

PBW DEFORMATIONS OF SKEW GROUP ALGEBRAS IN POSITIVE CHARACTERISTIC

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ABSTRACT. We investigate deformations of a skew group algebra that arise from a finite group acting on a polynomial ring. When the characteristic of the underlying field divides the order of the group, a new type of deformation emerges that does not occur in characteristic zero. This analogue of Lusztig's graded affine Hecke algebra for positive characteristic can not be forged from the template of symplectic reflection and related algebras as originally crafted by Drinfeld. By contrast, we show that in characteristic zero, for arbitrary finite groups, a Lusztig-type deformation is always isomorphic to a Drinfeld-type deformation. We fit all these deformations into a general theory, connecting Poincaré-Birkhoff-Witt deformations and Hochschild cohomology when working over fields of arbitrary characteristic. We make this connection by way of a double complex adapted from Guccione, Guccione, and Valqui, formed from the Koszul resolution of a polynomial ring and the bar resolution of a group algebra.

1. INTRODUCTION

Deformations of skew group algebras in positive characteristic provide a rich theory that differs substantially from the characteristic zero setting, as we argue in this article. Such deformations over fields of characteristic zero are classical. Lusztig used a graded version of the affine Hecke algebra over the complex numbers to investigate the representation theory of groups of Lie type. His graded affine Hecke algebra is a deformation of the natural semi-direct product algebra, i.e., skew group algebra, arising from a Coxeter group acting on a polynomial ring. Lusztig [19] described the algebra in terms of generators and relations that deform the action of the group but preserve the polynomial ring. Around the same time, Drinfeld [8] gave a deformation of the same skew group algebra, for arbitrary finite

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groups, by instead deforming the polynomial ring while preserving the action of the group. In characteristic zero, Ram and the first author [21] showed that these two deformations are isomorphic by producing an explicit conversion map. In this paper, we generalize that result from Coxeter groups to arbitrary finite groups and show that any algebra modeled on Lusztig's graded affine Hecke algebra in characteristic zero must arise via Drinfeld's construction. In fact, we explain that in the nonmodular setting, when the characteristic of the underlying field does not divide the order of the acting group, any deformation of the skew group algebra that changes the action of the group must be equivalent to one that does not. We also give explicit isomorphisms with which to convert between algebra templates.

These results fail in positive characteristic. Indeed, in the modular setting, the Hochschild cohomology of the acting group algebra is nontrivial and thus gives rise to deformations of the skew group algebra of a completely new flavor. Study has recently begun on the representation theory of deformations of skew group algebras in positive characteristic (see [1, 2, 3, 6, 20] for example) and these new types of deformations may help in developing a unified theory. In addition to combinatorial tools, we rely on homological algebra to reveal and to understand these new types of deformations that do not appear in characteristic zero.

We first establish in Section 2 a Poincaré-Birkhoff-Witt (PBW) property for nonhomogeneous quadratic algebras (noncommutative) that simultaneously generalize the construction patterns of Drinfeld and of Lusztig. We use the PBW property to describe deformations of the skew group algebra as nonhomogeneous quadratic algebras in terms of generators and relations, defining the algebras $\mathcal{H}_{\lambda,\kappa}$ that feature in the rest of the article. Theorem 3.1 gives conditions on the parameter functions λ and κ that are equivalent to $\mathcal{H}_{\lambda,\kappa}$ being a PBW deformation. In Section 4, we give an explicit isomorphism between the Lusztig style and the Drinfeld style deformations in characteristic zero. We give examples in Section 5 to show that such an isomorphism can fail to exist in positive characteristic. In the next two sections, we recall the notion of graded deformation and a resolution required to make a precise connection between our deformations and Hochschild cohomology. Theorem 8.3 shows that our conditions in Theorem 3.1, obtained combinatorially, correspond exactly to explicit homological conditions. In Proposition 6.5 and Theorem 6.11, we pinpoint precisely the class of deformations comprising the algebras $\mathcal{H}_{\lambda,\kappa}$.

Throughout this article, k denotes a field (of arbitrary characteristic) and tensor symbols without subscript denote tensor product over k : $\otimes = \otimes_k$. (Tensor products over other rings will always be indicated). We assume all k -algebras have unity and we use the notation $\mathbb{N} = \mathbb{Z}_{\geq 0}$ for the nonnegative integers.

2. FILTERED QUADRATIC ALGEBRAS

Let V be a finite dimensional vector space over the field k . We consider a finite group G acting linearly on V and extend the action so that G acts by automorphisms on the tensor algebra (free associative k -algebra) $T(V)$ and on the symmetric algebra $S(V)$ of V over k . We write ${}^g v$ for the action of any g in G on any element v in V , $S(V)$, or $T(V)$, to distinguish from the product of g and v in an algebra, and we let $V^g \subset V$ be the set of vectors fixed pointwise by g . We also identify the identity of the field k with the identity group element of G in the group ring kG , so that $1_k = 1_G$.

We consider an algebra generated by V and the group G with a set of relations that corresponds to deforming both the symmetric algebra $S(V)$ on V and the action of the group G . Specifically, let λ and κ be linear parameter functions,

$$\kappa : V \otimes V \rightarrow kG, \quad \lambda : kG \otimes V \rightarrow kG,$$

with κ alternating. We write $\kappa(v, w)$ for $\kappa(v \otimes w)$ and $\lambda(g, v)$ for $\lambda(g \otimes v)$ for ease with notation throughout the article (as λ and κ both define bilinear functions).

Let $\mathcal{H}_{\lambda, \kappa}$ be the associative k -algebra generated by a basis of V together with the group algebra kG , subject to the relations in kG and

$$(2.1) \quad \begin{array}{l} \text{a. } vw - wv = \kappa(v, w) \\ \text{b. } gv - {}^g v g = \lambda(g, v) \end{array}$$

for all v, w in V and g in G . Formally, $\mathcal{H}_{\lambda, \kappa}$ is a quotient of the tensor algebra on $kG \oplus V$:

$$\mathcal{H}_{\lambda, \kappa} := T_k(kG \oplus V) / I$$

where I is the two-sided ideal generated by

$$v \otimes w - w \otimes v - \kappa(v, w), \quad g \otimes v - {}^g v \otimes g - \lambda(g, v), \quad g \otimes h - gh, \quad 1_k - 1_G$$

for all g, h in G and $v, w \in V$, where we identify each v with $(0, v)$ in $kG \oplus V$ and each g with $(g, 0)$ in $kG \oplus V$.

Recall that the **skew group algebra** of G acting by automorphisms on a k -algebra R (for example, $R = S(V)$ or $R = T(V)$) is the semidirect product algebra $R \# G$: It is the k -vector space $R \otimes kG$ together with multiplication given by

$$(r \otimes g)(s \otimes h) = r({}^g s) \otimes gh$$

for all r, s in R and g, h in G . To simplify notation, we write rg in place of $r \otimes g$. Note that the skew group algebra $S(V) \# G$ is precisely the algebra $\mathcal{H}_{0,0}$.

We filter $\mathcal{H}_{\lambda, \kappa}$ by degree, assigning $\deg v = 1$ and $\deg g = 0$ for each v in V and g in G . Then the associated graded algebra $\text{gr } \mathcal{H}_{\lambda, \kappa}$ is a quotient of its homogeneous version $S(V) \# G$ obtained by crossing out all but the highest homogeneous part of each generating relation defining $\mathcal{H}_{\lambda, \kappa}$ (see Li [17, Theorem 3.2] or Braverman and Gaitsgory [5]). Recall that a ‘‘PBW property’’ on a filtered algebra indicates that the homogeneous version (with respect to some generating set of relations)

and its associated graded algebra coincide. Thus we say that $\mathcal{H}_{\lambda,\kappa}$ exhibits the **Poincaré-Birkhoff-Witt (PBW) property** (or is of **PBW type**) when

$$\text{gr}(\mathcal{H}_{\lambda,\kappa}) \cong S(V)\#G$$

as graded algebras. In this case, we call $\mathcal{H}_{\lambda,\kappa}$ a **PBW deformation** of $S(V)\#G$. Note that the algebra $\mathcal{H}_{\lambda,\kappa}$ exhibits the PBW property if and only if it has basis $\{v_1^{i_1}v_2^{i_2}\cdots v_m^{i_m}g : i_j \in \mathbb{N}, g \in G\}$ as a k -vector space, where v_1, \dots, v_m is any basis of V . In Theorem 3.1 below, we give necessary and sufficient conditions on λ and κ in order for the PBW property to hold.

Lusztig's graded version of the affine Hecke algebra (see [18, 19]) for a finite Coxeter group is a special case of a quotient algebra $\mathcal{H}_{\lambda,0}$ (with $\kappa \equiv 0$) displaying the PBW property; the parameter λ is defined using a simple root system (see Section 4). Rational Cherednik algebras and symplectic reflection algebras are also examples; they arise as quotient algebras $\mathcal{H}_{0,\kappa}$ (with $\lambda \equiv 0$) exhibiting the PBW property.

Example 2.2. Let k be a field of characteristic p and consider a cyclic group G of prime order p , generated by g . Suppose $V = k^2$ with basis v, w and let g act as the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

on the ordered basis v, w . Define an alternating linear parameter function $\kappa : V \otimes V \rightarrow kG$ by $\kappa(v, w) = g$ and a linear parameter function $\lambda : kG \otimes V \rightarrow kG$ by $\lambda(g^i, v) = 0$, $\lambda(g^i, w) = ig^{i-1}$ for $i = 0, 1, \dots, p-1$. Then the algebra $\mathcal{H}_{\lambda,\kappa}$ is a quotient of a free k -algebra on three generators:

$$\mathcal{H}_{\lambda,\kappa} = k\langle v, w, g \rangle / (gv - vg, gw - vg - wg - 1, vw - wv - g, g^p - 1).$$

We will see that $\mathcal{H}_{\lambda,\kappa}$ has the PBW property by applying Theorem 3.1 below (see Example 3.4). Alternatively, this may be checked directly by applying Bergman's Diamond Lemma [4] to this small example.

3. POINCARÉ-BIRKHOFF-WITT PROPERTY

In this section, we establish necessary and sufficient conditions for the PBW property to hold for the filtered quadratic algebra $\mathcal{H}_{\lambda,\kappa}$ defined by relations (2.1). Note that a PBW condition on $\mathcal{H}_{\lambda,\kappa}$ implies that none of the elements in kG collapse in $\mathcal{H}_{\lambda,\kappa}$. This in turn gives conditions on λ and κ when their images are expanded in terms of the elements of G . For each g in G , let $\lambda_g : kG \otimes V \rightarrow k$ and $\kappa_g : V \otimes V \rightarrow k$ be the k -linear maps for which

$$\lambda(h, v) = \sum_{g \in G} \lambda_g(h, v)g, \quad \kappa(u, v) = \sum_{g \in G} \kappa_g(u, v)g$$

for all h in G and u, v in V . We now record PBW conditions on the parameters λ and κ . Note that the first condition below implies that λ is determined by its

values on generators of the group G , and the left hand side of the second condition measures the failure of κ to be G -invariant.

