Peirce's Clarifications of Continuity

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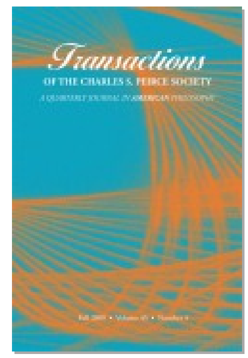

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JÉRÔME HAVENEL

Abstract:
This article aims to demonstrate that a careful examination of Peirce's original manuscripts shows that there are five main periods in Peirce's evolution in his mathematical and philosophical conceptualizations of continuity. The aim of this article is also to establish the relevance of Peirce's reflections on continuity for philosophers and mathematicians.[1]

Keywords: Continuity, Continuum, Cosmology, Infinite, Infinitesimal, Peirce, Perception, Potentiality, Principle of continuity, Realism, Synechism, Topology

The idea of continuity plays an important role throughout Peirce's philosophy since his rejection of nominalism in 1868, but Peirce "did not at first suppose that it was, as [he] gradually came to find it, the master-Key of philosophy" (MS 949 p. 1)[2]. In his mature thought continuity had become so important that he called his philosophy "Synechism", which he defines in 1893 as "the doctrine that continuity rules the whole domain of experience" (MS 946, p. 5)[3], and later on in a text written in 1901 as: "That tendency of philosophical thought which insists upon the idea of continuity as of prime importance in philosophy, and in particular, upon the necessity of hypotheses involving true continuity" (Baldwin, p. 657)[4]. Hence, in most aspects of Peirce's mature philosophy, the notion of continuity plays an important role.

In his 1903 *Principles of Mathematics*, Russell states that "as to what [philosophers] meant by continuity and discreteness, they preserved a discreet and continuous silence; . . . [and] whatever they did mean could not be relevant to mathematics"[5]. Russell's claim clearly does not apply to Peirce. Indeed, Peirce's gradual clarifications on continuity is a proof that mathematical and philosophical considerations can benefit from each other.

Hence, on the one hand, the theory of continuity needs a mathematical reflection, however, on the other hand, continuity is not only a question of pure mathematics:

> . . . He [Williams James] thought that the Achilles disproved Dedekind's theory of continuity, which I take to be generally believed by mathematicians, though it is beyond the jurisdiction of Pure Mathematics, which deals exclusively with the consequences deducible from hypotheses arbitrarily posited. (CP 6.182)[6]

Because it deals with Peirce's definitions of continuity, the present paper could have been called "Peirce's Definitions of Continuity". Indeed, Peirce states that Aristotle's "definition of continuity would be enough were it alone" (MS 816, 1905) to rank Aristotle at the highest grade of original work in logic. But for Peirce,

> . . . there are three grades of clearness in our apprehensions of the meanings of words. The first consists in the connexion of the word with familiar experience. . . . The second grade consists in the abstract definition, depending upon an analysis of just what it is that makes the word applicable. . . . The third grade of clearness consists in such a representation of the idea that fruitful reasoning can be made to turn upon it, and that it can be applied to the resolution of difficult practical problems. (CP 3.457, 1897)[7]

Since Peirce's aim was not only to give an abstract definition of continuity, but to develop a whole philosophy of continuity and thus to reach the third grade of clearness for continuity: not only familiarity, not only abstract definition, but also fruitful reasoning, I have chosen to speak of "Peirce's Clarifications of Continuity."

The fact that Peirce's aim was not only to give an abstract definition gives me a first argument in favor of my choice to use the term 'continuity' at least as much as 'continuum', to refer to Peirce's conception; unlike Putnam, 1995, p. 1: "Peirce's own notion of continuity, or, rather, of the continuum . . ." A second and close argument for my choice is that, as shown by Zalamea, 2001, Peirce's 'continuum' is not a unique well-determine structure, unlike Cantor's continuum. A third argument is the fact that during twenty years, in MS 1597, Peirce had been discussing Aristotle, Kant and Cantor's conceptions of continuity in the margin of his article "Continuity."[8]

The Peircean reader might think that in order to understand Peirce's philosophy of continuity, it could be enough to grasp an intuition of continuity that would be present throughout Peirce's writings on continuity.

For example, in his interesting attempt to reconstruct *The Continuity of Peirce's Thought*, Parker states that "Peirce's definition of the continuum went through several revisions, but it always involved the notion that there are no ultimate parts to a true continuum, and that infinitesimals are real" (Parker, 1998, p. 23). This claim is not true and it is misleading. Commenting on Parker's claim, Tiercelin writes that "as a result, the argumentative links Parker draws between the issues are very often artificial" (Tiercelin, 1999, p. 218).

Peirce's mathematical definitions of continuity are not just the struggle to translate into mathematical language a stable intuition; but his intuition itself evolves with its mathematical conceptualizations. Unlike Parker, I claim that if one compounds the different definitions of continuity Peirce gives from 1868 until the end of his intellectual life, the result does not constitute a consistent position. However, Parker's assertion is partially true. Indeed, if one excludes his Cantorian period (1884–1892), Peirce maintains in his four other periods the "Aristotelian" idea that "there are no ultimate parts to a true continuum".

One can find in MS 1597 five different modifications from his previous definition of continuity written for the *Century Dictionary*. Indeed, the evolution of Peirce's conception of continuity was not the smooth growth of a seminal idea. For example, we know by two rubber stamp marks in MS 278 that between February 7th, 1884 and April 1st, 1884, Peirce violently dismissed his previous conception of continuity.

> Continuity has never yet been defined. Kant's definition, to which I am ashamed to say I have hitherto given my adhesion, is ridiculous when you come to think of it. And without a definition of course all the reasoning about it is fallacious. (MS 278 D, 1884)

A division of Peirce's evolution in his definitions of continuity has been put forward by Potter and Shields (1977) who distinguish four periods. This article is still considered by most Peirce scholars as the reference on this topic. But despite its obvious qualities, when it was written their authors had access neither to *The New Elements of Mathematics* edited by Carolyn Eisele, nor to the important editorial work related to the publication of Peirce's *Writings*.

One purpose of the present paper is to update Potter and Shields' work.[9] I assert that a careful examination of Peirce's manuscripts shows that there are five main periods in Peirce's definitions of continuity, whereas Potter and Shields only distinguish four periods:

1) Pre-Cantorian Period: until 1884
2) Cantorian Period: 1884–1894
3) Kantian Period: 1895–1908
4) Post-Cantorian Period: 1908–1911.

For each of these five periods, I give a detailed description, and then explore the philosophical consequences of the main differences between Potter and Shields' position and my own proposal. This paper also offers some insights concerning the relevance of Peirce's definitions of continuity for philosophers and mathematicians.

1) Anti-nominalistic Period (1868–1884)[10]

The first proper definition of a continuum given by Peirce was written in 1868, the same year he rejected nominalism and "its denial of the reality of Thirds" (Fisch, 1971, p.2). This is not a coincidence. It is linked to Peirce's position, in 1868, that a continuum is not built out of its parts but that the parts of a *continuum* are built out of the continuum. "All the arguments of Zeno depend on supposing that a continuum has ultimate parts. But a *continuum* is precisely that, every part of which has parts, in the same sense." (W 2.256, CP 5.335, 1868)

This idea is essential in Peirce's theory of cognition developed in 1868. In "Some Consequences of Four Incapacities", Peirce explains that a cognitive process is not a succession of separated ideas at different instants, but a continuous flow. "At no one instant in my state of mind is there cognition or representation, but in the relation of my states of mind at different instants there is" (W 2.227, CP 5.289, 1868).[11] Therefore cognition or representation cannot exist at one specific instant in a state of mind, but it is a continuous flow of relations.

In his 1870 text about the logic of relatives, Peirce explains that an absolute individual cannot accept any logical division, like a point which cannot be divided. For him, every supposed absolute individual has to exist in reality or in thought and this implies temporality, but it follows that in each lapse of time an absolute individual can be logically divided; therefore it cannot be an absolute individual. Likewise, even in thought, a logical atom cannot exist. "A logical atom, then, like a point in space, would involve for its precise determination an endless process" (W 2.390, CP 3.93, 1870). Just as the division of a continuum cannot lead to an ultimate part like a point, a logical atom cannot exist even in thought.

On March 6th and March 8th, 1873, Peirce explains the necessity of the continuity of time to account for the association of ideas[12]. On March 6, he explains that between two distinct thoughts there must be an "element of consciousness" (CP 7.352), which he calls "the matter of thought", and which is continuously present within the process of thought. Two days later, Peirce explains that if the succession of ideas were unfolded by discreet steps, then the association of ideas would be impossible, and thought would be impossible.

