PASSIVITY OF A CLASS OF SAMPLED-DATA SYSTEMS: APPLICATION TO HAPTIC INTERFACES

J. Edward Colgate and Gerd Schenkel
Department of Mechanical Engineering
Northwestern University
Evanston, IL 60208-3111
(708) 491-4264 colgate@nwu.edu

Abstract

Passivity of systems comprising a continuous time plant and discrete time controller is considered. This topic is motivated by stability considerations arising in the control of robots and force-reflecting human interfaces ("haptic interfaces"). A necessary and sufficient condition for the passivity of a class of sampled-data systems is derived. An example — implementation of a "virtual wall" via a one degree-of-freedom haptic interface — is presented.

1. Introduction

Passivity is a powerful tool for the analysis of coupled stability problems arising in robotics and related disciplines. For instance, passivity methods have been used to establish conditions for the stability of a robot contacting an uncertain dynamic environment [5], to investigate the robustness of force feedback controllers [7], and to study the stability of telemanipulation with a time delay [1]. More recently, passivity techniques have been used in the design of "haptic interfaces" to virtual environments [4]. A haptic interface is a device which lets human operators touch, feel, and manipulate virtual (computer-generated) environments [9].

Haptic interface design provides direct motivation for the problem considered in this paper — passivity of sampled-data systems. This is because a haptic interface endowed with nearly ideal, collocated sensors and actuators, and implementing a virtual environment whose physical counterpart is passive, may nonetheless exhibit unstable oscillations when grasped by a human operator. Often, the frequency of these oscillations is outside the range of human voluntary movement and involuntary tremor, which would indicate that the energy required to sustain them is supplied by the interface. Thus, the interface is active, not passive. This is a direct consequence of the time delay and loss of information inherent in sampling.

Prior work on passivity has focused on continuous-time and discrete-time systems, but has not addressed sampled-data systems (continuous-time plant and discrete-time controller) [6, 11]. Recently, however, there has been a growing interest in the derivation of *norms* for sampled-data systems. This interest is motivated principally by the application of such norms to H_2 and H_∞ optimal controller design [10, 12, 14]. Conditions for passivity can, however, be converted via a bilinear transformation to conditions for the boundedness of an L_2 induced norm. A formula for the L_2 induced norm has been presented by Leung, et al. [12], although this formula assumes a bandlimited input (to eliminate aliasing), an assumption which may not be valid for robotic and haptic interface applications in which collisions create high frequencies. Kabamba and Hara [8] present conditions for the boundedness of the L_2 norm and a proposition concerning computation of the norm. Sivashankar and Khargonekar [14] present formulas for L_1 and L_∞ induced norms, and an upper bound for the L_2 induced norm.

2. Problem Statement

The problem will be described in terms of a prototypical one degree-of-freedom haptic interface, pictured in Figure 1. The interface basically consists of an actuator, such as a servomotor, which the operator grasps.

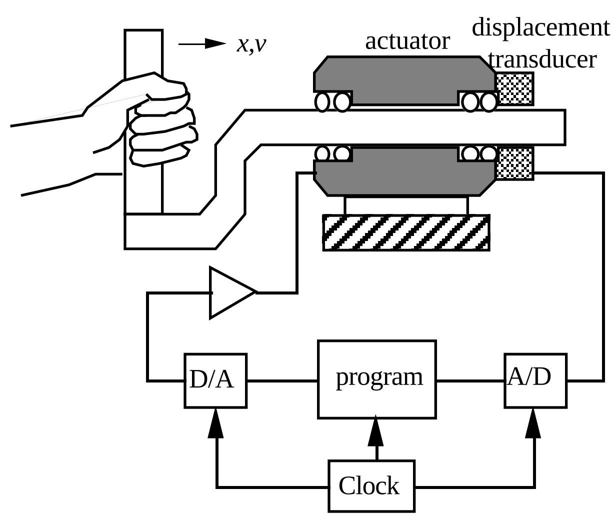

Figure 1. Schematic of a one degree-of-freedom haptic interface.

Feedback signals representing the state of the interface are input to a computer. In this paper, it is assumed that there is only one feedback signal, based on the displacement of the actuator. While no such restriction need be observed in practice, this is a common implementation. The computer calculates an actuator command according to its model of the virtual environment. This command is output, amplified, and sent to the actuator.

