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# PATH-INTEGRAL QUANTIZATION OF SPINNING PARTICLES INTERACTING WITH CROSSED EXTERNAL ELECTROMAGNETIC FIELDS

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## ABSTRACT

Describing the spin degree of freedom in terms of Grassmann variables, we study the quantization problem of non-relativistic and relativistic spinning particles interacting with crossed electric and magnetic fields by using the path-integral method. The problem presents some difficulties due to the odd-dimensional nature of the phase-space for the spin variables. We show how to deal with these difficulties by bringing back all the problems to the study of the quantum mechanics of a single Grassmann variable. Then it is straightforward to perform the path-integrals in all the cases considered here, simply by solving the classical equations of motion.

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## 1. INTRODUCTION

In recent years there has been a lot of interest in theories involving anticommuting variables. This interest was mainly raised by the supersymmetries[1], but very soon it was realized that the anticommuting variables play an important role in many other fields[2)-5)].

In particular it has been shown that anticommuting variables are suitable tools in order to get a "classical" description of spin[2),6)] and internal degrees of freedom of elementary particles, so as to reproduce the correct quantum spectrum after quantization[7)].

Having obtained a "classical" description of many interesting physical systems in terms of Grassmann variables, it is very natural to investigate the possibility of quantization of such systems by a path-integral performed on these variables. This can be done as a simple application of the techniques developed for the path-integral quantization of Fermi fields[8)].

This program has already been developed in some cases; for instance it has been shown that the Wilson loop can be reconstructed as a path-integral on Grassmann variables describing the colour degree of freedom[5),9)].

One can ask about the intrinsic interest of this approach apart from the pedagogical one. The interest is at least two-fold. First of all the development of new techniques to solve old problems can be very useful. For instance, one can hope to obtain new approximation methods to apply to new problems. Concerning this point, we would like to emphasize the use of these methods in statistical mechanics. In fact, there exists an impressive number of old problems in statistical mechanics (for example, 2-d Ising model correlation functions) which can be solved in an extremely fast way by using path-integrals over Grassmann variables. Furthermore, many new approximation methods have been developed in order to obtain approximate solutions to unsolvable models (d - dimensional Ising model, dimer problem, etc).[10)]

Secondly, one can apply these techniques to the realm of particle physics. In fact, it has been proposed that a convenient starting point for a quantum field theory is the underlying picture in terms of classical objects (relativistic scalar particle, string and so on)[11)].

Our paper is a preparation to this last kind of applications. In fact, we consider a certain number of physical systems like non-relativistic and relativistic spinning particles in external electromagnetic fields and we show how to evaluate the quantum propagator. In every case, the result is obtained in a very simple, quick and straightforward way (compared with the usual one). The method consists essentially in solving the classical equations of motion both for the position variables and the Grassmann variables describing the spin. The non-relativistic and the massive relativistic spinning particles are described by an odd number of Grassmann variables; this fact creates troubles in a naive application of the path-integral method[12].

After having recalled in Section 2 some general features of the Lagrangians involving Grassmann variables, in Section 3 we show how to extend the path integral techniques to systems which are described by an odd number of anticommuting variables. This is realized by noticing that such a case returns again to a system with an even dimensional phase space, plus a coupled one-dimensional system, and by studying in general this last case.

In Section 4 we study a non-relativistic spinning particle interacting with crossed electric and magnetic fields. In Section 5 we study the free relativistic spinning particle and we show that the problem can be factorized in two parts: the first one which describes a massless spinning particle (having an even dimensional phase space) and the second one, which is the one-dimensional problem studied in Section 3[12]. We evaluate the Feynman propagator for the massless case [in a different setting, this result has also been obtained in Ref. 13)] by integrating the classical equations of motion.

In Section 6 we obtain the Feynman propagator for the massive case by using the results of Sections 3 and 5[12]. This problem has been studied also by Ogielski and Sobczyk[14], but these authors evaluate rather the symbol of the evolution operator than the actual Feynman propagator.

In Section 7 we consider the propagation of a massive spinning particle in the presence of a crossed electromagnetic external field and we recover the Schwinger result. Also this last calculation is performed very simply by integrating the classical equations of motion.

In Appendix A we review the method for obtaining the physical propagator of a quantum system whose physical states obey a certain number of constraint equations. Finally in Appendix B we give an explicit representation for the transition functions between Dirac spinors and Fermi coherent states.

## 2. GENERAL FORMALISM

The typical Lagrangian for a pseudoclassical system is[2),3)]

$$L = \frac{i}{2}\sum_{a=1}^{N}\xi_a\dot{\xi}_a + \frac{1}{2}\sum_{i=1}^{M} m_i\dot{q}_i^2 + V(q_i,\xi_a). \qquad (2.1)$$

This Lagrangian gives rise to N second-class constraints[3),15)]

$$\chi_a = \pi_a + \frac{i}{2}\xi_a \approx 0. \tag{2.2}$$

Due to the constraints (2.2), one must use the following Dirac brackets

$$\{\xi_a, \xi_b\}^* = -i\,\delta_{ab}. \tag{2.3}$$

For the Lagrangian (2.1) there can be given two cases: let $N = 2n$ be an even number. In this case we can introduce the following complex combinations

$$\begin{aligned} \theta_\alpha &= \frac{1}{\sqrt{2}}\left(\xi_{2\alpha-1} + i\,\xi_{2\alpha}\right), \\ \theta^*_\alpha &= \frac{1}{\sqrt{2}}\left(\xi_{2\alpha-1} - i\,\xi_{2\alpha}\right), \\ \alpha &= 1, 2, \dots, n \end{aligned} \tag{2.4}$$

with Dirac brackets

$$\begin{aligned} \{\theta_\alpha, \theta^*_\beta\}^* &= -i\,\delta_{\alpha\beta}, \\ \{\theta_\alpha, \theta_\beta\}^* &= \{\theta^*_\alpha, \theta^*_\beta\}^* = 0. \end{aligned} \tag{2.5}$$

After quantization we get the following algebra for the operators corresponding to $\theta_\alpha$, $\theta^*_\alpha$ :

$$\begin{aligned} [\underset{\sim}{\theta}_\alpha, \underset{\sim}{\theta}^\dagger_\beta]_+ &= \delta_{\alpha\beta}, \\ [\underset{\sim}{\theta}_\alpha, \underset{\sim}{\theta}_\beta]_+ &= [\underset{\sim}{\theta}^\dagger_\alpha, \underset{\sim}{\theta}^\dagger_\beta]_+ = 0 \end{aligned} \tag{2.6}$$

That is, we get n Fermi oscillators. The coherent states relative to these oscillators have eigenvalues belonging to the Grassmann algebra spanned by $\theta_\alpha$, $\theta^*_\alpha$.

Now let $N = 2n + 1$ be an odd number. In this case, we find it convenient to isolate the variable $\xi_{2n+1}$ and rewrite (2.1) in the form

$$L=\frac{i}{2}\sum_{a=1}^{2n}\xi_a\dot{\xi}_a+\frac{1}{2}\sum_{i=1}^{M}m_i\dot{q}_i^2+V_0(q_i,\xi_a)+$$
$$+\frac{i}{2}\xi_{2n+1}\dot{\xi}_{2n+1}+iV_1(q_i,\xi_a)\xi_{2n+1}\,, \tag{2.7}$$

where $V_0$ and $V_1$ do not depend on $\xi_{2n+1}$. We see that this case goes back again to the even-dimensional case plus a problem depending on a single Grassmann variable.

## 3. QUANTUM MECHANICS OF A SINGLE GRASSMANN VARIABLE

In this section we will study the quantum problem associated to the pseudomechanics of a single Grassmann variable. The most general Lagrangian for this case is

$$L=\frac{i}{2}\xi\dot{\xi}+i\eta\xi\,, \tag{3.1}$$

where $\eta$ is an odd external source. The Dirac bracket for $\xi$ is

$$\{\xi,\xi\}^*=-i\,. \tag{3.2}$$

After quantization we get

$$\underset{\sim}{\xi}^2=\frac{1}{2}\,, \tag{3.3}$$

and we see that the algebra generated by $\underset{\sim}{\xi}$ has nothing to do with the Grassmann algebra generated by the pseudoclassical variable $\xi$.

This must be contrasted with the situation for $2n$ pseudoclassical variables, in which the algebra of the corresponding quantum operators is the algebra of $n$ Fermi oscillators. This algebra can be split into two subalgebras of creation and annihilation operators. These subalgebras are isomorphic to the original Grassmann algebras of the complex combinations of pseudoclassical variables. The same situation arises in ordinary theories, in which the quantum algebra of $\underset{\sim}{x}$ and $\underset{\sim}{p}$ can be split into two subalgebras isomorphic to the algebras generated by the classical variables $x$ and $p$.

We circumvent the problem represented by Eq. (3.3) by keeping the conjugate momentum throughout the calculation, and by requiring that the physical states satisfy the constraints (2.2).

The canonical anticommutators corresponding to the canonical Poisson brackets are

$$[\underset{\sim}{\pi}, \underset{\sim}{\xi}]_+ = -i \,,$$
$$[\underset{\sim}{\xi}, \underset{\sim}{\xi}]_+ = [\underset{\sim}{\pi}, \underset{\sim}{\pi}]_+ = 0 \,. \tag{3.4}$$

The algebra (3.4) can be realized on a space of functions on the Grassmann algebra $\mathcal{G}_1$ generated by $\xi$. We get

$$\underset{\sim}{\xi}\,\psi(\xi) = \xi\,\psi(\xi) \,,$$
$$\underset{\sim}{\pi}\,\psi(\xi) = -i \frac{\partial}{\partial \xi}\,\psi(\xi) \,. \tag{3.5}$$

A scalar product in this space is defined by

$$(\phi, \psi) = \int \phi^*(\xi)\,\psi(\xi)\,d\xi \,, \tag{3.6}$$

where the integration is the standard one on anticommuting variables[8].