> Every mind which reasons must have ideas which not only follow after others but are caused by them. . . But is it presupposed in the conception of the logical mind that the temporal succession in its

> ideas is continuous, and not by discreet steps? . . . If the succession of images in the mind is by discreet steps, time for that mind will be made up of indivisible instants. Any one idea will be absolutely distinguished from every other idea by its being present only in the passing moment. . . In short the resemblance of ideas implies that some two ideas are to be thought together which are present to the mind at different times. And this never can be, if instants are separated from one another by absolute steps. This conception is therefore to be abandoned. It is already presupposed in the conception of a logical mind that the flow of time should be continuous. . . (MS 377, March 8, 1873, p. 1–4)

As pointed out by Moore, these texts of 1873 are "an ancestor" of one of the main arguments in the "Law of Mind", and in these texts Peirce is already struggling "with issues that will compel him", many years later, to recognize that the points on a continuum "lack distinct identities"[13]. Thus, one key idea in this first period is that a logical mind involves the continuity of time.

On July 1st–2nd, 1873, Peirce added the idea that a continuum is something any part of which, however small it is, has parts of the same kind. A continuum is not only infinitely divisible, but its parts are of the same kind, which means that every part of a surface is a surface, and every part of a line is a line. But if a point in time or space is not a part of time or space, what is a point? To Peirce, in 1873, a point is nothing but the ideal limit towards which one approaches indefinitely close without ever reaching it in dividing time or space.

> To assert that something is true of a point is only to say that it is true of times and spaces however small or else that it is more and more nearly true the smaller the time or space and as little as we please from being true of a sufficiently small interval. . . nothing is true of a point which is not at least on the limit of what is true for spaces and times. (W 3.105–106, July 1–2, 1873)[14]

Peirce gives the example of a body moving in space and time. It is not true that a body occupies one position at any instant, because a body cannot exist but in time, and in any lapse of time it is moving. Nevertheless, the shorter the time for which the body's position is considered, the more determinate it is, and the variation of its position can be made less than any assigned difference.

Another important idea is the focus on the methodological principle of continuity which is obvious in 1878, when he explains that the dichotomy between union and separation can be overcome by distinguishing various degrees inside broadened conceptions. Moreover, this could lead to finding new solutions. Peirce rewrote this text in 1893 because in 1878 he did not make a clear distinction between continuity and infinite divisibility. Yet the idea of the methodological wonder of the application of the principle of continuity remained a key idea for Peirce:

> . . . it will be found everywhere that the idea of continuity is a powerful aid to the formation of true and fruitful conceptions. By means of it, the greatest differences are broken down and resolved into differences of degree, and the incessant application of it is of the greatest value in broadening our conceptions. I propose to make a great use of this idea. . . This application of continuity to cases where it does not really exist illustrates, also, another point which will hereafter demand a separate study, namely, the great utility which fictions sometimes have in science. (W 3.277–278, CP 2.646, 1878)[15]

In the same year of 1878, Peirce explains that although "we hear only what is present at the instant" (W3.262, CP 5.395, 1878), we are nevertheless able to perceive music. But such a capacity can only be explained if there is "some continuity of consciousness which makes the events of a lapse of time present to us" (*ibid*).

It is important within Peirce's intellectual evolution that in 1881, seven years before Dedekind's famous article[16], he gave a good criterion to distinguish a finite system from an infinite one[17]. However, in this brilliant article, Peirce still defines continuity using the notion of infinite divisibility. "A continuous system is one in which every quantity greater than another is also greater than some intermediate quantity greater than that other" (W4.300, CP 3.256, 1881).

I agree with Potter and Shields that before 1884 Peirce makes a blunder that he will later reject, that of confusing continuity with infinite divisibility; but this first period is nevertheless very interesting. It is before 1884 that Peirce claims that continuity has no ultimate parts, and that time is a continuum not made of instants[18]. Such a conception is deeply related with three important ideas: Peirce's claim that a logical mind involves the continuity of time, Peirce's rejection of an intuition that would be an absolute first, for this absolute first would have to take place in an instant occupying no time, and Peirce's conception of cognition as a continuous process that is not made of discontinuous steps. This is well explained by Peirce's superb argument of the inverted triangle[19].

2) Cantorian Period (1884–1892)

In several respects, a radical change occurs in 1884. A first sign of this change appears in the beginning of 1884, in a draft version of the lecture "Design and Chance" presented by Peirce at the Metaphysical Club, on January 17, 1884, in which he acknowledged that Darwin's theory of evolution is of great importance in the field of logic (MS 875, 1884). Evolution then becomes for Peirce a postulate of logic; evolution applies to things and to laws.[20] The importance of this new idea of evolution for Peirce's conception of continuity will gradually appear later on in his work.

In 1884–1885, another important event occurs. Whereas in his "New List of Categories" written in 1867 and published in 1868, the core work is the analysis of the sign relation to unify diversity, while the notion of continuity remains unessential, in the rewriting of his categories in 1884–1885, continuity corresponds to Thirdness, the category of rationality. "Continuity represents Thirdness almost to perfection" (MS 904 or CP 1.337, Summer - Fall of 1886)[21].

The central importance of the category of Thirdness for Peirce's philosophy appears clearly in "A Guess at the Riddle", written in 1887–1888. The goal of this paper is to show that triads are essential not only in logic, but in every department of philosophy. And for Peirce it is a great achievement of modern times to have understood this. For example, to describe approximately the facts of experience one can be satisfied with dual oppositions like "first and second, agent and patient, yes and no" (W 6.172, CP 1.359).[22] But a scientific mind will not be satisfied by these severe oppositions, and he will call for the third to bridge over the chasms, to bring into relation isolated cases; or, in other words, to try to make reality intelligible. This methodology is for Peirce essential in all scientific fields, for example in the development of modern geometry.

> The superiority of modern geometry, too, has certainly been due to nothing so much as to the bridging over of the innumerable distinct cases with which the ancient science was encumbered; and we may go so far as to say that all the great steps in the method of science in every department have consisted in bringing into relation cases previously discrete.[23]

From a historical point of view, the progress between discrete cases and the application of the principle of continuity corresponds to the evolution of science from its qualitative to its quantitative stage. Its qualitative stage is when the science can be described by dual distinctions, like whether or not a given subject has a given predicate. Its quantitative stage comes when such brutal distinctions are no longer enough for a scientist. For example, in ancient mechanics, forces were considered to be "causes which produced motions as their immediate effects"[24], but such an idea hindered the progress of dynamics during two thousands years. Roughly speaking, in ancient mechanics a force is related to a variation of velocity, whereas in modern mechanics it is related to a variation of acceleration.

> The work of Galileo and his successors lay in showing that forces are accelerations by which [a] state of velocity is gradually brought about . . . the old conceptions have been dropped . . . for the fact now known is that in certain relative positions bodies undergo certain

> accelerations. Now an acceleration, instead of being like a velocity a relation between two successive positions, is a relation between three; so that the new doctrine has consisted in the suitable introduction of the conception of threeness. On this idea, the whole of modern physics is built.[25]

And this is also true of modern mathematics. In "The Simplest Mathematics", written in 1902, Peirce explains that most success in mathematics comes from exchanging a smaller problem that involves exceptions for a larger one free from them.

> Thus, rather than suppose that parallel lines, unlike all other pairs of straight lines in a plane, never meet, he supposes that they intersect at infinity. Rather than suppose that some equations have roots while others have not, he supplements real quantity by the infinitely greater realm of imaginary quantity. . . (CP 4.236)

Now what conception of continuity does Peirce develop in his Cantorian Period? We have seen that in 1884, Peirce violently dismissed his previous conception of continuity. In a letter to Philip E. B. Jourdain, December 5, 1908, Peirce asserts that he first learned about Cantor's work "from the *Acta Mathematica* in the winter of 1883–4 or later" (NEM 3.883). But we know that it is in 1884 that the French translation of several articles by Cantor was published in the *Acta Mathematica*; thus, it cannot be before 1884.

Peirce explains in the article "Continuity" written for the *Century Dictionary*, that it is very difficult to give a good definition of continuity but that "The less unsatisfactory definition is that of G. Cantor, that continuity is the *perfect concatenation* of a system of points. . ." Cantor's definition of continuity is better than the previous ones because Cantor's definition gives a precise tool to clearly distinguish continuity and infinite divisibility.

For Cantor, a continuum is a set which is perfect and connected (*zusammenhängend*), in which 'connected' (*zusammenhängend*) means, according to §19 of the *Grundlagen*, that for every two elements t and t' of the set, and for every $\epsilon > 0$ there exist a finite number of elements $t_1 = t, t_2, \ldots, t_n = t'$ of the set such that for every i, $|t_i - t_{i+1}| < \epsilon$.