Theorem 3.1. *Let k be a field of arbitrary characteristic. The algebra $\mathcal{H}_{\lambda,\kappa}$ exhibits the PBW property if, and only if, the following conditions hold for all g, h in G and u, v, w in V .*

- (1) $\lambda(gh, v) = \lambda(g, {}^h v)h + g\lambda(h, v)$ in kG .
- (2) $\kappa({}^g u, {}^g v)g - g\kappa(u, v) = \lambda(\lambda(g, v), u) - \lambda(\lambda(g, u), v)$ in kG .
- (3) $\lambda_h(g, v)({}^h u - {}^g u) = \lambda_h(g, u)({}^h v - {}^g v)$ in V .
- (4) $\kappa_g(u, v)({}^g w - w) + \kappa_g(v, w)({}^g u - u) + \kappa_g(w, u)({}^g v - v) = 0$ in V .
- (5) $\lambda(\kappa(u, v), w) + \lambda(\kappa(v, w), u) + \lambda(\kappa(w, u), v) = 0$ in kG .

Proof. The result follows from careful application of Bergman's Diamond Lemma [4]. Choose a monoid partial order on $\{v_1, \dots, v_m, g : g \in G\}$ for some basis v_1, \dots, v_m of V compatible with the reduction system (see [7]) defined by the generating relations (2.1) of $\mathcal{H}_{\lambda,\kappa}$ which is a total order (see [16, Remarks 7.1 and 7.2] for an example). We check overlap ambiguities of the form ghv, guv, uvw for g, h in G and u, v, w in V . Setting ambiguities to zero (see [25]) results in the five conditions of the theorem.

For example, in $\mathcal{H}_{\lambda,\kappa}$,

$$\begin{aligned}
0 &= g(uv) - (gu)v \\
&= g(vu + \kappa(u, v)) - ({}^g u g + \lambda(g, u))v \\
&= ({}^g v g + \lambda(g, v))u + g\kappa(u, v) - {}^g u ({}^g v g + \lambda(g, v)) - \lambda(g, u)v \\
&= {}^g v ({}^g u g + \lambda(g, u)) + \lambda(g, v)u + g\kappa(u, v) \\
&\quad - ({}^g v {}^g u + \kappa({}^g u, {}^g v))g - {}^g u \lambda(g, v) - \lambda(g, u)v \\
&= ({}^g v \lambda(g, u) - \lambda(g, u)v) - ({}^g u \lambda(g, v) - \lambda(g, v)u) + (g\kappa(u, v) - \kappa({}^g u, {}^g v)g).
\end{aligned}$$

We expand:

$$\begin{aligned}
{}^g v \lambda(g, u) - \lambda(g, u) v &= \sum_{h \in G} {}^g v \lambda_h(g, u) h - \lambda_h(g, u) h v \\
&= \sum_{h \in G} {}^g v \lambda_h(g, u) h - \lambda_h(g, u) ({}^h v h + \lambda(h, v)) \\
&= \sum_{h \in G} ({}^g v \lambda_h(g, u) h - {}^h v \lambda_h(g, u) h) - \sum_{h \in G} \lambda_h(g, u) \lambda(h, v) \\
&= \sum_{h \in G} ({}^g v - {}^h v) \lambda_h(g, u) h - \lambda \left(\sum_{h \in G} \lambda_h(g, u) h, v \right) \\
&= \sum_{h \in G} ({}^g v - {}^h v) \lambda_h(g, u) h - \lambda(\lambda(g, u), v).
\end{aligned}$$

$$\text{Likewise, } {}^g u \lambda(g, v) - \lambda(g, v) u = \sum_{h \in G} ({}^g u - {}^h u) \lambda_h(g, v) h - \lambda(\lambda(g, v), u).$$

Hence, in $\mathcal{H}_{\lambda, \kappa}$, we may write $0 = g(uv) - (gu)v$ as

$$\begin{aligned}
&\sum_{h \in G} \left(({}^g v - {}^h v) \lambda_h(g, u) - ({}^g u - {}^h u) \lambda_h(g, v) \right) h \\
&\quad - \left(\lambda(\lambda(g, u), v) - \lambda(\lambda(g, v), u) \right) + (g\kappa(u, v) - \kappa({}^g u, {}^g v)g).
\end{aligned}$$

If $\mathcal{H}_{\lambda, \kappa}$ exhibits the PBW property, then the degree zero term vanishes as an element of kG yielding Condition (2) and the degree one term vanishes as an element of $kG \otimes V$ yielding Condition (3). The other conditions follow from similar considerations. \square

The conditions of Theorem 3.1 are quite restrictive. The following corollaries contain some identities derived from these conditions. These identities may be applied to narrow the possible parameter functions λ, κ in examples. We write $\text{Ker } \kappa_g$ for the set $\{v \in V : \kappa_g(v, w) = 0 \text{ for all } w \in V\}$. The following observation arose originally in some joint work with Stephen Griffeth.

Corollary 3.2. *Let k be a field of arbitrary characteristic. Suppose $\mathcal{H}_{\lambda, \kappa}$ is of PBW type. Then for every g in G , either $\kappa_g \equiv 0$ or one of the following statements holds.*

- (a) $\text{codim } V^g = 0$ (i.e., g acts as the identity on V).
- (b) $\text{codim } V^g = 1$ and $\kappa_g(v_1, v_2) = 0$ for all v_1, v_2 in V^g .
- (c) $\text{codim } V^g = 2$ with $\text{Ker } \kappa_g = V^g$
(and thus κ_g is determined by its values on a subspace complement to V^g).

Proof. Fix an element g of G that does not act as the identity on V . Suppose κ_g is not identically zero, that is, $\kappa_g(u, v) \neq 0$ for some u, v in V . We compare the dimensions of the kernel and image of the transformation T defined by $g - 1$

acting on V . (Note that V may lack a G -invariant inner product, and the action of T will replace classical arguments that involve the orthogonal complement of V^g .) Condition 4 of Theorem 3.1 implies that the image of T is spanned by ${}^g u - u$ and ${}^g v - v$ and thus

$$\text{codim } V^g = \text{codim Ker}(T) = \dim \text{Im}(T) \leq 2.$$

Now assume $\text{codim } V^g = 1$. Condition 4 of Theorem 3.1 then implies that $\kappa_g(v_1, v_2) = 0$ whenever both v_1, v_2 lie in V^g (just take some $w \notin V^g$). Now suppose $\text{codim } V^g = 2$ and thus ${}^g u - u, {}^g v - v$ are linearly independent. Then so are u, v , and $V = V^g \oplus ku \oplus kv$. Then

$$\kappa_g(u, w) = \kappa_g(v, w) = \kappa_g(v_1, w) = 0$$

for any v_1, w in V^g (again by Condition 4) and thus $\text{Ker } \kappa_g = V^g$ with κ_g supported on the 2-dimensional space spanned by u and v . \square

We say $\lambda(g, *)$ is **supported** on a set $S \subset G$ whenever $\lambda_h(g, v) = 0$ for all v in V and h not in S .

Corollary 3.3. *Let k be a field of arbitrary characteristic. Suppose $\mathcal{H}_{\lambda, \kappa}$ is of PBW type. Then for all g in G :*

- (1) $\lambda(1, *)$ is identically zero: $\lambda(1, v) = 0$ for all v in V ;
- (2) $\lambda(g, *)$ determines $\lambda(g^{-1}, *)$ by

$$g\lambda(g^{-1}, v) = -\lambda(g, {}^{g^{-1}}v)g^{-1};$$

- (3) $\lambda(g, *)$ can be defined recursively: For any $j \geq 1$,

$$\lambda(g^j, v) = \sum_{i=0}^{j-1} g^{j-1-i} \lambda(g, {}^{g^i}v) g^i;$$

- (4) $\lambda(g, *)$ is supported on h in G with $h^{-1}g$ either a reflection or the identity on V . If it is a reflection, then $\lambda_h(g, v) = 0$ for all v on the reflecting hyperplane.
- (5) If $V^g \neq V$, $\lambda_1(g, v) = 0$ unless g is a reflection and $v \notin V^g$.

Proof. Setting $g = h = 1$ in Theorem 3.1 (1), we obtain $\lambda(1, v) = \lambda(1, v) + \lambda(1, v)$, and thus $\lambda(1, v) = 0$. Setting $h = g^{-1}$, we find that

$$0 = \lambda(1, v) = \lambda(g, {}^{g^{-1}}v)g^{-1} + g\lambda(g^{-1}, v).$$

Recursively applying the same condition yields the expression for $\lambda(g^j, v)$.

Now suppose the action of group elements g and h on V do not coincide, so that the image of the transformation T on V defined by $h - g$ has dimension at least 1. Fix any nonzero u in V with ${}^g u \neq {}^h u$. Then Condition 3 of Theorem 3.1 implies that $\lambda_h(g, *)$ vanishes on $V^{h^{-1}g} = \text{Ker}(T)$. If $\dim \text{Im}(T) \geq 2$, then there is some w in V with ${}^h u - {}^g u$ and ${}^h w - {}^g w$ linearly independent in $\text{Im}(T)$ and

thus $\lambda_h(g, u) = 0$, i.e., $\lambda_h(g, *)$ vanishes on the set complement of $V^{h^{-1}g}$ as well. Therefore, $\lambda_h(g, *)$ is identically zero unless $\text{codim Ker}(T) = 1$ and $g^{-1}h$ is a reflection, in which case $\lambda_h(g, *)$ is supported off the reflecting hyperplane. The last statement of the corollary is merely the special case when $h = 1$. \square

Example 3.4. Applying Theorem 3.1, straightforward calculations show that the algebra $\mathcal{H}_{\lambda, \kappa}$ of Example 2.2 has the PBW property.

4. NONMODULAR SETTING AND GRADED AFFINE HECKE ALGEBRAS

Lusztig's graded version of the affine Hecke algebra is defined for a Coxeter group $G \subseteq \text{GL}(V)$ with natural reflection representation \mathbb{R}^m extended to $V = \mathbb{C}^m$. This algebra deforms the skew group relation defining $S(V)\#G$ and not the commutator relation defining $S(V)$. Indeed, the **graded affine Hecke algebra** $\mathcal{H}_{\text{gr aff}}$ (often just called the “graded Hecke algebra”) with parameter λ is the associative \mathbb{C} -algebra generated by the group algebra $\mathbb{C}G$ and the symmetric algebra $S(V)$ with relations

$$gv = {}^gvg + \lambda(g, v) \quad \text{for all } v, w \in V \text{ and } G$$

where $\lambda : \mathbb{C}G \otimes V \mapsto \mathbb{C}$ is a G -invariant map (for G acting on itself by conjugation) vanishing on pairs (g, v) with g a reflection whose reflecting hyperplane contains v .

