

Peak Estimation of Hybrid Systems with Convex Optimization

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Abstract

Peak estimation of hybrid systems aims to upper bound extreme values of a state function along trajectories, where this state function could be different in each subsystem. This finite-dimensional but nonconvex problem may be lifted into an infinite-dimensional linear program (LP) in occupation measures with an equal objective under mild finiteness/compactness and smoothness assumptions. This LP may in turn be approximated by a convergent sequence of upper bounds attained from solutions of Linear Matrix Inequalities (LMIs) using the Moment-Sum-of-Squares hierarchy. The peak estimation problem is extended to problems with uncertainty and safety settings, such as measuring the distance of closest approach between points along hybrid system trajectories and unsafe sets.

1 Introduction

This paper interprets and extends the peak estimation problem to dynamical systems with hybrid behavior. Peak estimation is the analysis problem of finding extremal values of state functions along system trajectories, such as finding the maximum speed of a craft given a set of initial conditions. A hybrid system is a dynamical system that possesses both continuous-time and discrete-time dynamics [1]. Hybrid systems have a wide array of applications, including walking robots [2], power converters [3], sampled-data control [4], and systems biology [5]. In this work (extending methods from [6]), the hybrid system is defined with respect to a series of spaces known as ‘locations’ in which the hybrid trajectory evolves according to per-location Ordinary Differential Equation (ODE) dynamics. When the hybrid trajectory encounters a guard surface, it will transition to a (possibly) new location according to a reset map and continue its ODE evolution. Peak estimation of hybrid systems equips each location with a state function, and the output of the peak estimation problem is the maximum state function value obtained across all locations by all hybrid systems trajectories starting from a set of initial conditions in a given time horizon.

The ODE peak estimation problem is an instance of an input-less Optimal Control Problem (OCP) with a free terminal time and zero running cost. In the ODE case, such OCPs are finite-dimensional but generally nonconvex to solve. This difficulty is exacerbated by the addition of hybrid dynamics. In both cases for a maximization objective, lower bounds on the true peak cost can be computed by approximate sampling, while upper bounds must be satisfied for all admissible trajectories. The foundational work in [7] represented ODE OCPs as infinite-dimensional Linear Programs (LPs) in nonnegative Borel Measures, and gave necessary conditions under which the OCP and its LP outer-approximation have the same optimal value (no relaxation gap). The LPs in [7] involve terminal measures and occupation measures, which describe all information needed to reconstruct families of trajectories ODE. The work in [8] treated the peak estimation problem as an infinite-dimensional LP in measures, and proposed approximations solved based on successively refined finite-dimensional gridded LPs. In the peak estimation setting, the necessary conditions for no relaxation gap are mild compactness and Lipschitz regularity requirements. Another approach to solving infinite-dimensional LPs is through the Moment-Sum of Squares (SOS) hierarchy, which truncates the LPs into Semidefinite Programs (SDPs) of increasing complexity to produce tightening outer approximations [9, 10]. The SOS side of this hierarchy was used in [11] to solve peak

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estimation problems. Other applications of the Moment-SOS hierarchy in analysis and control of ODEs includes OCPs [12], reachable set estimation [13], and maximum control invariant set estimation [14].

Measures and the Moment-SOS hierarchy have also been applied to solve problems featuring hybrid dynamical systems. Instances of these extensions include OCPs [6, 15, 16] and reachable sets [17, 18]. Barrier functions to certify safety of hybrid system trajectories with respect to unsafe sets may also be found by SOS programming [19].

The contributions of this paper are as follows:

- Application of measure techniques to peak estimation for hybrid systems
- Use of Zeno caps to prevent unbounded executions
- A modular MATLAB framework for posing peak estimation problems for hybrid systems
- Extensions of existing work on peak estimation for uncertainty and safety analysis to hybrid systems

The paper is organized as follows. Section 2 introduces the notation convention and reviews background information. Section 3 formulates an infinite-dimensional measure program for peak estimation of hybrid system and its associated Linear Matrix Inequality (LMI) relaxation. Section 4 extends the hybrid peak estimation framework to safety analysis and possibly uncertain dynamical systems. Numerical examples are presented in Section 5. The paper is concluded in Section 6.

2 Preliminaries

LMI	Linear Matrix Inequality
LP	Linear Program
OCP	Optimal Control Problem
ODE	Ordinary Differential Equation
PSD	Positive Semidefinite
SDP	Semidefinite Program
SOS	Sum of Squares

2.1 Notation

The set of real numbers is \mathbb{R} and of natural numbers is \mathbb{N} . The set of polynomials with real coefficients in indeterminates x is $\mathbb{R}[x]$. Every polynomial $g \in \mathbb{R}[x]$ may be uniquely expressed as $g = \sum_{\alpha \in \mathbb{N}^n} g_{\alpha} x^{\alpha}$ in multi-index notation $x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \dots$ for some finite number of nonzero coefficients g_{α} . The degree of a monomial x^{α} is $|\alpha| = \sum_i \alpha_i$, and the degree of a polynomial g is the maximum such $|\alpha|$ where $g_{\alpha} \neq 0$.

A nonnegative Borel measure supported in a set X is a function that assigns each element of the σ -algebra of sets over X with a nonnegative number (the ‘size’ or ‘measure’ of the set). The measure μ follows the rules $\mu(\emptyset) = 0$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A \cap B = \emptyset$ [20]. The support of a nonnegative Borel measure is the locus of points x where every open neighborhood $N(x)$ has $\mu(N(x)) > 0$. The set of all nonnegative Borel measures supported in X is $\mathcal{M}_+(X)$. The space of continuous functions is $C(X)$, and a pairing between a nonnegative measure $\mu \in \mathcal{M}_+(X)$ and a function $f \in C(X)$ may be defined by Lebesgue integration $\langle f, \mu \rangle = \int f d\mu = \int_X f(x) d\mu(x)$. The set $C^1(X) \subset C(X)$ is the set of continuous functions with continuous first derivatives. The indicator function I_A of a set $A \subseteq X$ takes on the value $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ otherwise, and satisfies the rule $\langle I_A, \mu \rangle = \mu(A)$ for all $A \subseteq X$. The mass of a measure μ is $\mu(X) = \langle 1, \mu \rangle$, and μ is a probability measure if $\langle 1, \mu \rangle = 1$. The Dirac delta $\delta_{x=x'} \in \mathcal{M}_+(X)$ is a probability measure supported only at the point $x = x'$, which follows the pairing rule $\langle f, \delta_{x=x'} \rangle = f(x')$ for all test functions $f \in C(X)$. A nonnegative Borel measure supported at r distinct points (atoms) is termed a rank- r atomic measure. Such an atomic measure may be formed by a conic combination of Dirac deltas. The projection $\pi^x : X \times Y \rightarrow X$ is the map $(x, y) \mapsto x$. Given a mapping $Q : X \rightarrow Y$ and a measure $\mu(x) \in \mathcal{M}_+(X)$, the pushforward $Q_{\#}\mu(y)$ is the unique measure satisfying $\forall f \in C(Y) : \langle f(Q(x)), \mu(x) \rangle = \langle f(y), Q_{\#}\mu(y) \rangle$.