It is important to note that, if 'connected' is the proper translation for '*zusammenhängend*', this differs from the modern topological concept of 'connectedness'. For modern terminology, a set S is connected if it cannot be split into two open non-empty subsets A and B, such that $A \cap B = \varnothing$; and $A \cup B = S$. The modern notion of 'connectedness' involves Cantor's notion of '*zusammenhängend*', but the opposite is not true, as is shown by the properties of the set of rational numbers. Therefore, according to Cantor's terminology the set of rational numbers is connected, whereas according to the modern terminology it is not.

Hence, when Peirce uses the term 'concatenation', it does not mean 'connectedness' but '*zusammenhängend*'. This is clear when Peirce, in the same article "Continuity", gives the following definition:

> Cantor calls a system of points concatenated when any two of them being given, and also any finite distance, however small, it is always possible to find a finite number of other points of the system through which by successive steps, each less than the given distance, it would be possible to proceed from one of the given points to the other. (CD, "Continuity")

In "The Law of Mind", written in 1892, in order to define 'concatenated series', Peirce uses the same definition[26].

As earlier mentioned, in 1881 Peirce showed independently of Dedekind how to distinguish clearly the finite and the infinite, but it is thanks to Cantor in 1884 that Peirce came to realize how to distinguish clearly continuity from infinite divisibility. Nevertheless, one can make the assumption that, for Peirce, Cantor's definition is not completely satisfactory because it does not meet this *requisit* in all Peirce's definitions since 1868 that a continuum is not composed of points but of parts that are themselves a divisible continuum.

But can one be sure that in his Cantorian period, Peirce has a good understanding of Cantor's theory of continuity? As Murphey pointed out, not only are we sure that Peirce did not read all Cantor's papers, but we also have good reason to believe that he read some of them superficially.

> There is good reason to believe, . . . that Peirce read selectively and skipped over a good deal of the content of these papers, for . . . he either did not know about some of Cantor's discoveries which are described in these papers or else he very badly misunderstood what he read. (Murphey, 1993, p. 241)

We have seen that in his article "Continuity" for the *Century Dictionary*, Peirce had correctly defined the property of concatenation (*zusammenhängend*), but as Potter and Shields pointed out, this is not so clear for the property of perfection[27].

Cantor had defined the derived set of a set S to be the set of limit points of S, where a limit point of S is a point of S with infinitely many points of S arbitrarily close to it. The set S is called 'perfect' if it coincides with its derived set. In modern terminology, a set is 'perfect' if it is closed and has no isolated points. But, for a set, being closed and having no isolated points, does not avoid being full of holes. For example, Cantor pointed out that what we call today 'Cantor's ternary set' does not verify the property of being connected (*zusammenhängend*), whereas it is a 'perfect' set. For Cantor, a continuous system is perfect and concatenated. "I believe that in these two predicates 'perfect' and 'connected' I have discovered the necessary and sufficient properties of a point-continuum." (Cantor, 1999a, p. 906)

In his article "Continuity", Peirce gives the following definition:

> He [Cantor] terms a system of points *perfect* when, whatever point not belonging to the system be given, it is possible to find a finite distance so small that there are not an infinite number of points of the system within that distance of the given point. (CD)

Peirce's hesitation concerning the definition of a perfect system can be seen in the fact that Peirce made a correction in MS 1597 by adding a negation, which he thereafter canceled. Here is the first modification: ". . . it is NOT possible to find a finite distance so small that there are not an infinite number of points of the system within that distance of the given point" (MS 1597, article "Continuity"). However, in the margin of this manuscript, Peirce canceled the addition of 'not'[28]. This version without the negation is more consistent with Cantor's definition of a perfect system. Nevertheless, in the above definition of a perfect system, Peirce only described the property that whatever point of the system being given, it must be a limit-point. Hence, it seems that in 1884, Peirce curtails Cantor's definition of a perfect system to the property of having no isolated points.

In a comment written between 1888 and April 1892, Peirce gave a correct definition of what a perfect system is, and he made a slight modification to his article "Continuity".

> I here slightly modify Cantor's definition of a perfect system. Namely, he defines it as such that it contains every point in the neighborhood of an infinity of points and no other. But the latter is a character of a concatenated system; hence I omit it as a character of a perfect system. (MS 1597, article "Continuity")

Peirce points out rightly that a set that has an isolated point cannot be concatenated, hence if a set is concatenated it has no isolated points. Now Peirce wants to omit the feature of having no isolated points in the definition of a perfect system for the reason that it is already implied in the property of being concatenated. Peirce's modification is a question of logical elegance, but I think it is not a real improvement, for what matters here is that whether or not two mathematical concepts (perfection and concatenation) involve continuity. One can sum up that from 1884 until 1892 Peirce thinks that Cantor's definition of continuity is better than all others because it allows us to distinguish continuity from infinite divisibility, but that it is still unsatisfactory.

"In his article "infinite" written for the CD, so in 1883–1889, Peirce states that for mathematicians there are two kinds of infinity, the infinity of the multitude of whole numbers, and the infinity of the multitude of points upon a line. Then he assumes that if $\propto$ represents the former infinity, then $10^{\propto}$ represents the latter infinity.

As pointed out by (Moore, 2007, p. 462), Peirce does not yet "use Cantor's transfinite numbers" in what I call his "Cantorian Period". However, when he starts using Cantor's transfinite numbers in his "Supermultitudinous Period", his definition of continuity is no longer close in spirit with Cantor's one, as this was the case in his "Cantorian Period".

In this article "infinite", Peirce also claims that "the multitude of points in a line is the greatest possible quantity", since "the points of a line . . . can be brought into a one-to-one correspondence with those of all space . . . and that although the space considered have an infinite multitude of dimensions". However, in the margin of this article, Peirce writes after his "Cantorian Period" that: "A continuum of an infinite number of dimensions, if such a thing can be conceived would have more points than an ordinary continuum" (MS 1597).

In a nutshell, in his "Cantorian Period", Peirce does not fully master Cantor's mathematical work, but he does mainly agree with what he knows of Cantor's conception of continuity.

Although I disagree with Potter and Shields that the Cantorian Period extends until 1894 rather than 1892, they have brilliantly explained Cantor's influence over Peirce's continuum. However, there are at least two important aspects in the Cantorian Period that cannot be explained by Cantor's influence.

First, the idea that continuity corresponds to thirdness, that triads are essential not only in logic, but in every department of philosophy and in scientific methodology, in order to bring into relation isolated cases, to try to make reality intelligible.

Secondly, the importance of the principle of continuity for the evolution of science. In particular, the application of the principle of continuity corresponds to the evolution of science from its qualitative to its quantitative stage. This means that qualitative distinctions do not provide enough accuracy, whereas quantitative distinctions allow a much more accurate description of phenomena.

But this could be somewhat puzzling for it seems that after his Cantorian Period, because of his rejection of the idea that continuity can be explained by metrical notions, and because of its Aristotelian position, Peirce's conception of continuity is qualitative rather than quantitative. However, the qualitative/quantitative opposition does have the same meaning for the evolution of science and for continuity.

In the case of the mathematical approach of continuity, as we will see in the subsequent periods, Peirce can be considered as a forerunner of René Thom's position (among others) that in order to understand the continuum mathematically, the qualitative approach of topology is better than quantitative mathematics.

3) Infinitesimal Period (1892–1897)

There are four new important aspects in 1892 concerning Peirce's conception of continuity. First, Peirce dismisses Cantor's definition of continuity. Second, it is in 1892 that Peirce makes for the first time an in-depth study of Aristotle's conception of continuity. Third, the year 1892 represents an important growth in Peirce's interest in the principle of continuity. Fourth, Peirce asserts that continuity implies infinitesimals.

Around April 1892, Peirce finds too many shortcomings in Cantor's definition of continuity, so he has to find something else.

> Cantor's definition of continuity is unsatisfactory as involving a vague reference to all the points, and one knows not what that may mean. It seems to me to point to this: that it is impossible to get the idea of continuity without two dimensions. An oval line is continuous, because it is impossible to pass from the inside to the outside without passing a point of the curve. (MS 1597, article "Continuity")

While looking for a new theory of continuity, Peirce was inspired by his in-depth study of Aristotle's conception. In September 1894, Peirce writes that he has "read and thought more about Aristotle than about any other man" (MS 1604). Nevertheless, as it has been pointed out by Robin, "it is difficult to tell whether this remark was meant to apply generally, since it was made in the context of his discussion of Greek philosophy." (Robin, 1967)[29]

Aristotle's influence already appears in "The Law of Mind", which Peirce completed on May 24, 1892. Although Peirce will later on dismiss this essay as his "blundering treatment of Continuity"[30], this article put forwards one of the key aspects of Peirce's mature conception of continuity that "the reality of continuity appears most clearly in reference to mental phenomena"[31]. For Peirce, the law of mind is that "ideas tend to spread continuously . . . and [to] become welded with other ideas"[32].