This scalar product is not positive definite. However, it will turn out to be such on the physical states satisfying the constraint $\chi = \pi + (i/2)\xi \approx 0$. Anyway, had we started with a positive definite scalar product, we would have found it impossible to get a solution of the constraint other than zero. The space of functions on $\mathcal{G}_1$ is isomorphic to the space of the two-dimensional complex vectors:

$$\psi(\xi) = \psi_1 + \xi\,\psi_2 \equiv \langle \xi | \psi \rangle \,,$$
$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \;; \quad \langle \xi | = [1, \xi] \,. \tag{3.7}$$

We do not solve the constraint $\chi = \pi + (i/2)\xi \approx 0$, therefore we have to use the extended Hamiltonian

$$H_E = \lambda\chi + i\xi\eta, \tag{3.8}$$

where $\lambda$ is fixed by requiring $\dot{\chi} = 0$ [15]. We find

$$H_E = \eta\left(\pi - \frac{i}{2}\xi\right). \tag{3.9}$$

The Schrödinger equation corresponding to this Hamiltonian is

$$\begin{gathered} i\frac{\partial}{\partial t}\psi(\xi,t) = H_E\psi(\xi,t), \\ H_E = -i\eta\left(\frac{\partial}{\partial\xi} + \frac{1}{2}\xi\right). \end{gathered} \tag{3.10}$$

The solution of (3.10) when $\dot{\eta} = 0$ is

$$\begin{aligned} \psi(\xi,t) &= (\psi_1 - \eta t\psi_2) + \xi\left(\psi_2 + \frac{1}{2}\eta t\psi_1\right) = \\ &= \langle\xi|\psi,t\rangle \equiv \langle\xi,t|\psi\rangle \end{aligned} \tag{3.11}$$

where

$$\langle\xi,t| = \left[1 + \frac{1}{2}\xi\eta t,\ \xi - \eta t\right].$$

The following completeness relations hold

$$\begin{aligned} &\int |\xi,t\rangle\langle\xi,t|\, C\, d\xi = 1, \\ &\int |\xi,t\rangle^{\#}\, {}^{\#}\langle\xi,t|\, C\, d\xi = -1, \end{aligned} \tag{3.12}$$

with $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and where the symbol $\#$ means that we have to change the sign of all the odd variables inside the state vector. It follows that the scalar product (3.6) can be rewritten as

$$(\psi,\phi) = \langle\psi|C|\phi\rangle^{\#}. \tag{3.13}$$

Therefore the propagator can be represented as follows

$$(\psi(t''),\phi(t')) = \langle \psi,t''|C|\phi,t'\rangle^{\#} =$$

$$= \int (\psi(\xi''))^{*} K(\xi'',t'';\xi',t')\,(\phi(\xi'))^{\#}\, d\xi'' d\xi', \tag{3.14}$$

where

$$(\phi(\xi'))^{\#} = {}^{\#}\langle \xi'|\phi\rangle^{\#}, \tag{3.15}$$

and

$$K(\xi'',t'';\xi',t') = \langle \xi'',t''|C|\xi',t'\rangle^{\#} =$$

$$= \xi'' - \xi' - \eta(t''-t')\, e^{-\frac{1}{2}\xi''\xi'} . \tag{3.16}$$

The kernel (3.16), or rather its Fourier transform with respect to $\xi''$, can be represented as a Feynman path-integral in phase space.

Define the Grassmann Fourier transform of the function $\phi(\xi)$ as

$$\phi(\pi) = \int \phi(\xi)\, e^{-i\xi\pi}\, d\xi, \tag{3.17}$$

the inverse transform being

$$\phi(\xi) = \int \phi(\pi)\, e^{i\xi\pi}\, \frac{d\pi}{i}, \tag{3.18}$$

then

$$K(\xi_f,t_f;\xi_i,t_i) = \int e^{i\xi_f\pi_f} \Big\{ \iint e^{\frac{i}{2}\pi_f\xi(t_f)} \cdot e^{i\int_{t_i}^{t_f}\left[\frac{1}{2}\dot{\pi}\xi + \frac{1}{2}\dot{\xi}\pi - \eta\left(\pi - \frac{i}{2}\xi\right)\right]dt}\, e^{\frac{i}{2}\pi(t_i)\xi_i} \cdot \mathcal{D}(\xi,\pi)\Big\}\, \frac{d\pi_f}{i} . \tag{3.19}$$

From the functional integral in (3.19) we see that $\xi(t)$ and $\pi(t)$ must be kept fixed for $t = t_i$ and $t = t_f$ respectively. Therefore the action

$$S = \int_{t_i}^{t_f} \left[ \frac{1}{2}\dot{\pi}\xi + \frac{1}{2}\dot{\xi}\pi - \eta\left(\pi - \frac{i}{2}\xi\right)\right] dt \tag{3.20}$$

must be varied with the previous boundary conditions in order to get the classical equations of motion

$$\dot{\xi} = \eta \quad ; \quad \dot{\pi} = -\frac{i}{2}\eta \tag{3.21}$$

The functional integral in (3.19) can be evaluated by the method of the classical shift

$$\begin{aligned} \xi(t) &= \xi_c(t) + \zeta(t), \\ \pi(t) &= \pi_c(t) + \sigma(t), \end{aligned} \tag{3.22}$$

with

$$\zeta(t_i) = \sigma(t_f) = 0, \tag{3.23}$$

where $\xi_c$ and $\pi_c$ are solutions of (3.21) satisfying the previously described boundary conditions. We get

$$\xi_c(t) = \xi_i + \eta(t - t_i), \tag{3.24}$$

$$\pi_c(t) = \pi_f - \frac{i}{2}\eta(t - t_f). \tag{3.25}$$

Inserting (3.22) into (3.19) we find

$$\int e^{\frac{i}{2}\pi_f\xi(t_f)}\, e^{iS}\, e^{\frac{i}{2}\pi(t_i)\xi_i}\, \mathcal{D}(\xi,\pi) = e^{i\pi_f\xi_i - i\eta\left(\pi_f - \frac{i}{2}\xi_i\right)(t_f - t_i)} \int e^{i\int_{t_i}^{t_f}\frac{1}{2}(\dot{\sigma}\zeta + \dot{\zeta}\sigma)dt}\, \mathcal{D}(\zeta,\sigma) \tag{3.26}$$

The integral over $\mathfrak{z}$ and $\sigma$ is equal to one, as can be seen by explicit evaluation using the time slicing procedure. Now inserting (3.26) into (3.19), we get

$$K(\xi_f, t_f; \xi_i, t_i) = \int e^{i\xi_f \pi_f - i\eta(\pi_f - \frac{i}{2}\xi_i)(t_f - t_i) + i\pi_f \xi_i} \cdot \frac{d\pi_f}{i} = \xi_f - \xi_i - \eta(t_f - t_i)\, e^{-\frac{1}{2}\xi_f \xi_i}. \tag{3.27}$$

Note that (3.27) can be written as a $\delta$ function*), in fact

$$K(\xi_f, t_f; \xi_i, t_i) = \left[\xi_f - \xi_i - \eta(t_f - t_i)\right] e^{-\frac{1}{2}\xi_f \xi_i} =$$

$$= \delta(\xi_f - \xi_c(t_f))\, e^{-\frac{1}{2}\xi_f \xi_i} =$$

$$= \delta(\xi_f - \xi_c(t_f))\, e^{-\frac{1}{2}\xi_c(t_f)\xi_i},$$

which gives a rather intuitive meaning to the propagation kernel. The constraint equation (2.2) in configuration space reads

$$\underset{\sim}{\chi}\, \psi(\xi) = -i\left(\frac{\partial}{\partial \xi} - \frac{1}{2}\xi\right)\psi(\xi) = 0. \tag{3.28}$$

From the relation $\underset{\sim}{\chi}^2 = 1/2$ we see that there is no solution for Eq. (3.28). However, we can find a solution requiring that the physical states are such that the expectation value of $\underset{\sim}{\chi}$ is zero:

$$\int \phi^*(\xi)\, \underset{\sim}{\chi}\, \phi(\xi)\, d\xi = 0. \tag{3.29}$$

Equation (3.29) is satisfied by

$$\phi(\xi) = \alpha\left(1 \pm \frac{1}{\sqrt{2}}\xi\right). \tag{3.30}$$

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*) We recall that $\xi' - \xi \equiv \delta_R(\xi', \xi)$ satisfies $\int f(\xi')\, \delta_R(\xi', \xi)\, d\xi' = f(\xi)$

We see that (3.30) splits the two-dimensional space of the functions $\phi(\xi)$ into two one-dimensional subspaces $C_1$ and $C_2$, which cannot be mixed together owing to the non-linear character of the condition (3.29). Furthermore, $C_1$ and $C_2$ have positive definite and negative definite norms respectively. In the positive definite sector $C_1$ we can demand the normalization condition

$$\int \phi^*(\xi)\phi(\xi)\,d\xi = 1 \tag{3.31}$$

from which we get

$$|\alpha|^2 = 1/\sqrt{2}\,. \tag{3.32}$$

The states (3.30) are energy eigenstates. In fact, the stationary Schrödinger equation corresponding to (3.10) is

$$-i\eta\left(\frac{\partial}{\partial\xi} + \frac{1}{2}\xi\right)\phi_n(\xi) = E_n\phi_n(\xi), \tag{3.33}$$

with solutions

$$\begin{gathered}\phi_n(\xi) = N_n\,(1 + i\epsilon_n\xi)\\ \epsilon_n^2 + \frac{1}{2} = 0\end{gathered} \tag{3.34}$$

which are nothing but (3.30). The energy eigenstates (3.34) satisfy the following orthogonality and completeness relations

$$\int \phi_m^*(\xi)\phi_n(\xi)\,d\xi = \rho_n\delta_{nm} \tag{3.35}$$

$$\sum_{n=1,2} \frac{\phi_n(\xi)\phi_n^*(\xi')}{\rho_n} = \xi + \xi' \equiv \delta_L(\xi,\xi') \tag{3.36}$$

where

$$\rho_n = 2i\epsilon_n|N_n|^2 \tag{3.38}$$

and $\delta_L(\xi,\xi')$ is such that

$$\int \delta_L(\xi,\xi')f(\xi')\,d\xi' = f(\xi), \tag{3.39}$$

for functions $f(\xi) = f_1 + \xi f_2$ with $f_1$ and $f_2$ even.