2.2 Hybrid Systems

The hybrid systems in this paper are posed over a set of L locations. Each location $\ell = 1..L$ has state variables x_{ℓ} contained in the space $X^{\ell} \subseteq \mathbb{R}^{n_{\ell}}$. The subsystems obey nominal locally Lipschitz dynamics

f_ℓ that satisfy

$$\dot{x}_\ell(t) = f_\ell(t, x_\ell(t)) \quad \forall \ell = 1..L. \quad (1)$$

Available transitions between subsystems may be represented by a directed multigraph. A multigraph is a graph where pairs of vertices may be connected by multiple distinct edges [21]. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a multigraph where each of the L vertices of \mathcal{V} corresponds to a location. Each edge $e \in \mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is a directed arc from a source $\text{src}(e)$ to a destination $\text{dst}(e)$. Self-loops with $\text{src}(e) = \text{dst}(e)$ are permitted in this class of multigraphs. Edges e are associated with a guard S_e and a reset map R_e . The guard S_e is a subset of $X_{\text{src}(e)}$, and the reset map $R_e : X_{\text{src}(e)} \rightarrow X_{\text{dst}(e)}$ effects the transition. The hybrid system is fully encoded by the tuple $\mathcal{H} = (X, f, \mathcal{G}, S, R)$ with attributes:

$X = \{X_\ell\}_{\ell=1}^L$	State Spaces
$f = \{f_\ell\}_{\ell=1}^L$	Dynamics
$\mathcal{G} = (\mathcal{V}, \mathcal{E})$	Transition Multigraph
$S = \{S_e\}_{e \in \mathcal{E}}$	Guard Surfaces
$R = \{R_e\}_{e \in \mathcal{E}}$	Reset Maps

Execution of a hybrid system with multigraph transitions is based on Algorithm 1 of [17]. An additional input is a set of Zeno caps $\{N_e\}_{e \in \mathcal{E}}$ which halt trajectory execution if any transition e is traversed at least N_e times [22]. The output of the following Algorithm 1 is a system trajectory $x(t)$, as well as records \mathcal{T}, \mathcal{C} containing information about the times and locations of state transitions respectively.

Algorithm 1 Execution of Hybrid System \mathcal{H}

Input

- x_0 Initial Point
- ℓ_0 Initial Location
- \mathcal{H} Hybrid System
- T Maximal Time
- N Transition Caps

Output

- $x(t)$ Trajectory of System
- \mathcal{T} Time Breaks
- \mathcal{C} Location Breaks
- \mathcal{N} Transition Counts

Initialize Trajectory $t \leftarrow 0, \ell \leftarrow \ell_0, x(0) \leftarrow x_0$

Initialize Traces $\mathcal{T} \leftarrow \{0\}, \mathcal{C} \leftarrow \{\ell\}, \mathcal{N} \leftarrow \{0\}_{e \in \mathcal{E}}$

loop

Follow dynamics $x'(s) = f_\ell(t, x(s))$ until $x(t)$ reaches a guard or $t = T$.

if $t = T$ **OR** $\nexists S_e : x(t) \in S_e$ and $\text{src}(e) = \ell$, **OR** $\exists e : \mathcal{N}_e = N_e$ **then**

halt

end if

Find a guard S_e with $x(t) \in S_e$ and $\text{src}(e) = \ell$

Append t to \mathcal{T} and $\text{dst}(e)$ to \mathcal{C}

Increment $\mathcal{N}_e \leftarrow \mathcal{N}_e + 1$

Transition to $\ell \leftarrow \text{dst}(e), x(t) \leftarrow R_e(x(t))$

end loop

The trajectory $x(t)$ is well-defined when the time horizon T and Zeno caps N_e for all $e \in \mathcal{E}$ are finite and the guard surfaces S_e are codimension-1. The trajectory $x(t)$ induces a function $\text{Loc} : [0, T] \rightarrow 1..L$ which returns the residing location of $x(t)$ at time t . Execution requires the following assumption of transversality,

Assumption 1. Let $x_\ell(t)$ be a segment of this trajectory that emerged from a transition (ℓ', ℓ) at time t^- . For all guards S^e with $\text{src}(e) = \ell$ such that $x_\ell(t) \in S^e$, the dynamics vector $f(t, x_\ell(t))$ possesses a normal component with respect to the tangent space of S^e at $x_\ell(t)$. This implies that the time elapsed between any two resets is bounded below by some $\delta > 0$.

2.3 Peak Estimation and Occupation Measures

The ODE (non-hybrid) peak estimation setting involves a trajectory $x(t | x_0)$, starting at the initial point $x_0 \in X_0 \subset X$, and evolving according to dynamics $\dot{x}(t) = f(t, x(t))$ in a space X . The program to find the maximum value of a state function $p(x)$ along trajectories is

$$P^* = \sup_{t, x_0 \in X_0} p(x(t | x_0)), \quad \dot{x}(t) = f(t, x(t)). \quad (2)$$

The extremum P^* may be bounded through the use of occupation measure relaxations [8]. An optimal trajectory satisfying $P^* = p(x^*) = p(x(t^* | x_0^*))$ is described by a triple (x_0^*, t^*, x^*) [23]. The initial probability measure $\mu_0 \in \mathcal{M}_+(X_0)$ is distributed over the set of initial conditions $x_0 \sim \mu_0$. The peak measure $\mu_p \in \mathcal{M}_+([0, T] \times X)$ is a terminal measure with free terminal time. For a stopping time t^* and subsets $A \subseteq [0, t^*]$, $B \subseteq X$, the occupation measure $\mu \in \mathcal{M}_+([0, T] \times X)$ has a definition [8]

$$\mu(A \times B) = \int_{X_0} \int_{t=0}^{t^*} I((t, x(t | x_0)) \in A \times B) dt d\mu_0(x_0).$$

The Lie derivative operator \mathcal{L}_f may be defined for all test functions $v \in C^1([0, T] \times X)$

$$\mathcal{L}_f v(t, x) = \partial_t v(t, x) + f(t, x) \cdot \nabla_x v(t, x). \quad (3)$$

The three measures (μ_0, μ_p, μ) are linked by Liouville's equation for all test functions v

$$\langle v(t, x), \mu_p \rangle = \langle v(0, x), \mu_0 \rangle + \langle \mathcal{L}_f v(t, x), \mu \rangle \quad (4)$$

$$\mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \quad (5)$$

Liouville's equation ensures that initial conditions distributed as μ_0 are connected to terminal points distributed as μ_p by trajectories following dynamics f . A convex measure relaxation of problem (2) is [8],

$$p^* = \sup \langle p(x), \mu_p \rangle \quad (6a)$$

$$\mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \quad (6b)$$

$$\langle 1, \mu_0 \rangle = 1 \quad (6c)$$

$$\mu, \mu_p \in \mathcal{M}_+([0, T] \times X) \quad (6d)$$

$$\mu_0 \in \mathcal{M}_+(X_0). \quad (6e)$$

Constraint (6c) ensures that both μ_0 and μ_p are probability measures with unit mass. The objective (6a) is the expectation of $p(x)$ with respect to μ_p . There will be no relaxation gap between problems (2) and (6c) ($P^* = p^*$) when $[0, T] \times X$ is compact and f is Lipschitz [7, 8, 11]. Program (6e) is a particular form of the optimal control program from [7] with zero running cost and free terminal time.

3 Peak estimation of hybrid systems

3.1 Peak Program

Let $X_0 = \{X_{0\ell}\}$ be the set of initial conditions for system trajectories. Each of these system trajectories lie inside the set $X = \{X_\ell\}$.

