Path Integrals for Continuous-Time Markov Chains

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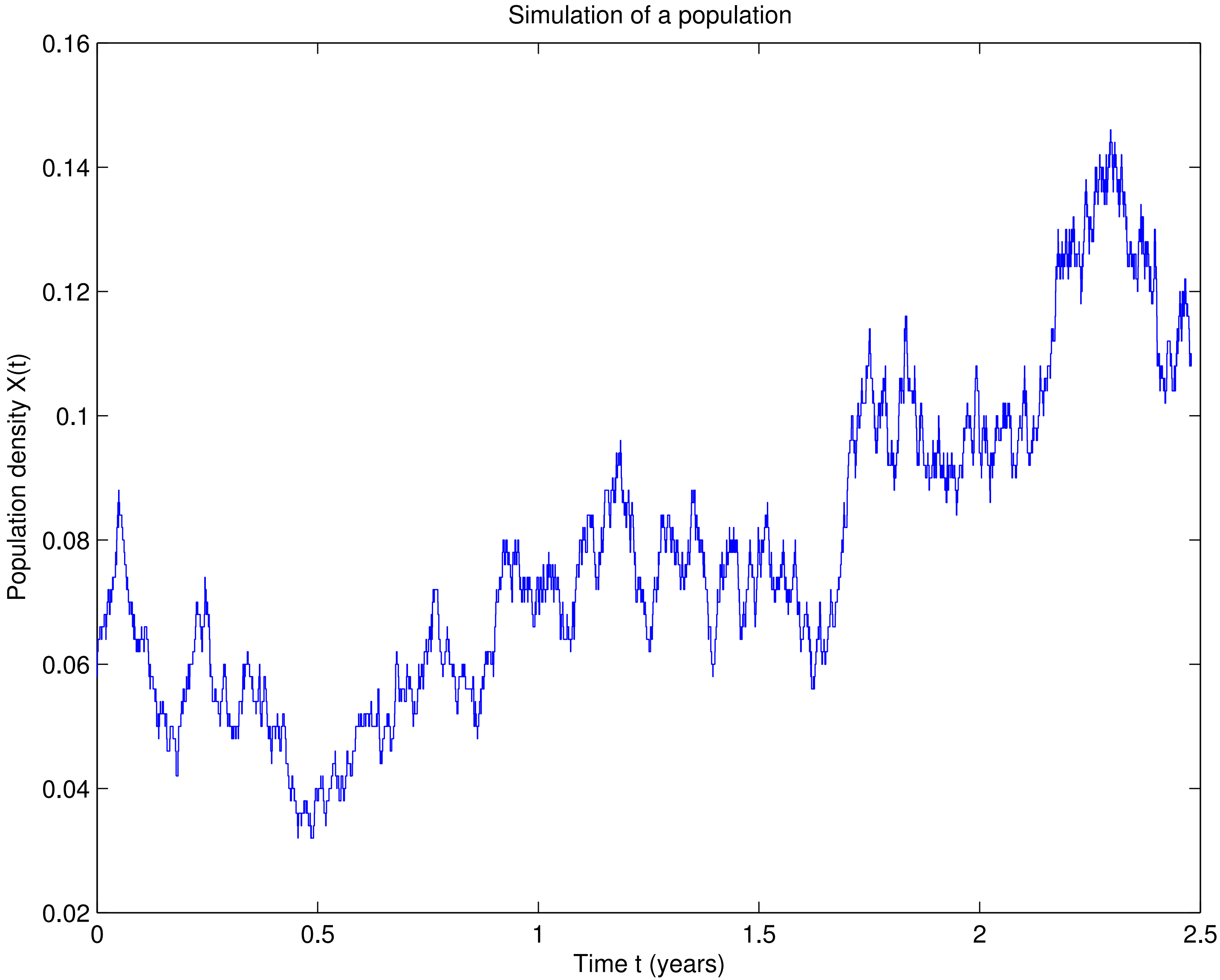
AUSTRALIAN RESEARCH COUNCIL
Centre of Excellence for Mathematics
and Statistics of Complex Systems

A population process



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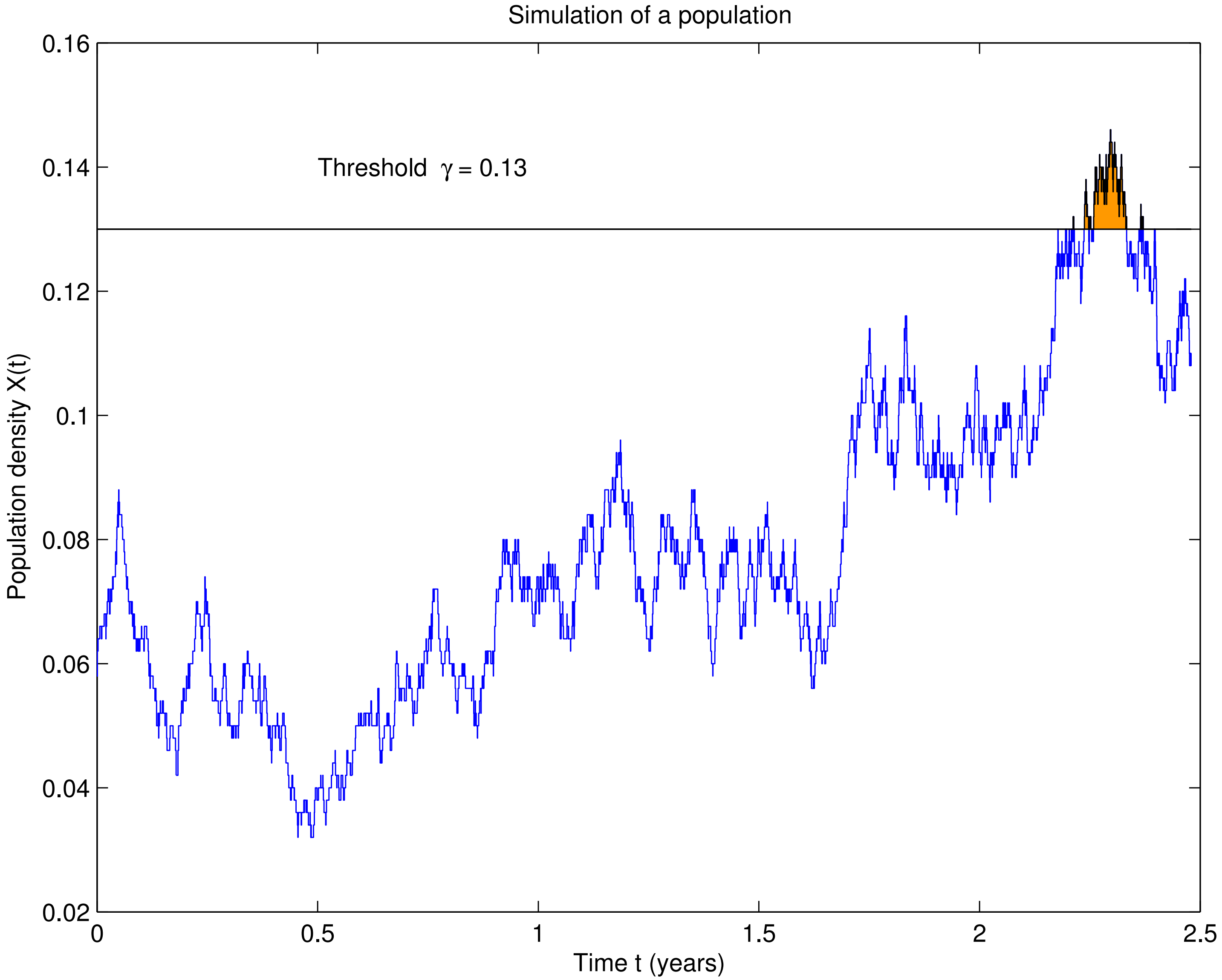


Total cost

Let $X(t)$ be the population density at time t.