In "The Law of Mind", Peirce defines continuity with two new terms: 'Kanticity' which means infinite divisibility and 'Aristotelicity' which corresponds to the modern property of completeness: "Aristotelicity is having every point between which and any that is a limit to an infinite series of points that belong to the system." (MS 1597, article "Continuity")[33]

In 1893–1894, Peirce explains that his intellectual education led him "at the very outset to think that one great desideratum in all theorizing was to make fuller use of the principle of continuit" (MS 949, p. 1). Indeed, we have seen that in 1878, Peirce has already developed the idea of the methodological wonder of the application of the principle of continuity for the growth of science and philosophy; and it has also been the case in "A Guess at the Riddle" written in 1887–1888.

Peirce maintains this idea in 1893–1894 by asserting that "before modern times, continuity was a recondite idea" (MS 949), and that one great achievement of the new way of doing science in the Renaissance is the broad use it made of the principle of continuity.

Hence, the most powerful methodology for the development of any kind of science is the application of the principle of continuity. But the importance of the principle of continuity is even wider because if one wants to facilitate any further research, one must leave all possibilities open. Now to say that anything is discontinuous is to close off possibilities:

> Accordingly a regulative principle of logic requires us to hold anything as continuous until it is proved discontinuous. But absolute discontinuity cannot be proved to be real, nor can any good reason for believing it real be alleged. We thus reach the conclusion that as a regulative principle, at least, ultimate continuity ought to be presumed everywhere. . . (CP 8 Bibliography General 1893 [G-1893–5])

Therefore, the idea of continuity "plays a great part in all scientific thought, and the greater the more scientific that thought is; and it is the master key which adepts tell us unlocks the arcana of philosophy"[34] (CP 1.163, Summer 1893).

In "Fallibilism, Continuity and Evolution", written in 1893, Peirce explains that the great opponent of his philosophy of continuity is infallibilism, the position of those men who fancy that part of their knowledge is perfectly exact and certain. Peirce rejects infallibilism because it blocks the road of inquiry, because it is related to a mechanical conception of the universe in which there is only transformation but no creation, and because it involves that thought is unable to modify reality.

On the contrary, Peirce thinks that:

> The principle of continuity is the idea of fallibilism objectified. For fallibilism is the doctrine that our knowledge is never absolute but always swims, as it were, in a continuum of uncertainty and of indeterminacy. (CP 1.171, Summer 1893)

Peirce insists that there is a deep link between his philosophy of continuity and evolution: "If all things are continuous, the universe must be undergoing a continuous growth from non-existence to existence" (CP 1.175, 1893).

Before explaining the reasons why, according to Peirce, continuity involves infinitesimals, it is necessary to understand why Peirce rejects the doctrine of limits used by Weierstrass, Dedekind and Cantor.

In "The Logic of Quantity" written in 1893, Peirce explains that in "many mathematical treatises the limit is defined as a point that can 'never' be reached." But for Peirce in mathematics "never" can only mean 'not consistently with—', so to say that a point can never be reached has no mathematical meaning unless it is specified with what it is not consistent. For example, the limit of the series 1 n is 0, and 0 is indeed not consistent with the value of 1 n for every finite value of n; but that is not the case for the infinite.

More generally, if we consider x[n] a converging series, it is not correct to say that the limit is never reached. Indeed, for finite values of [n], x[n] is a series that varies with [n]; but if we consider x[∝], we deal with an infinite series which no longer varies with [n], otherwise there would be no precise value for x[∝][35]. The precise boundary between the values of x[n] for finite values of [n], and the values of x[n] for infinite values of [n][36], is what constitutes the limit.

Peirce's conception of what is a limit, is to consider it as a border between two regions which is passed through at the limit. In other words, an isolated point that cannot be reached cannot be the limit of any process, since the specification of the limit of any process requires this process to take place in the neighborhood of the limit. "The metaphysicians have in this instance been clearer than the mathematicians—and that upon a point of mathematics; for they have always declared that a limit was inconceivable without a region beyond it" (CP 4.118, 1893). There is an important distinction here between continuous series and discrete ones. Each continuous series contains its limit, while the limit-point of a discrete series can be external.

To account for this conception, Peirce uses the example of midnight as the limit between today and tomorrow. Today is a series of instants with an end, therefore the instant of the midnight coming next must belong to today. But tomorrow is a series of instants with a beginning, therefore the instant of midnight must also belong to tomorrow. This is a rewording of Aristotle's idea that adjacent parts have their limits in common[37], and this is what constitutes the continuity of time. Hence, "a man cannot be broad awake quite through all the time before midnight, without being broad awake at midnight"[38], otherwise time would have to be discontinuous. "All the instants before one instant, *exclusive*, is in the continuous series a self-contradictory description" (NEM 3. 125–126)[39].

Of course, it is possible to separate a series of real numbers and its limiting element, as Dedekind and Cantor have shown, but this is not possible using Peirce's idea of what a truly continuous series consists in. This involves that a continuous series cannot be defined as the combination of all the elements before a certain element with all the elements after a certain element, because it would be a breach of continuity.

This property is found in a recent mathematical theory that I will hereafter compare with Peirce's own, *Smooth Infinitesimal Analysis* (SIA). Putnam has also shown that Kurt Gödel has developed a similar idea against the continuity of the set of real numbers.

> Kurt Gödel remarked (in a short unpublished note I was shown a number of years ago) that, at least intuitively, if you divide the geometrical line at a point, you would expect that the two halves of the line would be mirror images of each other. Yet, this is not the case if the geometrical line is isomorphic to the real numbers. (Putnam, 1995, p. 3)

This idea is indeed very close to Peirce's own, since for Peirce: "it is impossible to sever a continuum by separating the connections of the points, for the points only exist in virtue of those connections."[40]

Indeed, in Cantor's theory of continuity, grounded on the idea that the arithmetical and the geometrical continuum are isomorphic[41], we can write that $\mathbb{R} =]-\propto; 0[\cup [0; +\propto[$. But this formula is incompatible with the idea that the two halves of the line would be mirror images of each other.

Let's use $\mathbb{R}$ for the usual set of real numbers and $\mathbf{R}$ for the continuum domain for *Smooth Infinitesimal Analysis* (SIA). We can write that $\mathbb{R} = \{\mathbb{R} \setminus \{0\}\} \cup \{0\}$, but nevertheless one can demonstrate in SIA that $\mathbf{R} \neq \{\mathbf{R} \setminus \{0\}\} \cup \{0\}$. In other words, $\mathbf{R}$ contains more elements than $\{\mathbf{R} \setminus \{0\}\} \cup \{0\}$, because all the elements that are not distinguishable from zero belong to $\mathbf{R}$ but not to $\{\mathbf{R} \setminus \{0\}\} \cup \{0\}$. Therefore, it is wrong that $\forall\, x \in \mathbf{R}, (x = 0) \vee (x \neq 0)$; the law of excluded middle does not hold everywhere in SIA[42].

In the same way Peirce does not consider that the law of excluded middle and the principle of contradiction hold for every aspect of reality[43].

A limit can be a point, but it can also be a line, and Peirce gives the definition of a boundary-line as a "line on a surface returning into itself over which an area or limited surface in the whole surface is considered not to extend." (MS 1597, article "Boundary-line"). Peirce's conception of a line is enlightened by an odd psychological remark. In "The Logic of Quantity" written in 1893, Peirce makes a comment on a series of lectures given by Klein in August 1893 at the International Mathematical Congress in Chicago. For Klein, we imagine a line as a strip of a certain width, and I guess the reader will agree with Klein. Peirce asserts that he does not imagine the line as a strip, but he imagines the line as a curve being "the boundary between two regions pink and bluish grey" (CP 4.118).

Beyond this unexpected comment, what matters here for Peirce is "that absolute exactitude cannot be revealed by experience, and therefore every boundary of a figure which is to represent a possible experience ought to be blurred" (CP 4.118). In other words, because for Peirce in 1893 "every proposition must be interpreted as referring to a possible experience" (CP 4.118), the definition of a line as a boundary between two regions involves that it is impossible to determine its width, which is objectively blurred.

In the same text, Peirce suggests a thought experiment which consists in a drop of black ink on white paper. Are the points on the boundary between the black ink and the white paper black or white? If these points were existing realities, then it would be necessary for them to be either black or white. Notwithstanding, Peirce thinks that the mode of being of these points is not actual existence, but mere potentiality, so that it is not necessary for them to be either black or white. For Peirce, "it is only as they are connected together into a continuous surface that the points are colored; taken singly, they have no color, and are neither black nor white, none of them" (CP 4.127, 1893).

Peirce has changed his view on this question between 1892 and 1893. In 1892, he writes that on a surface divided into two parts, one red and one blue, the boundary between both "is half red and half blue" (CP 6.126, 1892)[44]. In 1893, he says that the parts in the immediate neighborhood of the boundary are half black and half white (CP 4.127), but that the points of the boundary are not existing points and as such are not determinate as to the property of being colored.