## 4. NON-RELATIVISTIC ELECTRON IN CROSSED ELECTRIC AND MAGNETIC FIELDS

As a first application of the techniques developed in the previous section, we are going to derive the propagator of a non-relativistic charged spin 1/2 particle in the presence of crossed external electric and magnetic fields[16].

The Lagrangian is[2),3)]

$$L = \frac{m}{2}\dot{\vec{x}}^2 + \frac{i}{2}\vec{\xi}\cdot\dot{\vec{\xi}} + \frac{e}{c}\dot{\vec{x}}\cdot\vec{A} - \frac{ie}{2mc}\epsilon_{ijk}\xi_i\xi_j H_k - e\phi \tag{4.1}$$

where

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi$$

$$H_k = \epsilon_{kij}\partial_i A_j$$

If we choose

$$\vec{A} = H\left(-\frac{1}{2}y, \frac{x}{2}, 0\right)$$

$$\phi = -\vec{E}\cdot\vec{x} = -Ey \tag{4.2}$$

so that

$$\vec{H} = (0,0,H) \,, \quad \vec{E} = (0,E,0), \tag{4.3}$$

with H and E constant in time we get

$$L = \frac{m}{2}\dot{\vec{x}}^2 + \frac{i}{2}\vec{\xi}\cdot\dot{\vec{\xi}} + \frac{eH}{2c}(x\dot{y} - \dot{x}y) - \frac{ie}{mc}\xi_1\xi_2 H + eEy \tag{4.4}$$

We see that L is the sum of two terms: $L = L_B + L_G$ where

$$L_B = \frac{m}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{eH}{2c}\left(x\dot{y}-\dot{x}y\right)+eEy \tag{4.5}$$

and

$$L_G = \frac{i}{2}\left(\xi_1\dot{\xi}_1+\xi_2\dot{\xi}_2\right)-\frac{ie}{mc}\xi_1\xi_2 H+\frac{i}{2}\xi_3\dot{\xi}_3 . \tag{4.6}$$

The Grassmann variables are decoupled from Bose variables, thus the propagator can be separated into the product of a bosonic propagator times a Grassmann propagator

$$K = K_B \cdot K_G$$

Via functional integration, $K_B$ is readily found to be

$$\begin{aligned} K_B(\vec{x}_f,t_f;\vec{x}_i,t_i) &= \left[\frac{m}{2\pi i(t_f-t_i)}\right]^{3/2}\frac{\mu_0 H(t_f-t_i)}{\sin(\mu_0 H(t_f-t_i))}\cdot \\ &\cdot\exp\left\{i\left[\frac{m}{2}\frac{(z_f-z_i)^2}{t_f-t_i}+\frac{m}{2}\mu_0 H \operatorname{ctg}(\mu_0 H(t_f-t_i))\cdot\right.\right. \\ &\left.\left.\cdot\left((x_f-x_i)^2+(y_f-y_i)^2\right)+m\mu_0 H(x_i y_f-x_f y_i)\right]\right\}\cdot \\ &\cdot\exp\left\{i\left[\frac{eE}{2\mu_0 H}\left(1-\mu_0 H(t_f-t_i)\operatorname{ctg}(\mu_0 H(t_f-t_i)\right)\cdot\right.\right. \\ &\left.\left.\cdot\left(x_f-x_i-\frac{eE(t_f-t_i)}{4m\mu_0 H}\right)+\frac{eE(t_f-t_i)}{2}(y_i+y_f)\right]\right\}. \end{aligned} \tag{4.7}$$

where $\mu_0 = e/2mc$ is the Bohr magneton.

As to $K_G$, in the Grassmann Lagrangian $L_G$ we make the following change of variables

$$\xi_1 = \frac{1}{\sqrt{2}}\left(\theta + \theta^*\right), \quad \xi_2 = -\frac{i}{\sqrt{2}}\left(\theta - \theta^*\right), \tag{4.8}$$

so that

$$\begin{aligned} L_G &= L_G^{(1)}(\theta, \theta^*) + L_G^{(2)}(\xi_3) = \\ &= \frac{i}{2}\left(\theta^*\dot{\theta} - \dot{\theta}^*\theta\right) - \frac{eH}{2mc}\left(\theta^*\theta - \theta\theta^*\right) + \\ &\quad + \frac{i}{2}\xi_3\dot{\xi}_3 . \end{aligned} \tag{4.9}$$

We see that, according to the discussion of the previous section, $L_G$ is naturally divided into two pieces, the first one depending on an even number of Grassmann variables and the second one depending on $\xi_3$ which describes a free motion. From $L_G^{(1)}$ we get the Poisson brackets

$$\{\theta, \Pi_\theta\} = -1, \quad \{\theta^*, \Pi_{\theta^*}\} = -1, \tag{4.10}$$

and the second-class constraints

$$\begin{aligned} \chi &= \Pi_\theta + \frac{i}{2}\theta^* \approx 0, \\ \bar{\chi} &= \Pi_{\theta^*} + \frac{i}{2}\theta \approx 0 . \end{aligned} \tag{4.11}$$

We then introduce Dirac brackets

$$\begin{aligned} \{\theta, \theta^*\}^* &= -i , \\ \{\theta, \theta\}^* = \{\theta^*, \theta^*\}^* &= 0 , \end{aligned} \tag{4.12}$$

and set the constraints (4.11) strongly equal to zero. After quantization we obtain

$$\begin{aligned} [\underset{\sim}{\theta}, \underset{\sim}{\theta}^\dagger]_+ &= 1 \\ [\underset{\sim}{\theta}, \underset{\sim}{\theta}]_+ = [\underset{\sim}{\theta}^\dagger, \underset{\sim}{\theta}^\dagger]_+ &= 0 \end{aligned} \tag{4.13}$$

$K_G$ factorizes in two pieces

$$K_G = K_G^{(1)}(\theta_f^*, t_f; \theta_i, t_i) \cdot K_G^{(2)}(\xi_{3f}, t_f; \xi_{3i}, t_i) \tag{4.14}$$

By putting $\eta = 0$ in (3.27) we get

$$K_G^{(2)}(\xi_{3f}, t_f; \xi_{3i}, t_i) = \xi_{3f} - \xi_{3i} = \delta_R(\xi_{3f}, \xi_{3i}) \tag{4.15}$$

which tells us nothing except that the third component of the spin vector is a constant of motion. For the evaluation of $K_G^{(1)}$, one can follow the general treatment described by Faddeev[17]. Here we recall that the kernel for a system of $n$ complex Grassmann variables with a normal ordered quantum Hamiltonian $\underset{\sim}{H}$ is given by

$$\begin{aligned} K(\vec{\theta}_f^*, t_f; \vec{\theta}_i, t_i) = \\ = \lim_{N\to\infty} \int \exp\Big[ & \vec{\theta}_N^* \cdot \vec{\theta}_{N-1} - \vec{\theta}_{N-1}^* \cdot \vec{\theta}_{N-1} + \cdots - \vec{\theta}_1^* \cdot \vec{\theta}_1 + \\ & + \vec{\theta}_1^* \cdot \vec{\theta}_0 - i\epsilon \left( H(\vec{\theta}_N^*, \vec{\theta}_{N-1}) + \cdots + H(\vec{\theta}_1^*, \vec{\theta}_0) \right) \Big] \cdot \\ & \cdot \prod_{k=1}^{N-1} \frac{d^n \vec{\theta}_k^* \, d^n \vec{\theta}_k}{2\pi i} , \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} \vec{\theta}^* \cdot \vec{\theta} &= \sum_{i=1}^{n} \theta^{(i)*} \theta^{(i)} , \\ \epsilon &= \frac{t_f - t_i}{N} , \\ \vec{\theta}_0 = \vec{\theta}_i , & \quad \vec{\theta}_N^* = \vec{\theta}_f^* , \end{aligned} \tag{4.17}$$

and $H$ is obtained by replacing the quantum operators in $\underset{\sim}{H}$ by the corresponding eigenvalues $\vec{\theta}$ and $\vec{\theta}^*$. In the formal limit $N \to \infty$, $\varepsilon \to 0$ we can write (4.16) as

$$K(\vec{\theta}_f^*, t_f; \vec{\theta}_i, t_i) = \int_{\vec{\theta}_i, t_i}^{\vec{\theta}_f^*, t_f} \mathcal{D}(\vec{\theta}^*, \vec{\theta})\, e^{\frac{1}{2}\left[\vec{\theta}_f^* \cdot \vec{\theta}(t_f) + \vec{\theta}^*(t_i)\cdot\vec{\theta}_i\right]} \cdot e^{i\int_{t_i}^{t_f}\left[\frac{i}{2}\left(\vec{\theta}^*(t)\cdot\dot{\vec{\theta}}(t) - \dot{\vec{\theta}}^*(t)\vec{\theta}(t)\right) - H(\vec{\theta}^*, \vec{\theta})\right]dt} . \tag{4.18}$$

The functional integration is over all the function $\vec{\theta}^*(t)$ and $\vec{\theta}(t)$ such that

$$\vec{\theta}^*(t_f) = \vec{\theta}_f^*, \quad \vec{\theta}(t_i) = \vec{\theta}_i . \tag{4.19}$$

Notice that there are no boundary conditions on $\vec{\theta}^*(t_i)$ and $\vec{\theta}(t_f)$. This can be intuitively understood because the classical variables $\vec{\theta}^*$ and $\vec{\theta}$ satisfy first order equations of motion.