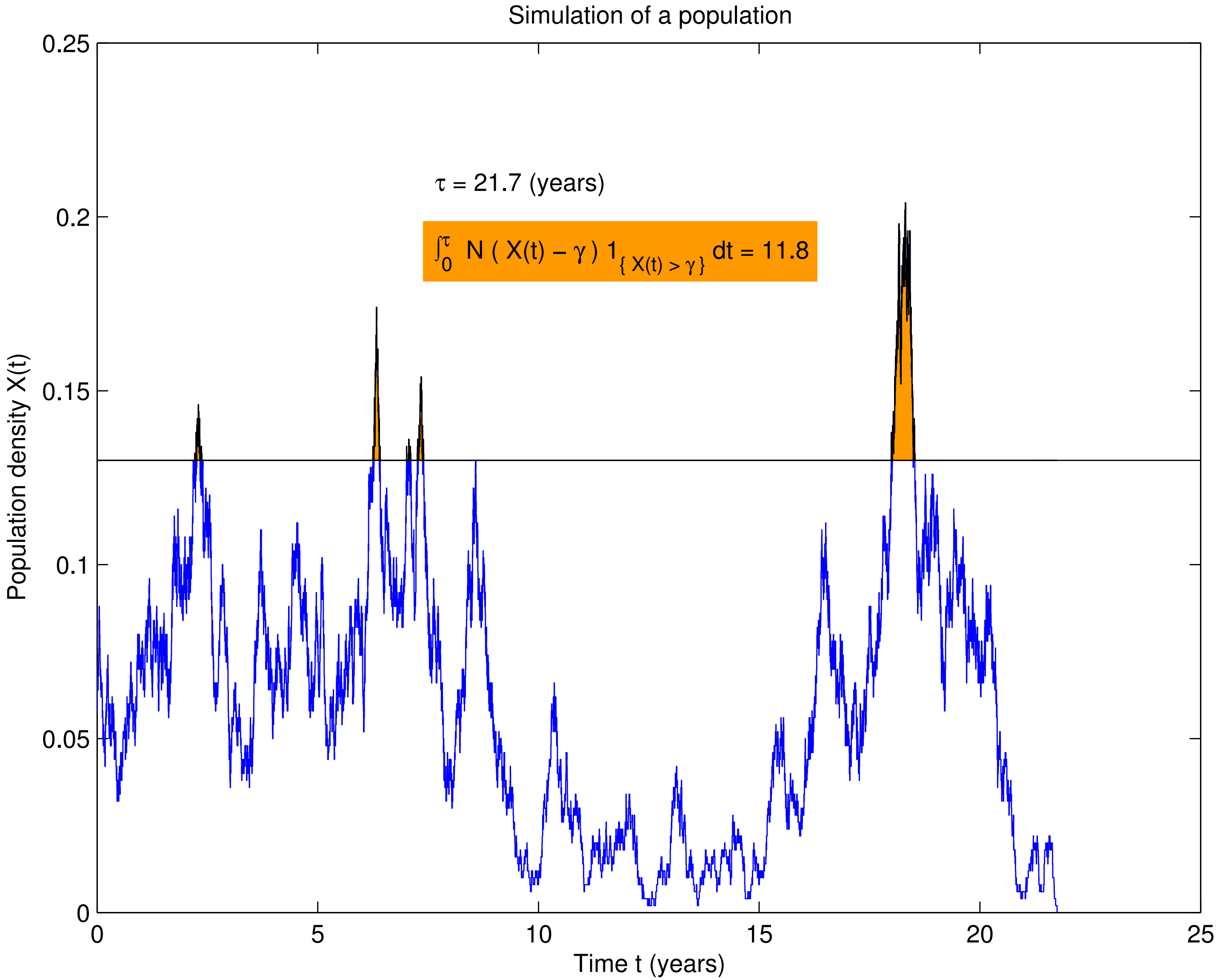
Let $c(x)$ be the cost per unit time of maintaining the population when its density is x units above a threshold γ.

Then, if τ is the time to extinction,

$$\int_0^\tau c(X(t) - \gamma) 1_{\{X(t) > \gamma\}} \, dt$$

is the total cost over the life of the population.

A population process



Ingredients

- A random process $(X(t),\, t \geq 0)$ in continuous time

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- A random process $(X(t),\, t \geq 0)$ in continuous time
- A set of states A
- The (random) time τ to first exit from A
- The cost (per unit time) f_x of being in state x
- The "path integral"

$$\Gamma = \int_0^\tau f_{X(t)}\, dt,$$

the total cost incurred before leaving A (also random)

Other examples

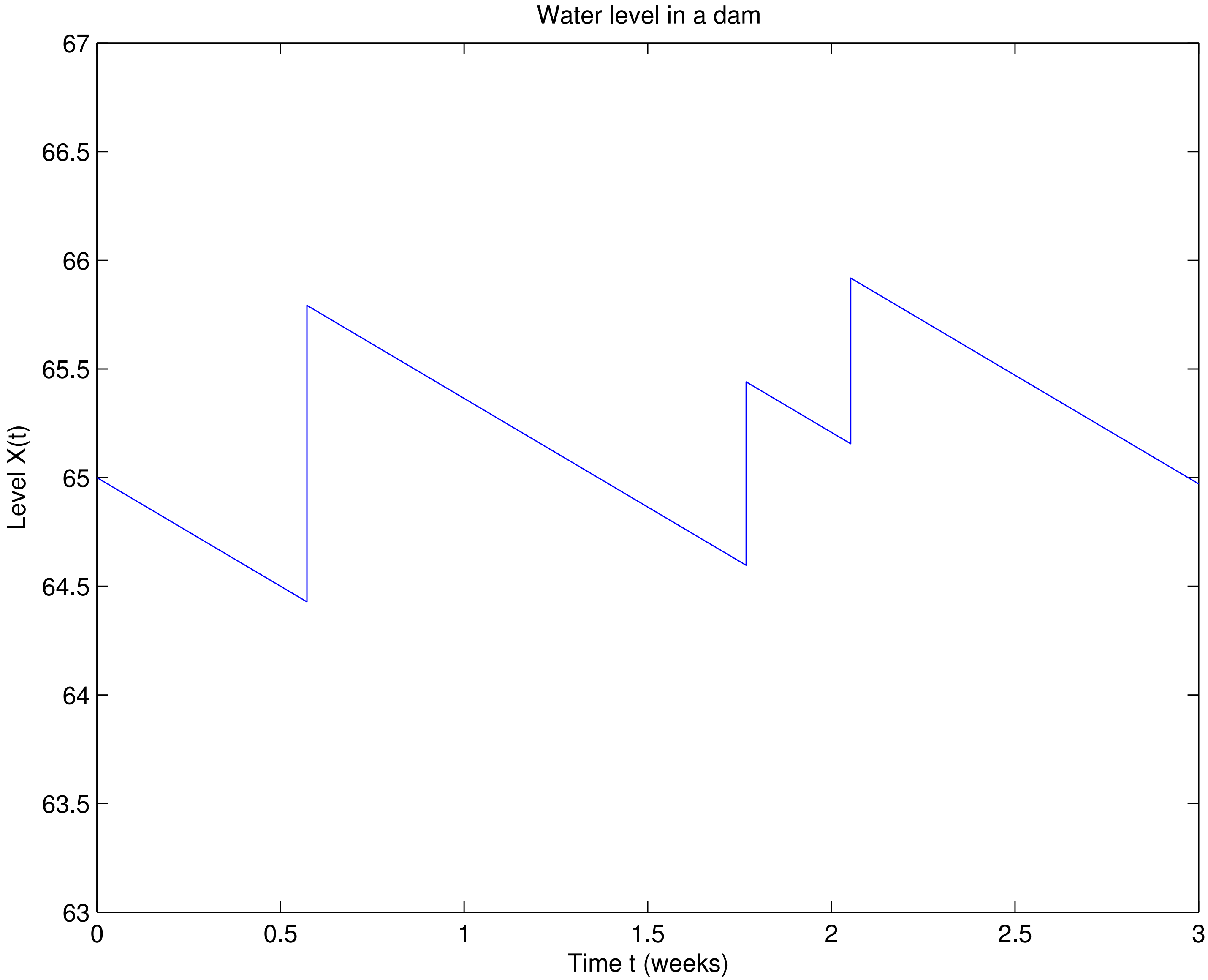
- Consider a dam with finite capacity V, and let $X(t)$ be the water level at time t.

