 interpolation points.

qualitatively, the approximation is quite bad.

In order to try a larger value for σ , the data was scaled down by 10. The scaled problem allows for larger values of σ . However, it is no longer guaranteed that the generated model will be passive. Figure 4-a) shows an approximation with $\sigma = 1.3 \times 10^{-3}$ and 45 interpolation points. The oscillations are smoother and the order is smaller, although still very high since a model with 90 states is quite large. Unfortunately, calculating the infinity norm it comes that $\|H(s)\|_{\infty} = 0.3024$. The model, in the original scale, is not passive since its infinity norm is larger than one. This may be confirmed by observing the out band behavior of the approximant in Figure 4-b).

This example shows that scaling the model so that a larger σ may be used is an unreliable technique, since pas-

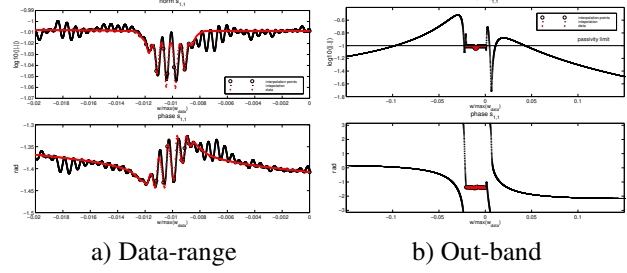


Figure 4. Approximation of a SAW filter's $s_{1,1}$ parameter, scaled by 0.1, with $\sigma = 1.3 \times 10^{-3}$ and 45 interpolation points.

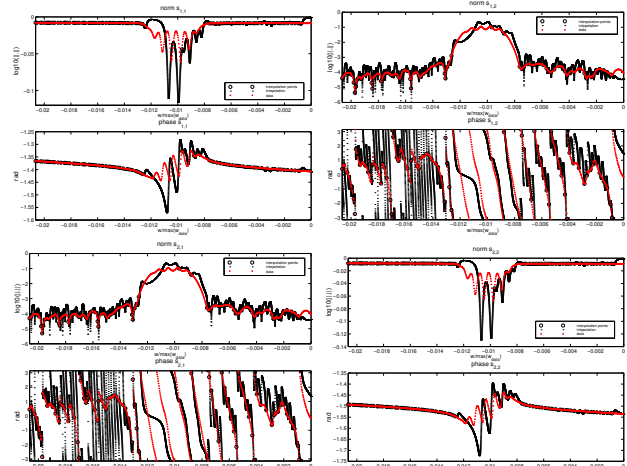


Figure 5. Multivariable approximant of the scattering matrix of a SAW filter using 80 interpolation points, $\sigma = 2.5 \times 10^{-4}$ and relative error norm.

sivity is no longer guaranteed. However, this is not always the case and, if a passive model is not required, scaling may be considered.

The problem of fitting data with different magnitudes is even more important in the multivariable fitting problem. For matrix function interpolation, the choice of the interpolation points and \mathbf{G} becomes harder as there are conflicting objectives. The choice of the point with the largest matrix error norm or even relative matrix error norm may not suffice, especially when the magnitude of the entries is different.

Our final example shows the approximation of the scattering parameter matrix of a SAW filter. In Figure 5 the frequency response of an approximant using 80 interpolation points, $\sigma = 2.5 \times 10^{-4}$ and the relative error norm is illustrated. In Figure 6 the frequency response of an approximant using 80 interpolation points, $\sigma = 1.25 \times 10^{-4}$ and the absolute error norm is illustrated. The results agree with what was expected after the previous example. The different shifts are due to the use of different interpolation point sets.

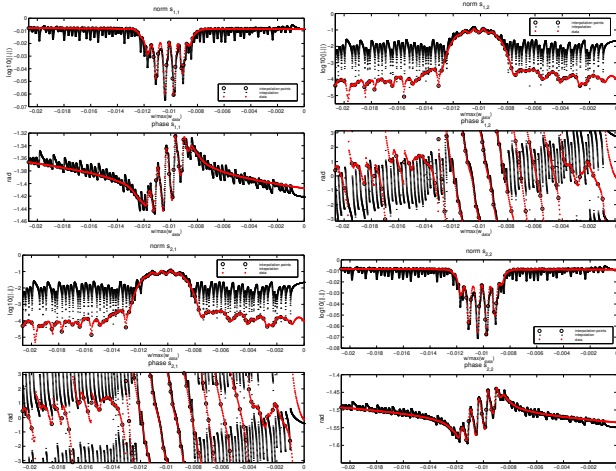


Figure 6. Multivariable approximant of the scattering matrix of a SAW filter using 80 interpolation points, $\sigma = 1.25 \times 10^{-4}$ and square error norm.

7 Conclusions

The Nevanlinna-Pick interpolation algorithm and the, more general, two-sided Nudelman problem and its solution are powerful mathematical tools that, appropriately used, may have an important role in frequency domain system identification. Since the interpolants are guaranteed to be stable and norm bounded by one, they are natural models for the scattering parameter matrices of passive systems.

In this paper, results from Nevanlinna-Pick interpolation theory were used to develop a bounded real matrix rational approximation algorithm. The algorithm was used to generate rational function models of passive systems by approximating their scattering parameter matrices. During the course of this research it was determined that factors such as the choice of \mathbf{G} as well as the choice of an appropriate σ are critical for approximation accuracy and efficiency. While the order of the generated models may in some cases be high, it is still much smaller than what would result from the direct application of the Nevanlinna-Pick interpolation algorithm to the full data set and the models meet the required passivity and accuracy requirements.

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