The representation (3.19) for the kernel describing the propagation of a single Grassmann variable can be obtained from (4.18) by the replacement $\theta^* \to$ $\to i\pi$ and integrating over $\pi_f$ with a weight given by the matrix element $\langle\xi_f|\pi_f\rangle = e^{i\xi_f\pi_f}$. $K_G^{(1)}$ can be evaluated directly from (4.16), or by means of a classical translation from (4.18)[5], by using

$$\underset{\sim}{H} = \frac{eH}{mc}\left(\underset{\sim}{\theta}^\dagger\underset{\sim}{\theta} - \frac{1}{2}\right) \tag{4.20}$$

we get

$$K_G^{(1)}(\theta_f^*, t_f; \theta_i, t_i) = e^{-i\mu_0 H(t_f - t_i)} + \theta_f^*\theta_i\, e^{i\mu_0 H(t_f - t_i)} \tag{4.21}$$

The result can be translated into the occupation number basis, by using the transition wave functions

$$\langle n|\theta\rangle = \theta^n , \quad n = 0,1 \tag{4.22}$$

where $|\theta\rangle$ is defined by

$$\underset{\sim}{\theta}\,|\theta\rangle = |\theta\rangle\,\theta \tag{4.23}$$

Multiplying (4.21) for $\langle n_f|\theta_f\rangle$ and $\langle\theta_i^*|n_i\rangle$ and integrating over the Grassmann variables we get the following matrix representation

$$K_G^{(1)}(t_f,t_i) = \begin{array}{c} n_f \downarrow \backslash\, n_i \rightarrow \\ \begin{bmatrix} e^{i\mu_0 H(t_f-t_i)} & 0 \\ 0 & e^{-i\mu_0 H(t_f-t_i)} \end{bmatrix} \end{array} \tag{4.24}$$

## 5. MASSLESS DIRAC PARTICLE

The Lagrangian describing a pseudoclassical massive spinning particle is[6)]

$$L = -\frac{i}{2}\xi_\mu\dot{\xi}^\mu - \frac{i}{2}\xi_5\dot{\xi}_5 - m\sqrt{\left(\dot{x}_\mu - \frac{i}{m}\xi_\mu\dot{\xi}_5\right)^2}. \tag{5.1}$$

This Lagrangian is singular and gives rise to the first class constraints

$$\chi = p^2 - m^2 \approx 0 \quad , \quad \chi_D = p\cdot\xi - i m\pi_5 - \frac{m}{2}\xi_5 \approx 0, \tag{5.2}$$

and to the second class constraints

$$\chi_\mu = \pi_\mu - \frac{i}{2}\xi_\mu \approx 0 \,, \tag{5.3}$$

where

$$p_\mu = -\frac{\partial L}{\partial \dot{x}^\mu},$$

$$\pi_\mu = \frac{\partial L}{\partial \dot{\xi}^\mu} \quad ; \quad \pi_5 = \frac{\partial L}{\partial \dot{\xi}_5}. \tag{5.4}$$

The Lagrangian (5.1) is not of the form (2.1). However, the separation into even and odd variables is still possible for the extended Hamiltonian of the system.

The second class constraints (5.3) require the introduction of Dirac brackets. The only non-zero Dirac brackets are

$$\{x_\mu, p_\nu\}^* = - g_{\mu\nu} ,$$

$$\{\xi_\mu, \xi_\nu\}^* = i g_{\mu\nu} , \quad \{\pi_5, \xi_5\}^* = -1 . \tag{5.5}$$

It turns out[6] that a further constraint must be imposed

$$\chi'_5 = \pi_5 + \frac{i}{2}\xi_5 \approx 0. \tag{5.6}$$

The extended Hamiltonian compatible with (5.6) is

$$H = \alpha_1 (p^2 - m^2) + i\alpha_2 (p\cdot\xi - im\pi_5 - \frac{1}{2} m \xi_5) =$$
$$= [\alpha_1 (p^2 - m^2) + i\alpha_2 p\cdot\xi] + m\alpha_2 (\pi_5 - \frac{i}{2}\xi_5) , \tag{5.7}$$

where $\alpha_2$ is an odd variable and $\alpha_1$ must be chosen to be definite negative in order to have a definite positive kinetic part.

As anticipated, we see that the problem is factorized in a part depending on the space-time and $\xi^\mu$ variables (which are in an even number) and in a one-dimensional (Grassmann) part [see Eq. (3.9)]. We notice also that the part of H not depending on $\xi_5$ and $\pi_5$ can be obtained (apart from the trivial constant term $-\alpha_1 m^2$) by putting $m = 0$ in (5.7). Therefore, the kernel will factorize in two pieces, the first one being the kernel for the massless case, and the second one corresponding to the one-dimensional problem in $\xi_5$. Let us start with the study of the massless case, for which the relevant Hamiltonian is

$$H_0 = \alpha_1 p^2 + i\alpha_2 p\cdot\xi . \tag{5.8}$$

Of course this Hamiltonian gives rise to the first class constraints

$$p^2 \approx 0 , \quad p\cdot\xi \approx 0. \tag{5.9}$$

The canonical quantization of the theory [after taking into account the further constraint (5.6)[6]] gives the relations

$$[\underset{\sim}{\xi}^\mu, \underset{\sim}{\xi}^\nu]_+ = - g^{\mu\nu} , \; [\underset{\sim}{\xi}_5 , \underset{\sim}{\xi}^\mu]_+ = 0 , \quad \underset{\sim}{\xi}_5^2 = \frac{1}{2} , \tag{5.10}$$

and the constraints (5.9) become the Klein-Gordon and Dirac equations. As discussed in Appendix A, it is convenient to choose $\alpha_1$ and $\alpha_2$ independent on $\tau$. The action corresponding to (5.8) is

$$S = \int_{\tau_i}^{\tau_f} \left[ -p\cdot\dot{x} - \frac{i}{2}\xi\cdot\dot{\xi} - \alpha_1 p^2 - i\alpha_2 p\cdot\xi \right] d\tau =$$

$$= \int_{\tau_i}^{\tau_f} \Big[ -p\cdot\dot{x} + \frac{i}{2}\left(\bar{\eta}_\alpha\dot{\eta}_\alpha - \dot{\bar{\eta}}_\alpha\eta_\alpha\right) - \alpha_1 p^2 + \tag{5.11}$$

$$+ i\alpha_2\left(\bar{\eta}_\alpha p_\alpha + \bar{p}_\alpha\eta_\alpha\right)\Big] d\tau \quad , \quad \alpha = 1,2$$

where

$$\eta_1 = \frac{1}{\sqrt{2}}\left(\xi^0 + \xi^3\right) \quad , \quad \bar{\eta}_1 = -\frac{1}{\sqrt{2}}\left(\xi^0 - \xi^3\right),$$

$$\eta_2 = \frac{1}{\sqrt{2}}\left(\xi^1 + i\xi^2\right) \quad , \quad \bar{\eta}_2 = \frac{1}{\sqrt{2}}\left(\xi^1 - i\xi^2\right), \tag{5.12}$$

and similar definitions for $p_\alpha$ and $\bar{p}_\alpha$. Starting from the action (5.11) we can write the kernel as a functional integral

$$K_0(x_f, \bar{\eta}_f, \tau_f; x_i, \eta_i, \tau_i) =$$

$$= \int_{x_i,\eta_i}^{x_f,\bar{\eta}_f} \exp\left\{ \frac{1}{2}\bar{\eta}_{\alpha,f}\eta_\alpha(\tau_f) + iS + \frac{1}{2}\bar{\eta}_\alpha(\tau_i)\eta_{\alpha,i} \right\} \cdot \tag{5.13}$$

$$\cdot\, \mathcal{D}\left(\frac{p(\tau)}{2\pi}\right) \mathcal{D}(x(\tau))\, \mathcal{D}\left(\eta_\alpha(\tau), \bar{\eta}_\alpha(\tau)\right)$$

We shall integrate (5.13) first over Bose and then over Fermi variables. Integrating over $p(\tau)$ we get

$$K_0 = \int_{x_i,\eta_i}^{x_f,\bar\eta_f} \exp\left\{ \frac{1}{2}\bar\eta_{\alpha,f}\eta_\alpha(\tau_f) + i\int_{\tau_i}^{\tau_f}\left[\frac{i}{2}\left(\bar\eta_\alpha\dot\eta_\alpha - \dot{\bar\eta}_\alpha\eta_\alpha\right) + \right.\right.$$

$$\left.\left. + \frac{1}{4\alpha_1}\left(\dot x + i\alpha_2\xi\right)^2\right]d\tau + \frac{1}{2}\bar\eta_\alpha(\tau_i)\eta_{\alpha,i}\right\}\mathcal{D}(x(\tau))\,\cdot \qquad (5.14)$$

$$\cdot\,\mathcal{D}\left(\eta_\alpha(\tau),\bar\eta_\alpha(\tau)\right)$$

Let us perform the following change of variable:

$$y^\mu(\tau) = x^\mu(\tau) + i\alpha_2\int_{\tau_0}^{\tau} d\tau'\,\xi^\mu(\tau')\,. \qquad (5.15)$$