Graded affine Hecke algebras are often defined merely using a class function on G and a set of simple reflections \mathcal{S} generating G . This allows one to record concretely the degrees of freedom in defining these algebras. Indeed, since kG is a subalgebra of $\mathcal{H}_{\text{gr aff}}$, the parameter function λ is determined by its values on \mathcal{S} (see Theorem 3.1 (1)) and since λ is linear, it can be expressed using linear forms in V^* defining the reflecting hyperplanes for G . We fix a G -invariant inner product on V . Then for any reflection g in \mathcal{S} and v in V ,

$$\lambda(g, v) = c_g \frac{2\langle v, \alpha_g \rangle}{\langle \alpha_g, \alpha_g \rangle} = c_g \langle v, \alpha_g^\vee \rangle = c_g \left(\frac{v - {}^g v}{\alpha_g} \right)$$

where $c : G \rightarrow \mathbb{C}$, $g \mapsto c_g$ is some conjugation invariant function (i.e., $c_g = c_{hgh^{-1}}$ for all g, h in G). Here, each α_g is a fixed “root vector” for the reflection g (i.e., a vector in V perpendicular to the reflecting hyperplane fixed pointwise by g) chosen so that conjugate reflections have root vectors in the same orbit. Note that the parameter λ can be extended to $\mathbb{C}G \otimes S(V)$ using Demazure-Lusztig (BGG) operators, see [15] for example.

Every graded affine Hecke algebra thus arises as an algebra $\mathcal{H}_{\lambda, \kappa}$ given by relations (2.1) with $\kappa \equiv 0$, in fact, an algebra with the PBW property (see [19]):

$$\mathcal{H}_{\text{gr aff}} = \mathcal{H}_{\lambda, 0}$$

and $\mathcal{H}_{\text{gr aff}} \cong S(V)\#G$ as \mathbb{C} -vector spaces. Ram and the first author [21] showed that every graded affine Hecke algebra arises as a special case of a **Drinfeld Hecke**

algebra, i.e., every $\mathcal{H}_{\text{gr aff}}$ is isomorphic (as a filtered algebra) to $\mathcal{H}_{0,\kappa}$ for some parameter κ :

$$\mathcal{H}_{\lambda,0} = \mathcal{H}_{\text{gr aff}} \cong \mathcal{H}_{0,\kappa}.$$

Indeed, they constructed an explicit isomorphism between any graded affine Hecke algebra (for G a Coxeter group) and a Drinfeld Hecke algebra $\mathcal{H}_{0,\kappa}$ with parameter κ defined in terms of the class function c and positive roots.

In this section, we prove that this result follows from a more general fact about the algebras $\mathcal{H}_{\lambda,\kappa}$ for arbitrary finite groups (not just Coxeter groups) and we give a proof valid over arbitrary fields in the nonmodular setting: Any $\mathcal{H}_{\lambda,0}$ (deforming just the skew group relations) with the PBW property is isomorphic to some $\mathcal{H}_{0,\kappa}$ (deforming just the relations for the commutative polynomial ring $S(V)$). In the proof, we construct an explicit isomorphism (of filtered algebras) converting between the two types of nonhomogeneous quadratic algebras. In the next section, we give an example showing that the theorem fails in positive characteristic.

Theorem 4.1. *Suppose G acts linearly on a finite dimensional vector space V over a field k whose characteristic is coprime to $|G|$. If the algebra $\mathcal{H}_{\lambda,0}$ exhibits the PBW property for some parameter $\lambda : kG \otimes V \rightarrow kG$, then there exists a parameter $\kappa : V \otimes V \rightarrow kG$ with*

$$\mathcal{H}_{0,\kappa} \cong \mathcal{H}_{\lambda,0}$$

as filtered algebras.

Proof. Define $\gamma : V \rightarrow kG$ by

$$\gamma(v) = \frac{1}{|G|} \sum_{a,b \in G} \lambda_{ab}(b, b^{-1}v)a$$

and set

$$\gamma_a(v) := \frac{1}{|G|} \sum_{b \in G} \lambda_{ab}(b, b^{-1}v)$$

so that $\gamma = \sum_a \gamma_a a$. Define a parameter function $\kappa : V \otimes V \rightarrow kG$ by

$$\kappa(u, v) = \gamma(u)\gamma(v) - \gamma(v)\gamma(u) + \lambda(\gamma(u), v) - \lambda(\gamma(v), u).$$

Let F be the associative k -algebra generated by v in V and the algebra kG , i.e.,

$$F = T_k(kG \oplus V) / (g \otimes h - gh, 1_k - 1_G : g, h \in G)$$

(where we identify each g with $(g, 0) \in kG \oplus V$). Consider an algebra homomorphism $f : F \rightarrow \mathcal{H}_{\lambda,0}$ defined by

$$f(v) = v + \gamma(v) \quad \text{and} \quad f(g) = g \quad \text{for all } v \in V \text{ and } g \in G.$$

We claim that Theorem 3.1 forces the relations defining $\mathcal{H}_{0,\kappa}$ to lie in the kernel of f and thus this map extends to a filtered algebra homomorphism

$$f : \mathcal{H}_{0,\kappa} \rightarrow \mathcal{H}_{\lambda,0}.$$

We first argue that Condition 3 of Theorem 3.1 implies that $uv - vu - \kappa(u, v)$ is mapped to zero under f for all u, v in V . Indeed, applying f to the commutator of u and v gives the commutator of u and v added to the commutator of $\gamma(u)$ and $\gamma(v)$ plus cross terms. But the commutator of u and v is zero in $\mathcal{H}_{\lambda,0}$, and by definition of κ , the commutator of $\gamma(u)$ and $\gamma(v)$ is just

$$\kappa(u, v) - \lambda(\gamma(u), v) + \lambda(\gamma(v), u).$$

We expand the cross terms:

$$\begin{aligned} & u\gamma(v) - \gamma(v)u - v\gamma(u) + \gamma(u)v \\ &= \sum_{g \in G} \left(\gamma_g(v)(u - {}^g u) - \gamma_g(u)(v - {}^g v) \right) g - \gamma_g(v)\lambda(g, u) + \gamma_g(u)\lambda(g, v) \\ &= -\lambda(\gamma(v), u) + \lambda(\gamma(u), v) + \sum_{g \in G} \left(\gamma_g(v)(u - {}^g u) - \gamma_g(u)(v - {}^g v) \right) g. \end{aligned}$$

We rewrite Condition 3 to show that the above sum over g is zero: Relabel ($h = ab, g = b, {}^g u \mapsto u$, etc.) to obtain an equivalent condition,

$$0 = \lambda_{ab}(b, {}^{b^{-1}}v)(u - {}^a u) - \lambda_{ab}(b, {}^{b^{-1}}u)(v - {}^a v),$$

and then divide by $|G|$ and sum over all b in G to obtain

$$0 = \gamma_a(v)(u - {}^a u) - \gamma_a(u)(v - {}^a v)$$

for all a in G and u, v in V . Hence the commutator of u and v maps to $\kappa(u, v)$ in $\mathcal{H}_{\lambda,0}$ under f , and $uv - vu - \kappa(u, v)$ lies in the kernel of f .

We now argue that Condition 1 of Theorem 3.1 implies that $cv - {}^cvc$ lies in the kernel of f for all c in G and v in V , i.e.,

$$f(cv - {}^cvc) = c\gamma(v) - \gamma({}^c v)c + \lambda(c, v)$$

is zero in $\mathcal{H}_{\lambda,0}$. We expand $c\gamma(v)$ as a double sum over a, b in G , reindex (with $ca \mapsto a$) and apply Corollary 3.3 (2). We likewise expand $\gamma({}^c v)c$ also as a double sum (but reindex with $ac \mapsto a$ and $b \mapsto cb$) and obtain

$$\begin{aligned} c\gamma(v) - \gamma({}^c v)c &= \frac{1}{|G|} \sum_{a,b} \lambda_{ab}(b, {}^{b^{-1}}v)ca - \frac{1}{|G|} \sum_{a,b} \lambda_{ab}(b, {}^{b^{-1}}c)vac \\ &= \frac{1}{|G|} \sum_{a,b} \lambda_{c^{-1}ab}(b, {}^{b^{-1}}v)a - \frac{1}{|G|} \sum_{a,b} \lambda_{ab}(cb, {}^{b^{-1}}v)a \\ &= \frac{-1}{|G|} \sum_{a,b} \lambda_{b^{-1}c^{-1}a}(b^{-1}, v)a - \frac{1}{|G|} \sum_{a,b} \lambda_{ab}(cb, {}^{b^{-1}}v)a. \end{aligned}$$

But Condition 1 of Theorem 3.1 implies that

$$\lambda_{b^{-1}c^{-1}a}(b^{-1}, v) + \lambda_{ab}(cb, {}^{b^{-1}}v) = \lambda_a(c, v)$$

and hence

$$c\gamma(v) - \gamma({}^c v)c = -\frac{1}{|G|} \sum_{a,b \in G} \lambda_a(c, v)a = -\lambda(c, v),$$

i.e., $f(cv - {}^c vc) = 0$.

Thus $f : \mathcal{H}_{0,\kappa} \rightarrow \mathcal{H}_{\lambda,0}$ is a surjective homomorphism of filtered algebras. An inverse homomorphism is easy to define ($v \mapsto v - \gamma(v)$ and $g \mapsto g$) and thus f is a filtered algebra isomorphism which in turn induces an isomorphism of graded algebras,

$$\text{gr } \mathcal{H}_{0,\kappa} \cong \text{gr } \mathcal{H}_{\lambda,0} \cong S(V) \# G.$$

Thus $\text{gr } \mathcal{H}_{0,\kappa}$ exhibits the PBW property as well. \square

5. MODULAR SETTING

We now observe that in positive characteristic, the algebras $\mathcal{H}_{\lambda,0}$ (modeled on Lusztig's graded affine Hecke algebra but for arbitrary groups) do not always arise as algebras $\mathcal{H}_{0,\kappa}$ (modeled on Drinfeld Hecke algebras), in contrast to the zero characteristic setting. Indeed, the conclusion of Theorem 4.1 fails for Example 2.2, as we show next. However, allowing a more general parameter function κ , it is possible to obtain the conclusion of Theorem 4.1 for this example (see Example 5.2 below).

Example 5.1. Consider the cyclic 2-group generated by a unipotent upper triangular matrix. Say $G = \langle g \rangle \leq \text{GL}(V)$ where g acts as the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

on the ordered basis v, w of $V = k^2$ (see Example 2.2) and k is the finite field of order 2.