  We might wish to estimate the total time for which the level was below a given value γ,

$$\Gamma = \int_0^\tau 1_{\{X(t)<\gamma\}}\, dt,$$

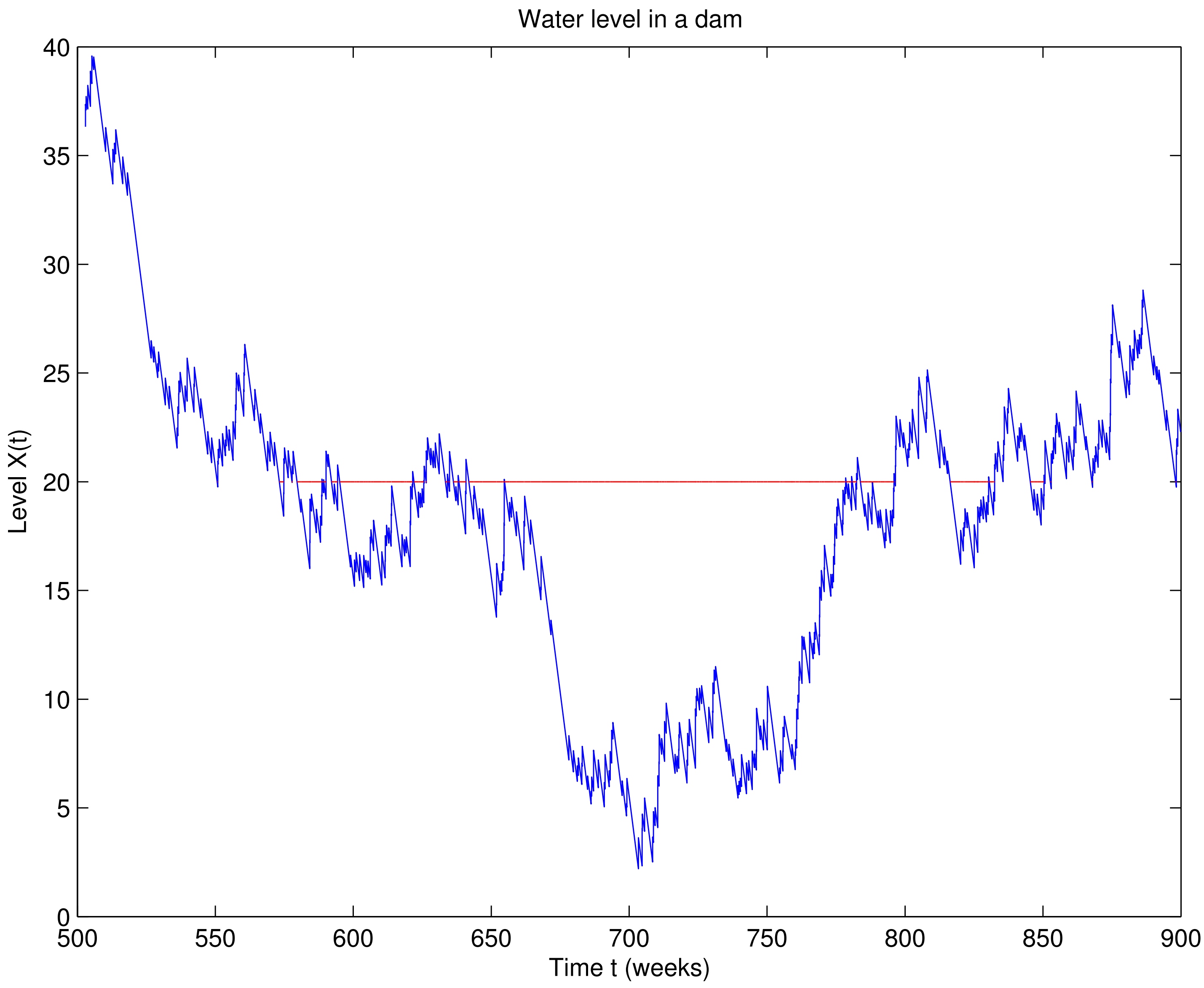
  where τ is (say) the time to reach capacity or to empty (whichever occurs first).

Dam



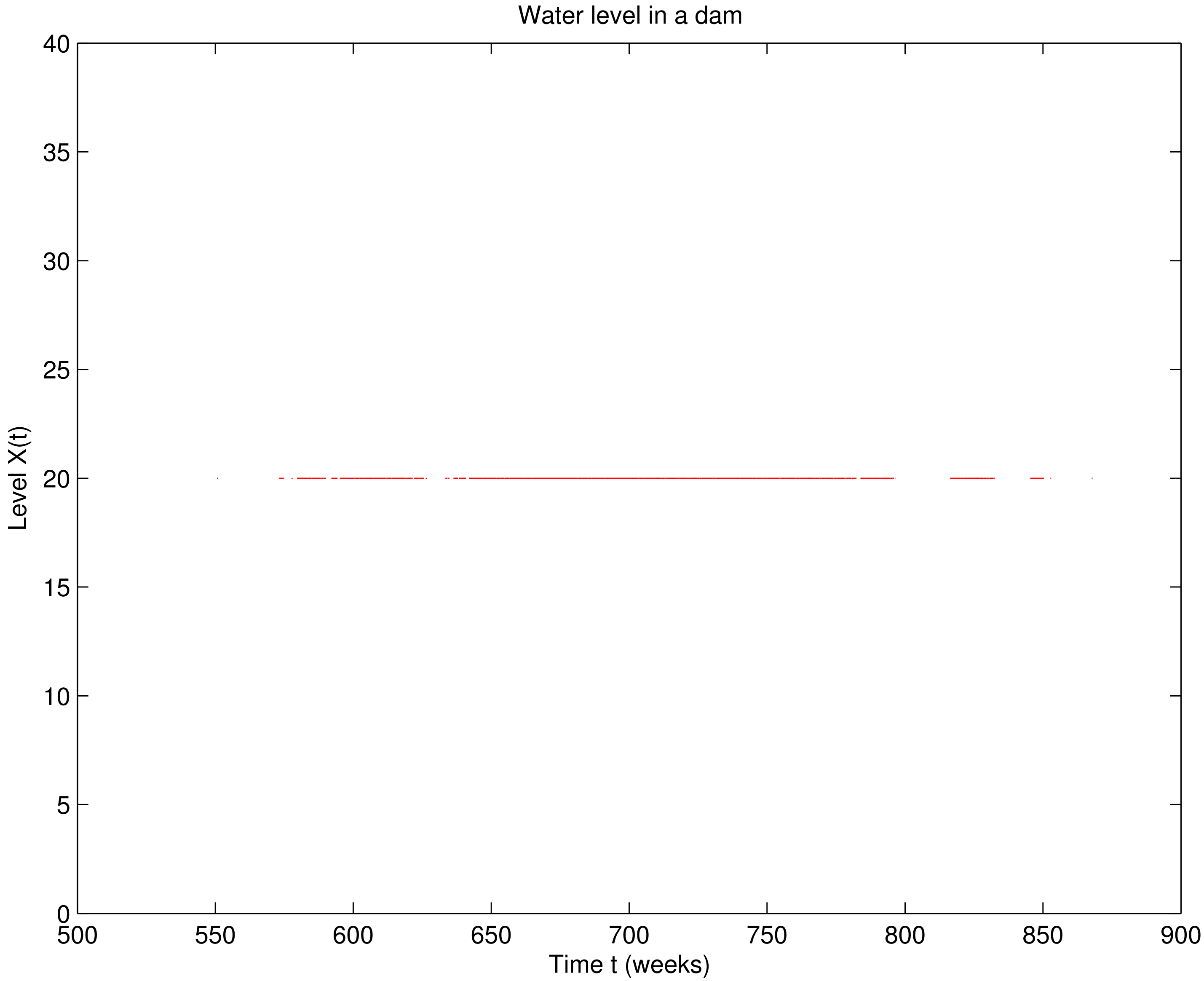
Water level in a dam
67
66.5
66
65.5
65
64.5
64
63.5
63
Level X(t)
0
0.5
1
1.5
2
2.5
3
Time t (weeks)

