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Path decompositions of digraphs

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A *path decomposition* of a digraph G (having no loops or multiple arcs) is a family of simple paths such that every arc of G lies on precisely one of the paths of the family. The *path number*, $pn(G)$ is the minimal number of paths necessary to form a path decomposition of G.

We show that $pn(G) \geq \sum_{v} \max\{0, \mathrm{od}(v)-\mathrm{id}(v)\}$ the sum taken over all vertices v of G, with equality holding if G is acyclic. If G is a subgraph of a tournament on n vertices we show that $pn(G) \leq [\![n^2/4]\!]$ with equality holding if G is transitive.

We conjecture that $pn(G) \leq [\![n^2/4]\!]$ for any digraph G on n vertices if n is sufficiently large, perhaps for all $n \geq 4$.

In this paper the word "digraph" will be reserved for directed graphs having neither loops nor multiple arcs. A *path decomposition* of a digraph G is a family of simple paths such that every arc of G lies on precisely one of the paths of the family. The *path number* of G is the minimal number of paths necessary to form a path decomposition of G; it is denoted by $pn(G)$.

Counterparts of these ideas for undirected graphs were studied by Lovász [*3*] in response to a problem posed by Erdös (see also Erdös [*1*], Harary, Schwenk [*2*] and Stanton, Cowan and James [*4*]).

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We obtain a lower bound for the path number of a digraph involving only the in- and out-degrees of its vertices (Theorem 1). In Theorem 3 we show that $[\![n^2/4]\!]$ is an upper bound for $pn(G)$ when G is a tournament or a subgraph of a tournament and in Corollary 4.2 we show that this bound is sharp. A formula is given in Theorem 4 expressing the path number of any acyclic digraph entirely in terms of its in- and out-degrees.

We conjecture that $[\![n^2/4]\!]$ is an upper bound for $pn(G)$ for any digraph G on n vertices if n is sufficiently large, perhaps for all $n \geq 4$.

Notation and preliminaries

The set of arcs of a digraph G will be denoted by $A(G)$, the set of vertices by $V(G)$ and vw will be the notation for the arc from the vertex v to the vertex w. For every vertex v of G, if $\mathrm{od}(v)$ arcs begin at v and $\mathrm{id}(v)$ arcs terminate at v we define:

$x(v) = \max\{0, \mathrm{od}(v)-\mathrm{id}(v)\}$ called the *excess* at v,

$d(v) = |\mathrm{od}(v)-\mathrm{id}(v)|$ called the *net degree* at v, and

$\mu(v) = \max\{\mathrm{od}(v), \mathrm{id}(v)\}$ called the *maximum degree* at v.

The graph obtained from G by deleting a vertex v and all its adjacent arcs will be denoted by G/v. If a vertex on a path is neither the initial nor the terminal vertex of the path we will call it an *intermediate* vertex of the path. Hereafter "path" will mean *simple* path.

The following inequality is useful for estimating path numbers.

THEOREM 1. *For every digraph* G,

$$pn(G) \geq \sum_{v \in V(G)} x(v) . \tag{1}$$

Proof. Let P be any path decomposition of G and $P(v)$ be the set of paths belonging to P which begin at v. Suppose $x(v) > 0$. The vertex v is an intermediate vertex for at most $\mathrm{id}(v)$ paths of P so at least $\mathrm{od}(v) - \mathrm{id}(v)$ paths begin at v. Therefore $|P(v)| \geq x(v)$ but $|P| = \sum_{v \in V(G)} |P(v)|$ and hence a minimal path decomposition has at least

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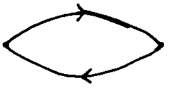
$\sum_{v \in V(G)} x(v)$ members.

EXAMPLE 1. The digon below is an example of a digraph in which strict inequality holds in (1):



Another proposition useful in estimating the path number of a digraph is:

THEOREM 2. *If v is any vertex of an arbitrary digraph G then* $pn(G) \leq pn(G/v) + \mu(v)$.