Each location ℓ has a state cost $p_\ell : X_\ell \rightarrow \mathbb{R}$ and a set of initial conditions $X_{0\ell} \subset X_\ell$. Each p_ℓ is either bounded below or constant at $-\infty$, and at least one p_ℓ is bounded. The goal of peak estimation is to find the trajectory $x(t)$ which maximizes the state cost across all trajectories and locations:

$$P^* = \sup_{t, \ell_0} \max_{x_0} \max_{\ell} p_\ell(x(t | x_0)) \quad x(t) \in X_\ell$$

Dynamics follow Algorithm 1 with input $(\ell_0, x_0, \mathcal{H}, T)$

$$x_0 \in X_{0\ell_0}. \quad (7)$$

The optimization variables of (7) are the peak time t , initial location ℓ_0 , and initial state $x_0 \in X_{\ell_0}$. The inner maximization runs over all location-objective functions p_ℓ .

The following assumptions will be posed on problem (7):

Assumption 2. The sets $[0, T]$, $X_\ell, X_{\ell 0}$ are compact for all $\ell = 1..L$.

Assumption 3. Each function p_ℓ is continuous inside X_ℓ .

Assumption 4. Each dynamics function $f_\ell(t, x_\ell)$ is Lipschitz over the compact set $[0, T] \times X_\ell$.

Assumption 5. Trajectories stay in X_ℓ for all $t \in [0, T]$ when starting inside $X_{0\ell}$.

3.2 Measures for Hybrid Systems

The control and reachability set programs in [17, 18, 24] define measures ρ_e supported over the guard $\mathcal{M}_+(S_e)$ for each transition $e \in \mathcal{E}$. For subsets $A \subset [0, T]$, $C_e \subset S_e$ and an initial condition x_0 , the counting measure ρ_e records the number of times the trajectory, starting from location $\text{src}(e)$, enters the patch C_e of the guard S_e with

$$\rho_e(A \times C_e) = \int_A \text{card} \left(\lim_{t' \rightarrow t^-} x(t' \mid x_0) \in C_e \right) dt. \quad (8)$$

The mass of the counting measure ρ_e is the expected number of times a trajectory will traverse the transition with arc e . In a Zeno execution of transition e , the mass $\langle 1, \rho_e \rangle$ will be unbounded, and constraints such as $\langle 1, \rho_e \rangle \leq N_e$ may be imposed to cap the maximum number of transitions on arc e . Let $X_{0\ell} \subseteq X_\ell$ be a set of initial conditions defined on each space X_ℓ in X . A distribution of initial conditions over each location is $\mu_{0\ell} \in \mathcal{M}_+(X_{0\ell})$ for $\ell = 1..L$. Let $T < \infty$ be a final time, and $\mu_{p\ell} \in \mathcal{M}_+([0, T] \times X_\ell)$ be peak measures supported over each location-space. Trajectories following dynamics $x'(t) = f_\ell(t, x(t))$ in each space X_ℓ are tracked by occupation measures $\mathcal{M}_+([0, T] \times X_\ell)$. Counting measures $\rho_e \in \mathcal{M}_+(S_e)$ are set up over all guards to handle state transitions. The Liouville equation with guard measures holding for all test functions $v_\ell \in C^1([0, T] \times X_\ell)$ and locations $\ell = 1..L$ is

$$\begin{aligned} \mu_{p\ell} &= \delta_0 \otimes \mu_{0\ell} + \mathcal{L}_{f_\ell}^\dagger \mu_\ell \\ &+ \sum_{\text{src}(e)=\ell} R_{e\#} \rho_e - \sum_{\text{dst}(e)=\ell} \rho_e. \end{aligned} \quad (9)$$

For a location ℓ and edge e with $\text{src}(e) = \ell$, the pushforward term $R_{e\#}$ in (9) should be understood as

$$\langle v_\ell, R_{e\#} \rho_e \rangle = \langle v_\ell(t, R_e(x_\ell)), \rho_e \rangle. \quad (10)$$

The mass of the peak measure $\mu_{p\ell}$ is equal to the mass of the initial measure $\mu_{0\ell}$ plus the net flux due to state transitions.

3.3 Measure Program

Problem (7) may be relaxed through an infinite-dimensional linear program in occupation measures. The measures $\mu_{0\ell}$ are distributions of initial conditions, and ρ_e are transition counting measures, just as in the Liouville equation (9). The peak measures $\mu_{p\ell}$ are final measures with free terminal time between $t \in [0, T]$. The measure program in terms of $(\mu_0, \mu_p, \mu, \rho)$ for hybrid peak estimation (where $\forall \ell$ and $\forall e$ may be expanded to $\forall \ell = 1..L$ and $\forall e \in \mathcal{E}$) is

$$p^* = \sup \sum_{\ell=1}^L \langle p_\ell, \mu_{p\ell} \rangle \quad (11a)$$

$$\mu_{p\ell} = \delta_0 \otimes \mu_{0\ell} + \mathcal{L}_{f_\ell}^\dagger \mu_\ell \quad \forall \ell \quad (11b)$$

$$+ \sum_{\text{dst}(e)=\ell} R_{e\#} \rho_e - \sum_{\text{src}(e)=\ell} \rho_e$$

$$\sum_{\ell=1}^L \langle 1, \mu_{0\ell} \rangle = 1 \quad (11c)$$

$$\langle 1, \rho_e \rangle \leq N_e \quad \forall e \quad (11d)$$

$$\mu_\ell, \mu_{p\ell} \in \mathcal{M}_+([0, T] \times X_\ell) \quad \forall \ell \quad (11e)$$

$$\mu_{0\ell} \in \mathcal{M}_+(X_{0\ell}) \quad \forall \ell \quad (11f)$$

$$\rho_e \in \mathcal{M}_+(S_e) \quad \forall e. \quad (11g)$$

Theorem 6. Solutions to (11) and (7) satisfy $p^* \geq P^*$

Proof. Let $(x(t \mid x_0, \ell_0), \mathcal{T}, \mathcal{C})$ be a trajectory from the execution of Algorithm 1 that stops at time $t^* \in [0, T]$, and $\text{Loc}(t)$ be the function returning the residing location of $x(t)$ at time t . This trajectory may be described by a tuple (ℓ_0, x_0, t^*) . Measures $\forall \ell : \mu_{0\ell}, \mu_{p\ell}, \mu_\ell$ and $\forall e : \rho_e$ that are feasible solutions to constraints (11b)-(11g) may be formed from the trajectory $x(t)$. The initial measure $\mu_{0\ell}$ is $\delta_{x=x_0}$ for

$\ell = \ell_0$ and is the zero measure for $\ell \neq \ell_0$. The peak measure $\mu_{p\ell}$ is $\delta_{t=t^*} \otimes \delta_{x=x(t^*|x_0, \ell_0)}$ for $\ell = \text{Loc}(t^*)$ and is also the zero measure for all other ℓ . Let T_ℓ be the set $T_\ell = \{t \mid t \in [0, t_p^*], \text{Loc}(t) = \ell\}$ of times where $x(t)$ is in location ℓ . Each relaxed occupation measure μ_ℓ may respectively be set to the occupation measure of $t \mapsto (t, x(t \mid x_0, \ell_0))$ in the times $t \in T_\ell$. If the transition with edge $e \in \mathcal{E}$ is traversed N_e times along the trajectory $x(t)$ at points $\{(t_i^e, x_i^e)\}_{i=1}^{N_e}$ for $x_i^e \in X_{\text{src}(e)}$, the guard measure ρ_e may be defined as $\rho_e = \sum_{i=1}^{N_e} \delta_{t=t_i^e} \otimes \delta_{x=x_i^e}$. The objective p^* is an upper bound on P^* because a set of measures $(\mu_{0\ell}, \mu_{p\ell}, \mu_\ell, \rho_e)$ constructed from every trajectory $x(t)$ satisfy the constraints of (11) with objective P^* . \square

Remark 1. *Setting a peak objective to $p_\ell(x) = -\infty$ is equivalent to constraining $\mu_{p\ell}$ to the zero measure, because trajectories to maximize $p(x)$ will not terminate in location ℓ . Likewise, a measure $\mu_{0\ell} \in \mathcal{M}_+(X_{0\ell})$ where $X_{0\ell} = \emptyset$ is the zero measure.*

Theorem 7. *All measures involved in a solution to (11) are bounded.*

Proof. Sufficient conditions for a measure to be bounded are that its mass is finite and its support is compact. This setting satisfies the compact support requirement.