Dam



Water level in a dam
Level X(t)
Time t (weeks)
40
35
30
25
20
15
10
5
0
500
550
600
650
700
750
800
850
900

Dam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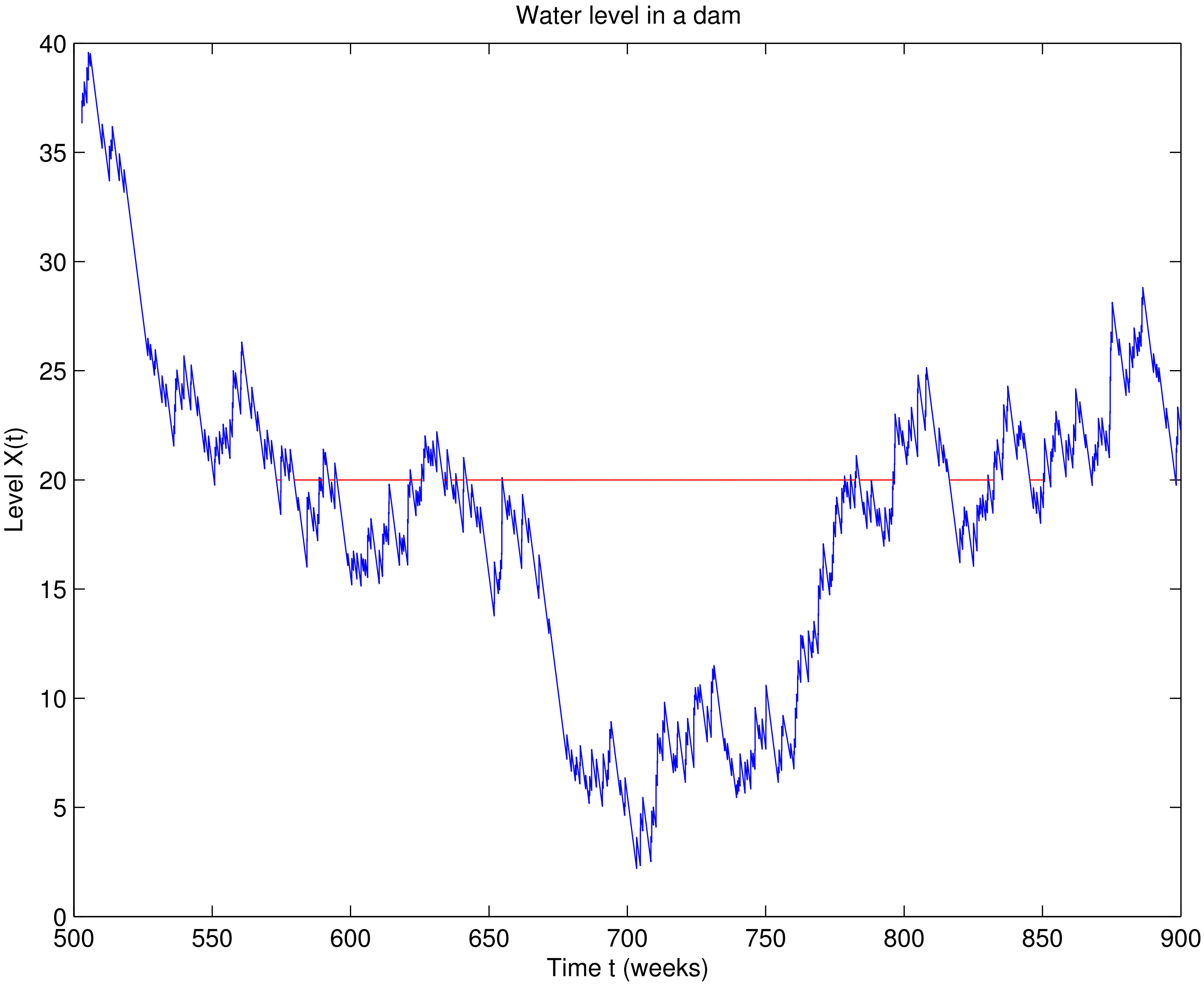

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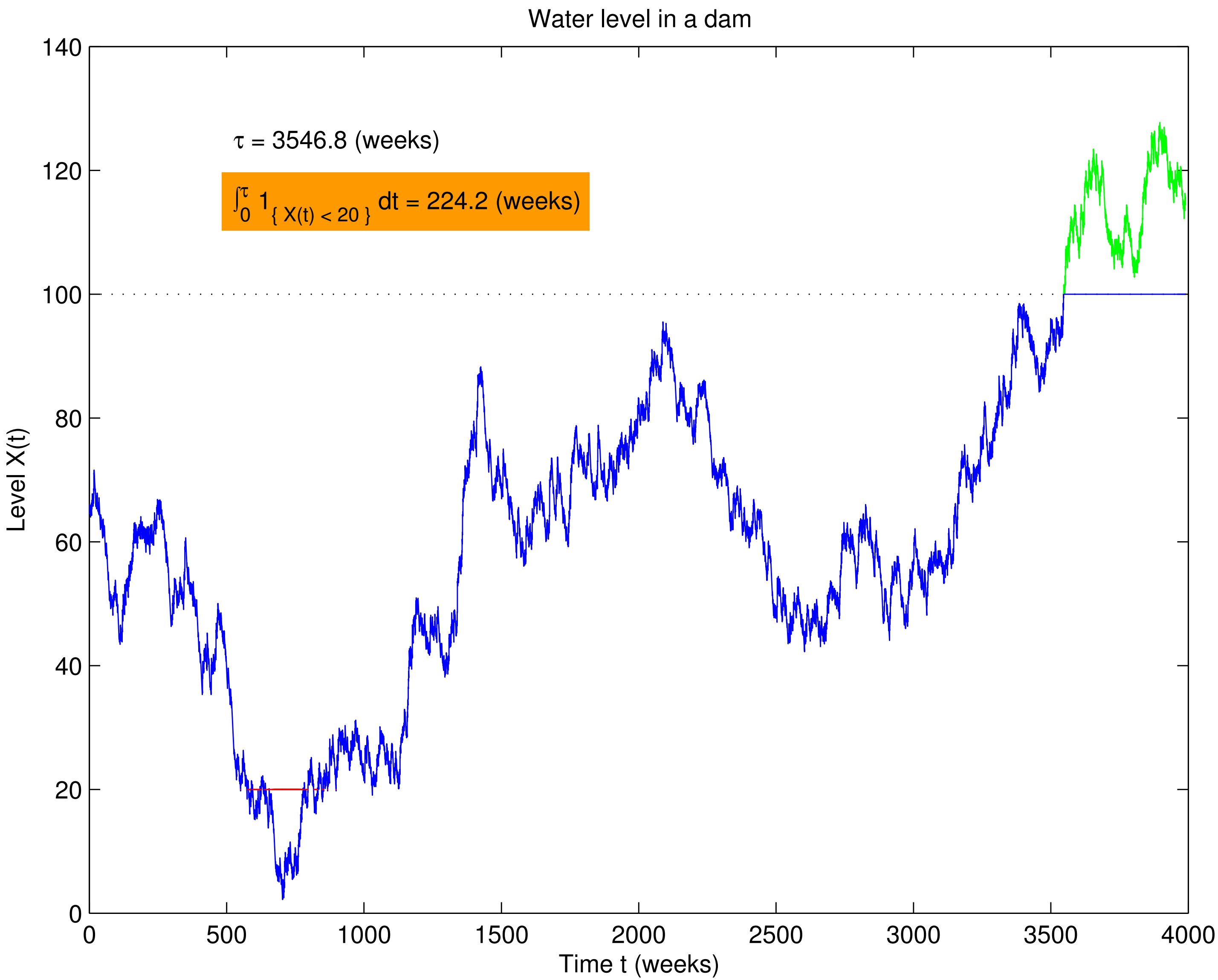