The notion of parts in the immediate neighborhood of a continuous surface involves for Peirce the notion of infinitesimals. Suppose that a surface is split into two parts by a thunderbolt, and that each new part is also split into two parts, in a process which strikes in a minute (CP 4.125, 1893). The initial surface is neither a point nor an infinite region; it is a finite one. Because this initial surface is divided into new parts that are all finite surfaces, these parts have to be infinitesimals, for they are surfaces smaller than every finite surface. Therefore, "these parts are neighborhoods or infinitesimals" (CP 4.125).

Peirce was fully aware that in his time, the notion of infinitesimal was strongly rejected by most mathematicians, especially in analysis, with the works of Weierstrass, Dedekind and Cantor. However, his father, the great mathematician Benjamin Peirce, was himself a supporter of infinitesimals both for their scientific value and for pedagogy.

Charles Sanders Peirce argued for the method of infinitesimals in his article "Limit, Doctrine of" for the *Century Dictionary*, and we have good reasons to think that this text was written in 1883, according to MS 238. In a draft of "The Law of Mind", written in 1891, Peirce explains that most contemporary mathematicians think that we practically cannot reason about infinitesimals with confidence and assurance, and that some mathematicians even think that an infinitesimal quantity is an absurdity.

> . . . the doctrine of limits has been invented to evade the difficulty, or according to some as an exposition of the signification of the word infinitesimal; that this doctrine, in one form or another, is taught in all the text-books; it is satisfactory enough for the purposes of the calculus.

> I was myself of the opinion that the conception of an infinitesimal involved contradiction, until I had applied to the subject a notation for the logic of relations which seemed to me against all danger of fallacy, when I found that opinion was erroneous. (NEM 3.122, MS 961, 1891)

Later on, Peirce goes on defending the superiority of the method of infinitesimals over the doctrine of limits. For him, the idea that the limit of an increasing converging series being the least quantity which is greater than all the approximations of the series, limits the system of possible quantities to those considered by Cantor to constitute an arithmetical continuum, namely what Peirce calls a primipostnumeral multitude, which means $2^{\aleph_0}$. For Peirce the idea that there is just one point that can be the limit is just a *petitio principii*, an arbitrary limitation of possibilities[45].

In "The Bedrock beneath Pragmaticism" written in 1906, Peirce maintains that the core of the doctrine of limits is the following principle that:

> . . . two values, that differ at all, differ by a finite value, which would not be true if the ω-th place of decimals were supposed to be included in their exact expressions; and indeed the whole purpose of the doctrine of limits is to avoid acknowledging that that place is concerned. (CP 6.176)

Nevertheless, the set $\mathbb{R}$ of real numbers is said to be 'complete'. This sense of completeness is related to the construction of the real numbers from Cauchy sequences, which starts with an Archimedean field (the rational numbers) and forms the uniform completion of it. One can then demonstrate that $\mathbb{R}$ is the only uniformly complete Archimedean field. But the original use of the phrase "complete Archimedean field" was by David Hilbert, who meant that the real numbers form the largest Archimedean field in the sense that every other Archimedean field is a subfield of $\mathbb{R}$. Thus $\mathbb{R}$ is 'complete' in the sense that nothing further can be added to it without making it no longer an Archimedean field. Hence, Peirce's theory is related to the idea that the geometrical line is non Archimedean. As a matter of fact, Peirce was contemporary to the rise of non-Archimedean mathematics.

If Peirce's interest in infinitesimals begins at least in 1884, lasting until at least 1906, it is in 1892 that he establishes a relation between continuity and infinitesimals. In "The Law of Mind", Peirce says that as the boundary of the red part and the blue part of a surface "is half red and half blue", likewise, the boundary between the past and the future is half past and half to come; therefore, "the present is half past and half to come" (CP 6.126, 1892). Thus, immediate consciousness occupies a time that is not a point but that lasts between past and future. For

Peirce, my immediate feeling is my feeling during an infinitesimal period of time which contains the present: ". . . in the present we are conscious of the flow of time. There is no flow in an instant. Hence, the present is not an instant." (NEM 3.126)[46] This idea is also in the physical world where: "the velocity of a particle at any instant of time is its mean velocity during an infinitesimal instant in which that time is contained" (NEM 3.126).

In a manuscript written in the summer of 1893, while explaining that one cannot reason the same way with finite quantities and with infinitesimal ones, Peirce asserts "that in a continuous expanse, say a continuous line, there are continuous lines infinitely short. In fact, the whole line is made up of such infinitesimal parts" (MS 955).

Thus, in his third period Peirce dismisses Cantor's definition of continuity and he strives to give to continuity another definite meaning than that involved by the doctrine of limits[47]. The new conception of continuity Peirce is developing in his third period involves that a continuous line is made up of infinitesimal parts. This is the reason why I have chosen to call his third period the infinitesimal one, although Peirce will refine in his fourth period the relation between continuity and infinitesimals.

My infinitesimal period is very different from Potter and Shields' Kantian Period (1895–1908). I consider that the main justification of this Kantian Period comes from an editorial blunder in the *Collected Papers*. This blunder was first noticed in 1971 by Max Fisch: "The order of development of Peirce's own theory of continuity has been confused by editorial notes in CP 6.164–168. The term defined is 'continuity', not 'continuous'. . ." (Fisch, 1971, note 20, p. 23)

This editorial mistake explains why Potter and Shields determine a third period between 1895 and 1908, and why they call it "Kantistic". This so-called "Kantistic" third period is a very misleading characterization of Peirce's thought, although Potter and Shields make a clear distinction between a Kantian definition of continuity and the notion of 'Kanticity', which is mere infinite divisibility.

In fact, in 1884, Peirce rejects Kant's definition of continuity because infinite divisibility is not enough to characterize continuity. In 1892 and 1893, Peirce used the notion of 'Kanticity' as a necessary but not sufficient property of continuity[48]. Indeed, from 1884 until 1900 Peirce rejects Kant's definition of continuity, but after 1900 Peirce holds that infinite divisibility should not be considered as the main definition of continuity given by Kant. In a letter to the Editor of *Science*, written on March 16, 1900, Peirce states that:

> Although Kant confuses continuity with infinite divisibility, yet it is noticeable that he always defines a continuum as that of which every part . . . has itself parts. This is a very different thing from infinite

divisibility, since it implies that the continuum is not composed of points. (CP 3.569, 1900)

A few years later, Peirce goes on rehabilitating Kant's definition by stating that "Kant's real definition implies that a continuous line contains no points" (MS 1597, CP 6.168, September 18, 1903).

So, why would Potter and Shields call 'Kantistic' the period from 1895 until 1908, whereas Peirce rejects Kant's definition of continuity from 1884 until 1900? The reason is that they follow the editorial affirmation in the *Collected Papers* that CP 6.165–167 was written by Peirce on September 18, 1903. So when Peirce asserts that "continuity consists in Kanticity and Aristotelicity"[49], Potter and Shields are led to believe it is Peirce's conception of continuity in 1903 (Potter and Shields, p. 25), whereas a careful examination of the manuscript shows that it was written in 1892–1893.

Hence, I think that Potter and Shields emphasize too much Kant's alleged influence in this third period, whereas Aristotle's influence is indeed greater. I cannot, in this paper, deal with the intricate relations between Aristotle and Peirce, but the relation between continuity and potentiality in Peirce's mature conception bears a deep Aristotelian influence.

But in this period, Peirce develops ideas that are clearly different from those of Aristotle, such as his philosophy of continuity, which has deep links with fallibilism, mental phenomena and evolutionary cosmology. Peirce also widen his principle of continuity because to assume that anything is discontinuous is to close off possibilities, whereas in order to facilitate any further scientific research, one must leave all possibilities open.

It is important to understand the philosophical context in which Peirce's infinitesimals are elaborated, in order to avoid the somewhat misleading claim made by people such as Dauben or Eisele, that Peirce's infinitesimals are a forerunner of Non Standard Analysis. We will see in the following section that Peirce's infinitesimals have much more similarities with *Smooth Infinitesimal Analysis* (SIA) than with Robinson's Non Standard Analysis.

4) Supermultitudinous Period (1897–1907)

What constitutes the beginning of the Supermultitudinous Period is the impact of a logico-mathematical discovery that has influenced Peirce's continuum, and his conception of the mode of being of possibilities[50]. Supermultitudinality is a greatness beyond any discrete multitude; it is beyond any Cantorian transfinite cardinal[51].

Cantor had proved in 1891 that the power of the set of all subsets of a given set is always greater than the power of the original set itself[52]. This result implies that one can produce increasingly larger sets of greater and greater power; or, in other words, that there are various kinds of infinities.