Given that all measures $(\mu_0, \mu_p, \mu, \rho)$ are nonnegative, their masses will also be nonnegative numbers. The mass of the transition measures ρ are upper bounded by the Zeno constraints (11d) under the assumption that all N_e are finite. Constraint (11f) upper bounds each mass $\langle 1, \mu_{0\ell} \rangle$. For each location ℓ , choosing a test function $v_\ell(t, x_\ell) = 1$ for Liouville equation (11b) yields

$$\langle 1, \mu_{p\ell} \rangle = \langle 1, \mu_{0\ell} \rangle + \sum_{\text{dst}(e)=\ell} \langle 1, \rho_e \rangle - \sum_{\text{src}(e)=\ell} \langle 1, \rho_e \rangle. \quad (12)$$

Every term on the right-hand side of (12) is finite and $\langle 1, \mu_{p\ell} \rangle \geq 0$ by measure nonnegativity, so each peak measure $\mu_{p\ell}$ has bounded mass. Utilizing a test function of $v_\ell(t, x_\ell) = t$ with $\mathcal{L}_{f_\ell} t = 1$ results in

$$\langle t, \mu_{p\ell} \rangle = \langle 1, \mu_\ell \rangle + \sum_{\text{dst}(e)=\ell} \langle t, \rho_e \rangle - \sum_{\text{src}(e)=\ell} \langle t, \rho_e \rangle. \quad (13)$$

The terms $\langle t, \mu_{p\ell} \rangle, \langle t, \rho_e \rangle$ are all finite due to bounded masses and compact support, so the occupation measures μ_ℓ also have finite mass and are bounded. \square

Theorem 8. *The objectives in (7) and (11) will satisfy $p^* = P^*$ when $[0, T] \times \prod_\ell X_\ell$ is compact, each f_ℓ is Lipschitz, and p^* is bounded above.*

Proof. This statement may be proved by extending arguments from [6]. Theorem 17 of [6] states there is no relaxation gap in measure LPs of an optimal control program with appropriate assumptions, extending the ODE result of [7]. Free final time is already accounted for in [6] by reference to Remark 2.1 of [12]. The ODE problem in [7] can handle initial conditions lying in a set X_0 , so the method in [6] can similarly work with sets of initial conditions $\{X_{0\ell}\}_{\ell=1}^L$ as demonstrated by [15]. The work in [15] has ‘switching’ costs (possibly differing running and terminal costs in each location), which is realized by the costs p_ℓ . The final modification between this work and [6] is that problem (7) has finite Zeno caps N_e , while Assumption 3 of [6] forbids Zeno trajectories. The allowance for free terminal time permits consequence 4 of Theorem 12 of [6] to read that there exists a constant C such that $\sum_e \langle 1, \rho_e \rangle \leq \sum_e N_e = C$. The three modifications of [6] (free terminal time, multiple initial conditions, Zeno caps) are all cleared, so $p^* = P^*$ under the compactness and Lipschitz assumptions. \square

3.4 Function Program

The measure program (11) is dual to an infinite-dimensional linear program in continuous functions. The Lagrangian \mathcal{L} of problem (11) with dual variables $v_\ell \in C^1([0, T] \times X_\ell)$, $\gamma \in \mathbb{R}$, $\alpha \in \mathbb{R}_+^{|\mathcal{E}|}$ is

$$\begin{aligned} \mathcal{L} = & \sum_{\ell=1}^L \langle p_\ell, \mu_{p\ell} \rangle + \langle v_\ell(t, x), \delta_0 \otimes \mu_{0\ell} + \mathcal{L}_{f_\ell}^\dagger \mu_\ell \rangle \\ & + \langle v_\ell(t, x), \sum_{\text{dst}(e)=\ell} R_{e\#} \rho_e - \sum_{\text{src}(e)=\ell} \rho_e - \mu_{p\ell} \rangle \\ & + \gamma(1 - \sum_{\ell=1}^L \langle 1, \mu_{0\ell} \rangle) + \sum_{e \in \mathcal{E}} \alpha_e (N_e - \langle 1, \rho_e \rangle). \end{aligned} \quad (14)$$

The dual function program of (11) is

$$\begin{aligned} d^* &= \inf_{\gamma, \alpha, v} \sup_{\mu_{0\ell}, \mu_{p\ell}, \mu_\ell, \rho_e} \mathcal{L} \\ d^* &= \inf_{\gamma \in \mathbb{R}, \alpha \in \mathbb{R}_+^{|\mathcal{E}|}} \gamma + \sum_{e \in \mathcal{E}} N_e \alpha_e \end{aligned} \quad (15a)$$

$$\begin{aligned} \forall \ell : \forall x_\ell \in X_{0\ell} : \\ \gamma &\geq v_\ell(0, x_\ell) \end{aligned} \quad (15b)$$

$$\begin{aligned} \forall \ell : \forall (t, x_\ell) \in [0, T] \times X_\ell : \\ 0 &\geq \mathcal{L}_{f_\ell} v_\ell(t, x_\ell) \end{aligned} \quad (15c)$$

$$\begin{aligned} \forall e : \forall (t, x_{\text{src}(e)}) \in [0, T] \times X_{\text{src}(e)} : \\ v_{\text{src}(e)}(t, x_{\text{src}(e)}) - v_{\text{dst}(e)}(t, R_e(x_{\text{src}(e)})) &\geq -\alpha_e \end{aligned} \quad (15d)$$

$$\begin{aligned} \forall \ell : \forall (t, x_\ell) \in [0, T] \times X_\ell : \\ v_\ell(t, x_\ell) &\geq p_\ell(x_\ell) \end{aligned} \quad (15e)$$

$$\forall \ell : v_\ell(t, x_\ell) \in C^1([0, T] \times X_\ell). \quad (15f)$$

The dual variables v_ℓ are auxiliary functions that decrease along trajectories (15c) and along transitions (15d). The auxiliary functions upper bound the location-costs by (15e). The dual variable α_e will be zero if transition e is traveled at most $N_e - 1$ times (complementary slackness of (11g)).

Theorem 9. *Programs (11) and (15) will possess equal objectives $p^* = d^*$ when each X_ℓ is compact and (T, N_e) are each finite.*

Proof. $p^* = d^*$: Strong duality follows by arguments from Theorem 2.6 of [10], specifically from boundedness of measures (Theorem 7) and compactness (Assumption A2). \square

3.5 Linear Matrix Inequality Program

The Moment-SOS hierarchy is a method to produce upper bounds to measure LPs by a sequence of Linear Matrix Inequalities (LMIs) of increasing size [9]. Let $X \in \mathbb{R}^n$ be a basic semialgebraic set $X = \{x \mid g_i(x) \geq 0, i = 1..N_c\}$, which is the locus of a finite number of finite-degree polynomial inequality constraints. An α -moment of a measure $\mu \in \mathcal{M}_+(X)$ for $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}$ is $\mathbf{y}_\alpha = \langle x^\alpha, \mu \rangle$. To each moment sequence \mathbf{y} , there is an associated Riesz linear functional $\mathbb{L}_\mathbf{y}$ acting as $\mathbb{L}_\mathbf{y}[\sum_{\alpha, \beta} c_{\alpha\beta} x^\alpha] \rightarrow \sum_\alpha c_\alpha \mathbf{y}_\alpha$.