Set $\lambda(g^i, v) = 0$ and $\lambda(g^i, w) = ig^{i-1}$ and let κ be arbitrary (we are particularly interested in the case that κ is zero so that $\mathcal{H}_{\lambda,\kappa}$ fits the pattern of Lusztig's graded affine Hecke algebras). We argue that the algebra $\mathcal{H}_{\lambda,\kappa}$ is not isomorphic to $\mathcal{H}_{0,\kappa'}$ for *any* parameter κ' .

Indeed, suppose $f : \mathcal{H}_{0,\kappa'} \rightarrow \mathcal{H}_{\lambda,\kappa}$ were an isomorphism of filtered algebras for some choice of parameter κ' with $\mathcal{H}_{0,\kappa'}$ exhibiting the PBW property:

$$\begin{aligned} f : \mathcal{H}_{0,\kappa'} &\longrightarrow \mathcal{H}_{\lambda,\kappa}, \\ k[v, w] \# G / \begin{pmatrix} gv - vg, \\ gw - wg - vg, \\ vw - wv - \kappa' \end{pmatrix} &\longrightarrow k[v, w] \# G / \begin{pmatrix} gv - vg, \\ gw - wg - vg - 1, \\ vw - wv - \kappa \end{pmatrix}, \end{aligned}$$

where κ, κ' both lie in $kG = \{0, 1, g, 1 + g\}$. Then

$$\begin{aligned} f(g) &= g, \\ f(v) &= v\gamma_1 + w\gamma_2 + \gamma_3, \text{ and} \\ f(w) &= v\gamma_4 + w\gamma_5 + \gamma_6 \end{aligned}$$

for some γ_i in kG . We apply f to the relation $gv - vg$ in $\mathcal{H}_{0, \kappa'}$ and use the relations in $\mathcal{H}_{\lambda, \kappa}$, the PBW property of $\mathcal{H}_{\lambda, \kappa}$, and the fact that G is abelian to see that $\gamma_2 = 0$. Likewise, the relation $gw - wg - vg$ in $\mathcal{H}_{0, \kappa'}$ implies that $\gamma_1 = \gamma_5 = \gamma_3g$. One may check that this gives a contradiction when we apply f to the relation $vw - wv - \kappa'$ in $\mathcal{H}_{0, \kappa'}$. Hence, f can not be an algebra isomorphism.

Next we will show that if one generalizes the possible parameter functions κ to include those of higher degree, i.e., with image in $kG \oplus (V \otimes kG)$, one does obtain an isomorphism $\mathcal{H}_{\lambda, \kappa} \cong \mathcal{H}_{0, \kappa'}$:

Example 5.2. Let k, G, V be as in Example 2.2. Let κ' be the function defined by $\kappa'(v, w) = g - vg^{-1}$. Let $f : \mathcal{H}_{\lambda, \kappa} \rightarrow \mathcal{H}_{0, \kappa'}$ be the algebra homomorphism defined by

$$f(g) = g, \quad f(v) = v - g^{-1}, \quad f(w) = w.$$

We check that f is well-defined by showing that it takes the relations of $\mathcal{H}_{\lambda, \kappa}$ to 0:

$$\begin{aligned} f(gw - vg - wg - 1) &= gw - (v - g^{-1})g - wg - 1 \\ &= gw - vg - wg = 0 \end{aligned}$$

in $\mathcal{H}_{0, \kappa'}$. Similarly we find

$$\begin{aligned} f(vw - wv - g) &= (v - g^{-1})w - w(v - g^{-1}) - g \\ &= vw - g^{-1}w - wv + wg^{-1} - g \\ &= g - vg^{-1} - g^{-1}w + wg^{-1} - g \\ &= g - vg^{-1} - (-v + w)g^{-1} + wg^{-1} - g = 0 \end{aligned}$$

in $\mathcal{H}_{0, \kappa'}$. See [25] for the general theory of such algebras with more general parameter functions κ' , in particular, the second proof of [25, Theorem 3.1].

One might expect there to be examples $\mathcal{H}_{\lambda, \kappa}$ for which there is no κ' such that $\mathcal{H}_{\lambda, \kappa} \cong \mathcal{H}_{0, \kappa'}$, even allowing more general values for κ' as in the above example. In the next section, we begin analysis of the relevant Hochschild cohomology, which will clarify the distinction between the modular and nonmodular settings, and will help to understand better such examples. In the remainder of this paper, we continue to focus largely on the modular setting.

6. GRADED DEFORMATIONS AND HOCHSCHILD COHOMOLOGY

We recall that for any algebra A over a field k , the Hochschild cohomology of an A -bimodule M in degree n is

$$\mathrm{HH}^n(A, M) = \mathrm{Ext}_{A^e}^n(A, M),$$

where $A^e = A \otimes A^{op}$, and M is viewed as an A^e -module. In case $M = A$, we abbreviate $\mathrm{HH}^n(A) := \mathrm{HH}^n(A, A)$. This cohomology can be investigated by using the bar resolution, that is, the following free resolution of the A^e -module A :

$$(6.1) \quad \cdots \xrightarrow{\delta_3} A \otimes A \otimes A \otimes A \xrightarrow{\delta_2} A \otimes A \otimes A \xrightarrow{\delta_1} A \otimes A \xrightarrow{\delta_0} A \rightarrow 0,$$

where

$$\delta_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^n (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for all $n \geq 0$ and $a_0, \dots, a_{n+1} \in A$. If A is a graded algebra, its Hochschild cohomology inherits the grading.

A **deformation of A over $k[t]$** is an associative $k[t]$ -algebra on the vector space $A[t]$ whose product $*$ is determined by

$$(6.2) \quad a_1 * a_2 = a_1 a_2 + \mu_1(a_1 \otimes a_2)t + \mu_2(a_1 \otimes a_2)t^2 + \cdots$$

where $a_1 a_2$ is the product of a_1 and a_2 in A and each $\mu_j : A \otimes A \rightarrow A$ is a k -linear map (called the j -th *multiplication map*) extended to be linear over $k[t]$. (Only finitely many terms may be nonzero for each pair a_1, a_2 in the above expansion.) The multiplicative identity 1_A of A is assumed to be the multiplicative identity with respect to $*$. (Every deformation of A is equivalent to one for which 1_A is the multiplicative identity; see [11, p. 43].)

Under the isomorphism $\mathrm{Hom}_k(A \otimes A, A) \cong \mathrm{Hom}_{A^e}(A \otimes A \otimes A \otimes A, A)$, we may identify μ_1 with a 2-cochain on the bar resolution (6.1). The associativity of the multiplication $*$ implies that μ_1 is a Hochschild 2-cocycle, that is,

$$a_1 \mu_1(a_2 \otimes a_3) + \mu_1(a_1 \otimes a_2 a_3) = \mu_1(a_1 a_2 \otimes a_3) + \mu_1(a_1 \otimes a_2) a_3$$

for all $a_1, a_2, a_3 \in A$ (equivalently $\delta_3^*(\mu_1) = 0$), and that

$$(6.3) \quad \delta_3^*(\mu_2)(a_1 \otimes a_2 \otimes a_3) = \mu_1(\mu_1(a_1 \otimes a_2) \otimes a_3) - \mu_1(a_1 \otimes \mu_1(a_2 \otimes a_3))$$

for all $a_1, a_2, a_3 \in A$. Note that this last equality can be written as $\delta_3^*(\mu_2) = \frac{1}{2}[\mu_1, \mu_1]$, where $[\cdot, \cdot]$ is the Gerstenhaber bracket; see [10] for a definition of the Gerstenhaber bracket in arbitrary degrees. We call a Hochschild 2-cocycle μ for A a **noncommutative Poisson structure** if the Gerstenhaber square bracket $[\mu, \mu]$ vanishes in cohomology; the first multiplication map μ_1 of any deformation of A satisfies this condition.

There are further conditions on the μ_j for higher values of j implied by the associativity of $*$. We will need just one more:

$$(6.4) \quad \delta_3^*(\mu_3)(a_1 \otimes a_2 \otimes a_3) = \mu_1(\mu_2(a_1 \otimes a_2) \otimes a_3) - \mu_1(a_1 \otimes \mu_2(a_2 \otimes a_3))$$

$$+\mu_2(\mu_1(a_1 \otimes a_2) \otimes a_3) - \mu_2(a_1 \otimes \mu_1(a_2 \otimes a_3))$$

for all $a_1, a_2, a_3 \in A$. The right side defines the Gerstenhaber bracket $[\mu_1, \mu_2]$ as evaluated on $a_1 \otimes a_2 \otimes a_3$, and thus equation (6.4) may also be written as $\delta_3^*(\mu_3) = [\mu_1, \mu_2]$.

Assume now that A is \mathbb{N} -graded. Let t be an indeterminate and extend the grading on A to $A[t]$ by assigning $\deg t = 1$. A **graded deformation of A over $k[t]$** is a deformation of A over $k[t]$ that is graded, i.e., for which each $\mu_j : A \otimes A \rightarrow A$ in (6.2) is a k -linear map that is homogeneous of degree $-j$.

We next show explicitly how the algebra $\mathcal{H}_{\lambda, \kappa}$ arises as a graded deformation of $S(V)\#G$ under the assumption that it has the PBW property. In fact, every algebra $\mathcal{H}_{\lambda, \kappa}$ having the PBW property is the fiber (at $t = 1$) of some deformation A_t of $S(V)\#G$ over $k[t]$, as stated in the next proposition. We give a converse afterwards in Theorem 6.11.

Proposition 6.5. *Let k be a field of arbitrary characteristic. If $\mathcal{H}_{\lambda, \kappa}$ exhibits the PBW property, then there is a graded deformation A_t of $A = S(V)\#G$ over $k[t]$ for which*

$$\mathcal{H}_{\lambda, \kappa} \cong A_t|_{t=1}$$

and whose first and second multiplication maps μ_1 and μ_2 satisfy

$$(6.6) \quad \lambda(g, v) = \mu_1(g \otimes v) - \mu_1({}^g v \otimes g), \quad \text{and}$$

$$(6.7) \quad \kappa(v, w) = \mu_2(v \otimes w) - \mu_2(w \otimes v)$$

for all v, w in V and g in G .

Proof. Assume that $\mathcal{H}_{\lambda, \kappa}$ has the PBW property. Let $\mathcal{H}_{\lambda, \kappa, t}$ be the associative $k[t]$ -algebra generated by kG and a basis of V , subject to the relations

$$(6.8) \quad \begin{aligned} gv - {}^g v g - \lambda(g, v)t, \\ vw - wv - \kappa(v, w)t^2 \end{aligned}$$

for all v, w in V and g in G .

Since $\mathcal{H}_{\lambda, \kappa}$ has the PBW property, $A_t = \mathcal{H}_{\lambda, \kappa, t}$ is a graded associative algebra (with $\deg t = 1$) which is isomorphic to $S(V)\#G[t]$ as a vector space and thus defines a graded deformation of $S(V)\#G$. We need only verify that its first and second multiplication maps μ_1 and μ_2 depend on κ and λ as claimed.