This logico-mathematical discovery is of prime importance for Peirce because one requisite for achieving a "perfectly satisfactory logical account of the conception of continuity" (CP 3.526, 1897)[53] is to develop the logical doctrine of infinite multitude. But in 1897 Peirce was not satisfied with this doctrine, which from his point of view still remained in an inchoate condition, even after the works of Cantor, Dedekind, and others.

Peirce wondered in 1893 whether or not there was "a higher degree of multitude than that of the points upon a line" (CP 4.121, 1893)[54]. This is a mathematical question, but it matters for Peirce not so much for its mathematical significance, as for its logical implications. Peirce's intuition is that ". . . a continuum is merely a discontinuous series with additional possibilities" (MS 955, CP 1.170, 1893). Consequently, if it could be shown that there is a higher degree of multitude than that of the points upon a line, this would prove to be a huge difficulty for Peirce's conception.

It is in 1896–1897 that, independently of Cantor, Peirce rediscovered Cantor's result.

> We now come to a theorem of prime importance in reference to multitudes. It is that the multitude of partial multitudes composed of individuals of a given multitude is always greater than the multitude itself, it being understood that among these partial multitudes we are to include *none* and also the total multitude. (NEM 3.51, MS 14)[55]

One can have some doubt about the independence of this discovery. Carolyn Eisele asserts that Peirce did possess the Italian translation of Cantor's argument[56], "Sopra una questione elementare della teoria degli aggregati", published in 1892 by *Rivista di Mathematica*.

However, Matthew Moore informed me that Eisele is wrong, for the Italian text she is referring to is a translation of the first part of Cantor's *Beiträge*, published in 1895, (the second part being published in 1897). Moore's convincing hypothesis is that when Peirce received in 1896, most likely from Schröder, the first part of Cantor's *Beiträge*, then arises in his mind the question of the relationship between the multitude of a collection and the multitude of the collection of all its subcollections[57].

Another argument in favor of the independence of Peirce's rediscovery of Cantor's result is that in the text in which Cantor proved this theorem in 1891, there is another important theorem of which, in 1897, Peirce was unaware:

> Peirce was apparently unfamiliar with a paper Cantor had published in 1891, in which he showed by his famous method of diagonalization that the set of all continuous functions on [0, 1]—the unit

> interval—was indeed of a power, or cardinality, greater than the set of all real numbers. (Dauben, 1982, p. 317)

Nevertheless, Dauben makes a mistake when he later asserts that Peirce maintained that in fact mathematics did not offer the opportunity to consider multitudes as great as secundopostnumeral multitudes. It is true that Peirce claims that: "The multitude of all the numbers considered in the calculus and theory of functions is the first abnumeral multitude" (MS 1597, article 'abnumeral')[58]. But in "Multitude and Number", written in 1897, Peirce said that despite being yet unable to construct a mathematical collection that would be of a secundopostnumeral multitude, he had no doubt concerning "the existence in the world of mathematical ideas, of the secundopostnumeral multitude. . ." (CP 4.216)

It is surprising that even at the end of his life, Peirce still claims that he was the first to prove the theorem about the possibility to produce an infinite series of infinite multitudes. In a draft of a letter to Cantor, Peirce writes as if Cantor was unaware of this theorem[59]. In 1906, Peirce still thought himself to be "the author of the first proof of the general proposition that there is a multitude greater than any given multitude" (NEM 3.785).[60]

With the theorem Peirce demonstrated in 1896–1897, there is certainly a higher degree of multitude than that of which is the power of points upon a line according to Cantor. The only way for Peirce to combine this new theorem with his idea that a continuum is merely a discontinuous series with additional possibilities, is to make the assumption that the multiplicity of a continuum is beyond all degrees of multitude. For Peirce, ". . . the collection of possible ways of distributing the individuals of a supermultitudinous collection into two abodes equals that collection itself" (NEM 3.86).

Some commentators, like Murphey, have argued that here Peirce falls for Cantor's paradox, namely to assume that there is a greater multitude[61]. But as Potter and Shields remark, Peirce never states that there is a greater multitude; he states that the multiplicity (and not the multitude) of continuity is beyond all possible multitude, which are all discrete[62].

Whereas the term 'multitude' is assigned to discrete collections, Peirce uses the term 'multiplicity' as the greatness of a collection, discrete or continuous: "I have myself carefully abstained from using the word multitude in connection with supermultitudinous collections. Multitude implies an independence in the individuals of one another which is not found in the supermultitudinous" (NEM 3.97). There is no doubt in Peirce's mind that there is no greater multitude. In MS 1597, for the article 'abnumeral', Peirce writes that:

> There is no highest abnumeral multitude, any more than there is a highest enumerable multitude. Nor is there any multitude greater

> than all abnumeral multitudes, since beyond them the individuals members of the collection lose their separate identity and merge into one another in true continuity[63].

According to Peirce, what we now call Cantor's paradox is not a difficulty but on the contrary an argument in favor of his conception. For Peirce, there is no multitude greater than all those in the series obtained by successive exponentiation. For him, the fact that Cantor's theorem does not apply for a continuous collection indicates that such a collection is no longer discrete, since a logical law suitable for discrete collections does not hold for a supermultitudinous one.

> The formula $2^n > n$ which I have proved holds for all discrete collections cannot hold for this. In fact [this aggregate] is evidently so great that this formula ceases to hold and it represents a collection no longer discrete. (CP 4.218, "Multitude and Number", 1897)

Indeed, if one assumes that a continuous collection has a definite multitude, since for every multitude there is a higher one, a continuous collection could be a part of a discrete collection; and such a situation would be incompatible with Peirce's idea that a continuum series contains more than a discontinuous one. Hence, for Peirce, on a continuous line there is room for a collection of whatsoever multitudes.

But saying that a continuous series must contain more points than a discontinuous one, seems to involve that it must contain points. However, if it contains points, then it is not continuous, for an actual point on a continuum is a breach of continuity. Why is it that an actual point on a continuum is a breach of continuity? The reason is that such an actual point ". . . breaks the continuity at that point, because it is a part which does not consist of parts" (NEM 3.748)[64]. Therefore, Peirce's solution is that a continuous series cannot contain actual points, but it contains potential points.

The distinction between actual points and potential points is a key concept Peirce has drawn from his theorem on multitudes. A continuous line is made of potential points; it is a potential aggregate of points.

> Thus the potential aggregate is, with the strictest exactitude, greater in multitude than any possible multitude of individuals. But being a potential aggregate only, it does not contain any individuals at all. It only contains general conditions which permit the determination of individuals.[65]

Thus, the relation between a continuum and its elements is that a continuum contains no actual but only potential elements, and that it is impossible to exhaust all the elements, in the sense that whatever elements of the multitude have been given existence, there is always the possibility to give existence to more and more, *ad infinitum*.

It is unfortunate that Peirce is not always clear about the meaning of terms such as: elements, points, determinable points, actual points, potential points, parts, and infinitesimal parts. This lack of precision was already there in his definition of continuity for the CD, in which he makes no clear distinction between elements, points and instants: "[Continuity is] a connection of points (or other elements) as intimate as that of the instants or points of an interval of time." But within the Supermultitudinous Period, I think that some clarification can be made by distinguishing, in around 1900, a significative change in Peirce's conception of the meaning of "collection".

Although Peirce often uses "point" without specifying if it is a potential point or an actual one, from 1897 until around 1900, Peirce argues that a continuous line is a collection of potential points. Being a continuum for a collection of potential points means that the potential points are welded together.

However, contrary to his position in his Cantorian Period, Peirce does not mean that a line is a set of points, but that a line consists of all the potential points that correspond to the movement of a moving particle along this line. In other words, a moving particle would pass through all its virtual positions, but in its movement, it does not "jump", step by step, from one position to the following. Mathematically, this means that a continuum cannot be a set of distinct points.

From around 1900 and afterwards, Peirce argues that a collection is made of discrete elements, so that a collection can in no way be continuous. A collection is always made of individuals distinct from each other. All the determinable points on a continuum are of a multiplicity so great that those points cannot be actualized together, since their supermultitudinality involves that they are welded together[66]. Then Peirce asks himself, if "the totality of the points determinable on a line does not constitute a collection, what shall we call it?"[67]

The terminology of "potential aggregate" seems to have been dismissed in 1900. Peirce apparently did not find a good terminological alternative, but the evolution in his thinking is clear if one compare his position in 1893 in which he states that a line is a continuous collection of points[68], and his position in 1903 in which he states that: "A continuous line contains no points" (MS 1597, CP 6.168, 1903 Sep 18). Unfortunately, Peirce is not clear in the previous quotation concerning what he means by "point". However, in 1904, Peirce sums up more clearly the position he has developed around 1900:

> A collection is a whole whose being consists in the independent being of its members; a line, on the contrary, has a being from which the being of its points is derived and in which they, as possibilities, are involved. (NEM 2.531)

In order to disentangle the meaning of "point" for Peirce in 1898, Putnam claims that, according to the following passage, Peirce implicitly considers that a point can have parts[69]:

> The end of a line might burst into any discrete multitude of points whatever, and they would all have been one point before the explosion. Points might fly off, in multitude and order like all the real irrational quantites from 0 to 1; and they might all have had that order of succession in the line and yet all have been at one point. (RLT, p. 160)

Putnam's hypothesis that a point can have parts is at least consistent with a text in which Peirce states that: "the whole series of numbers, rational and irrational, . . . do not constitute a continuous series" (NEM 3. 125). In this text, Peirce argues that there is "a certain kind of *next-ness*" (*ibid*) in the series of rational and irrational numbers. His idea is that being a continuous series involves that there is a unity between an element and its successor, whereas this is not the case for the discrete series of numbers, rational and irrational.