Figure 2 is a model of this system. It is assumed that the actuator and handle behave as a rigid body (m) with some viscous friction (b), acted upon by a controller force (u). Amplifier and sensor dynamics, nonlinearity, and noise are ignored. The feedback signal is sampled at the rate T, and the control signal is passed through a zero order hold. The virtual environment (feedback controller) is represented by a stable linear, shift-invariant transfer function, $H(z)$.

Useful virtual environments cannot be composed strictly of linear operators, however. At a minimum, it is necessary to include the nonlinear element pictured in Figure 2. This element, the unilateral constraint, is ubiquitous in the physical world. An example is the constraint experienced by a ball dropped on a floor — vastly different equations of motion apply when the ball is and is not in contact with the floor. Unilateral constraints are needed, in general, to account for collisions and contact. They are, however, nonlinear. In this paper, necessary and sufficient conditions for passivity will first be found for the linear case. It will then be shown that the sufficient conditions apply equally if a unilateral constraint is incorporated as in Figure 2.

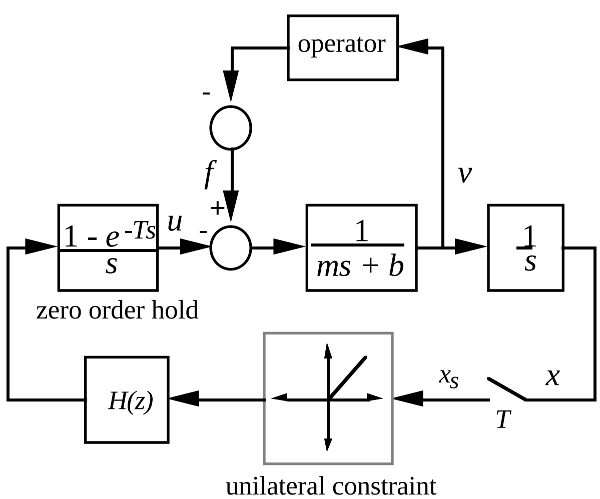

Figure 2. Model of a one degree-of-freedom haptic interface.

If the haptic display behaves passively, then the operator can never extract energy from it. Here, we will use the slightly more stringent statement that the energy input to the haptic display from the operator must be positive for all admissible force histories $f(t)$ (see discussion in Section 3.2) and all times greater than zero:

$$\int_0^t f(\tau)v(\tau)d\tau > 0, \;\; \forall\, t > 0, \text{ admissible } f(t) \qquad (2.1)$$

A system which does not satisfy 2.1 is said to be "active."

3. An Analytical Passivity Criterion

The major result of this paper is given by the following theorem:

<u>*Theorem*</u>
A necessary and sufficient condition for passivity of the sampled data system in Figure 2 is:

$$b > \frac{T}{2}\,\frac{1}{1 - \cos\omega T}\,\mathrm{Re}\left\{(1 - e^{-j\omega T})\,H(e^{j\omega T})\right\} \qquad 0 \le \omega \le \omega_N \qquad (3.1)$$

where $\omega_N = \pi/T$ is the Nyquist frequency.

3.1. Proof of Necessity

One of the well-known consequences of passivity is the following: a strictly passive system, connected to any passive environment, is necessarily stable. Thus, stability when connected to a linear time-invariant, passive, but otherwise arbitrary environment may be considered a necessary condition for passivity. This idea is the basis of the necessity proof.

Suppose that the unilateral constraint in Figure 2 is removed and the operator is replaced with a passive, but otherwise arbitrary impedance $Z_o(s)$. The closed loop characteristic equation of the resulting system is:

$$1 + H(e^{sT})G^*(s) = 0 \qquad (3.2)$$

where:

$$G^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} G(s + jn\omega_s) \qquad (3.3)$$

$$G(s) = \frac{1-e^{-Ts}}{s^2}\,\frac{1}{ms + b + Z_o(s)} \qquad (3.4)$$

and $\omega_s = 2\pi/T$. It will be proved that 3.1 is necessary to ensure 3.2 contains no unstable roots. The proof requires the use of coupled stability theory [13]. For clarity, the approach is first outlined: To begin, the constraint that $Z_o(s)$ is passive is used to identify the region of the Nyquist plane, R_{G^*}, within which $G^*(j\omega)$ must lie at each frequency. Next, a linear fractional transformation[1], $M\{j\omega, G^*(j\omega)\}$ (defined below), is found which will map this region to the *complete* interior of the unit disk, pointwise in frequency. The results of [2] are then used to find a related LFT, $N\{s, H(e^{sT})\}$, through which $H(e^{sT})$ can be mapped such that the closed loop characteristic equation, written in terms of transformed quantities, contains the same unstable poles as 3.2. This ensures that the transformation does not alter closed loop stability. The Small Gain Theorem then leads directly to a necessary and sufficient condition for closed loop stability: $\|N\{s, H(e^{sT})\}\|_\infty < 1$.