Then the functional integral over $x(\tau)$ becomes

$$\int_{x_i}^{x_f} e^{\,i\int_{\tau_i}^{\tau_f} d\tau\,\frac{1}{4\alpha_1}\left(\dot x + i\alpha_2\xi\right)^2}\,\mathcal{D}(x(\tau)) =$$

$$= \int_{x_i + i\alpha_2\int_{\tau_0}^{\tau_i}d\tau'\xi(\tau')}^{x_f + i\alpha_2\int_{\tau_0}^{\tau_f}d\tau'\xi(\tau')} \mathcal{D}(y(\tau))\; e^{\,i\int_{\tau_i}^{\tau_f} d\tau\left[\frac{1}{4\alpha_1}\dot y^2\right]} = \qquad (5.16)$$

$$= \frac{-i}{16\pi^2\alpha_1^2(\tau_f-\tau_i)^2}\exp\left\{\frac{i}{4\alpha_1}\frac{1}{\tau_f-\tau_i}\left[(x_f-x_i)^2 + \right.\right.$$

$$\left.\left. + 2i\alpha_2(x_f-x_i)_\mu\int_{\tau_i}^{\tau_f} d\tau\,\xi^\mu(\tau)\right\}.$$

Finally we must evaluate the Grassmann functional integral

$$K_0(x_f,\bar\eta_f,\tau_f;x_i,\eta_i,\tau_i) =$$

$$= \frac{-i}{16\pi^2\alpha_1^2(\tau_f-\tau_i)^2}\exp\left[\frac{i}{4\alpha_1}\frac{(x_f-x_i)^2}{\tau_f-\tau_i}\right]. \qquad (5.17)$$

$$\cdot \int_{\eta_i}^{\bar{\eta}_f} \exp\left\{\frac{1}{2}\bar{\eta}_{\alpha,f}\eta_\alpha(\tau_f) + \frac{1}{2}\bar{\eta}_\alpha(\tau_i)\eta_{\alpha,i} + i\int_{\tau_i}^{\tau_f} d\tau\left[\frac{i}{2}\left(\bar{\eta}_\alpha\dot{\eta}_\alpha - \dot{\bar{\eta}}_\alpha\eta_\alpha\right) - \frac{i\alpha_2}{2\alpha_1(\tau_f-\tau_i)}\left(\bar{\eta}_\alpha z_\alpha + \eta_\alpha\bar{z}_\alpha\right)\right\} \mathcal{D}(\eta_\alpha,\bar{\eta}_\alpha), \tag{5.17 cont.}$$

where

$$(x_f - x_i)_\mu \xi^\mu = -\left(\bar{\eta}_\alpha z_\alpha + \eta_\alpha \bar{z}_\alpha\right), \tag{5.18}$$

$$z^\mu = (x_f - x_i)^\mu, \tag{5.19}$$

and $z_\alpha$, $\bar{z}_\alpha$ are defined as $\eta_\alpha$, $\bar{\eta}_\alpha$, see Eq. (5.12). The action in (5.17) is quadratic in $\eta$, $\bar{\eta}$ with the effective Lagrangian

$$L_{eff.} = \frac{i}{2}\left(\bar{\eta}_\alpha\dot{\eta}_\alpha - \dot{\bar{\eta}}_\alpha\eta_\alpha\right) - \frac{i\alpha_2}{2\alpha_1(\tau_f-\tau_i)}\left(\bar{\eta}_\alpha z_\alpha + \eta_\alpha\bar{z}_\alpha\right). \tag{5.20}$$

The equations of motion corresponding to (5.20) are

$$\begin{aligned} \dot{\eta}_\alpha &= -\frac{\alpha_2}{2\alpha_1\Delta\tau} z_\alpha , \\ \dot{\bar{\eta}}_\alpha &= -\frac{\alpha_2}{2\alpha_1\Delta\tau}\bar{z}_\alpha , \qquad \Delta\tau = \tau_f - \tau_i \end{aligned} \tag{5.21}$$

with solutions

$$\begin{aligned} \eta_\alpha(\tau) &= \eta_{\alpha,i} - \frac{\alpha_2}{2\alpha_1\Delta\tau} z_\alpha\,(\tau-\tau_i), \\ \bar{\eta}_\alpha(\tau) &= \bar{\eta}_{\alpha,f} - \frac{\alpha_2}{2\alpha_1\Delta\tau}\bar{z}_\alpha\,(\tau-\tau_f). \end{aligned} \tag{5.22}$$

The classical action is

$$S_c = \int_{\tau_i}^{\tau_f} d\tau\, L_{eff.}(\tau) = -\frac{i\alpha_2}{4\alpha_1}\left(\bar{\eta}_{\alpha,f}\, z_\alpha + \eta_{\alpha,i}\, \bar{z}_\alpha\right). \tag{5.23}$$

Now go back to (5.17) and let

$$\begin{aligned} \eta_\alpha(\tau) &= \eta_{\alpha,c}(\tau) + \zeta_\alpha(\tau), \\ \bar{\eta}_\alpha(\tau) &= \bar{\eta}_{\alpha,c}(\tau) + \bar{\zeta}_\alpha(\tau), \end{aligned} \tag{5.24}$$

where $\eta_{\alpha,c}$ and $\bar{\eta}_{\alpha c}$ are the classical solutions (5.22) and

$$\bar{\zeta}_\alpha(\tau_f) = \zeta_\alpha(\tau_i) = 0. \tag{5.25}$$

Then the functional integral is readily found to be,

$$\begin{aligned} &K_0(x_f, \bar{\eta}_f, \tau_f; x_i, \eta_i, \tau_i) = \\ &= \frac{-i}{16\pi^2\alpha_1^2(\tau_f-\tau_i)^2} \exp\left\{\frac{i}{4\alpha_1}\frac{(x_f-x_i)^2}{\tau_f-\tau_i} + \bar{\eta}_{\alpha,f}\,\eta_{\alpha,i} + \right. \\ &\qquad \left. + \frac{\alpha_2}{2\alpha_1}\left(\bar{\eta}_{\alpha,f}\, z_\alpha + \eta_{\alpha,i}\, \bar{z}_\alpha\right)\right\}. \end{aligned} \tag{5.26}$$

We are interested in the propagator in momentum space. By taking the Fourier transform we get

$$\begin{aligned} &K_0(p', \bar{\eta}_f, \tau_f; p, \eta_i, \tau_i) = \\ &= \delta^4(p'-p)\, e^{-i\alpha_1(\tau_f-\tau_i)p^2 - \alpha_2(\tau_f-\tau_i)(\bar{p}_\alpha\eta_{\alpha,i} + p_\alpha\bar{\eta}_{\alpha,f})} \cdot \\ &\qquad \cdot e^{\bar{\eta}_{\alpha,f}\,\eta_{\alpha,i}}. \end{aligned} \tag{5.27}$$

This expression is nothing but the matrix element of the evolution operator between coherent states of definite momentum. In fact, if we define operators $\bar{\underset{\sim}{\eta}}_\alpha$, $\underset{\sim}{\eta}_\alpha$ related to $\underset{\sim}{\xi}^\mu$ by the relation (5.12), it turns out that they satisfy the following algebra

$$[\bar{\eta}_\alpha, \eta_\beta]_+ = \delta_{\alpha\beta},$$
$$[\eta_\alpha, \eta_\beta]_+ = [\bar{\eta}_\alpha, \bar{\eta}_\beta]_+ = 0. \tag{5.28}$$

Then we define coherent states associated to these oscillators

$$\eta_\alpha |\eta\rangle = |\eta\rangle \eta_\alpha,$$
$$\langle\bar{\eta}| \bar{\eta}_\alpha = \bar{\eta}_\alpha \langle\bar{\eta}|. \tag{5.29}$$

From this definition we get

$$\langle\bar{\eta}'|\eta\rangle = e^{\bar{\eta}'\eta}, \tag{5.30}$$

where the normalization has been chosen according to Ref. 17). Therefore we have

$$K_0(p', \bar{\eta}_f, \tau_f; p, \eta_i, \tau_i) =$$
$$= \langle p', \bar{\eta}_f | e^{-i(\tau_f - \tau_i)[\alpha_1 p^2 + i\alpha_2 p\cdot\xi]} | p, \eta_i\rangle \tag{5.31}$$

As explained in Appendix B, in order to obtain the physical kernel we must put $\beta_1 = \alpha_1(\tau_f - \tau_i)$, $\beta_2 = \alpha_2(\tau_f - \tau_i)$ and integrate the kernel $K_0$ times $\theta(\tau_f - \tau_i)$ over $\beta_1$ and $\beta_2$. Recalling that $\alpha_2$ is an odd variable, and that the Berezin integration over such variables is defined by[8)]

$$\int \eta \, d\eta = 1 \tag{5.32}$$

we get

$$K_{0\,phys.}(p', \bar{\eta}_f; p, \eta_i) = \int_{-\infty}^{0} d\beta_1 \int d\beta_2 \, e^{-i\beta_1(p^2 + i\epsilon)} \cdot$$
$$\cdot e^{-\beta_2(\bar{p}_\alpha \eta_{\alpha,i} + p_\alpha \bar{\eta}_{\alpha,f})} \, e^{\bar{\eta}_{\alpha,f}\eta_{\alpha,i}} \, \delta^4(p'-p) = \tag{5.33}$$

$$= \frac{i}{p^2+i\epsilon}\,\delta^4(p'-p)\,e^{\bar\eta_{\alpha,f}\eta_{\alpha,i}}\left[\bar p_\alpha \eta_{\alpha,i} + p_\alpha \bar\eta_{\alpha,f}\right] =$$

$$= \langle p', \bar\eta_f |\, \frac{-i}{p^2+i\epsilon}\, p\cdot \underset{\sim}{\xi}\, | p, \eta_i \rangle . \tag{5.33 cont.}$$

The propagator can also be defined on a spinor basis (see Appendix B). Of course varying this basis one varies the explicit matrix representation for the operator $\underset{\sim}{\xi}_\mu$. Two of such possible representations are

$$\underset{\sim}{\xi}^\mu = \frac{i}{\sqrt{2}}\gamma^\mu , \qquad \underset{\sim}{\xi}^\mu = \frac{1}{\sqrt{2}}\gamma_5\gamma^\mu , \tag{5.34}$$

and we see that, as expected, the physical kernel is nothing but the propagator for a Dirac massless particle.