Let v_1, \dots, v_m be a basis of the vector space V , so that the monomials $v_1^{i_1} \cdots v_m^{i_m}$ (with i_1, \dots, i_m ranging over nonnegative integers) form a basis of $S(V)$ and $S(V)\#G$ has basis consisting of all $v_1^{i_1} \cdots v_m^{i_m} g$ (with g ranging over elements of G). We identify $\mathcal{H}_{\lambda, \kappa, t}$ with $S(V)\#G[t]$ as a $k[t]$ -module by writing all elements of $\mathcal{H}_{\lambda, \kappa, t}$ in terms of this basis and then translate the multiplication on $\mathcal{H}_{\lambda, \kappa, t}$

to a multiplication on $S(V)\#G[t]$ (under this identification) to find μ_1 and μ_2 explicitly.

Denote the product on $\mathcal{H}_{\lambda,\kappa,t}$ now by $*$ to distinguish from the multiplication in $S(V)\#G$. We apply the relations in $\mathcal{H}_{\lambda,\kappa,t}$ to expand the product $r*s$ of arbitrary $r = v_1^{i_1} \cdots v_m^{i_m} g$ and $s = v_1^{j_1} \cdots v_m^{j_m} h$ (for g, h in G) uniquely as a polynomial in t whose coefficients are expressed in terms of the chosen basis:

$$r * s = rs + \mu_1(r \otimes s)t + \mu_2(r \otimes s)t^2 + \cdots + \mu_l(r \otimes s)t^l,$$

where rs denotes the product in $S(V)\#G$. We set $r = v_1$ and $s = v_2$ and compare to the case $r = v_2$ and $r = v_1$. Since

$$v_2 * v_1 = v_1 v_2 + \kappa(v_2, v_1)t^2 \quad \text{and} \quad v_1 * v_2 = v_1 v_2,$$

the first multiplication map μ_1 vanishes on $v_1 \otimes v_2$ and on $v_2 \otimes v_1$, and so we have $v_2 * v_1 - v_1 * v_2 = \kappa(v_2, v_1)t^2$. Thus

$$(6.9) \quad \kappa(v_2, v_1) = \mu_2(v_2 \otimes v_1) - \mu_2(v_1 \otimes v_2),$$

and the same holds if we exchange v_1 and v_2 as κ is alternating. Similarly, for any v in V , $g * v = {}^g v g + \lambda(g, v)t$ and ${}^g v * g = {}^g v g$ so that $g * v - {}^g v * g = \lambda(g, v)t$ and

$$(6.10) \quad \lambda(g, v) = \mu_1(g \otimes v) - \mu_1({}^g v \otimes g).$$

□

We saw in the last proposition that the algebras $\mathcal{H}_{\lambda,\kappa}$ with the PBW property correspond to graded deformations of $S(V)\#G$ whose first and second multiplication maps define the parameters λ and κ explicitly. In fact, in the proof of Proposition 6.5, we see that the first multiplication maps μ_1 of such deformations all vanish on $V \otimes V$. But precisely *which* class of deformations of $S(V)\#G$ comprise the algebras $\mathcal{H}_{\lambda,\kappa}$? We answer this question by giving a converse to Proposition 6.5.

If we leave off the assumption, in the theorem below, that $V \otimes V$ lies in the kernel of μ_1 , then we obtain deformations of $S(V)\#G$ generalizing the algebras $\mathcal{H}_{\lambda,\kappa}$ investigated in this article. For example, some of the other graded deformations of $S(V)\#G$ are the Drinfeld orbifold algebras [25], and there may well be other possibilities in positive characteristic, such as analogs of Drinfeld orbifold algebras which in addition deform the group action.

Theorem 6.11. *Let k be a field of arbitrary characteristic. Let A_t be a graded deformation of $A = S(V)\#G$ with multiplication maps μ_i . Assume $V \otimes V$ lies in the kernel of μ_1 . Define parameters $\lambda : kG \otimes V \rightarrow kG$ and $\kappa : V \otimes V \rightarrow kG$ by*

$$\lambda(g, v) = \mu_1(g \otimes v) - \mu_1({}^g v \otimes g) \quad \text{and} \quad \kappa(v, w) = \mu_2(v \otimes w) - \mu_2(w \otimes v)$$

for all v, w in V and g in G . Then the algebra $\mathcal{H}_{\lambda,\kappa}$ defined by relations (2.1) exhibits the PBW property and

$$A_t|_{t=1} \cong \mathcal{H}_{\lambda,\kappa}.$$

Proof. Let v_1, \dots, v_m be a basis of V . Let F_t be the $k[t]$ -algebra generated by v_1, \dots, v_m and all g in G subject only to the relations of G . Define a $k[t]$ -linear map $f : F_t \rightarrow A_t$ by

$$f(x_{i_1} \cdots x_{i_n}) = x_{i_1} * \cdots * x_{i_n}$$

for all words $x_{i_1} \cdots x_{i_n}$ in F_t , where $*$ is the multiplication on A_t and each x_{i_j} is an element of the generating set $\{v_1, \dots, v_m\} \cup G$. Then f is an algebra homomorphism since the relations of G hold in A_t , as A_t is graded.

We claim that f is surjective. To see this, first note that $f(g) = g$, $f(v) = v$, and $f(vg) = vg + \mu_1(v \otimes g)t$ for all $v \in V$, $g \in G$, so $vg = f(vg - \mu_1(v \otimes g)t)$ in A_t . Now since A_t is a graded deformation of $S(V)\#G$, it has $k[t]$ -basis given by monomials of the form $v_{i_1} \cdots v_{i_r}g$ where $g \in G$. We show that $v_{i_1} \cdots v_{i_r}g$ lies in the image of f , by induction on r : We may assume $v_{i_2} \cdots v_{i_r}g = f(x)$ for some element x of F_t . Then by writing x as a linear combination of monomials, we find

$$\begin{aligned} f(v_{i_1}x) &= f(v_{i_1}) * f(x) \\ &= v_{i_1} * v_{i_2} \cdots v_{i_r}g \\ &= v_{i_1}v_{i_2} \cdots v_{i_r}g + \mu_1(v_{i_1}, v_{i_2} \cdots v_{i_r}g)t + \mu_2(v_{i_1}, v_{i_2} \cdots v_{i_r}g)t^2 + \cdots \end{aligned}$$

By induction, since each μ_j is of degree $-j$, each $\mu_j(v_{i_1}, v_{i_2} \cdots v_{i_r}g)$ is in the image of f . So $v_{i_1} \cdots v_{i_r}g$ is in the image of f , and f is surjective.

Next we consider the kernel of f . Since each μ_j has degree $-j$,

$$\begin{aligned} f(gv) &= g * v = gv + \mu_1(g \otimes v)t, \\ f({}^gvg) &= {}^g v * g = {}^gvg + \mu_1({}^g v \otimes g)t, \end{aligned}$$

and subtracting we obtain

$$f(gv - {}^gvg) = (\mu_1(g \otimes v) - \mu_1({}^g v \otimes g))t$$

since in $S(V)\#G$, $gv = {}^gvg$. It follows that

$$gv - {}^gvg - \lambda(g, v)t = gv - {}^gvg - (\mu_1(g \otimes v) - \mu_1({}^g v \otimes g))t$$

lies in the kernel of f for all $g \in G$, $v \in V$.

Similarly we find that for all $v, w \in V$, since $V \otimes V$ is in the kernel of μ_1 and μ_j has degree $-j$ for all j ,

$$\begin{aligned} f(vw) &= v * w = vw + \mu_2(v \otimes w)t^2, \\ f(wv) &= w * v = wv + \mu_2(w \otimes v)t^2, \end{aligned}$$

and subtracting we obtain

$$f(vw - wv) = (\mu_2(v \otimes w) - \mu_2(w \otimes v))t^2$$

since in $S(V)\#G$, $vw = wv$. Therefore

$$vw - wv - \kappa(v, w)t^2 = vw - wv - (\mu_2(v \otimes w) - \mu_2(w \otimes v))t^2$$

lies in the kernel of f for all $v, w \in V$.

Now let I_t be the ideal of F_t generated by all

$$\begin{aligned} gv - {}^gvg - \lambda(g, v)t, \\ vw - wv - \kappa(v, w)t^2, \end{aligned}$$

for g in G and v, w in V , so that $I_t \subseteq \text{Ker } f$. We claim that $I_t = \text{Ker } f$. To see this, first note that by its definition, F_t/I_t is isomorphic to $\mathcal{H}_{\lambda, \kappa, t}$, where $\mathcal{H}_{\lambda, \kappa, t}$ is the associative $k[t]$ -algebra generated by kG and a basis of V subject to the relations (6.8). Therefore we have a surjective homomorphism of graded algebras (with $\deg t = 1$)

$$\mathcal{H}_{\lambda, \kappa, t} \cong F_t/I_t \twoheadrightarrow F_t/\text{Ker } f \cong A_t.$$

However, in each degree, the dimension of $\mathcal{H}_{\lambda, \kappa, t}$ is at most that of $S(V)\#G[t]$, which is the dimension of A_t . This forces the dimensions to agree, so that the surjection is also injective, $I_t = \text{Ker } f$, and $\mathcal{H}_{\lambda, \kappa, t} \cong A_t$. Thus, as filtered algebras,

$$\mathcal{H}_{\lambda, \kappa} = \mathcal{H}_{\lambda, \kappa, t} |_{t=1} \cong A_t |_{t=1}.$$

In addition, $\mathcal{H}_{\lambda, \kappa}$ has the PBW property, since the associated graded algebra of the fiber of A_t at $t = 1$ is just $A = S(V)\#G$ (see, for example, [5, Section 1.4]). \square

Remark 6.12. Note that Theorem 6.11 generalizes a result for Drinfeld Hecke algebras (also called graded Hecke algebras) in characteristic zero to a larger class of algebras existing over fields of arbitrary characteristic. In characteristic zero, Drinfeld Hecke algebras correspond exactly to those deformations A_t of $S(V)\#G$ whose i -th multiplication map has graded degree $-2i$ (up to isomorphism) by [22, Proposition 8.14] (see also [28, Theorem 3.2] for a more general statement). If we replace the deformation parameter t by t^2 , we can convert each deformation A_t to a graded deformation (whose odd index multiplication maps all vanish), obtaining a special case of Theorem 6.11 with $\lambda \equiv 0$. See Corollary 8.5.

In the next section, we develop the homological algebra needed to make more precise the connections among the maps λ, κ, μ_1 , and μ_2 and to express the conditions of Theorem 3.1 homologically. We will identify λ and κ with cochains on a particular resolution X_\bullet of $S(V)\#G$, and show that $\lambda = \phi_2^*(\mu_1)$ and $\kappa = \phi_2^*(\mu_2)$ for a choice of chain map ϕ_\bullet from X_\bullet to the bar resolution of $S(V)\#G$.