R_{G^*} is found via a series of transformations on the region corresponding to $Z_o(j\omega)$, as illustrated in Figure 3. The latter is simply the closed right half plane: $\mathrm{Re}\{Z_o(j\omega)\} \ge 0$, $\mathrm{Im}\{Z_o(j\omega)\}$ arbitrary. Consider first the

[1] A linear fractional transformation (LFT) is a type of conformal mapping having the property that it maps circular regions in the complex plane to other circular regions (this includes half planes, which are considered circles of infinite extent).

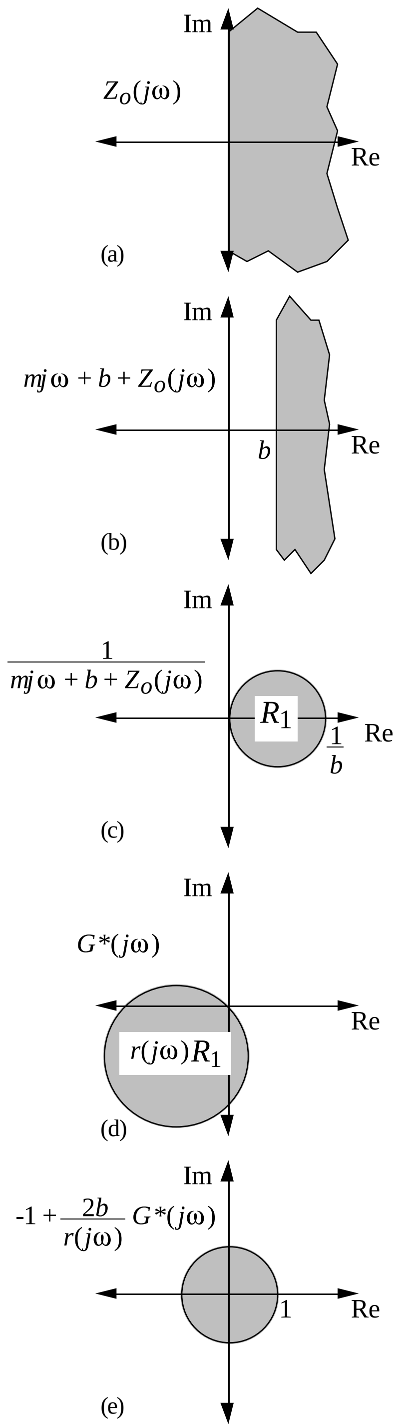

Figure 3.

term $mj\omega + b + Z_o(j\omega)$. The purely imaginary contribution of the mass will have no effect on the region, while the damping will shift the entire region to the right by b units. The resulting half plane is shown in Figure 3b. Next consider the term $(mj\omega + b + Z_o(j\omega))^{-1}$. The half plane is mapped, via the inverse, to a closed disk centered on the real axis at $1/2b$ with a radius of $1/2b$. Note that this is true at every frequency. Let this region be denoted R_1.

Because R_1 is frequency-independent, it can be removed from the infinite sum when computing the region corresponding to $G^*(j\omega)$:

$$R_{G^*}(\omega) = r(j\omega)\, R_1 \tag{3.5}$$

where:

$$r(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \frac{1 - e^{-(j\omega - jn\omega_s)T}}{(j\omega + jn\omega_s)^2} \tag{3.6a}$$

$$= \frac{1 - e^{-j\omega T}}{T} \sum_{n=-\infty}^{\infty} \frac{1}{(j\omega + jn\omega_s)^2} \tag{3.6b}$$

$$= \frac{T}{2} \, \frac{e^{-j\omega T} - 1}{1 - \cos \omega T} \tag{3.6c}$$

The factor $r(j\omega)$ may be viewed as a frequency-dependent rotation and scaling. Thus, $R_{G^*}(\omega)$ is a disk at each frequency (Figure 3d), and it may be mapped to the unit disk by a translation and scaling. An appropriate LFT, applied to $G^*(s)$, is:

$$M\{s, G^*(s)\} = -1 + \frac{2b}{r(s)} G^*(s) \tag{3.7}$$

Thus, the region corresponding to $M\{j\omega, G^*(j\omega)\}$ is the closed unit disk centered at the origin (Figure 3e). Details for finding the associated transformation $N\{s, H(e^{sT})\}$, are given in [2]. The result is:

$$N\{s, H(e^{sT})\} = \frac{r(s)H(e^{sT})}{2b + r(s)H(e^{sT})} \tag{3.8}$$

It can be verified by direct computation that the closed loop characteristic equation of the transformed system has the same unstable roots as those of the original system. Because $M\{j\omega, G^*(j\omega)\}$ is completely uncertain in phase and may have a magnitude as great as one, the Small Gain Theorem [6] gives a necessary and sufficient condition for stability, which is $\|N\{s, H(e^{sT})\}\|_\infty < 1$, or:

$$\left| \frac{r(j\omega)H(e^{j\omega T})}{2b + r(j\omega)H(e^{j\omega T})} \right| < 1 \quad \forall\, \omega \tag{3.9}$$

Straightforward manipulation then leads to the alternate expression in 3.1. The periodicity of $r(j\omega)H(e^{j\omega T})$ has also been used to narrow the range of frequencies which must be examined to $0 \le \omega \le \omega_N$.

3.2. Proof of Sufficiency

Consider again the system in Figure 2, and suppose the mass is initially at rest. An intuitive statement of passivity is that the kinetic energy of the mass never be as great as the total energy input by the source $f(t)$:

$$\frac{1}{2} m v^2(t) < \int_0^t f(\tau) v(\tau) d\tau, \;\; \forall\; t > 0, \text{ admissible } f(t) \tag{3.10}$$

Because kinetic energy is positive definite, satisfaction of 3.10 is clearly a sufficient condition for passivity according to 2.1. Also, if kinetic energy ever exceeded total energy input, energy could be extracted simply by applying a force pulse sufficient to bring the mass to rest in an arbitrarily short time period. We conclude, therefore, that 3.10 is equivalent to 2.1.

A force balance on the mass leads to the following:

$$\frac{1}{2}mv^2(t) = \int_0^t f(\tau)v(\tau)d\tau - \int_0^t u(\tau)v(\tau)d\tau - \int_0^t b\ v(\tau)v(\tau)d\tau \quad (3.11)$$

Subtracting 3.11 from 3.10 gives:

$$\int_0^t u(\tau)v(\tau)d\tau + \int_0^t b\ v^2(\tau)d\tau > 0, \quad \forall\ t > 0, \text{ admissible } v(t), \dot{v}(t) \quad (3.12)$$

An admissible signal is one for which the truncated L_2 norm is non-zero and finite for all t. The restriction to admissible velocity and acceleration in 3.12 ensures that the same class of signals is covered as in 3.10.

The passivity condition (3.12) may be converted to a frequency domain condition using Parseval's Theorem. First, define a class of truncated signals:

$$v_\theta(\tau) = \begin{cases} 0 & \tau < 0 \\ v(\tau) & 0 \le \tau \le \theta \\ 0 & \tau > \theta \end{cases} \quad (3.13)$$

Equation 3.12 can be rewritten as:

$$\int_{-\infty}^{\infty} u(\tau)v_t(\tau)d\tau + \int_{-\infty}^{\infty} b\ v_t^2(\tau)d\tau > 0, \quad \forall\ t, \text{ admissible } v(t), \dot{v}(t) \quad (3.14)$$

Parseval's Theorem gives an equivalent inequality:

$$\int_{-\infty}^{\infty} U(j\omega)V^*(j\omega)d\omega + \int_{-\infty}^{\infty} b\ V(j\omega)V^*(j\omega)d\omega > 0, \quad \forall\ \omega, \text{ admissible } V(j\omega) \quad (3.15)$$

where $U(j\omega)$ and (admissible) $V(j\omega)$ are Fourier Transforms of $u(\tau)$ and $v_t(\tau)$, respectively.