## 6. MASSIVE DIRAC PARTICLE

In this section we will evaluate the propagator for the massive case. From the expression (5.7) for the Hamiltonian, we see that before the projection on the physical states, the propagator is given by (5.27) times the phase $e^{-i\alpha_1 m^2(\tau_f-\tau_i)}$ times the propagator associated to $\xi_5$. This last one has been evaluated in Section 3, Eq. (3.27). Collecting everything together we get

$$\begin{aligned} K(p',\bar\eta_f,\xi_{5f},\tau_f;\,p,\eta_i,\xi_{5i},\tau_i) &= \\ = \delta^4(p'-p)\, e^{-i\alpha_1(\tau_f-\tau_i)(p^2-m^2)} &\cdot \\ \cdot\, e^{-\alpha_2(\tau_f-\tau_i)(\bar p_\alpha\eta_{\alpha,i}+p_\alpha\bar\eta_{\alpha,f})}\, e^{\bar\eta_{\alpha,f}\eta_{\alpha,i}} &\cdot \\ \cdot \left[\xi_{5f}-\xi_{5i}-m\alpha_2(\tau_f-\tau_i)\,e^{-\frac{1}{2}\xi_{5f}\xi_{5i}}\right]. \end{aligned} \tag{6.1}$$

Now we proceed as in the previous section, integrating (6.1) over $\beta_1 = \alpha_1(\tau_f-\tau_i)$ and $\beta_2 = \alpha_2(\tau_f-\tau_i)$. We get

$$\begin{aligned} \tilde K(p',\bar\eta_f,\xi_{5f};\,p,\eta_i,\xi_{5i}) &= \frac{i}{p^2-m^2+i\epsilon}\,\delta^4(p'-p)\,\cdot \\ \cdot\, e^{\bar\eta_{\alpha,f}\eta_{\alpha,i}} &\left[(\bar p_\alpha\eta_{\alpha,i}+p_\alpha\bar\eta_{\alpha,f})(\xi_{5f}-\xi_{5i}) + m\, e^{-\frac{1}{2}\xi_{5f}\xi_{5i}}\right]. \end{aligned} \tag{6.2}$$

From this expression we can obtain the propagator in a spinor basis by multiplying (6.2) for the transition wave functions and integrating over $\eta_i$ and $\eta_f$ with the appropriate measure (see Appendix B)

$$\tilde{K}(p',p) = \int \psi_f^*(\xi_{5f})\,\bar{\psi}_f(\eta_f)\,K(p',\bar{\eta}_f,\xi_{5f};\,p,\eta_i,\xi_{5i})\cdot$$
$$\cdot\,[\psi_i(\bar{\eta}_i)]^{\#}\,[\psi_i(\xi_{5i})]^{\#}\,d\mu(\eta_f)\,d\mu(\eta_i)\,d\xi_{5f}\,d\xi_{5i} =$$

$$= \frac{i}{p^2-m^2+i\epsilon}\,\delta^4(p'-p)\,\langle\psi_f|\,\gamma_0\Big[(-p\cdot\underset{\sim}{\xi})\otimes\begin{bmatrix}0&1\\1&0\end{bmatrix} - \tag{6.3}$$
$$- m\gamma_5\otimes\begin{bmatrix}\frac{1}{2}&0\\0&1\end{bmatrix}\Big]|\psi_i\rangle ,$$

where $|\psi\rangle$ is a tensor product of a four-component Dirac spinor times a two-dimensional spinor. The appearance of $\gamma_5$ in (6.3) is due to the fact that in our basis (see Appendix B)

$$\gamma_5|\eta\rangle = -|-\eta\rangle . \tag{6.4}$$

The expression (6.3) is not yet the physical propagator. In fact as discussed in Section 3, we have still to use the second-class constraint (5.6). Therefore the physical states are characterized by the condition

$$\int \bar{\psi}(\eta_\alpha,\xi_5)\,\chi_5'\,\psi(\bar{\eta}_\alpha,\xi_5)\,d\xi_5 = 0 \tag{6.5}$$

which in the positive norm sector implies [see Section 3, Eqs (3.30) and (3.31)]

$$|\psi\rangle = |u\rangle\otimes\frac{1}{\sqrt[4]{2}}\begin{bmatrix}1\\1/\sqrt{2}\end{bmatrix} , \tag{6.6}$$

where $|u\rangle$ is an arbitrary four-component spinor. Finally by using Eq. (6.6) we get the physical propagator

$$K_{phys}(p',p) = \frac{-i}{p^2 - m^2 + i\epsilon}\, \delta^4(p'-p) \langle u_f | \left( p\cdot \underset{\sim}{\xi} + \frac{m}{\sqrt{2}} \gamma_5 \right) | u_i \rangle . \tag{6.7}$$

As for the massless case, one can choose a spinor basis such that

$$\underset{\sim}{\xi}^\mu = \frac{i}{\sqrt{2}} \gamma^\mu . \tag{6.8}$$

Furthermore, one can perform a Pauli-Gürsey[18)] transformation on the physical spinors in such a way as to wash out the $\gamma_5$ dependence on the mass term. The transformation is

$$\begin{aligned} |u_i\rangle &\to e^{i\frac{\pi}{4}\gamma_5} |u_i\rangle \\ \langle u_f | \gamma_0 &\to \langle u_f | \gamma_0 \, e^{i\frac{\pi}{4}\gamma_5} \end{aligned} \tag{6.9}$$

and we get

$$\begin{aligned} K_{phys}(p',p) = \frac{1}{\sqrt{2}} \frac{1}{p^2 - m^2 + i\epsilon} \delta^4(p'-p) \,\cdot \\ \cdot \; \langle u_f | \gamma_0 \, (\hat{p} + m) | u_i \rangle \end{aligned} \tag{6.10}$$

which is the massive Dirac propagator, except for the factor $1/\sqrt{2}$ which can be reabsorbed into the definition of the states.

## 7. RELATIVISTIC ELECTRON IN A CROSSED ELECTROMAGNETIC FIELD

In this last section we compute the Green's function for a Dirac particle interacting with an external electromagnetic crossed field[19)]. The corresponding pseudoclassical Lagrangian is[6)]

$$\begin{aligned} L = &-\frac{i}{2} \xi_\mu \dot{\xi}^\mu - \frac{i}{2} \xi_5 \dot{\xi}_5 - \\ &- \sqrt{m^2 - ie F_{\mu\nu} \xi^\mu \xi^\nu} \sqrt{\left(\dot{x}_\mu - \frac{i}{m} \xi_\mu \dot{\xi}_5\right)^2} - e \dot{x}_\mu A^\mu . \end{aligned} \tag{7.1}$$

The Lagrangian (7.1) gives rise to the first-class

$$\chi = (p-eA)^2 - m^2 + ie F_{\mu\nu}\xi^\mu\xi^\nu \approx 0,$$
$$\chi_D = (p-eA)\cdot\xi - im\pi_5 - \frac{m}{2}\xi_5 \approx 0, \tag{7.2}$$

and to the second-class constraints

$$\chi_\mu = \pi_\mu - \frac{i}{2}\xi_\mu \approx 0. \tag{7.3}$$

As in Section 6, we get the extended Hamiltonian

$$H_E = \alpha_1\left[(p-eA)^2 - m^2 + ie F_{\mu\nu}\xi^\mu\xi^\nu\right] +$$
$$+ i\alpha_2 (p-eA)\cdot\xi + \alpha_2 m \left(\pi_5 - \frac{i}{2}\xi_5\right), \tag{7.4}$$

with $\alpha_2$ an odd variable. It is convenient to choose $\alpha_1 = -1/2m$ (but, of course, the result is independent from this choice). Then, the part of the extended Lagrangian containing Bose and $\xi^\mu$ variables is

$$L_E = -p\cdot\dot{x} - \frac{i}{2}\xi\cdot\dot{\xi} + \frac{1}{2m}\left[(p-eA)^2 - m^2 +\right.$$
$$\left. + ie F_{\mu\nu}\xi^\mu\xi^\nu\right] - i\alpha_2 (p-eA)\cdot\xi. \tag{7.5}$$