Water level in a dam
τ = 3546.8 (weeks)
$\int_0^\tau 1_{\{X(t) < 20\}}\, dt$ = 224.2 (weeks)
Level X(t)
Time t (weeks)
0
20
40
60
80
100
120
140
0
500
1000
1500
2000
2500
3000
3500
4000

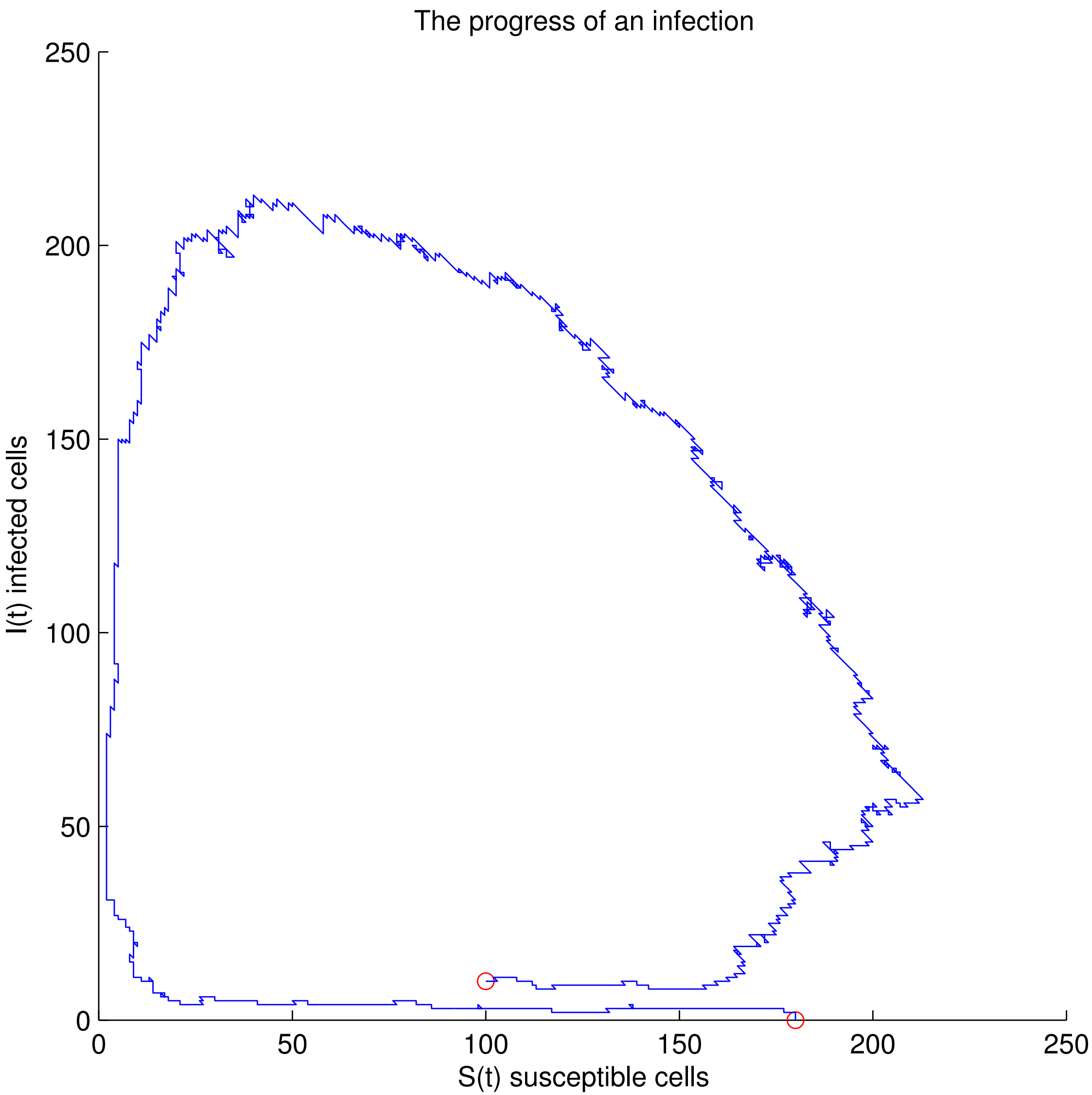
Other examples

- Let $(S(t), I(t))$ be the number of susceptibles and infectives in an epidemic at time t.

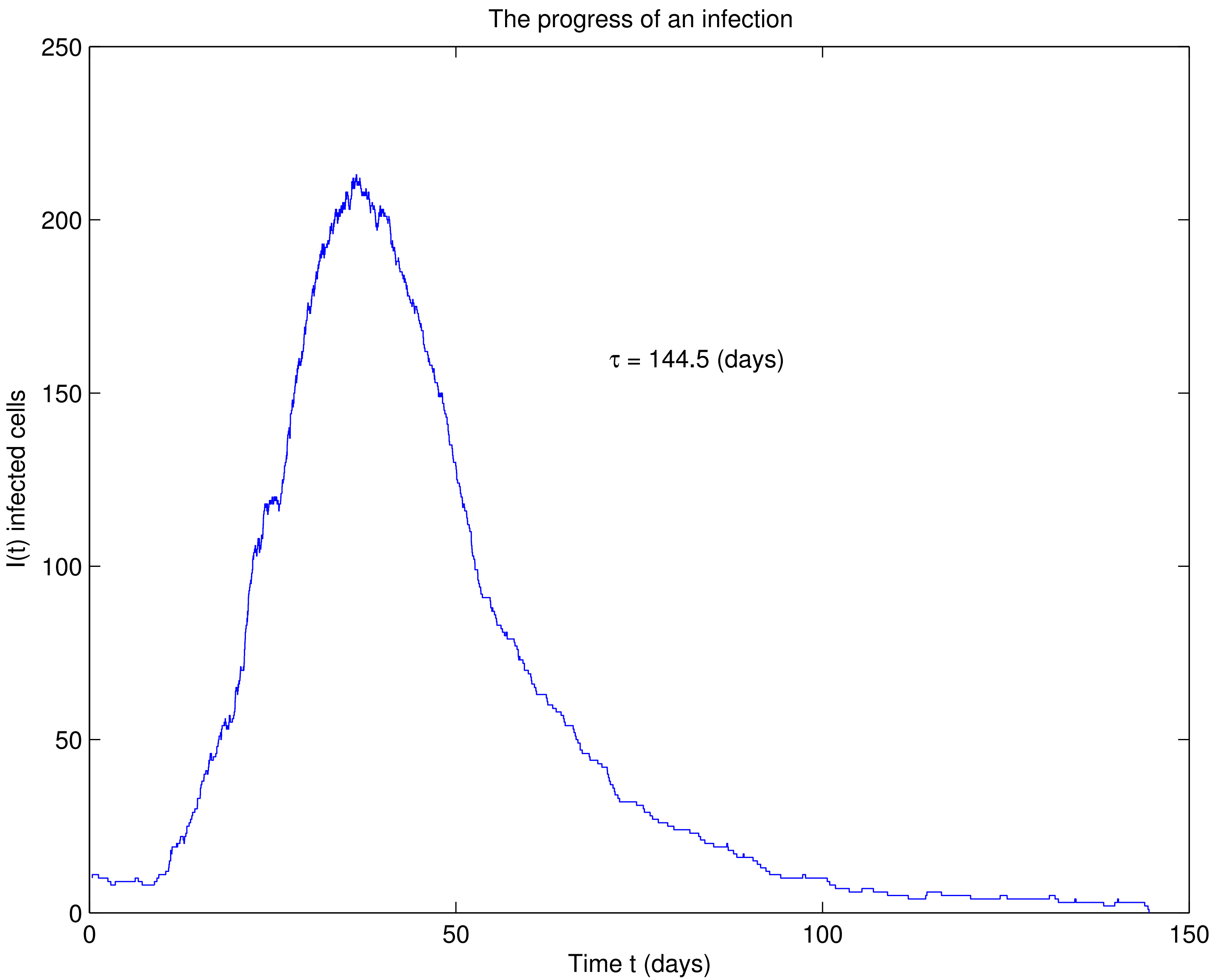
  If τ is the period of infection and $f_{(s,i)} = i$, then Γ is the total amount of infection:

$$\Gamma = \int_0^\tau I(t)\, dt.$$

Epidemic

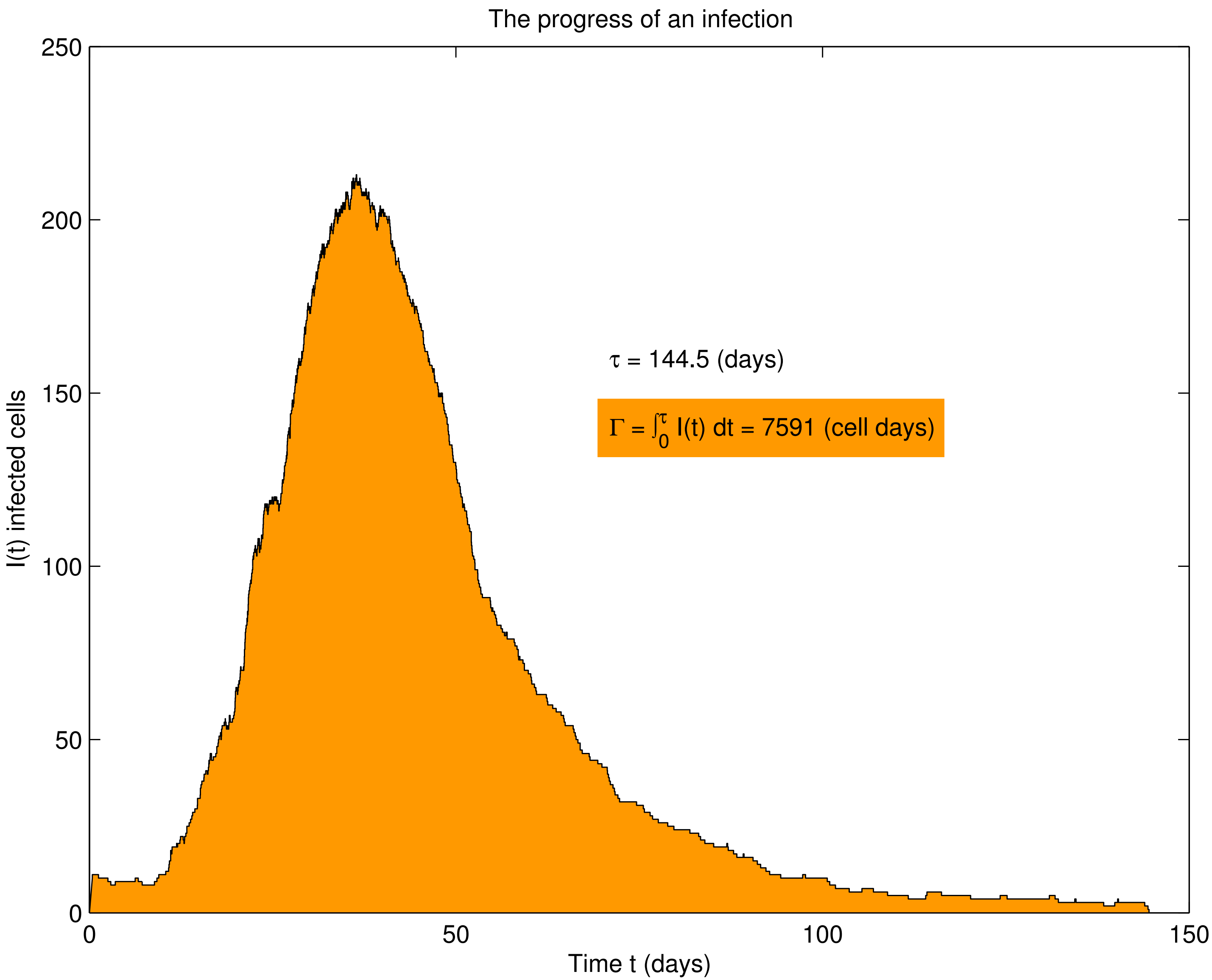

The progress of an infection
250
200
150
100
50
0
I(t) infected cells
0
50
100
150
200
250
S(t) susceptible cells

Epidemic



The progress of an infection
τ = 144.5 (days)
I(t) infected cells
Time t (days)
0
50
100
150
200
250
0
50
100
150

Epidemic


The progress of an infection
250
200
150
100
50
0
I(t) infected cells
0
50
100
150
Time t (days)
τ = 144.5 (days)
Γ = ∫₀^τ I(t) dt = 7591 (cell days)

The problem

Our problem is to determine the *expected value*, and the *distribution* of the total cost

$$\Gamma = \int_0^\tau f_{X(t)}\, dt,$$

where recall that τ is the time to first exit from a set A and f_x is cost per unit time of being in state x.

For simplicity, suppose that $X(t)$ takes values in $S = \{0, 1, \ldots\}$.

For example, $X(t)$ might be the number in a population at time t, and $A = \{1, 2, \ldots\}$, so that τ is the time to extinction.

A first attempt at evaluating $E(\Gamma)$

Let T_j be the total time that the process spends in state j during the period up to time τ and let N_j be the number of visits to j during that period. Then,

$$\Gamma = \sum_{j \in A} f_j T_j$$

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where X_{jn}, $n = 1, 2, \ldots$, are the successive occupancy times for state j.

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where X_{jn}, $n = 1, 2, \ldots$, are the successive occupancy times for state j. Then, under mild conditions, $E(\Gamma) = \sum_{j \in A} f_j E(N_j)\mu_j$, where μ_j is the mean occupancy time for state j.

Markovian models

We will assume that $(X(t),\, t \geq 0)$ is a *Markov chain* with *transition rates*

$$Q = (q_{ij},\; i, j \in S),$$

so that q_{ij} represents the rate of transition from state i to state j, for $j \neq i$, and $q_{ii} = -q_i$, where

$$q_i := \sum_{j \neq i} q_{ij} \;\; (< \infty)$$

represents the total rate out of state i.

Markovian models

An example is the *birth-death process*, which has

$$q_{i,i+1} = \lambda_i \qquad \text{(birth rates)}$$

$$q_{i,i-1} = \mu_i \qquad \text{(death rates)},$$

with $\mu_0 = 0$ and otherwise 0 ($q_i = \lambda_i + \mu_i$):

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \cdots \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix}$$

Example

The *Stochastic Logistic Model* (simulated earlier) is a birth-death process on $S = \{0, 1, \ldots, N\}$, with

$$\lambda_i = \frac{\lambda}{N} i(N - i) \quad \text{and} \quad \mu_i = \mu i,$$

where $\lambda, \mu > 0$.