The signal U can be written in terms of V using the sampler, pulse transfer function, and zero order hold. Refer to Figure 2. Using the impulse modulation model of a sampler:

$$U(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega T} H(e^{j\omega T}) \sum_{n=-\infty}^{\infty} \frac{V(j\omega + jn\omega_s)}{j\omega + jn\omega_s} \quad (3.16)$$

The following definition is made to simplify notation:

$$\bar{H}(\omega) = -\frac{1 - e^{-j\omega T}}{T} H(e^{j\omega T}) \quad (3.17)$$

$\bar{H}(\omega)$ is periodic with a period equal to the sampling rate, T. Equation 3.15 may now be rewritten as:

$$\int_{-\infty}^{\infty} \bar{H}(\omega) \sum_{n=-\infty}^{\infty} \frac{V(j\omega + jn\omega_s)}{j\omega + jn\omega_s} \left[\frac{V(j\omega)}{j\omega}\right]^* d\omega + \int_{-\infty}^{\infty} b\ V(j\omega)V^*(j\omega)d\omega > 0 \quad (3.18)$$

It can be shown that the value of the first integral in 3.18 is unchanged if $\bar{H}(\omega)$ is replaced by $\mathrm{Re}\{\bar{H}(\omega)\}$. The requisite manipulations are omitted.

The next step is to identify an analytical lower bound to the sum of integrals in equation 3.18. While details are omitted due to space restrictions, a lower bound can be found, leading to the sufficient condition:

$$\int_{-\infty}^{\infty} \left[b + \mathrm{Re}\{\bar{H}(\omega)\} \sum_{n=-\infty}^{\infty} \frac{1}{(\omega + n\omega_s)^2} \right] V(j\omega)V^*(j\omega)d\omega > 0 \quad (3.19)$$

This inequality will be satisfied for all admissible $V(j\omega)$ if and only if the expression in brackets is positive at every frequency. This is because the power in $v(t)$ can be concentrated in an arbitrarily narrow frequency band. Using the same equality as in equation 3.6, the sufficient condition may be rewritten:

$$b + \mathrm{Re}\{\bar{H}(\omega)\} \frac{T^2}{2} \frac{1}{1 - \cos\omega T} > 0 \qquad \forall\ \omega \quad (3.20)$$

Finally, it can be seen that equations 3.20 and 3.1 are equivalent. This completes the proof in the absence of a unilateral constraint.

3.3. Sufficiency with a Unilateral Constraint

Because the output of the sampled data controller is fixed during each period, one can always construct a function $v(t)$, $kT \le t < (k+1)T$, which will extract an arbitrarily large amount of energy from the actuator. By selecting large enough b (according to 3.1), however, one is assured that at least as much energy will be lost to friction. Thus, one implication of the sufficiency proof is that, while moving, the haptic interface will consume energy during each and every sample period. Because, in addition, the haptic interface is passive in the absence of a feedback loop, we may conclude that $u(t)$ can be set to zero during any sample period without affecting the sufficiency result. In other words, the kinetic energy of the mass will at no point be greater than the total energy input by $f(t)$. Thus, the sufficiency proof guarantees that the incorporation of a unilateral constraint will not affect the passivity result obtained with a given linear controller.

4. Example

This section considers a common implementation of a "virtual wall," composed of a virtual spring and damper in mechanical parallel, together with a unilateral constraint operator [4]. A velocity estimate is obtained via backward difference differentiation of position, giving the following transfer function within the wall:

$$H(z) = K + B\frac{z - 1}{Tz} \quad (4.1)$$

where $K > 0$ is a virtual stiffness, and $B > 0$ is a virtual damping coefficient. From 3.1, the condition for passivity is:

$$b > \frac{T}{2}\frac{1}{1 - \cos\omega T} \mathrm{Re}\left\{ (1 - e^{-j\omega T}) \left(K + B\frac{e^{j\omega T} - 1}{Te^{j\omega T}} \right) \right\} \quad 0 \le \omega \le \omega_N \quad (4.2)$$

This relation can be reduced by straightforward algebraic manipulation to:

$$b > \frac{KT}{2} - B\cos\omega T \quad 0 \le \omega \le \omega_N \tag{4.3}$$

The right hand side is maximized at $\omega = \omega_N$, leading to the condition:

$$b > \frac{KT}{2} + B \tag{4.4}$$

This result shows that, to achieve passivity, some physical dissipation is essential. It also shows that, given fixed physical and virtual damping, the maximum achievable virtual stiffness is proportional to the sampling rate. Further, the achievable virtual damping is independent of the sampling rate.