The kernel corresponding to (7.5) can be expressed as a functional integral as follows

$$K(x_f, \bar{\eta}_f, \tau_f; x_i, \eta_i, \tau_i) =$$
$$= \int_{x_i,\eta_i}^{x_f,\bar{\eta}_f} \mathcal{D}(x(\tau))\,\mathcal{D}\left(\frac{p(\tau)}{2\pi}\right)\mathcal{D}(\eta(\tau),\bar{\eta}(\tau)) \cdot$$
$$\cdot e^{\frac{1}{2}\bar{\eta}_{\alpha,f}\eta_\alpha(\tau_f) + \frac{1}{2}\bar{\eta}_\alpha(\tau_i)\eta_{\alpha,i}} \cdot e^{i\int_{\tau_i}^{\tau_f} d\tau\, L_E}, \tag{7.6}$$

where $\eta_\alpha$, $\bar{\eta}_\alpha$ are defined as in Section 6. If we integrate over the momenta in (7.6) we get

$$K(x_f, \bar{\eta}_f, \tau_f; x_i, \eta_i, \tau_i) =$$

$$= \int_{x_i, \eta_i}^{x_f, \bar{\eta}_f} \mathcal{D}(x(\tau))\, \mathcal{D}(\eta(\tau), \bar{\eta}(\tau))\; e^{\frac{1}{2}\bar{\eta}_{\alpha,f}\eta_\alpha(\tau_f) + \frac{1}{2}\bar{\eta}_\alpha(\tau_i)\eta_{\alpha,i}} \,\cdot$$

$$\cdot \exp\left\{ i \int_{\tau_i}^{\tau_f} d\tau \left[ -\frac{m}{2}(\dot{x}^2 + 1) - e\,\dot{x}\cdot A - \frac{i}{2}\xi\cdot\dot{\xi} - \right.\right. \tag{7.7}$$

$$\left.\left. - i m \alpha_2\, \xi\cdot\dot{x} + \frac{ie}{2m} F_{\mu\nu}\xi^\mu \xi^\nu \right]\right\}.$$

Let us now specify the external electromagnetic field $A_\mu$ as a crossed field

$$A_\mu(x) = a\,(\epsilon_\mu + \epsilon^*_\mu)\, k\cdot x\,, \tag{7.8}$$

where $a$ is a constant and

$$k^2 = k\cdot\epsilon = k\cdot\epsilon^* = \epsilon^2 = \epsilon^{*2} = 0, \qquad \epsilon\cdot\epsilon^* = -1. \tag{7.9}$$

From (7.8) we get that the field strength is

$$F_{\mu\nu} = a\,[\,k_\mu(\epsilon_\nu + \epsilon^*_\nu) - k_\nu(\epsilon_\mu + \epsilon^*_\mu)\,], \tag{7.10}$$

hence

$$F_{\mu\nu}F^{\mu\nu} = 0\,, \qquad F^*_{\mu\nu}F^{\mu\nu} = 0\,, \tag{7.11}$$

where $F^*_{\mu\nu}$ is the dual field; that is

$$\vec{E}^2 - \vec{H}^2 = 0, \qquad \vec{E}\cdot\vec{H} = 0\,, \tag{7.12}$$

and indeed we have constant electric and magnetic fields, of equal amplitude and mutually orthogonal. Let us now introduce a fourth four-vector $\nu_\mu$ with the properties

$$\begin{aligned} \nu^2 = \nu\cdot\epsilon = \nu\cdot\epsilon^* = 0\,, \\ k\cdot\nu = 1\,. \end{aligned} \tag{7.13}$$

A convenient choice for $k_\mu$, $\nu_\mu$, $\varepsilon_\mu$, $\varepsilon^*_\mu$ is the following

$$\begin{aligned} k_\mu &= \frac{1}{\sqrt{2}}(1,0,0,-1)\,, \\ \nu_\mu &= \frac{1}{\sqrt{2}}(1,0,0,1)\,, \\ \epsilon_\mu &= \frac{1}{\sqrt{2}}(0,1,-i,0)\,, \\ \epsilon^*_\mu &= \frac{1}{\sqrt{2}}(0,1,i,0)\,, \end{aligned} \tag{7.14}$$

With these positions we can easily evaluate the functional integral (7.7) with the method of classical translation. The extended Lagrangian following from (7.7):

$$\begin{aligned} L_E = -\frac{m}{2}(\dot{x}^2+1) - e\,\dot{x}\cdot A - \frac{i}{2}\,\xi\cdot\dot{\xi} - i\,m\alpha_2\,\xi\cdot\dot{x} + \\ + \frac{ie}{2m}F_{\mu\nu}\xi^\mu\xi^\nu\,, \end{aligned} \tag{7.15}$$

gives rise to the classical equations of motion

$$\begin{aligned} m\ddot{x}^\mu &= e\,F^{\mu\nu}\dot{x}_\nu - i\,m\alpha_2\,\dot{\xi}^\mu - \frac{ie}{2m}\xi^\nu\xi^\rho\,\partial^\mu F_{\nu\rho}\,, \\ \dot{\xi}^\mu &= m\alpha_2\,\dot{x}^\mu + \frac{e}{m}F^{\mu\nu}\xi_\nu\,. \end{aligned} \tag{7.16}$$

By projecting these equations along the various axes, we get

$$
\begin{aligned}
& k\cdot\ddot{x} = 0 , \\
& k\cdot\dot{\xi} = m\alpha_2\, k\cdot\dot{x} , \\
& \epsilon\cdot\ddot{x} = \frac{ea}{m}\left[ k\cdot\dot{x} - i\alpha_2\, k\cdot\xi \right], \\
& \epsilon^{*}\cdot\ddot{x} = \frac{ea}{m}\left[ k\cdot\dot{x} - i\alpha_2\, k\cdot\xi \right], \\
& \epsilon\cdot\dot{\xi} = m\alpha_2\,\epsilon\cdot\dot{x} + \frac{ea}{m}\, k\cdot\xi , \\
& \epsilon^{*}\cdot\dot{\xi} = m\alpha_2\,\epsilon^{*}\cdot\dot{x} + \frac{ea}{m}\, k\cdot\xi , \\
& \nu\cdot\ddot{x} = \frac{ea}{m}\,(\epsilon+\epsilon^{*})\cdot(\dot{x} - i\alpha_2\xi) , \\
& \nu\cdot\dot{\xi} = m\alpha_2\,\nu\cdot\dot{x} + \frac{ea}{m}\,(\epsilon+\epsilon^{*})\cdot\xi ,
\end{aligned}
\tag{7.17}
$$

where we have used $\alpha_2^2 = 0$. The reason for writing the equations of motion in this form is that we are able to integrate the $K^{th}$ equation if we know the integrals of the first $(k-1)$ equations. By integrating these equations with the initial conditions

$$
\begin{aligned}
& x(\tau_i) = x_i , \quad x(\tau_f) = x_f , \\
& \eta_1(\tau_i) = \nu\cdot\xi(\tau_i) = \eta_{1,i} , \\
& \eta_2(\tau_i) = \epsilon^{*}\cdot\xi(\tau_i) = \eta_{2,i} , \\
& \bar{\eta}_1(\tau_f) = -\,k\cdot\xi(\tau_f) = \bar{\eta}_{1,f} , \\
& \bar{\eta}_2(\tau_f) = \epsilon\cdot\xi(\tau_f) = \bar{\eta}_{2,f} ,
\end{aligned}
\tag{7.18}
$$

and using the classical solutions to translate the integration variables in (7.7), we get

$$K(x_f, \bar{\eta}_f, \tau_f ; x_i, \eta_i, \tau_i) =$$

$$
\begin{aligned}
&= F(\tau_f,\tau_i)\exp\Big\{-i\frac{m}{2}\Delta\tau\Big[\Big(\frac{\Delta x}{\Delta\tau}\Big)^2+1\Big]-i\frac{ea}{2}\Big[(\epsilon+\epsilon^*)\cdot\Delta x \\
&\quad k\cdot(x_f+x_i)\Big]-\frac{ie^2a^2}{12m}(k\cdot\Delta x)^2\Delta\tau+\frac{ea}{m}\bar\eta_{1,f}(\eta_{2,i}+\bar\eta_{2,f})\Delta\tau+ \\
&\quad +\bar\eta_{\alpha,f}\eta_{\alpha,i}+\alpha_2\Big[\frac{ea}{2}\bar\eta_{1,f}(\epsilon-\epsilon^*)\cdot\Delta x\,\Delta\tau+ \\
&\quad +\frac{ea}{2}(\bar\eta_{2,f}+\eta_{2,i})\,k\cdot\Delta x\,\Delta\tau+\frac{1}{3}\frac{e^2a^2}{m}\bar\eta_{1,f}\,k\cdot\Delta x(\Delta\tau)^2- \\
&\quad -m\bar\eta_{1,f}\,\nu\cdot\Delta x+m\eta_{1,i}\,k\cdot\Delta x-m\eta_{2,i}\,\epsilon\cdot\Delta x- \\
&\quad -m\bar\eta_{2,f}\,\epsilon^*\cdot\Delta x\Big]\Big\},
\end{aligned}
\tag{7.19}
$$

where $\Delta x = (x_f - x_i)$, $\Delta\tau = (\tau_f - \tau_i)$. $F(\tau_f,\tau_i)$ is the usual normalization factor which is determined by the condition

$$
\lim_{\Delta\tau\to 0} K(x_f,\bar\eta_f,\tau_f;x_i,\eta_i,\tau_i)=\delta^4(x_f-x_i)\,\delta(\eta_f,\eta_i), \tag{7.20}
$$

where $\delta(\eta_f,\eta_i)$ is defined in Eq. (B.18). Then, we get

$$
F(\tau_f,\tau_i)=\frac{-im^2}{4\pi^2(\Delta\tau)^2}, \tag{7.21}
$$

as for the free scalar case.