Assume that each polynomial $g_i(x) = \sum_\gamma g_{i\gamma} x^\gamma$ in the definition of X has a finite degree d_i . If the set X satisfies an Archimedean condition (all compact sets may be made Archimedean by adding a redundant ball constraint) [25], then necessary and sufficient conditions for the sequence \mathbf{y} of putative moments (pseudo-moments) up to degree $2d$ to be moments of a measure $\mu \in \mathcal{M}_+(X)$ are that the following matrices indexed by monomials $\alpha, \beta \in \mathbb{N}^n$ are Positive Semidefinite (PSD):

$$\mathbb{M}_d(\mathbf{y})_{\alpha\beta} = \mathbf{y}_{\alpha+\beta}, \quad \mathbb{M}_{d-d_i}(g_i \mathbf{y})_{\alpha\beta} = \sum_\gamma g_{i\gamma} \mathbf{y}_{\alpha+\beta+\gamma}. \quad (16)$$

The measure μ is referred to as the *representing measure* of the pseudo-moments \mathbf{y} . The symbol $\mathbb{M}_d(X\mathbf{y})$ will denote a block-diagonal matrix formed by the matrices in (16).

The basic semialgebraic sets containing measures in (11) are

$$\begin{aligned} \forall \ell : \quad X_\ell &= \{x_\ell \mid g_{\ell i}(x_\ell) \geq 0 \mid i = 1..N_c^\ell\} \\ \forall \ell : \quad X_{0\ell} &= \{x_\ell \mid g_{0\ell i}(x_\ell) \geq 0 \mid i = 1..N_c^{0\ell}\} \\ \forall e : \quad S_e &= \{x_{\text{src}(e)} \mid g_{ei}(x_{\text{src}(e)}) \geq 0 \mid i = 1..N_c^e\}. \end{aligned} \quad (17)$$

Polynomials $g_{\ell i}(x)$, $g_{0\ell i}(x_\ell)$, $g_{ei}(x_{\text{src}(e)})$ have finite degrees $d_{\ell i}$, $d_{0\ell i}$, d_{ei} respectively for each i, ℓ, e as appropriate. Let $(\mathbf{y}^{0\ell}, \mathbf{y}^{p\ell}, \mathbf{y}^\ell, \mathbf{r}^e)$ be pseudo-moments of the measures $(\mu_{0\ell}, \mu_{p\ell}, \mu_p, \rho_e)$. The Liouville equation (11b) may be expressed as a collection of affine constraints in the pseudo-moments. Substituting the test function $v(t, x_\ell) = x_\ell^\alpha t^\beta$ into (11b) yields a relation for each $\alpha \in \mathbb{N}^{n_\ell}$, $\beta \in \mathbb{N}$, $\ell \in 1..L$:

$$\begin{aligned} 0 &= -\langle x_\ell^\alpha t^\beta, \mu_{p\ell} \rangle + \langle x_\ell^\alpha t^\beta, \delta_{t=0} \otimes \mu_{0\ell} \rangle + \langle \mathcal{L}_{f_\ell} x_\ell^\alpha t^\beta, \mu_\ell \rangle \\ &+ \sum_{\text{dst}(e)=\ell} \langle R_e(x_\ell)^\alpha t^\beta, \rho_e \rangle - \sum_{\text{src}(e)=\ell} \langle x_\ell^\alpha t^\beta, \rho_e \rangle. \end{aligned} \quad (18)$$

The expression $\text{Liou}_{\alpha\beta}^\ell(\mathbf{y}^{0\ell}, \mathbf{y}^{p\ell}, \mathbf{y}^\ell, \mathbf{r}^{\mathcal{E}_\ell}) = 0$ may be defined to abbreviate the affine constraint in pseudo-moments induced by (18), where $\mathcal{E}_\ell = \{e \in \mathcal{E} \mid \text{src}(e) = \ell \text{ or } \text{dst}(e) = \ell\}$ is the set of arcs including

location ℓ . For a constant degree $d \in \mathbb{N}$, define the quantities $d'_\ell = d + \lceil \deg f_\ell / 2 \rceil - 1$ and $k_e = \deg R_e$. The degree- d LMI relaxation of (11) with variables $(\mathbf{y}^{0\ell}, \mathbf{y}^{p\ell}, \mathbf{y}^\ell, \mathbf{r}^e)$ is

$$p_d^* = \max \sum_\ell \sum_\alpha p_{\ell\alpha} \mathbf{y}_\alpha^{p\ell} \quad (19a)$$

$$\sum_\ell \mathbf{y}_0^{0\ell} = 1 \quad (19b)$$

$$\forall \ell : \alpha \in \mathbb{N}^{n_\ell}, \beta \in \mathbb{N}, |\alpha| + |\beta| \leq 2d$$

$$\text{Liou}_{\alpha\beta}^\ell(\mathbf{y}^{0\ell}, \mathbf{y}^{p\ell}, \mathbf{y}^\ell, \mathbf{r}^{\varepsilon_\ell}) = 0 \text{ by (18)} \quad (19c)$$

$$\forall e : \mathbf{y}_0^e \leq N_e \quad (19d)$$

$$\forall \ell : \mathbb{M}_d(X^{0\ell} \mathbf{y}^{0\ell}), \mathbb{M}_d([0, T] \times X^\ell \mathbf{y}^{p\ell}), \mathbb{M}_{d'_\ell}([0, T] \times X^\ell \mathbf{y}^\ell) \succeq 0 \quad (19e)$$

$$\forall e : \mathbb{M}_{k_e d_e}(S_e \mathbf{r}^e) \succeq 0. \quad (19f)$$

The affine constraints (19c)-(19d) implement a truncation of constraints (11d)-(11d) in terms of finite-length pseudo-moments. Constraints (19e)-(19f) ensure that there exist representing measures for the pseudo-moments. Solutions to the SDP generated from the LMI (19) by raising the degree d will form a chain of upper bounds $p_d^* \geq p_{d+1}^* \geq \dots \geq p^*$.

Theorem 10. *The sequence of upper bounds will satisfy $\lim_{d \rightarrow \infty} p_d^* = P^*$ when $\forall \ell = 1..L : [0, T] \times X_\ell$ and $X_{0\ell}$ are Archimedean, $f_\ell(t, x)$ are polynomial, and $\forall e : N_e$ are finite.*

Proof. The upper bound sequence will converge to p^* when all sets are Archimedean, there exists an interior point to constraints (11b)-(11g), and all measures $(\mu_{0\ell}, \mu_{p\ell}, \mu_\ell, \rho_e)$ have bounded moments ([10], Theorem 5 of [26] and Theorem 4.4 of [9]).