These findings have certain implications for haptic interface design. In order to implement very stiff, dissipative constraints (high K, B), it is helpful to maximize b and minimize T. Fast sampling is a standard objective, but maximizing damping goes against conventional wisdom. It is generally argued that the dynamics of a haptic interface should be dominated by the virtual environment (which is the programmed behavior we wish to display) rather than any inherent dynamics (which is considered parasitic). Unfortunately, this ignores the effect of sampling. Sampling ensures a certain disparity between the actual and intended behaviors of the virtual environment which will result in active behavior and the potential for coupled instability unless accompanied by a sufficient degree of inherent damping (b). It is interesting to note, however, that the passivity condition does not rule out the use of *negative* virtual damping. For instance, if $B<0$ is permitted, passivity condition 4.4 changes to:

$$b > \frac{KT}{2} + |B| \tag{4.5}$$

Thus, as much negative virtual damping is permissible as positive virtual damping. In the case of K=0, it should be possible to eliminate almost completely the effect of inherent damping.

In a recent set of psychophysical experiments performed in the authors' laboratory, the benefits of physical damping and negative virtual damping for virtual wall implementation have been demonstrated [3].

5. Conclusions

A necessary and sufficient condition for the passivity of a class of sampled-data systems has been derived. The example of a "virtual wall" characterized by virtual stiffness and damping coefficients has been given and investigated with the aid of the passivity condition.

A related problem deserving careful investigation is the effect of quantization (or sensor resolution) on passivity. Quantization is as fundamental a consequence of digital control as sampling. The roundoff generated by quantization may be viewed as a form of high frequency noise. This noise may be amplified by differentiation, leading to sustained oscillations in a haptic interface.

Finally, the application of this theory, leading to the improved design and control of robots and haptic interfaces, is an important area for research.

Acknowledgements

The authors gratefully acknowledge the support of the National Science Foundation, grants MSS-9022513 and IRI-9213234, and valuable discussions with Mike Stanley and Paul Millman.

References

[1] Anderson, R. J. and M. W. Spong. *Bilateral Control of Teleoperators with a Time Delay.* IEEE Trans. on Automatic Control, vol. 34, no. 5, pp. 494-501, 1989.

[2] Colgate, J. E. *Coordinate Transformations and Logical Operations for Minimizing Conservativeness in Coupled Stability Criteria.* Journal of Dynamic Systems, Measurement and Control, in press.

[3] Colgate, J. E. and J. M. Brown. *Factors Affecting the Z-Range of a Haptic Display.* To appear in IEEE International Conference on Robotics and Automation. 1994.

[4] Colgate, J. E., P. E. Grafing, M. C. Stanley and G. Schenkel. *Implementation of Stiff Virtual Walls in Force-Reflecting Interfaces.* IEEE Virtual Reality Annual International Symposium. Seattle, WA. pp. 202-207, 1993.

[5] Colgate, J. E. and N. Hogan. *Robust control of dynamically interacting systems.* International Journal of Control, vol. 48, no. 1, pp. 65-88, 1988.

[6] Desoer, C. A. and M. Vidyasagar. *Feedback Systems: Input-Output Properties.* Academic Press. New York, 1975.

[7] Hogan, N. *On the Stability of Manipulators Performing Contact Tasks.* IEEE Journal of Robotics and Automation, vol. 4, no. 6, pp. 677-686, 1988.

[8] Kabamba, P. and S. Hara. *On Computing the Induced Norm of Sampled Data Systems.* ACC. pp. 319-320, 1990.

[9] Kazerooni, H., J. E. Colgate and B. D. Adelstein. *Advances in Robotics, Mechatronics and Haptic Interfaces, 1993.* ASME Winter Annual Meeting, vol. DSC-49, 1993.

[10] Khargonekar, P. P. and N. Sivashankar. *H 2 Optimal Control for Sampled-Data Systems.* Systems and Control Letters, vol. 17, no. 6, pp. 425-436, 1991.

[11] Landau, I. D. *Adaptive Control — The Model Reference Approach.* Marcel Dekker. New York, 1979.

[12] Leung, G. M. H., T. P. Perry and B. A. Francis. *Performance Analysis of Sampled-data Control Systems.* Automatica, vol. 27, no. 4, pp. 699-704, 1991.

[13] Safonov, M. G., E. A. Jonckheere, M. Verma and D. J. N. Limebeer. *Synthesis of Positive Real Multivariable Feedback Systems.* International Journal of Control, vol. 45, no. 3, pp. 817-842, 1987.

[14] Sivashankar, N. and P. Khargonekar. *Induced Norms for Sampled-Data Systems.* Automatica, vol. 28, no. 6, pp. 1267-1272, 1992.