7. A RESOLUTION AND CHAIN MAPS

We investigate deformations of $S(V)\#G$ using homological algebra by implementing a practical resolution for handling Hochschild cohomology over fields of arbitrary characteristic, as well as well-behaved chain maps between this resolution and the bar resolution. The resolution X_\bullet of $A = S(V)\#G$ we describe next is from [26], a generalization of a resolution given by Guccione, Guccione, and Valqui [13, §4.1]. It arises as a tensor product of the Koszul resolution of $S(V)$ with the bar resolution of kG , and we recall only the outcome of this construction here. For details, see [26]. In the case that the characteristic of k does not divide the

order of G , one may simply use the Koszul resolution of $S(V)$ to obtain homological information about deformations, as in that case the Hochschild cohomology of $S(V)\#G$ is isomorphic to the G -invariants in the Hochschild cohomology of $S(V)$ with coefficients in $S(V)\#G$. See, e.g., [22].

The complex X_\bullet is the total complex of the double complex $X_{\bullet,\bullet}$, where

$$(7.1) \quad X_{i,j} := A \otimes (kG)^{\otimes i} \otimes \bigwedge^j(V) \otimes A,$$

and A^e acts by left and right multiplication on the outermost tensor factors A . The horizontal differentials are defined by

$$\begin{aligned} d_{i,j}^h(1 \otimes g_1 \otimes \cdots \otimes g_i \otimes x \otimes 1) \\ = g_1 \otimes \cdots \otimes g_i \otimes x \otimes 1 + \sum_{l=1}^{i-1} (-1)^l \otimes g_1 \otimes \cdots \otimes g_l g_{l+1} \otimes \cdots \otimes g_i \otimes x \otimes 1 \\ + (-1)^i \otimes g_1 \otimes \cdots \otimes g_{i-1} \otimes g^i x \otimes g_i \end{aligned}$$

for all g_1, \dots, g_i in G and x in $\bigwedge^j(V)$, and the vertical differentials are defined by

$$\begin{aligned} d_{i,j}^v(1 \otimes g_1 \otimes \cdots \otimes g_i \otimes v_1 \wedge \cdots \wedge v_j \otimes 1) \\ = (-1)^i \sum_{l=1}^j (-1)^l (g_1 \cdots g_i v_l \otimes g_1 \otimes \cdots \otimes g_i \otimes v_1 \wedge \cdots \wedge \hat{v}_l \wedge \cdots \wedge v_j \otimes 1 \\ - 1 \otimes g_1 \otimes \cdots \otimes g_i \otimes v_1 \wedge \cdots \wedge \hat{v}_l \wedge \cdots \wedge v_j \otimes v_l), \end{aligned}$$

for all g_1, \dots, g_i in G and v_1, \dots, v_j in V . Setting $X_n = \bigoplus_{i+j=n} X_{i,j}$ for each $n \geq 0$ yields the total complex X_\bullet :

$$(7.2) \quad \cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow A \rightarrow 0,$$

with differential in degree n given by $d_n = \sum_{i+j=n} (d_{i,j}^h + d_{i,j}^v)$, and in degree 0 by the multiplication map. It is proven in [26], in a more general context, that X_\bullet is a free resolution of the A^e -module A .

We will need to extend some results on chain maps found in [26]. In the following lemma, we let $A^{\otimes(\bullet+2)}$ denote the bar resolution of A . We consider elements of $\bigwedge^j(V)$ to have degree j and elements of $(kG)^{\otimes i}$ to have degree 0.

Lemma 7.3. *Let k be a field of arbitrary characteristic. There exist chain maps $\phi_\bullet : X_\bullet \rightarrow A^{\otimes(\bullet+2)}$ and $\psi_\bullet : A^{\otimes(\bullet+2)} \rightarrow X_\bullet$ that are of degree 0 as maps and for which $\psi_n \phi_n$ is the identity map on X_n for $n = 0, 1, 2$.*

As a start, recall that there is an embedding of the Koszul resolution into the bar resolution of $S(V)$. The chain maps $\phi_\bullet, \psi_\bullet$ in the lemma may be taken to be extensions of such maps on $X_{0,\bullet}$ and $S(V)^{\otimes(\bullet+2)}$. Indeed, it is proven in [26, Lemma 4.7], for more general Koszul algebras, that there are chain maps $\phi_\bullet : X_\bullet \rightarrow A^{\otimes(\bullet+2)}$ and $\psi_\bullet : A^{\otimes(\bullet+2)} \rightarrow X_\bullet$ of degree 0 (as maps) for which $\psi_n \phi_n$ is the identity

on the subspace $X_{0,n}$ of X_n for each $n \geq 0$. See [23], for example, for explicit choices of chain maps involving the Koszul and bar resolutions of $S(V)$.

Proof. We show that the maps may be chosen so that $\psi_0\phi_0$, $\psi_1\phi_1$, and $\psi_2\phi_2$ are identity maps on all of X_0 , X_1 , and X_2 , respectively. In degree 0, ψ_0 and ϕ_0 are defined to be identity maps from $A \otimes A$ to itself. It is noted in the proof of [26, Lemma 4.7] that one may choose ϕ_1 and ψ_1 to be defined by

$$\begin{aligned} \phi_1(1 \otimes g \otimes 1) &= 1 \otimes g \otimes 1, & \phi_1(1 \otimes v \otimes 1) &= 1 \otimes v \otimes 1, \\ \psi_1(1 \otimes g \otimes 1) &= 1 \otimes g \otimes 1, & \psi_1(1 \otimes v \otimes 1) &= 1 \otimes v \otimes 1, \\ \text{and} \quad \psi_1(1 \otimes vg \otimes 1) &= 1 \otimes v \otimes g + v \otimes g \otimes 1, \end{aligned}$$

for all g in G and v in V . Extend ψ_1 to $A^{\otimes 3}$ by extending these elements to a free A^e -basis and defining ψ_1 on the remaining basis elements arbitrarily subject to the condition $d_1\psi_1 = \psi_0\delta_1$. It follows that $\psi_1\phi_1$ is the identity map on X_1 .

We now proceed to degree 2. We define $\phi_2 : X_2 \rightarrow A^{\otimes 4}$ by setting

$$(7.4) \quad \begin{aligned} \phi_2(1 \otimes g \otimes h \otimes 1) &= 1 \otimes g \otimes h \otimes 1, \\ \phi_2(1 \otimes g \otimes v \otimes 1) &= 1 \otimes g \otimes v \otimes 1 - 1 \otimes {}^g v \otimes g \otimes 1, \quad \text{and} \\ \phi_2(1 \otimes v \wedge w \otimes 1) &= 1 \otimes v \otimes w \otimes 1 - 1 \otimes w \otimes v \otimes 1 \end{aligned}$$

for all g, h in G and v, w in V . One may check directly that ϕ_2 continues the chain map ϕ_\bullet to degree 2. We construct a map $\psi_2 : A^{\otimes 4} \rightarrow X_2$ by first setting

$$(7.5) \quad \begin{aligned} \psi_2(1 \otimes g \otimes h \otimes 1) &= 1 \otimes g \otimes h \otimes 1 \text{ in } X_{2,0}, \\ \psi_2(1 \otimes g \otimes v \otimes 1 - 1 \otimes {}^g v \otimes g \otimes 1) &= 1 \otimes g \otimes v \otimes 1 \text{ in } X_{1,1}, \\ \psi_2(1 \otimes v \otimes g \otimes 1) &= 0, \quad \text{and} \\ \psi_2(1 \otimes v \otimes w \otimes 1 - 1 \otimes w \otimes v \otimes 1) &= 1 \otimes v \wedge w \otimes 1 \text{ in } X_{0,2} \end{aligned}$$

for all v, w in V and g in G . (In fact, we set $\psi_2(1 \otimes v_i \otimes v_j \otimes 1)$ equal to $1 \otimes v_i \wedge v_j \otimes 1$ if $i < j$ and zero if $j < i$ given a fixed basis v_1, \dots, v_m of V .) The elements of the form $1 \otimes g \otimes v \otimes 1 - 1 \otimes {}^g v \otimes g \otimes 1$ and $1 \otimes v \otimes g \otimes 1$, where v runs over a basis of V and g runs over all elements of G , are linearly independent, and the same is true if we take their union with all $1 \otimes v \otimes w \otimes 1$ where v, w run over a basis of V and all $1 \otimes g \otimes h \otimes 1$ where g, h run over the elements of G . We extend to a free A^e -basis of $A^{\otimes 4}$ and define ψ_2 arbitrarily on the remaining basis elements subject to the condition $d_2\psi_2 = \psi_1\delta_2$. By construction, $\psi_2\phi_2$ is the identity map on X_2 . \square

Remark 7.6. An explicit choice of chain map ϕ_3 is often helpful in checking PBW conditions homologically. We may define ϕ_3 on a free A^e -basis of X_3 : Let

$g, h, \ell \in G$, $v, w, v_1, v_2, v_3 \in V$, and set

$$\begin{aligned}
\phi_3(1 \otimes g \otimes h \otimes \ell \otimes 1) &= 1 \otimes g \otimes h \otimes \ell \otimes 1, \\
\phi_3(1 \otimes g \otimes h \otimes v \otimes 1) &= 1 \otimes g \otimes h \otimes v \otimes 1 - 1 \otimes g \otimes {}^h v \otimes h \otimes 1 \\
&\quad + 1 \otimes {}^{gh} v \otimes g \otimes h \otimes 1, \\
(7.7) \quad \phi_3(1 \otimes g \otimes v \wedge w \otimes 1) &= 1 \otimes g \otimes v \otimes w \otimes 1 - 1 \otimes g \otimes w \otimes v \otimes 1 \\
&\quad + 1 \otimes {}^g v \otimes {}^g w \otimes g \otimes 1 - 1 \otimes {}^g w \otimes {}^g v \otimes g \otimes 1 \\
&\quad - 1 \otimes {}^g v \otimes g \otimes w \otimes 1 + 1 \otimes {}^g w \otimes g \otimes v \otimes 1, \\
\phi_3(1 \otimes v_1 \wedge v_2 \wedge v_3 \otimes 1) &= \sum_{\sigma \in \text{Sym}_3} \text{sgn } \sigma \otimes v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)} \otimes 1.
\end{aligned}$$

It can be checked that $\delta_3 \phi_3 = \phi_2 d_3$ on X_3 .

In the remainder of this article, we will always choose to work with chain maps $\phi_\bullet, \psi_\bullet$ satisfying the above conditions and formulas.