Proof. Suppose there are $t \geq 0$ digons adjacent to v . If $t \neq 1$ then the digons' arcs can be partitioned into t paths of length 2 . The remaining arcs adjacent at v (if any) can be partitioned into $\max\{\mathrm{od}(v)-t, \mathrm{id}(v)-t\}$ paths and hence

$$\begin{aligned} pn(G) &\leq pn(G/v) + t + \max\{\mathrm{od}(v)-t, \mathrm{id}(v)-t\} \\ &\leq pn(G/v) + \mu(v) . \end{aligned}$$

If $t = 1$ and $\mathrm{od}(v)$ or $\mathrm{id}(v)$ isn't 1 it is easy to verify that $\mu(v)$ paths suffice to form a decomposition of the arcs adjacent at v . If $\mathrm{od}(v) = \mathrm{id}(v) = t = 1$ let w be the vertex of G/v adjacent to v and $\mathcal{P}$ be a minimal path decomposition of G/v . If w is an initial (or terminal) vertex of some path of $\mathcal{P}$ then we can augment that path by vw (or wv) to obtain a path decomposition for G having $pn(G/v) + 1$ members. If w is an intermediate vertex of some path $\mathfrak{p}$ of $\mathcal{P}$ augment the part of $\mathfrak{p}$ from its beginning to w by wv (call the resulting path $\mathfrak{r}$), augment the part of $\mathfrak{p}$ from w to the end of $\mathfrak{p}$ by vw (call the resulting path $\mathfrak{s}$), then $(\mathcal{P} \sim \{\mathfrak{p}\}) \cup \{\mathfrak{r}, \mathfrak{s}\}$ is a path decomposition of G having $pn(G/v) + 1$ members.

Asymmetric digraphs

A digraph G is *asymmetric* iff wv is not an arc of G whenever vw is an arc of G for every arc vw of G . Alternatively, G is asymmetric iff there exists a tournament T such that G is a subgraph of T .

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THEOREM 3. *If G is an asymmetric digraph with n vertices then* $pn(G) \le [\![n^2/4]\!]$.

Proof (by induction on n). It is easy to check the proposition is true for $1 \le n \le 3$.

Suppose $n = 2m > 2$. If $d(\dot{v}) \le 1$ for some $v \in V(G)$ then $\mu(v) \le m$ because G is asymmetric. Assuming inductively that

$$pn(G/v) \le [\![(n-1)^2/4]\!]$$

we have $pn(G) \le m^2$ by Theorem 2. If on the other hand, $d(w) \ge 2$ for all $w \in V(G)$ we may assume that $x(w) > 0$ for at least m vertices w of G . Let v be one of these vertices. If $\text{od}(v) \le m$ then $\mu(v) \le m$ and we have $pn(G) \le m^2$ by Theorem 2 as before. Therefore we may assume that $\text{od}(v) > m$. Notice that the vertices of G other than v having positive excess as vertices of G have positive excess as vertices of G/v because their net degrees (relative to G) exceed 1 . If P is a minimal path decomposition of G/v then these vertices of positive excess are initial vertices of paths of P . Let $k = \text{od}(v) - m$ and $W = \{w \in V(G) : vw \in A(G) \text{ and } x(w) > 0\}$. W has at least k members. Let p_w denote a path of P beginning at w and $\hat{\text{p}}_w$ be the path obtained by augmenting p_w by vw . The arcs of G which are adjacent at v but do not end in W can be decomposed into $\text{id}(v)$ paths of length 2 and at most $m - \text{id}(v)$ paths of length 1 . Calling this path decomposition P^* we see that

$$\left(P \sim \{\text{p}_w : w \in W\}\right) \cup \{\hat{\text{p}}_w : w \in W\} \cup P^*$$

is a path decomposition of G of cardinality not exceeding

$$\left(pn(G/v) - |W|\right) + |W| + m$$

and hence

$$\begin{aligned} pn(G) &\le pn(G/v) + m \\ &\le m^2 \quad \text{by our inductive assumption.} \end{aligned}$$

A similar argument shows that $pn(G) \le m^2 + m$ when $n = 2m + 1 > 1$ and hence

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$$pn(G) \leq [\![n^2/4]\!] .$$

COROLLARY 3.1. *For every tournament* T ,

$$pn(T) \leq [\![n^2/4]\!] .$$

Later in Corollary 4.2 we will show that the path number of the transitive tournament on n vertices is $[\![n^2/4]\!]$ so the bound given by Corollary 3.1 is best possible with respect to the class of tournaments as well as the class of asymmetric digraphs. In Example 3 below we present an arc-minimal family of digraphs realizing this upper bound for each n .