Let x_0 be an initial point starting in some nonempty location X_ℓ . The set of measures where $\mu_{0\ell} = \delta_{x=x_0}$, $\mu_{p\ell} = \delta_{t=0} \otimes \delta_{x=x_0}$ and all other measures are the zero measure is an interior point to (11b)-(11g) (trajectory starting at x_0 with zero elapsed time). Given that each $[0, T] \times X_\ell$ is compact, it is sufficient that all measures have bounded masses in order for the measures to have bounded moments. The masses of ρ_e are each upper bounded by the finite quantity N_e by constraint (11g), and the sum of the masses of $\mu_{0\ell}$ are upper bounded by 1 through (11c). The sum of constraint (11b) with test function $v_\ell = 1$ along all ℓ is $\sum_{\ell=1}^L \langle 1, \mu_{p\ell} \rangle = \sum_{\ell=1}^L \langle 1, \mu_{0\ell} \rangle = 1$, so each mass of $\mu_{p\ell}$ is finite. Lastly, the use of a test function of $v_\ell = t$ on each Liouville equation in (11b) yields the finite expression $\langle 1, \mu_\ell \rangle = \langle t, \mu_p \rangle - \sum_{\text{dst}(e)=\ell} \langle t, \rho_e \rangle + \sum_{\text{src}(e)=\ell} \langle t, \rho_e \rangle$. The sequence of upper bounds will therefore converge to p^* as $d \rightarrow \infty$ with $p^* = P^*$ from Theorem 8. \square

The sizes of the moment matrices in problem (19) are listed in Table 1. The computational complexity of numerical SDP solvers scale in a polynomial manner with the size of the largest PSD matrix [12]. These PSD matrix sizes may be reduced if extant structure such as symmetry, quotient, or sparsity structure is present in (11).

Table 1: Sizes of moment matrices in LMI (19)

Moment	$\mathbb{M}_d(\mathbf{y}^{0\ell})$	$\mathbb{M}_d(\mathbf{y}^{p\ell})$	$\mathbb{M}_{d'_\ell}(\mathbf{y}^\ell)$	$\mathbb{M}_{dk_e}(\mathbf{r}^e)$
Size	$\binom{n_\ell+d}{d}$	$\binom{1+n_\ell+d}{d}$	$\binom{1+n_\ell+d'_\ell}{d'_\ell}$	$\binom{1+n_\ell+k_e d}{k_e d}$

Remark 2. *Guards with codimension-1 sets S_e may replace their PSD localizing constraints with linear equality constraints or quotient ring reductions in \mathbf{r}^e .*

Remark 3. *Algorithm 1 of [23] may be used to attempt extraction of near-optimal trajectories if the moment matrices $\forall \ell : \mathbb{M}_d(\mathbf{y}^{0\ell}), \mathbb{M}_d(\mathbf{y}^{p\ell})$ are low-rank.*

4 Extensions

This section details extensions to the previously presented peak estimation framework for hybrid systems.

4.1 Safety

This section verifies safety of hybrid system trajectories with respect to a group of unsafe sets, based on prior (ODE) work in [27]. Let $X_{u\ell} = \{x_\ell \mid p_{\ell i}(x_\ell) \geq 0, i = 1..N_u\}$ be an unsafe basic semialgebraic set for each location $\ell = 1..L$.

Letting $c_\ell(x_\ell, y_\ell)$ be a distance function (cost) with a point-unsafe-set distance

$$c_\ell(x_\ell; X_{u\ell}) = \inf_{y_\ell \in X_{u\ell}} c_\ell(x_\ell, y_\ell), \quad (20)$$

the distance estimation problem for hybrid systems is

$$\begin{aligned} Q^* &= \inf_{\ell' \in 1..N_u, t \in [0, T], \ell_0, x_0} c(x(t \mid x_0); X_{u\ell'}) \\ \text{Dynamics} &\text{ follow Algorithm 1 with input } (\ell_0, x_0, \mathcal{H}, T) \\ x_0 &\in X_{0\ell}. \end{aligned}$$

Following the procedure from [27], a joint-measure $\eta_\ell(x_\ell, y_\ell) \in \mathcal{M}_+(X_\ell \times X_{u\ell})$ is added for each unsafe set. The distance objective in (21) is replaced with an equivalent expectation over the joint probability measure $\langle c_\ell(x_\ell, y_\ell), \eta_\ell \rangle$.

The measure program for distance estimation with variables $(\mu_{p\ell}, \mu_{0\ell}, \mu_\ell, \eta_\ell, \rho_e)$ is

$$q^* = \inf \sum_{\ell=1}^L \langle c_\ell(x_\ell, y_\ell), \eta_\ell \rangle \quad (21a)$$

$$\pi_{\#}^{x_\ell} \eta_\ell = \pi_{\#}^{x_\ell} \mu_{p\ell} \quad \forall \ell \quad (21b)$$

$$\text{Constraints (11b)-(11d)} \quad (21c)$$

$$\text{Variables from (11e)-(11g)} \quad (21d)$$

$$\forall \ell : \eta_\ell(x_\ell, y_\ell) \in \mathcal{M}_+(X_\ell \times X_{u\ell}). \quad (21e)$$

4.2 Uncertainty

Peak estimation for hybrid systems may be applied to systems with uncertainty, extending the ODE case in [28]. Let $W_\ell \subset \mathbb{R}^{N_{w\ell}}$ be a compact set of time-dependent disturbances for each location. Each location obeys dynamics $\dot{x}_\ell = f(t, x_\ell(t), w_\ell(t))$, $\forall t : w_\ell(t) \in W_\ell$, in which there is no prior assumption of continuity on the process $w(\cdot)$. The uncertainty act as adversarial optimal controls attempting to raise the peak functions (p_ℓ) .

Uncertainty in this manner may be realized by adjusting the Liouville equation in (11b) and occupation measure definitions in (11e) (where $w_\ell(t)$ acts as a Young measure/relaxed control [29]) as in

$$\mu_{p\ell} = \delta_0 \otimes \mu_{0\ell} + \pi_{\#}^{tx} \mathcal{L}_{f_\ell}^\dagger \mu_\ell \quad \forall \ell \quad (22a)$$

$$\begin{aligned} &+ \sum_{\text{dst}(e)=\ell} R_{e\#} \rho_e - \sum_{\text{src}(e)=\ell} \rho_e \\ \mu_\ell &\in \mathcal{M}_+([0, T] \times X_\ell \times W_\ell) \quad \forall \ell. \end{aligned} \quad (22b)$$

A particular form of time-dependent uncertainty is switching/polytopic structure. If the system model is $\dot{x}_\ell = \sum_k^{N_s} w_{k\ell} f_{k\ell}(t, x)$ for N_s switching modes and $w_{k\ell} \geq 0, \sum_k w_{k\ell} = 1$, then the Liouville equation in (22) may be expressed for occupation measures $\mu_{k\ell} \in \mathcal{M}_+([0, T] \times X)$ as

$$\mu_{p\ell} = \delta_0 \otimes \mu_{0\ell} + \sum_k^{N_s} \mathcal{L}_{f_{k\ell}}^\dagger \mu_{k\ell} \quad \forall \ell \quad (23a)$$

$$\begin{aligned} &+ \sum_{\text{dst}(e)=\ell} R_{e\#} \rho_e - \sum_{\text{src}(e)=\ell} \rho_e \\ \mu_\ell &\in \mathcal{M}_+([0, T] \times X_\ell) \quad \forall \ell. \end{aligned} \quad (23b)$$

Time-independent uncertainty restricted to a compact set $\Theta \subset \mathbb{R}^{N_\theta}$ may also be added by adjoining to dynamics a new state $\dot{x}_\ell = f(t, x_\ell(t), \theta, w_\ell(t))$, $\dot{\theta} = 0$. This new state θ is preserved between transition jumps, inducing lifted reset maps $\tilde{R}_{s \rightarrow t}(x_s, \theta) = (R(x_t), \theta)$.