As a result, Peirce claims that: "When the scale of numbers, rational and irrational, is applied to a line . . . the environs of each number is called a point. Thus, a point is the hazily outlined part of the line whereon is placed a single number" (NEM 3. 127). This means that, if the actual points on a line are the neighborhood of rational and irrational numbers, and if it is "intrinsically doubtful" where "each number is placed" (*ibid*), then it follows that these actual points are "the hazily outlined part of the line" whereon the numbers are placed. In other words, these actual points can have parts.

However, I think that NEM 3. 125–127 was written before 1900. Now, does the fact that, in around 1900, Peirce dismissed the notion of a "continuous collection", have implications for Putnam's hypothesis that a point can have parts? In NEM 3.748, written in 1900, Peirce clearly states that an actual point cannot be divided in parts:

> Kant defines a continuum as that of which every part consists of parts . . . [this] may . . . be accepted as an approximate definition of a continuum. For it is the same as to say that it is not a collection of individuals. . . . Each point put upon it, if it be regarded as a part of the figure, breaks the continuity at that point, because it is a part which does not consist of parts. (NEM 3.748)

Nevertheless, I think that this previous quotation is rather awkward because it entails that an actual point is a part ("it is a part which does not consist of parts"), whereas it seems to me that what Peirce really wants to say is that each part of a continuum is itself a continuum that has continuous parts.

In a nutshell, I think that in 1897–1900, Peirce still wants to use the mathematical tools of sets in order to characterize continuity, so that he claims that a continuum consists of (potential) points. But such a position leads to logical difficulties, because a collection should be made of actual elements, and a point should not have parts. Such problems are parts of the reasons why Peirce will gradually develop logical and topological tools in order to grasp continuity.[70]

Now, when Peirce claims (in 1898) that a continous line is a potential aggregate of points, what does he mean? To explain this idea, Peirce considers a similar case, our conception of whole numbers. Nobody has a clear conception of each whole number, but if our conception is in that sense indeterminate, it is nevertheless determinable. When we consider the collection of whole numbers, we do not see a complete object but the possibility of building it. As Putnam pointed out, there is here a proximity between Peirce and Brouwer, despite the fact that for Peirce complete infinite processes are perfectly conceivable[71].

> But though the aggregate of all whole numbers cannot be completely counted, that does not prevent our having a distinct idea of the multitude of all whole numbers. We have a conception of the entire collection of whole numbers. It is a potential collection, indeterminate yet determinable. (RLT, p. 248, CP 6.186)

Whereas the actual elements of a discrete collection are subject to the principle of excluded middle, this is not true of the potential points or "possible, or potential, point-place wherever a point might be placed" (CP 6.182) of a continuum.

> A continuous line contains no points or we must say that the principle of excluded middle does not hold of these points. The principle of excluded middle only applies to an individual. . . But places being mere possible, without actual existence, are not individuals. Hence a point or indivisible place really does not exist unless there actually be something there to mark it, which, if there is, interrupts the continuity. (MS 1597, CP 6.168, 1903 Sep 18)

As we have seen, whereas for Cantor it is enough to add irrational numbers to rational ones in order to get continuity, Peirce thinks that the collection of real numbers is a discrete one, and that in order to get closer to continuity, one should add infinitesimals of all kinds of orders. However, Cantor has demonstrated that the set $\mathbb{R}$ of real numbers can be said to be "complete", in the sense that nothing further can be added to it without making it no longer an Archimedean field. Hence, Peirce's theory is related to the idea that the geometrical line is non-Archimedean[72].

Finally, Peirce considers that although Cantor's real numbers do not form a continuum, for they lack infinitesimals of all kinds of orders, it is nevertheless not possible to construct a true continuum by adding infinitesimals for: "Numbers express nothing whatsoever except order, *discrete order*. . . Number cannot express continuity" (NEM 3.93).[73] This does not mean that Peirce dismisses infinitesimals, there are infinitesimal parts in a continuum. As Putnam suggests, one cannot construct the continuum by adding infinitesimals of all kinds of orders, for such a construction would require a number of steps not "less than Peirce's ideal limiting cardinal" (RLT, p. 50).

Since at least 1893, Peirce considers that a continuous line "is made up of . . . infinitesimal parts." (MS 955). This entails that contrary to Cantor's continuum but like Veronese's one, Peirce's continuum is not Archimedean. Now is an infinitesimal a constant (Nieuwentijt's position) or a variable (Leibniz's position), and what is its ontological status? According to Hegel's classification, an infinitesimal could be of the mode of: Being (Cavalieri's position), Nothingness (Euler's position) or Becoming (Hegel's position). As for Peirce's own position, he seems to have changed his mind. Whereas in his definition for the CD an infinitesimal is "a fictitious quantity", Peirce writes in 1900 that: "the infinitesimals must be actual real distances" (CP 3.570). More precisely, in NEM 3.989, written in 1906, Peirce distinguishes different orders of infinitesimals, and he says that: "the diameter of the soul-stuff atoms will be . . . an infinitesimal of the *infinite order*" (NEM 3.898, 1908). However, at least in comparison with contemporary theories, Peirce's theory of infinitesimals is not fully developed.

What are the relations between Peirce's continuum in his supermultitudinous period and modern mathematics? We have already seen that it is related to non-Archimedean mathematics. Since Peirce defended infinitesimals, many commentators, like Eisele or Dauben have maintained that Peirce was a forerunner of Non-standard Analysis. But this is only partially true. Unlike *Smooth Infinitesimal Analysis* (SIA) and Veronese, for Robinson and most of Non Standard Analysis theories after him, the continuum is built out of points, and this is deeply incompatible with Peirce's conception. It is useful to notice that Robinson's ANS is a development of mathematical logic, whereas SIA is grounded on the mathematical theory of category. Moreover, it is strange that most readers have understood Putnam's "Peirce's Continuum" as advocating a Non-standard reading of Peirce's conception, whereas Putnam clearly states:

> . . . although I have used the terminology of contemporary Non-Standard Analysis to explain Peirce's conception of the line, that terminology is in a way extremely misleading. When one does Non-Standard Analysis, one starts by expanding the real number system by adding non-standard real numbers that are infinitesimally close to the standard real numbers (as well as infinite non-standard

> real numbers) and then one assumes that the non-standard geometric line is isomorphic to the non-standard real numbers. But this is not Peirce's view at all. Peirce did not propose to add non-standard numbers to the real number system. He simply proposed that there are non-standard points on the geometrical line. (Putnam, 1995, p.12)

Since a Hausdorff space is a space in which points can be separated by neighbourhoods, Peirce's continuum is not a Hausdorff space. But there is another way to do mathematical analysis. Within the mathematical theory of category, in *Smooth Infinitesimal Analysis* (SIA), the elements of a continuum are not all distinguishable. In SIA, all the elements that are not distinguishable from zero belong to the continuum, but not to the continuum deprived of zero. Therefore, the law of excluded middle does not hold everywhere in SIA, for it is wrong that $\forall\, x \in \mathbf{R}, (x = 0) \vee (x \neq 0)$[74].

Whereas the notion of "point" is somewhat different, there are similar properties for both SIA and Peirce in his Supermultitudinous Period: the points in a continuum have no distinct identities but they are welded together; the law of excluded middle does not apply to points of a continuum, the continuum "contains" infinitesimals, the parts of a continuum are themselves continuous and every continuous function is differentiable.

However, a deeper comparison between the philosophical background of SIA and Peirce's continuum remains to be done.

Now the property of supermultitudinality for the possible points on a continuum has influenced Peirce's conception of the mode of being of possibilities and his realism.