Acyclic digraphs

LEMMA. *If* $\mathcal{P}$ *is a path decomposition of an acyclic digraph* G *and there exist distinct paths* p *and* r *of* $\mathcal{P}$ *such that* p *ends at the vertex where* r *begins then* $\mathcal{P}$ *is not minimal.*

Proof. Augment p by r to obtain a path decomposition smaller than $\mathcal{P}$.

THEOREM 4. *If* G *is an acyclic digraph then* $pn(G) = \sum_{v\in V(G)} x(v)$.

Proof. Suppose $\mathcal{P}$ is a minimal path decomposition of G . If $x(v) = 0$ then $\mathrm{id}(v) \geq \mathrm{od}(v)$ and hence if some path of $\mathcal{P}$ began at v then some other path ends at v . This would imply by the Lemma that $\mathcal{P}$ is not minimal. Therefore $\mathcal{P}(v) = \emptyset$ whenever $x(v) = 0$. Consequently $\{\mathcal{P}(v) : x(v) > 0\}$ partitions $\mathcal{P}$ and we have

$$pn(G) = \sum_{v\in V(G)} |\mathcal{P}(v)| . \tag{2}$$

Now suppose $x(v) > 0$. No path of $\mathcal{P}$ entering v ends at v otherwise as $\mathrm{od}(v) > \mathrm{id}(v)$, some other path of $\mathcal{P}$ would begin at v which is impossible by the Lemma. Therefore v is an intermediate vertex for $\mathrm{id}(v)$ paths and hence $\mathrm{od}(v) - \mathrm{id}(v)$ paths begin at v . Therefore $|\mathcal{P}(v)| = x(v)$ for all $v \in V(G)$. Applying (2) we have

$$pn(G) = \sum_{v\in V(G)} x(v) . \tag{3}$$

EXAMPLE 2. That the converse to Theorem 4 is false is shown by the

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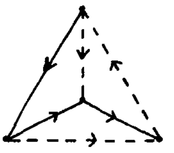
digraph below:



COROLLARY 4.1. *If G is an acyclic digraph on n vertices, precisely k of which have positive excess then* $pn(G) \le k(n-k)$.

Proof. Let P denote the vertices of G of positive excess. If an arc of G joins two vertices of P then it may be removed from G without altering $\sum_{v \in V(G)} x(v)$ which is $pn(G)$ by Theorem 4. Therefore we may assume, without loss of generality, that any arc of G beginning at a vertex of P must end in a vertex which is not in P . Thus $x(v) \le n - k$ for all $v \in P$ and hence $pn(G) \le k(n-k)$.

This gives us an alternate proof of

COROLLARY 4.2. *If G is an acyclic digraph on n vertices then* $pn(G) \le [\![n^2/4]\!]$.

COROLLARY 4.3. $pn(T) = [\![n^2/4]\!]$ *if T is a transitive tournament.*

EXAMPLE 3. Define a bipartite digraph B_k on n vertices as follows: let $V(B_k) = \{1, 2, \ldots, n\}$ and let $vw \in A(B_k)$ iff $1 \le v \le k$ and $k < w \le n$. It follows that $x(v) = n - k$ if $1 \le v \le k$ and $x(v) = 0$ if $k < v \le n$. Theorem 4 implies that $pn(B_k) = k(n-k)$.

If we take $q := [\![(n+1)/2]\!]$ then B_q is an acyclic digraph whose path number is $[\![n^2/4]\!]$. This shows that the inequalities of Corollary 4.2 and Theorem 3 are best possible. No digraph G can have fewer than $pn(G)$ arcs. Therefore B_q is an arc-minimal digraph with path number $[\![n^2/4]\!]$. In fact, for each $1 \le k \le [\![n^2/4]\!]$ we can exhibit an arc-minimal connected digraph on n vertices having path number k , because if $k \ge n - 1$ a suitable bipartite digraph with k arcs can always be chosen, and if $1 \le k < n - 1$ one can orient the arcs on a hamiltonian path suitably to obtain the required path number.

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