Interlude

These birth and death rates can be written

$$\frac{\lambda_i}{N} = \lambda \left(\frac{i}{N} \right) \left(1 - \frac{i}{N} \right) \quad \text{and} \quad \frac{\mu_i}{N} = \mu \left(\frac{i}{N} \right)$$

Intuition: for large N the population *density* $X(t)/N$ becomes more deterministic (non-random):

$$\frac{dx}{dt} = \lambda(x) - \mu(x),$$

where

$$\lambda(x) = \lambda x \left(1 - x\right) \quad \text{and} \quad \mu(x) = \mu x.$$

Interlude

Soit p la population : représentons par dp l'accroissement infiniment petit qu'elle reçoit pendant un temps infiniment court dt. Si la population croissait en progression géométrique, nous aurions l'équation $\frac{dp}{dt} = mp$. Mais comme la vitesse d'accroissement de la population est retardée par l'augmentation même du nombre des habitans, nous devrons retrancher de mp une fonction inconnue de p; de manière que la formule à intégrer deviendra

$$\frac{dp}{dt} = mp - \varphi(p).$$

L'hypothèse la plus simple que l'on puisse faire sur la forme de la fonction φ, est de supposer $\varphi(p) = np^2$. On trouve alors pour intégrale de l'équation ci-dessus

$$t = \frac{1}{m}\left[\log.\, p - \log.(m - np)\right] + \text{constante},$$

et il suffira de trois observations pour déterminer lés deux coefficiens constans m et n et la constante arbitraire.

Interlude

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En résolvant la dernière équation par rapport à p, il vient

$$p = \frac{mp' e^{mt}}{np' e^{mt} + m - np'} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

en désignant par p' la population qui répond à $t = o$, et par e la base des logarithmes népériens. Si l'on fait $t = \infty$, on voit que la valeur de p correspondante est $P = \frac{m}{n}$. Telle est donc *la limite supérieure de la population.*

Au lieu de supposer $\varphi p = np^2$, on peut prendre $\varphi p = np^\alpha$, α étant quelconque, ou $\varphi p = n \log. p$. Toutes ces hypothèses satisfont également bien aux faits observés ; mais elles donnent des valeurs très-différentes pour la limite supérieure de la population.

J'ai supposé successivement

$$\varphi p = np^2, \ \varphi p = np^3, \ \varphi p = np^4, \ \varphi p = n \log. p;$$

et les différences entre les populations calculées et celles que fournit l'observation ont été sensiblement les mêmes.

Interlude

This is from ...

P.F. Verhulst, Notice sur la loi que la population suit dans son accroisement, *Corr. Math. et Phys*. X (1838), 113–121.

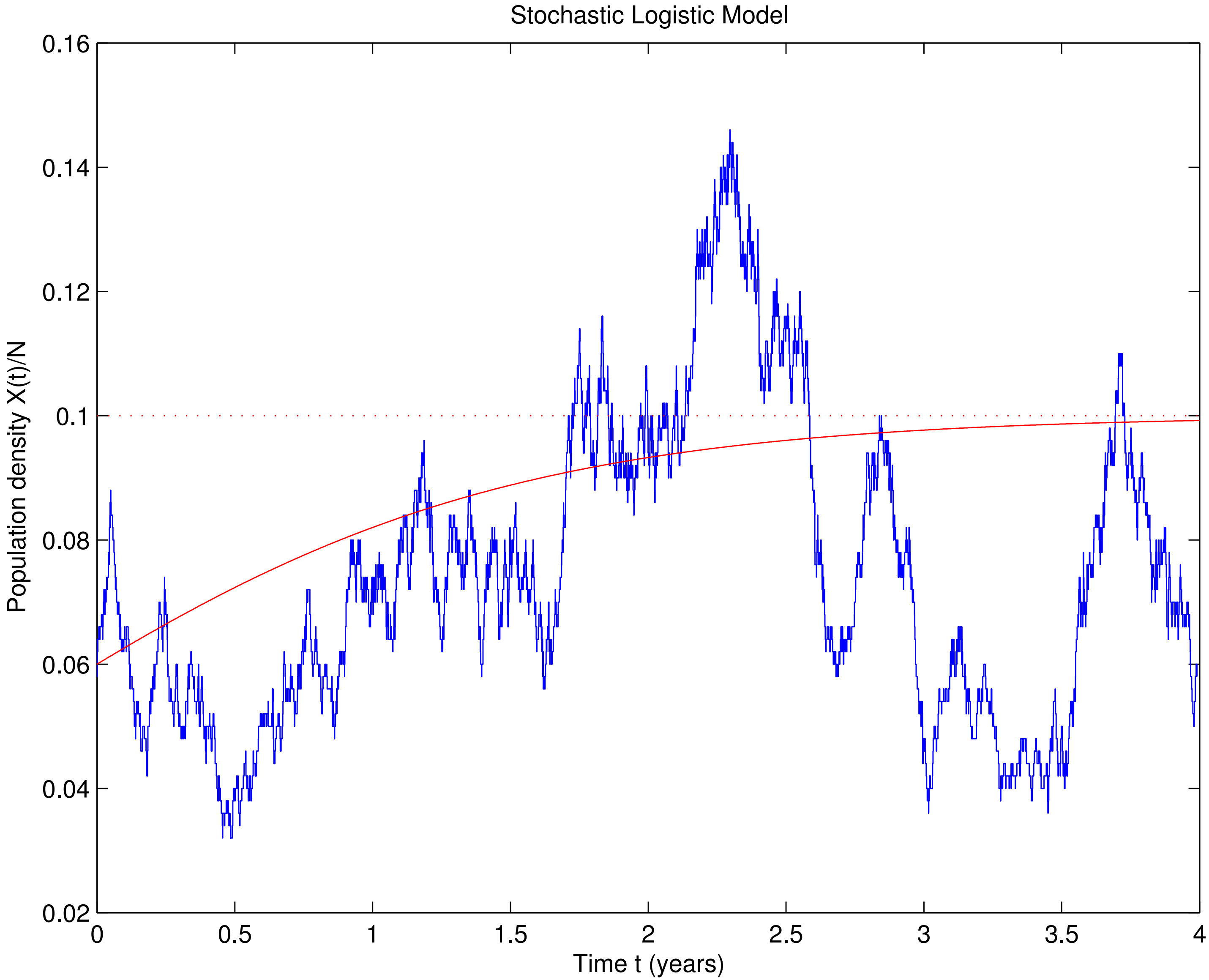
We learn that

$$p(t) = \frac{mp_0}{np_0 + (m - np_0)e^{-mt}}, \qquad t \geq 0.$$

For us,

$$\frac{X(t)}{N} \sim \frac{(1-\rho)x_0}{x_0 + (1 - \rho - x_0)e^{-\lambda(1-\rho)t}}, \quad \text{where } \rho = \frac{\mu}{\lambda}.$$

A population process



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$$\lambda_i = \frac{\lambda}{N} i(N - i) \quad \text{and} \quad \mu_i = \mu i,$$

where $\lambda, \mu > 0$.

The *epidemic model* mentioned earlier is a two-dimensional Markov chain with transition rates

$$q_{(s\ i),(s+1\ i)} = \alpha s, \qquad q_{(s\ i),(s\ i-1)} = \gamma i,$$

$$q_{(s\ i),(s-1\ i+1)} = \beta s i,$$

where $\alpha, \gamma, \beta > 0$ are the *splitting*, *removal* and *infection* rates.