5 Numerical Examples

Experiments are available at https://github.com/Jarmill/hybrid_peak_est, and were written in MATLAB 2021a. Dependencies include Gloptipoly3 [30], Yalmip interface [31], and Mosek [32]. All experiments were run on an 2.30 GHz Intel i9 CPU with 64.0 GB of RAM.

5.1 Two-Mode

This system is a modification of Example 2 of [19] to ensure improved numerical conditioning. The two locations correspond to modes of ‘No Control’ ($\ell = 1$) and ‘Control’ ($\ell = 2$) with dynamics

$$\begin{aligned} f_1(t, x) &= [x_2; -x_1 + x_3; x_1 + (1 + x_3)^2(2x_2 + 3x_3)] \\ f_2(t, x) &= [x_2; -x_1 + x_3; -x_1 - (2x_2 + 3x_3)]. \end{aligned} \quad (24a)$$

5.1.1 Two-Mode: Standard

Trajectories start in the initial set $X_{01} = \{x \mid \|x\|_2^2 = 0.2^2\}$ ($X_{02} = \emptyset$), and evolve for a time horizon of $T = 20$. The transition edges are $\mathcal{E} = \{(1, 2), (2, 1)\}$ with guard surfaces

$$\begin{aligned} S_{(1,2)} &= \{x \mid x_1^2/4 + x_2^2 + x_3^2 = 1.5^2\} \\ S_{(2,1)} &= \{x \mid x_1^2 + x_2^2 + x_3^2 = 0.2^2\}, \end{aligned} \quad (25)$$

and each transition has a trivial reset map $R_e(x) = x$. The Zeno caps used in simulation were $N_{(1,2)} = N_{(2,1)} = 5$ with total spaces of $X_1 = X_2 = [-1.5, 1.5]^3$. Figure 1 plots system trajectories in location 1 (left) and 2 (right), starting from the initial set X_0 (gray region). The peak estimation task for this system is to upper bound extreme values of $p_2(x) = x_1^2$ along system trajectories ($p_1(x) = -\infty$). Solving the SDP generated from LMI (19) at orders 1-5 produces the sequence of upper bounds, $p_{1:5}^* = [2.250, 0.6514, 0.4643, 0.4076, 0.3958]$.

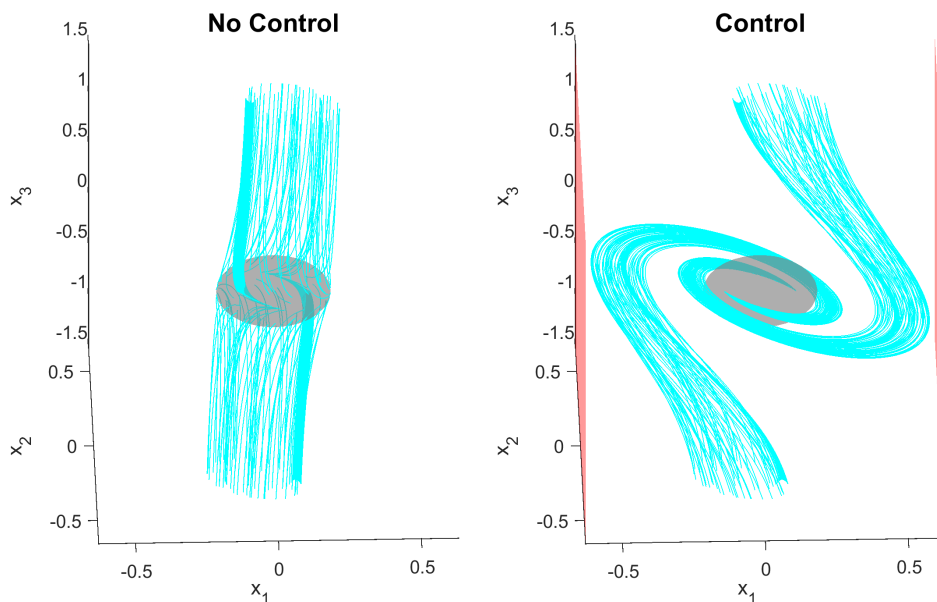


Figure 1: Deterministic Two-Mode Bound of $x_1^2 \leq 0.3958 = p_5^*$

5.1.2 Two-Mode: Distance Estimation

Distance estimation is conducted for the deterministic two-mode system (24) with respect to the half-sphere unsafe set

$$X_u = \{x \mid 0.4^2 \geq (x_1 + 0.5)^2 + (x_2 + 0.5)^2 + (x_3 - 0.5)^2, x_3 \geq 0.5\}. \quad (26)$$

The distance penalty $c(x, y) = \|x - y\|_2^2$ is used in locations $\ell = 1, 2$ with the unsafe set X_u . SDP lower bounds for the distance $\min_{\ell} \min_{y \in X_u} \|x_{\ell} - y\|_2^2$ (via (21)) are $p_{1:5}^* = [0, 0, 0, 2.799 \times 10^{-3}, 7.942 \times 10^{-3}]$. The output of distance estimation is plotted in Figure 2. The solid red half-sphere is the set X_u , and the corona surrounding X_u is the set of all points with an L_2 distance at most $0.0891 = \sqrt{p_5^*}$ away from X_u .

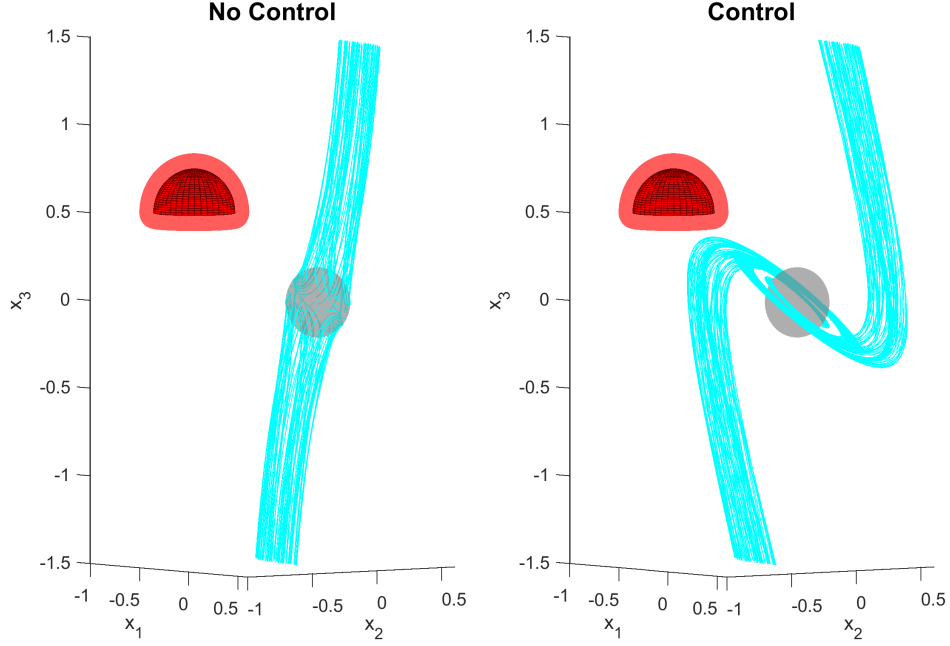


Figure 2: Deterministic Two-Mode Distance Bound of $\min_{y \in X_u} \|x - y\|_2 \leq 0.0891 = \sqrt{p_3^*}$

5.1.3 Two-Mode: Uncertainty

Time-dependent uncertainty may be added to dynamics in (24) by defining a process $w(t) \in [-1, 1]$ under the dynamics $\tilde{f}_\ell(t, x) = f_\ell(t, x) + [0; 0; w]$. The SDP bounds for x_1^2 when w is realized as switching-type uncertainty is $p_{1.5}^* = [2.250, 1.4029, 1.0350, 0.9790, 0.9660]$. System trajectories and the order-5 bound of this noisy system are plotted in Figure 3.