8. HOMOLOGICAL CONDITIONS ARISING FROM THE PBW PROPERTY

We now reverse the flow of information and use homological theory to explain which quotient algebras $\mathcal{H}_{\lambda, \kappa}$, defined by relations (2.1), exhibit the PBW property. We show that the PBW conditions of Theorem 3.1 have a natural and elegant homological interpretation.

We saw in Section 6 that if the algebra $\mathcal{H}_{\lambda, \kappa}$ has the PBW property, then it corresponds to a graded deformation of $S(V) \# G$ with first multiplication maps μ_1 and μ_2 related to the parameters λ and κ (see Proposition 6.5). Since the multiplication maps μ_1 and μ_2 lift to a deformation, say with third multiplication map μ_3 , they satisfy the following homological conditions:

$$\begin{aligned}
(8.1) \quad &1) \quad \delta^*(\mu_1) = 0, \\
&2) \quad [\mu_1, \mu_1] = 2\delta^*(\mu_2), \quad \text{and} \\
&3) \quad [\mu_1, \mu_2] = \delta^*(\mu_3)
\end{aligned}$$

as cochains (on the bar complex) for the Hochschild cohomology of $S(V) \# G$, where $[,]$ denotes the Gerstenhaber bracket (see Section 6). These general homological conditions are necessary, but not sufficient, for some random maps to arise as the multiplication maps of a deformation of an algebra; the conditions merely guarantee that the initial obstructions to constructing a deformation from some candidate multiplication maps vanish.

We seek homological conditions which completely characterize the algebras $\mathcal{H}_{\lambda, \kappa}$ with the PBW property. If we begin with *arbitrary parameters* κ and λ , can we find homological conditions analgous to those above which are both necessary *and sufficient* to imply the parameters define a deformation of $S(V) \# G$? We begin

with a lemma that shows how to extend parameters λ and κ to the A^e -resolution X_\bullet of $A = S(V)\#G$ defined in Section 7.

Lemma 8.2.

- (1) *Any k -linear parameter function $\lambda : kG \otimes V \rightarrow kG$ extends uniquely to a graded 2-cochain on X_\bullet of degree -1 that vanishes on $X_{0,2}$.*
- (2) *Any alternating k -linear parameter function $\kappa : V \otimes V \rightarrow kG$ extends uniquely to graded 2-cochain on X_\bullet of degree -2 .*

Proof. The second degree term of the complex X_\bullet is $X_2 = X_{2,0} \oplus X_{1,1} \oplus X_{0,2}$. The k -linear map $\lambda : kG \otimes V \rightarrow kG$ extends to an A -bimodule map λ' on the summand $X_{1,1} = A \otimes kG \otimes V \otimes A$ of X_2 ,

$$\lambda' : A \otimes kG \otimes V \otimes A \rightarrow kG,$$

which may be extended to a map on the rest of X_2 by setting it to zero on $X_{2,0}$ and $X_{0,2}$. Thus λ' is a cochain on X_\bullet of degree -1 that extends λ :

$$\lambda(g, v) = \lambda'(1 \otimes g \otimes v \otimes 1)$$

for all g in G and v in V . Any other degree -1 cochain on X_2 that extends λ in this way and also vanishes on $X_{0,2}$ would define the same cochain (since it would automatically vanish on $X_{2,0}$ given its degree).

Likewise, the alternating k -linear map $\kappa : V \otimes V \rightarrow kG$ naturally defines an A -bimodule map κ' on the summand $X_{0,2} = A \otimes V \wedge V \otimes A$ of X_2 ,

$$\kappa' : A \otimes V \wedge V \otimes A \rightarrow kG,$$

which may be extended to a map on the rest of X_2 by setting it to zero on $X_{2,0}$ and $X_{1,1}$. Thus κ' is a cochain on X_\bullet of degree -2 that extends κ :

$$\kappa(v, w) = \kappa'(1 \otimes v \otimes w \otimes 1)$$

for all v, w in V . Any other degree -2 cochain on X_2 that extends κ in this way would define the same cochain since it must vanish on $X_{2,0}$ and $X_{1,1}$ given its degree. \square

We now express the PBW conditions of Theorem 3.1 as homological conditions on the parameters λ, κ . In fact, the conditions of the theorem are concisely articulated using the graded Lie bracket on Hochschild cohomology. We show that if $\mathcal{H}_{\lambda, \kappa}$ has the PBW property, then λ defines a noncommutative Poisson structure on $S(V)\#G$ and the Gerstenhaber bracket of λ and κ is zero in cohomology. The converse also holds, and we will make this statement precise at the cochain level. The bracket in the following theorem is the Gerstenhaber bracket (graded Lie bracket) on Hochschild cohomology induced on the complex X_\bullet from the bar complex where it is defined (via the chain maps $\phi_\bullet, \psi_\bullet$):

Theorem 8.3. *Let k be a field of arbitrary characteristic. A quotient algebra $\mathcal{H}_{\lambda,\kappa}$ defined by relations (2.1) exhibits the PBW property if and only if*

- $d^*(\lambda) = 0$,
- $[\lambda, \lambda] = 2d^*(\kappa)$, and
- $[\lambda, \kappa] = 0$

as cochains, where λ and κ are viewed as cochains on the resolution X_\bullet via Lemma 8.2.

Proof. We choose chain maps $\phi_\bullet, \psi_\bullet$ satisfying the conditions of Lemma 7.3 and equations (7.4) and (7.5). We convert λ and κ to cochains on the bar resolution of $A = S(V)\#G$ by simply applying these chain maps. Explicitly, we set

$$\mu_1 = \psi_2^*(\lambda) \quad \text{and} \quad \mu_2 = \psi_2^*(\kappa).$$

Since $\psi_2\phi_2$ is the identity map by Lemma 7.3, we find that $\lambda = \phi_2^*(\mu_1)$ and $\kappa = \phi_2^*(\mu_2)$, so that by formula (7.4),

$$\begin{aligned} \lambda(g, v) &= \mu_1(g \otimes v) - \mu_1({}^g v \otimes g), \\ \kappa(v, w) &= \mu_2(v \otimes w) - \mu_2(w \otimes v) \end{aligned}$$

for all v, w in V and g in G .

The Gerstenhaber bracket on X_\bullet is lifted from the bar complex using the chain maps ψ and ϕ as well:

$$[\lambda, \lambda] = \phi^*[\psi^*(\lambda), \psi^*(\lambda)] \quad \text{and} \quad [\lambda, \kappa] = \phi^*[\psi^*(\lambda), \psi^*(\kappa)].$$

We show next that the conditions in the theorem are equivalent to the PBW conditions of Theorem 3.1.

The Hochschild cocycle condition on λ . We first show that λ is a cocycle if and only if Conditions 1 and 3 of Theorem 3.1 hold. The cochain λ on the complex X_\bullet is only nonzero on the summand $X_{1,1}$ of X_2 . We therefore check the values of λ on the image of the differential on $X_{2,1} \oplus X_{1,2}$.

Note that $X_{2,1}$ is spanned by all $1 \otimes g \otimes h \otimes v \otimes 1$ where g, h range over all pairs of group elements in G and v ranges over vectors in V . Since λ is 0 on $X_{2,0}$,

$$\begin{aligned} d^*(\lambda)(1 \otimes g \otimes h \otimes v \otimes 1) &= \lambda(g \otimes h \otimes v \otimes 1 - 1 \otimes gh \otimes v \otimes 1 + 1 \otimes g \otimes {}^h v \otimes h) \\ &= g\lambda(h, v) - \lambda(gh, v) + \lambda(g, {}^h v)h. \end{aligned}$$

Thus $d^*(\lambda)$ vanishes on $X_{2,1}$ exactly when Condition 1 of Theorem 3.1 holds.

Next we note that $X_{1,2}$ is spanned by all $1 \otimes g \otimes v \wedge w \otimes 1$, where g ranges over all group elements and v, w range over vectors in V . Then, since λ is 0 on $X_{0,2}$,

$$\begin{aligned} d^*(\lambda)(1 \otimes g \otimes v \wedge w \otimes 1) &= -\lambda({}^g v \otimes g \otimes w \otimes 1 - 1 \otimes g \otimes w \otimes v - {}^g w \otimes g \otimes v \otimes 1 + 1 \otimes g \otimes v \otimes w) \\ &= -{}^g v \lambda(g, w) + \lambda(g, w)v + {}^g w \lambda(g, v) - \lambda(g, v)w. \end{aligned}$$

Expand $\lambda(g, v)$ as $\sum_{h \in G} \lambda_h(g, v)h$ and rearrange terms to see that $d^*(\lambda)$ vanishes on $X_{1,2}$ exactly when Condition 3 of Theorem 3.1 holds. Thus λ is a cocycle exactly when both Conditions 1 and 3 hold.

The first obstruction to integrating. We next show that the condition

$$[\lambda, \lambda] = 2d^*(\kappa)$$

corresponds to (6.3), the condition that the first obstruction to lifting (or integrating) an infinitesimal deformation must vanish. Note that this condition gives an equality of maps on X_3 . But both $[\lambda, \lambda]$ and $d^*\kappa$ vanish on $X_{3,0}$ and $X_{2,1}$ by degree reasons, thus we need only check the condition on $X_{1,2}$ and $X_{0,3}$.

We first calculate $2d^*(\kappa)$ on $X_{1,2}$. Since κ is nonzero only on $X_{0,2}$, a direct calculation shows that

$$d^*(\kappa)(1 \otimes g \otimes v \wedge w \otimes 1) = g\kappa(v, w) - \kappa({}^g v, {}^g w)g$$

for all v, w in V and g in G . Next we calculate $[\lambda, \lambda] = \phi_3^*[\mu_1, \mu_1]$ on such input. We apply (7.7):

$$\begin{aligned} & \frac{1}{2}[\mu_1, \mu_1](\phi_3(1 \otimes g \otimes v \wedge w \otimes 1)) \\ &= \mu_1(\mu_1(g \otimes v - {}^g v \otimes g) \otimes w) - \mu_1(\mu_1(g \otimes w - {}^g w \otimes g) \otimes v) \\ & \quad + \mu_1({}^g v \otimes \mu_1(g \otimes w - {}^g w \otimes g)) - \mu_1({}^g w \otimes \mu_1(g \otimes v - {}^g v \otimes g)) \\ & \quad - \mu_1(g \otimes \mu_1(v \otimes w - w \otimes v)) + \mu_1(\mu_1({}^g v \otimes {}^g w - {}^g w \otimes {}^g v) \otimes g) . \end{aligned}$$

Since $\mu_1 = \psi_2^*(\lambda)$ and λ vanishes on $X_{0,2}$, the last two summands above are zero. We use (7.5) to simplify the remaining terms:

$$\begin{aligned} & \mu_1(\lambda(g, v) \otimes w) - \mu_1(\lambda(g, w) \otimes v) + \mu_1({}^g v \otimes \lambda(g, w)) - \mu_1({}^g w \otimes \lambda(g, v)) \\ &= \sum_{h \in G} \lambda_h(g, v)\mu_1(h \otimes w) - \lambda_h(g, w)\mu_1(h \otimes v) \\ & \quad + \lambda_h(g, w)\mu_1({}^g v \otimes h) - \lambda_h(g, v)\mu_1({}^g w \otimes h) \\ &= \sum_{h \in G} \lambda_h(g, v)\mu_1(h \otimes w - {}^g w \otimes h) + \lambda_h(g, w)\mu_1({}^g v \otimes h - h \otimes v) \\ &= \sum_{h \in G} \lambda_h(g, v)\mu_1(h \otimes w - {}^h w \otimes h + {}^h w \otimes h - {}^g w \otimes h) \\ & \quad + \lambda_h(g, w)\mu_1({}^g v \otimes h - {}^h v \otimes h + {}^h v \otimes h - h \otimes v). \end{aligned}$$

We rewrite $\mu_1 = \psi_2^*(\lambda)$ and apply (7.5) to this last expression:

$$\sum_{h \in G} (\lambda_h(g, v)\lambda(h, w) - \lambda_h(g, w)\lambda(h, v)) = \lambda(\lambda(g, v), w) - \lambda(\lambda(g, w), v).$$

Thus, $[\lambda, \lambda] = 2d^*(\kappa)$ on $X_{1,2}$ if and only if Condition 2 of Theorem 3.1 holds.