As in the previous section, we have to multiply this kernel for the kernel relative to the $\xi_5$ part [given in Eq. (3.27)] and integrate over $\beta_2 = \alpha_2\,\Delta\tau$. Going to a spinor basis and using the physical states (6.5) we get:

$$K(x_f,\tau_f;x_i,\tau_i) = -\frac{i}{\sqrt{2}}\, F(\tau_f,\tau_i)\, e^{-\frac{im}{2}\Delta\tau\left[\left(\frac{\Delta x}{\Delta\tau}\right)^2+1\right]} \cdot$$

$$\cdot\, e^{-\frac{iea}{2}[(\epsilon+\epsilon^*)\cdot\Delta x][k\cdot(x_i+x_f)] - \frac{ie^2a^2}{12m}(k\cdot\Delta x)^2\Delta\tau} \cdot$$

$$\cdot\left[ m\left(\frac{\Delta x_\mu \gamma^\mu}{\Delta\tau}+1\right) + \frac{e}{2} F_{\mu\nu}\gamma^\mu \Delta x^\nu + \right. \tag{7.22}$$

$$\left. + \frac{e^2}{12m}\Delta\tau F_{\mu\nu} F^{\nu\rho}\gamma^\mu \Delta x_\rho \right] e^{-\frac{ie}{4m} F_{\mu\nu}\sigma^{\mu\nu}\Delta\tau} .$$

Furthermore, by integrating this expression over $\beta_1 = -(1/2m)\Delta\tau$ we recover the Schwinger result[19].

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APPENDIX A

In this Appendix we discuss the problem of a dynamical system satisfying $m$ first class constraints $\chi_i$. The extended Hamiltonian of such a system is

$$H_E = H_c + \sum_{i=1}^{m} \alpha_i \chi_i , \tag{A.1}$$

where $H_c$ is the canonical Hamiltonian and $\alpha_i$ are arbitrary Lagrangian multipliers. Corresponding to the Hamiltonian (A.1) we have the following equation for the evolution in $\tau$:

$$i \frac{\partial}{\partial \tau} |\psi, \tau\rangle = H_E |\psi, \tau\rangle . \tag{A.2}$$

The physical states must satisfy the conditions following from the constraints $\chi_i$. If we define $H_E$ with $\dot{\alpha}_i = 0$, it is easy to integrate (A.2) and we get

$$|\psi, t, \alpha_i\rangle = e^{-iH_c \tau - i \sum_{i=1}^{m} \alpha_i \tau \chi_i} |\psi\rangle . \tag{A.3}$$

Now let $\beta_i = \alpha_i \tau$. Then the states (A.3) satisfy the following set of Schrödinger-like equations*)

$$\begin{aligned} i \frac{\partial}{\partial \beta_i} |\psi, \tau, \beta_i\rangle &= \chi_i |\psi, \tau, \beta_i\rangle , \\ i \frac{\partial}{\partial \tau} |\psi, \tau, \beta_i\rangle &= H_c |\psi, \tau, \beta_i\rangle . \end{aligned} \tag{A.4}$$

From Eq. (A.4) we see that the physical states are obtained by integrating (A.3) over $\beta_i$

$$|\psi, \tau\rangle_{phys.} = \int \prod_{i=1}^{m} d\beta_i \; |\psi, \tau, \beta_i\rangle . \tag{A.5}$$

---

*) Of course this set of partial differential equations is integrable due to the first class character of the $\chi_i$'s.

When $H_c = 0$ we could fix one of the $\alpha_i$'s, say $\alpha_1$, to a given value and replace the integration over $\beta_1$ with an integration over $\tau$. In this way one gets back the Feynman[20] prescription for the Klein-Gordon operator.

In this last case, a solution of (A.2) in configuration space satisfies the integral equation

$$\psi(x', \alpha_i, \tau') = \int d^4x \, K(x', x; \alpha_i; \tau', \tau) \, \theta(\tau' - \tau) \, \psi(x, \alpha_i, \tau) \tag{A.6}$$

We are interested in deriving the expression for the physical kernel, that is, the one propagating physical states in physical states. By choosing the wave function at $\tau = 0$ to be a physical one, we get

$$\psi(x', \alpha_i, \tau') = \int d^4x \, K(x', x; \alpha_i; \tau') \, \theta(\tau') \, \psi_{phys}(x) . \tag{A.7}$$

In general $\psi(x', \alpha_i, \tau)$ will not be a physical state, however from (A.5) (in the case $H_c = 0$) we get

$$\begin{aligned} \psi_{phys}(x') &= \int \prod_{i=1}^{m} d\beta_i' \, \psi(x'; \beta_i') = \\ &= \int \prod_{i=1}^{m} d\beta_i' \, \theta(\tau') \, K(x', x, \beta') \, \psi_{phys}(x) \, d^4x , \end{aligned} \tag{A.8}$$

where we have used the fact that for $H_c = 0$, the wave function depends on $\tau$ only through the combinations $\alpha_i \tau$. In (A.8) $\tau'$ can be expressed in terms of one of the $\beta_i$'s, say $\beta_1$. Therefore, the expression for the physical kernel is

$$K_{phys}(x', x) = \int_0^\infty d\beta_1 \int \prod_{2}^{m} {}_k \, d\beta_k \, K(x', x; \beta_i) \tag{A.9}$$

$$i = 1, 2, \ldots, m$$

APPENDIX B

In this Appendix we want to construct an explicit representation for the transition functions between spinors and the generalized coherent states $|\eta\rangle$ defined by

$$\underset{\sim}{\eta}_\alpha |\eta\rangle = |\eta\rangle \eta_\alpha , \tag{B.1}$$

where the $\underset{\sim}{\eta}_\alpha$'s are defined in Eq. (5.28). This equation can be satisfied by putting

$$|\eta\rangle = \begin{bmatrix} A\eta_1 \\ B\eta_2 \\ C \\ D\eta_1\eta_2 \end{bmatrix} , \tag{B.2}$$

with A,B,C,D complex parameters satisfying

$$|A|^2 = |B|^2 = |C|^2 = |D|^2 = 1. \tag{B.3}$$

in agreement with Eq. (5.30). The corresponding representation for $\underset{\sim}{\eta}_\alpha$ is

$$\underset{\sim}{\eta}_1 = \begin{bmatrix} 0 & -\frac{B}{D}\,\frac{1-\sigma_3}{2} \\ \frac{C}{A}\,\frac{1+\sigma_3}{2} & 0 \end{bmatrix} ,$$

$$\underset{\sim}{\eta}_2 = \begin{bmatrix} 0 & \frac{A}{D}\,\frac{\sigma_1+i\sigma_2}{2} \\ \frac{C}{B}\,\frac{\sigma_1+i\sigma_2}{2} & 0 \end{bmatrix} . \tag{B.4}$$

In a Weyl basis, we have the following representation for the $\gamma$ matrices:

$$\gamma^\mu = \begin{bmatrix} 0 & \tilde{\sigma}^\mu \\ \sigma^\mu & 0 \end{bmatrix} , \tag{B.5}$$

where

$$(\sigma^\mu)_{\dot\alpha\beta} = (1, \vec{\sigma}) \quad ; \quad (\tilde{\sigma}^\mu)^{\alpha\dot\beta} = (1, -\vec{\sigma}) . \tag{B.6}$$

The algebra (5.10) of the $\underset{\sim}{\xi}_\mu$ operators can be realized in various ways in terms of the $\gamma$ matrices. Correspondingly, we get different choices for the parameters A,B,C,D of Eq. (B.4). Here, we are interested in the following two choices

$$\underset{\sim}{\xi}_\mu = \frac{1}{\sqrt{2}} \gamma_5 \gamma_\mu , \quad \underset{\sim}{\xi}_\mu = \frac{i}{\sqrt{2}} \gamma_\mu . \tag{B.7}$$

For the first choice, by using (5.12) we get

$$A = B = 1, \quad C = D = -1 , \tag{B.8}$$

and therefore

$$|\eta\rangle = \begin{bmatrix} \eta_1 \\ \eta_2 \\ -1 \\ -\eta_1\eta_2 \end{bmatrix} . \tag{B.9}$$

For the second choice we get

$$A = B = 1, \quad C = D = i , \tag{B.10}$$

from which

$$|\eta\rangle = \begin{bmatrix} \eta_1 \\ \eta_2 \\ i \\ i\eta_1\eta_2 \end{bmatrix} . \tag{B.11}$$

According to Eqs (B.5) and (B.6), a Dirac spinor in a Weyl basis is defined by

$$\psi_A = \begin{bmatrix} \phi^\alpha \\ \chi_{\dot\alpha} \end{bmatrix} . \tag{B.12}$$

The representation of this spinor in the $\eta$ space is given by the wave function

$$\psi(\bar{\eta}) = \langle \bar{\eta} | \psi \rangle = A^* \bar{\eta}_1 \phi^1 + B^* \bar{\eta}_2 \phi^2 + C^* \chi_{\dot{1}} + D^* \bar{\eta}_2 \bar{\eta}_1 \chi_{\dot{2}} \tag{B.13}$$

The adjoint wave function is

$$\bar{\psi}(\eta) = \langle \psi | \gamma_0 | \eta \rangle = (\chi_{\dot{1}})^* A \eta_1 + (\chi_{\dot{2}})^* B \eta_2 + (\phi^1)^* C + (\phi^2)^* D \eta_1 \eta_2 . \tag{B.14}$$

By using the completeness relation

$$\int | \eta \rangle \langle \bar{\eta} | \, d\mu(\eta) = 1 \tag{B.15}$$

with

$$d\mu(\eta) = e^{-\sum_{\alpha=1}^{2} \bar{\eta}_\alpha \eta_\alpha} \prod_{\alpha=1,2} d\bar{\eta}_\alpha \, d\eta_\alpha \tag{B.16}$$

we have

$$\bar{\psi}\psi \equiv \langle \psi | \gamma_0 | \psi \rangle = \int \bar{\psi}(\eta) \psi(\bar{\eta}) \, d\mu(\eta) . \tag{B.17}$$

We recall also that the $\delta$ function for the Fermi coherent states is defined by

$$\delta(\eta_f, \eta_i) = e^{\sum_{\alpha=1}^{2} \bar{\eta}_{\alpha,f} \eta_{\alpha,i}} \tag{B.18}$$

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