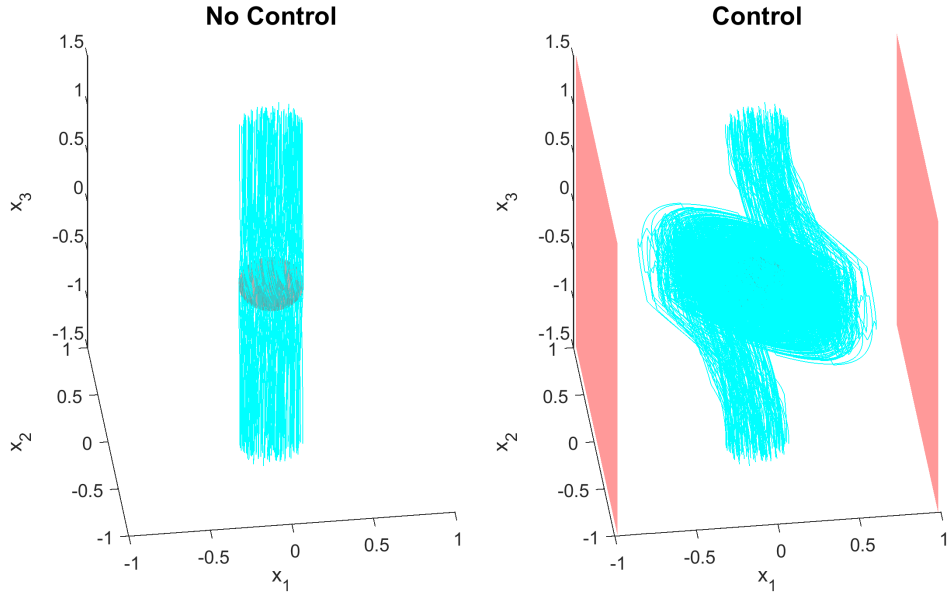


Figure 3: Noisy Two-Mode Bound of $x_1^2 \leq 0.9660 = p_5^*$

5.2 Right-Left Wrap

This example has a single location $X = [-1, 1]^2$ with nontrivial reset maps. The dynamics in the single location are

$$\dot{x} = \begin{bmatrix} -x_2 + x_1 * x_2 + 0.5 \\ -x_2 - x_1 + x_1^3 \end{bmatrix}. \quad (27)$$

System (27) has a stable equilibrium point at $(-0.8128, 0.2758)$ and a saddle point at $(-0.4288, 0.3499)$. The following right \rightarrow top and left \rightarrow bottom transitions are defined with Zeno caps of $N = 5$

$$\begin{aligned} S_{\text{right}\rightarrow\text{top}} &= \{x \mid x_1 = 1, x_2 \in [-1, 1]\} & R_{\text{right}\rightarrow\text{top}} &= [x_2, x_1]^T \\ S_{\text{left}\rightarrow\text{bottom}} &= \{x \mid x_1 = -1, x_2 \in [-1, 1]\} & R_{\text{left}\rightarrow\text{bottom}} &= [1 - x_2, x_1]^T. \end{aligned} \quad (28)$$

The set X is invariant under these state transitions. A peak estimation task to maximize $p(x) = -(x_1 + 0.5)^2 + (x_2 + 0.5)^2$ is defined on system dynamics (27) starting from the initial set $X_0 = \{x \mid 0.2^2 = (x_1 - 0.5)^2 + (x_2 + 0.3)^2\}$ for a $T = 5$ time horizon. SDP upper bounds for this objective are $p_{1:6}^* = [0, 0, -0.3644, -0.5259, -0.5659, -0.5721]$.

Figure 4 plots the ODE system dynamics in (27). Figure 4b plots hybrid system dynamics in cyan, starting from the black-circle X_0 . The p_6^* bound is indicated in the red circle of radius $\sqrt{-p_6^*} = 0.7564$, in which no considered hybrid system trajectory is contained.

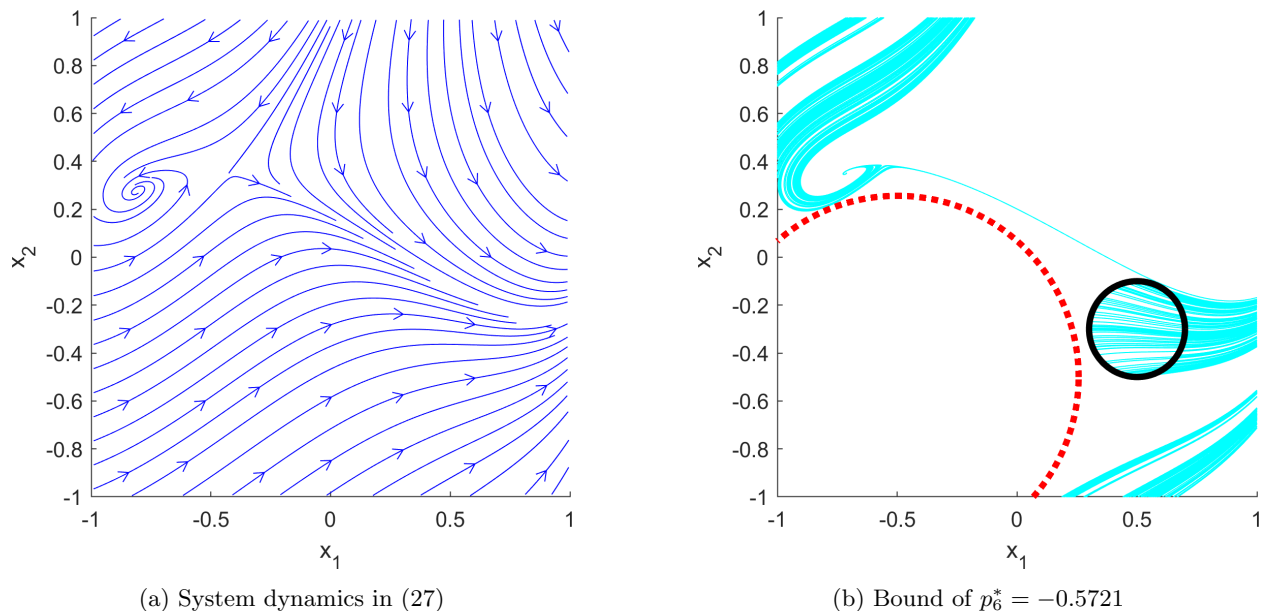


Figure 4: Peak Estimation of Right-Left Wrap Dynamics (27) and (28)

6 Conclusion

An existing peak estimation framework for ODE systems was extended in this paper to hybrid systems. A hierarchy of SDPs result in a (convergent) decreasing sequence of upper bounds to the true peak value. Extensions to the hybrid peak estimation framework, such as bounding the distance to unsafe sets [27] and estimation of systems with uncertainty [28], can be accomplished by modifying equations in the LP. Future work includes performing peak-minimizing (L_1 -optimal) control of hybrid systems [33], data-driven peak estimation of hybrid systems [34], applying numerical techniques to perform peak estimation on more complicated systems (e.g., rigid body dynamics in robotics), and generalizing analysis of deterministic hybrid dynamics to Markov Decision Processes.

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