Next we calculate $[\lambda, \lambda]$ and $d^*(\kappa)$ on $X_{0,3}$. On one hand,

$$[\mu_1, \mu_1](\phi_3(1 \otimes u \wedge v \wedge w \otimes 1)) = 0$$

for all u, v, w in V by a degree argument. On the other hand,

$$d^*(\kappa)(1 \otimes u \wedge v \wedge w \otimes 1) = u\kappa(v, w) - \kappa(v, w)u - v\kappa(u, w) + \kappa(u, w)v + w\kappa(u, v) - \kappa(u, v)w.$$

We express κ in terms of its components κ_g to see that $[\lambda, \lambda] = 2d^*(\kappa)$ on $X_{0,3}$ if and only if Condition 4 of Theorem 3.1 holds. Thus, $[\lambda, \lambda] = 2d^*(\kappa)$ on X_3 if and only if Conditions 2 and 4 of Theorem 3.1 hold.

The second obstruction to integrating. We finally show that the condition

$$[\lambda, \kappa] = 0$$

corresponds to (6.4), the condition that the second obstruction to lifting (or integrating) an infinitesimal deformation must vanish.

We first express $[\lambda, \kappa] = \phi^*[\mu_1, \mu_2]$ as a map on X_3 . By degree considerations, $[\lambda, \kappa]$ vanishes everywhere except on the summand $X_{0,3}$ of X_3 and it thus suffices to evaluate $[\mu_1, \mu_2]$ on $\phi_3(1 \otimes v_1 \wedge v_2 \wedge v_3 \otimes 1)$ for v_i in V . In fact, we need only compute the values of the first two terms of the right side of (6.4), since the other terms will be 0 on this input. We obtain

$$\begin{aligned} & \sum_{\sigma \in \text{Sym}_3} \text{sgn } \sigma \left(\mu_1(\mu_2(v_{\sigma(1)} \otimes v_{\sigma(2)}) \otimes v_{\sigma(3)}) - \mu_1(v_{\sigma(1)} \otimes \mu_2(v_{\sigma(2)} \otimes v_{\sigma(3)})) \right) \\ &= \sum_{\sigma \in \text{Alt}_3} \mu_1(\mu_2(v_{\sigma(1)} \otimes v_{\sigma(2)} - v_{\sigma(2)} \otimes v_{\sigma(1)}) \otimes v_{\sigma(3)}) \\ & \quad - \mu_1(v_{\sigma(1)} \otimes \mu_2(v_{\sigma(2)} \otimes v_{\sigma(3)} - v_{\sigma(3)} \otimes v_{\sigma(2)})) \\ &= \sum_{\sigma \in \text{Alt}_3} \mu_1(\kappa(v_{\sigma(1)}, v_{\sigma(2)}) \otimes v_{\sigma(3)}) - \mu_1(v_{\sigma(1)} \otimes \kappa(v_{\sigma(2)}, v_{\sigma(3)})) \\ &= \sum_{\sigma \in \text{Alt}_3} \sum_{g \in G} \kappa_g(v_{\sigma(1)}, v_{\sigma(2)}) (\mu_1(g \otimes v_{\sigma(3)}) - \mu_1(v_{\sigma(3)} \otimes g)) \\ &= \sum_{\sigma \in \text{Alt}_3} \sum_{g \in G} \kappa_g(v_{\sigma(1)}, v_{\sigma(2)}) (\mu_1(g \otimes v_{\sigma(3)}) - \mu_1({}^g v_{\sigma(3)} \otimes g)) \\ & \quad + \kappa_g(v_{\sigma(1)}, v_{\sigma(2)}) (\mu_1({}^g v_{\sigma(3)} \otimes g) - \mu_1(v_{\sigma(3)} \otimes g)). \end{aligned}$$

But $\mu_1 = \psi_2^*(\lambda)$ and $\psi_2(1 \otimes v \otimes g \otimes 1 - 1 \otimes {}^g v \otimes g \otimes 1) = 0$, while

$$\psi_2(1 \otimes g \otimes v \otimes 1 - 1 \otimes {}^g v \otimes g \otimes 1) = 1 \otimes g \otimes v \otimes 1$$

for all v in V and g in G , both by (7.5). Thus the sum is just

$$\sum_{\sigma \in \text{Alt}_3} \sum_{g \in G} \kappa_g(v_{\sigma(1)}, v_{\sigma(2)}) \lambda(g, v_{\sigma(3)}) = \sum_{\sigma \in \text{Alt}_3} \lambda(\kappa(v_{\sigma(1)}, v_{\sigma(2)}), v_{\sigma(3)}).$$

Hence, $[\lambda, \kappa] = 0$ as a cochain if and only if Condition 5 of Theorem 3.1 holds. \square

We close with a few observations about a central question in deformation theory: Which Hochschild cocycles of an algebra A lift (or integrate) to deformations, i.e., when is a Hochschild 2-cocycle the first multiplication map of some deformation of A ? In characteristic zero, we showed that every Hochschild 2-cocycle of $A = S(V)\#G$ of graded degree -2 lifts to a deformation, in fact, to a Drinfeld Hecke algebra (also called a “graded Hecke algebra”). See [22, Theorem 8.7] for the original argument and [24, Section 11] for additional discussion. We give two analogs in arbitrary characteristic implied by Theorem 8.3, now with two parameters corresponding to two different types of relations that may define deformations.

But first recall that every deformation of an algebra arises from a noncommutative Poisson structure. As the converse is false in general, we often seek a description of those noncommutative Poisson structures that actually lift to deformations. Indeed, the conditions (8.1) are in general necessary but not sufficient for a multiplication map to arise from a deformation. However in some cases, such as when the algebra in question is a skew group algebra $S(V)\#G$, we may interpret these classical obstruction conditions on a different resolution and obtain a complete characterization of multiplication maps that lift to certain graded deformations, as seen in the last theorem. Note that by focusing on graded deformations, we restrict attention to Hochschild 2-cocycles of graded degree -1 . The following straightforward corollary of Theorem 8.3 explains which Hochschild 2-cocycles can be lifted to graded deformations of $S(V)\#G$ isomorphic to algebras $\mathcal{H}_{\lambda,\kappa}$ with the PBW condition.

Corollary 8.4. *Let k be a field of arbitrary characteristic. Let λ be a noncommutative Poisson structure on $S(V)\#G$ of degree -1 as a graded map, and let κ be a 2-cochain for which $[\lambda, \lambda] = 2d^*(\kappa)$. As cochains on the resolution X_\bullet , suppose the Gerstenhaber bracket $[\lambda, \kappa]$ is zero, and suppose that λ vanishes on $X_{0,2}$. Then λ lifts to a graded deformation of $S(V)\#G$.*

Proof. Since $\deg \lambda = -1$, the A -module map λ takes $X_{1,1} = A \otimes kG \otimes V \otimes A$ to kG , thus defining a k -linear parameter function (by restricting to $1 \otimes kG \otimes V \otimes 1$),

$$\lambda : kG \otimes V \rightarrow kG.$$

Since $[\lambda, \lambda] = 2d^*\kappa$, the A -module map κ has degree -2 and thus takes $X_{0,2} = 1 \otimes V \wedge V \otimes 1$ to kG , defining an alternating k -linear parameter function

$$\kappa : V \otimes V \rightarrow kG.$$

By Lemma 8.2, the parameters λ and κ extend uniquely to cochains on X_\bullet of the same name, and since

$$d^*(\lambda) = 0, \quad [\lambda, \lambda] = 2d^*(\kappa), \quad \text{and} \quad [\lambda, \kappa] = 0,$$

the algebra $\mathcal{H}_{\lambda,\kappa}$ satisfies the PBW condition by Theorem 8.3.

By Proposition 6.5, $\mathcal{H}_{\lambda,\kappa}$ is isomorphic to the fiber at $t = 1$ of a deformation A_t of $A = S(V)\#G$ with first multiplication map μ_1 given by $\mu_1 = \psi^*(\lambda)$, i.e.,

$$\lambda(g, v) = \mu_1(g \otimes v) - \mu_1({}^g v \otimes g)$$

for all v in V and g in G . Thus the cocycle λ , whose realization on the bar complex is just μ_1 , lifts to a deformation of $S(V)\#G$. \square

Next we generalize [22, Theorem 8.7] to fields of arbitrary characteristic (see also [24, Theorem 11.4]).

Corollary 8.5. *Let k be a field of arbitrary characteristic. Let κ be a Hochschild 2-cocycle for $S(V)\#G$ of graded degree -2 . Then κ lifts to a deformation of $S(V)\#G$.*

Proof. As in the last proof, we may regard κ as an alternating k -linear parameter map, $\kappa : V \otimes V \rightarrow kG$. Then the conditions of Theorem 8.3 are satisfied with $\lambda \equiv 0$ and thus κ defines an algebra $\mathcal{H}_{0,\kappa}$ with the PBW condition. By Proposition 6.5, $\mathcal{H}_{0,\kappa}$ is isomorphic to the fiber at $t = 1$ of a graded deformation A_t of $A = S(V)\#G$ with first multiplication map zero and second multiplication map μ_2 given by $\mu_2 = \psi^*(\kappa)$, i.e.,

$$\kappa(v, w) = \mu_2(v \otimes w) - \mu_2(w \otimes v)$$

for all v, w in V . We replace t^2 by t in the graded deformation A_t to obtain an (ungraded) deformation with first multiplication map defined by κ . \square

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