The expected value of Γ

Returning to our general Markov chain, let $e_i = E_i(\Gamma) := E(\Gamma | X(0) = i)$, and condition on the time of the first jump and the state visited at that time, to get

$$E_i(\Gamma) = \int_0^\infty \sum_{k \neq i} \left(\frac{f_i}{q_i} + E_k(\Gamma) \right) \frac{q_{ik}}{q_i} q_i e^{-q_i u} \, du,$$

which leads to

$$q_i e_i = f_i + \sum_{k \neq i} q_{ik} e_k,$$

so that

$$\sum_k q_{ik} e_k + f_i = 0.$$

The expected value of Γ

We can do better:

Theorem 1 $\boldsymbol{e} = (e_i,\, i \in A)$, where $e_i = E_i(\Gamma)$, is the *minimal* non-negative solution to

$$\sum_{k \in A} q_{ik} z_k + f_i = 0, \quad i \in A,$$

in the sense that $\boldsymbol{e}$ satisfies these equations, and, if $\boldsymbol{z} = (z_i,\, i \in A)$ is any non-negative solution, then $e_i \leq z_i$ for all $i \in A$.

The expected value of Γ

So, we solve a system of linear equations to obtain the vector of expected total costs starting in the various states:

$$Q\boldsymbol{z} = -\boldsymbol{f}$$

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Transition rates restricted to A (the model)

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Transition rates restricted to A (the model)

Unit costs

The expected value of Γ

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$$Q\boldsymbol{z} = -\boldsymbol{f}$$

Transition rates restricted to A (the model)

Expected total cost (minimal solution)

Unit costs

Birth-death processes

Let's apply this to *birth-death processes*:

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \cdots \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix}$$

Assume that the birth rates $(\lambda_i,\ i \geq 1)$ and the death rates $(\mu_i,\ i \geq 0)$ are all strictly positive, except that $\lambda_0 = 0$. So, all states in $A = \{1, 2, \ldots\}$ intercommunicate, and 0 is an absorbing state (corresponding to population extinction).

Birth-death processes

Define $(\pi_i,\ i \geq 1)$ by $\pi_1 = 1$ and

$$\pi_i = \prod_{j=2}^{i} \frac{\lambda_{j-1}}{\mu_j}, \qquad i \geq 2,$$

and assume that

$$\sum_{i=1}^{\infty} \frac{1}{\mu_i \pi_i} = \infty,$$

a condition that corresponds to extinction being certain.

Birth-death processes

On applying Theorem 1 we get:

Proposition The expected cost up to the time of extinction, starting in state i (≥ 1), is given by

$$E_i(\Gamma) = \sum_{j=1}^{i} \frac{1}{\mu_j \pi_j} \sum_{k=j}^{\infty} f_k \pi_k,$$

this being finite if and only if $\sum_{k=1}^{\infty} f_k \pi_k < \infty$.

Birth-death processes

In the finite state-space case ($S = \{0, 1, \dots, N\}$), we get

$$E_i(\Gamma) = \sum_{j=1}^{i} \frac{1}{\mu_j \pi_j} \sum_{k=j}^{N} f_k \pi_k, \qquad i = 1, 2, \dots, N.$$

For the Stochastic Logistic Model,

$$E_i(\Gamma) = \frac{1}{\mu} \sum_{j=1}^{i} \sum_{k=0}^{N-j} \left(\frac{1}{N\rho}\right)^k \frac{f_{j+k}}{j+k} \frac{(N-j)!}{(N-j-k)!},$$

where $\rho = \mu/\lambda$. If $\rho < 1$ (the interesting case),

$$E_i(\Gamma) \sim \frac{\rho}{\mu(1-\rho)} \left(\frac{e^{-(1-\rho)}}{\rho}\right)^N \sqrt{\frac{2\pi}{N}} \sum_{j=1}^{i} f_j \rho^j \quad \text{as } N \to \infty.$$

The distribution of Γ

Can we evaluate the *distribution* of Γ, that is,

$$\Pr(\Gamma \leq x | X(0) = i)\,?$$

The distribution of Γ

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$$\Pr(\Gamma \leq x | X(0) = i)\,?$$

I will explain how to evaluate $y_i(\theta) = E_i(e^{-\theta\Gamma})$, the Laplace-Steiltjes Transform (LST) of the distribution:

$$y_i(\theta) = \int_0^\infty e^{-\theta x}\, d\Pr(\Gamma \leq x | X(0) = i).$$

The distribution of Γ

An argument similar to that used to evaluate $E_i(\Gamma)$ leads to:

Theorem 2 For each $\theta > 0$, $\boldsymbol{y}(\theta) = (y_i(\theta), i \in S)$ is the *maximal* solution to

$$\sum_{k \in S} q_{ik} z_k = \theta f_i z_i, \quad i \in A,$$

with $0 \leq z_i \leq 1$ for $i \in A$ and $z_i = 1$ for $i \notin A$.

A catastrophe process

Assume that the transition rates have the form

$$q_{ij} = \begin{cases} i\rho a, & i \geq 0,\ j = i+1, \\ -i\rho, & i \geq 0,\ j = i, \\ i\rho d_{i-j}, & i \geq 2,\ 1 \leq j < i, \\ i\rho \sum_{k \geq i} d_k, & i \geq 1,\ j = 0, \end{cases}$$

with all other transition rates equal to 0. Here ρ and a are positive, d_i is positive for at least one i in $A = \{1, 2, \ldots\}$ and $a + \sum_{i=1}^{\infty} d_i = 1$.

Clearly 0 is an absorbing state for the process and A is a communicating class.

A catastrophe process

We will consider only the *subcritical case*, where the drift D, given by $D = a - \sum_{i=1}^{\infty} i d_i$, is strictly negative and extinction is certain.

Let $b(s) = d(s) - s$, where d is the probability generating function $d(s) = a + \sum_{i=1}^{\infty} d_i s^{i+1}$, $|s| < 1$.

There is a unique solution, σ, to $b(s) = 0$ on the interval $0 < s < 1$.

A catastrophe process

We can evaluate $E_i(e^{-\theta\Gamma})$ for specific choices of f.

For example, take $f_i = i$.

We seek the maximal solution to

$$\sum_{j=0}^{\infty} q_{ij} z_j = \theta i z_i, \qquad i \geq 1,$$

satisfying $0 \leq z_i \leq 1$ for $i \geq 1$ and $z_0 = 1$.

A catastrophe process

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We seek the maximal solution to

$$\rho a z_{i+1} - \rho z_i + \rho \sum_{j=1}^{i-1} d_{i-j} z_j + \rho z_0 \sum_{j=i}^{\infty} d_j = \theta z_i, \quad i \geq 1,$$

satisfying $0 \leq z_i \leq 1$ for $i \geq 1$ and $z_0 = 1$.

A catastrophe process

Multiplying by s^{i-1} and summing over i gives

$$\sum_{i=1}^{\infty} E_i(e^{-\theta\Gamma})s^{i-1} = \frac{1}{1-s} - \frac{\theta(\gamma_\theta - s)}{(1-\gamma_\theta)(1-s)(\rho b(s) - \theta s)},$$

where γ_θ is the unique solution to $\rho b(s) = \theta s$ on the interval $0 < s < \sigma$, where σ itself is the unique solution to $b(s) = 0$ on the interval $0 < s < 1$.

A catastrophe process

In the case of "geometric catastrophes" ($d_i = d(1-q)q^{i-1}$, $i \geq 1$, where $d > 0$ satisfies $a + d = 1$, and $0 \leq q < 1$), we get

$$E_i(e^{-\theta\Gamma}) = \frac{\beta(\theta) - q}{1 - q} \left(\beta(\theta)\right)^{i-1}, \quad i \geq 1,$$

where $\beta(\theta)$ is the smaller of the two zeros of $a\rho s^2 - (\rho(1 + qa) + \theta)s + \rho(d + qa) + q\theta$.

Workshop

ARC Centre of Excellence for Mathematics and Statistics of Complex Systems

Workshop on Metapopulations

The University of Queensland
Thursday 2nd September 2004

Invited speakers: Andrew Barbour (University of Zürich) Ben Cairns, Phil Pollett, Hugh Possingham, Tracey Regan, Joshua Ross, Severine Vuilleumier and Chris Wilcox (University of Queensland).

URL: http://www.maths.uq.edu.au/~pkp/MetaPop04.html