In his Infinitesimal Period (1892–1897), Peirce was already aware that the relation between a general concept and its individuals is similar to the relation between a continuum and its points. Whereas within Aristotelian logic the question of realism is mainly related with the ontological status of the properties of objects, within Peirce's stronger logic of relatives, the question of realism extends beyond the ontological status of the properties of objects and concerns also relations: ". . . in the light of the logic of relatives, the general is seen to be precisely the continuous. Therefore, the doctrine of the reality of continuity is simply that doctrine the scholastics called realism . . ." (MS 398, quoted by Murphey, p.397).[75]

In his article "universal" written for the *CD*, Peirce states that: "the dispute concerning universals chiefly concerns the universals *in re*", and he explains that:

> any tendency in the things themselves toward generalizations of their characters constitutes what is termed a universal *in re*. Before the laws of physics were established it was particularly the uniformities of heredity, and consequent commonness of organic forms, which spe-

> cially attracted attention; so that *man* and *horse* are the traditional examples of universals in re. (*ibid*)

In the beginning of his Supermultitudinous Period, with the idea that a continuum is a potential aggregate of points, Peirce has found a way to describe real but non-actualized possibilities. Whereas a logical possibility just means that it is not impossible, a real possibility depends: "... on a power residing in a thing, whether active or passive. Opposed to mere logical possibility" (MS 1169 A). A real possibility corresponds to an: "... indeterminacy in things as to the future happening or non-happening of something which lies within the power of a free agent" (MS 1166 A).

Thus, with his metaphysical conception of the reality of possibilities, Peirce can give a better account of the meaning of the hardness of a diamond. To say that hardness is a real property of the diamond, it has to be true even when it is not tested. It is not only true while its hardness is being tested by being scratched; it is always true; it is a law that is a continuum in reference to its future manifestations.

Taking into account his logic of relations, the question of realism has for Peirce finally taken the following shape: "Are any continua real?"[76] Peirce's aim was to demonstrate that there are continua in reality. Thus, he claims that there are: "three categories of being; ideas of feelings, acts of reaction, and habits" (CP 4.157, c.1897). And within Peirce's objective idealism, laws of Nature correspond to habits.

For him, the relation between a law and its future manifestations is similar to the relation between a continuum and its points. Indeed, for Peirce:

> The possible is necessarily general; and no amount of general specification can reduce a general class of possibilities to an individual case. It is only actuality, the force of existence, which bursts the fluidity of the general and produces a discrete unit ... the possible is general, and continuity and generality are two names for the same absence of distinction of individuals. (CP 4.157, c.1897)

Sfendoni-Mentzou, in "Peirce on Continuity and Laws of Nature" (*TCSPS*, 1997), has put forward the idea of laws of Nature as continua:

> [T]he actual manifestations of law are the discrete units which burst the flux of what is essentially a continuum, the very nature of which involves the absence of distinct individuality. And since there is no limit as to the number of instances; a law of nature can be described both in an Aristotelian and in a Peircean fashion as "the potential though not the realized whole", which embraces all phenomena as a continuous spectrum of its possible future manifestations. (Sfendoni-Mentzou, 1997, p. 660–661)

Peirce thinks that laws of Nature are real continua, and that they are "reasonableness energizing in the world"[77]. For him, this belongs to an Aristotelian "evolutionary metaphysics"[78], for which laws of Nature are essentially instantiatable. It involves a realism that considers laws of Nature as living realities, whereas they do not have the ontological status of existence. "The extreme form of realism which I myself entertain that every true universal, every continuum, is a living and conscious being. . ." (NEM 4.345, 1898).

Another philosophical aspect in his supermultitudinous period, is the relation between perception, continuity, and the rejection of nominalism. If in our perceptions there are firstness and secondness but no thirdness, then it would be a strong argument for nominalism.

To prove that there is thirdness in our perceptions, Peirce begins with the idea that a perceptual judgement cannot occur in an absolute instant since time is a continuum that is not made of instants or ultimate parts. Then, Peirce argues that as a consequence, the "present moment will be a lapse of time . . . its earlier parts being somewhat of the nature of memory, a little vague, and its later parts somewhat of the nature of anticipation, a little generalized" (CP 7.653, "Telepathy and Perception", 1903). Hence, in all perceptual judgement there is the vague of memory and the generality of anticipation. Moreover, every perception is the coalescence of quasi-percepts, it is "a generalized percept" (CP 8. 144, Jan 1901). Therefore, there is thirdness in our perceptions, and this is a strong argument against nominalism and in favor of Peirce's realism.

We have seen that in 1868 Peirce was already aware that thought is a semiotic process which is dynamical and continuous[79]. From 1884–1885, thought corresponds to Thirdness and continuity, and from 1898 Peirce claims that "the generality of meaning is but a special aspect of its continuity" (MS 1109, p.2), and that such a continuity "transcends all multitude" (MS 1109, p.2).

> The meaning is something which belongs to a proto-acted series of events as a whole. How can this series of events . . . have any wholeness? In the sense of reaction they cannot. They can only do so by filling a continuous time. (MS 1109, p.4)

Peirce's conceptualization of the continuum also has an important cosmological aspect. For Peirce, his Synechism involves evolution; therefore, Peirce tries to explain the Universe as an evolutionary process. In particular, Peirce has to explain how the continuum of laws of Nature can have been derived. Objective idealism is a corollary of Peirce's Synechism, so he postulates that as a logical process proceeds from the vague to the definite, likewise cosmological evolution should proceeds from the vague to the definite.

> In Spencer's phrase the undifferentiated differentiates itself. The homogeneous puts on heterogeneity. However it may be in special cases, then, we must suppose that as a rule the continuum has been derived from a more general continuum, a continuum of higher generality . . . If this be correct, we cannot suppose the process of derivation, a process which extends from before time and from before logic, we cannot suppose that it began elsewhere than in the utter vagueness of completely undetermined and dimensionless potentiality. (RLT, p. 258)

Why does this process extend before time, space and logic? Peirce says that if we try to explain how the universe could have arisen from nothingness, then we must "suppose a state of things before time was organized" (CP 6.214, 1898), for "time is itself an organized something, having its law or regularity" (*ibid*). More generally, Peirce thinks that time and space are belated realities in the evolution of the Universe, because from a topological point of view, time has but one dimension, and likewise our physical space, according to Peirce, only has three dimensions; whereas the original continuum must have many more dimensions, and so, originally, time and physical space were not realities. As for evolution taking place before logic, it is not only in the sense that our logic is a result of evolution, but also in the sense that there were no regularities at the beginning.

Now, how could the universe have arisen from "nothing, pure zero, . . . prior to every first"? (CP 6.217, 1898). Contrary to Hegel's logic of events, Peirce considers that no deduction "necessarily resulted from the Nothing of boundless freedom" (CP 6.219), but that the "logic may be that of the inductive or hypothetic inference" (CP 6.218).

But at which stage did the "nothing, pure zero", becomes a continuum? Whereas the zero is mere "germinal possibility" (NEM 4.345), the continuum is "developed possibility" (*ibid*). According to Floyd Merrell, before any existing thing could have arisen in the universe, the nothingness has become: "the continuous flux of Firstness"[80]. Indeed, Peirce states that:

> The whole universe of true and real possibilities forms a continuum, upon which this Universe of Actual Existence is, by virtue of the essential Secondness of Existence, a discontinuous mark . . . There is room in the world of possibility for any multitude of such universes of Existence. (NEM 4.345)

Peirce thinks that the "original potentiality is the Aristotelian matter or indeterminacy from which the universe is formed" (RLT, p. 263), and this "original potentiality is essentially continuous" (RLT, p. 262). Since the definitions of otherness and of identity proper presuppose a universe of individuals, Peirce considers that in the original continuum,

". . . the principle of excluded middle, or that of contradiction, ought to be regarded as violated" (NEM 3.747)[81].

Although for Peirce the dimension of a continuum may be of any discrete multitude, there is an exception for the original continuum, whose number of dimensions is no longer discrete.

> If the multitude of dimensions surpasses all discrete multitudes there cease to be any distinct dimensions. I have not as yet obtained a logically distinct conception of such a continuum. Provisionally, I identify it with the *uralt* vague generality of the most abstract potentiality. (RLT, p. 253–254)

Yet, Douglas R. Anderson has pointed out the following difficulty: how can evolution be a continuous process if chance and spontaneity are discontinuous events[82]? According to Peirce's Tychism, there can be no rational continuity between past events and spontaneity. The answer is that discontinuity is not absolute but is relative. For example, if one draws a new curve on a blackboard, it is a discontinuity. Nevertheless, "although it is new in its distinctive character, yet it derives its continuity from the continuity of the blackboard itself" (RLT, p. 263).

Moreover, Peirce's theory of evolution involves the notion of a final continuum. The universe evolves not only by chance and necessity, but also towards a final continuum, which is final less as a result than as a principle. For Peirce, "continuity is Thirdness in its full entelechy" (RLT, p. 190), and as a final cause, there is an end of History, but as a final result there is not. Thus, the ultimate good lies in the evolutionary process but not in individual reactions in their isolation; it lies in the growth of sympathy with others.

> Synechism is founded on the notion that the coalescence, the becoming continuous, the becoming governed by laws, the becoming instinct with general ideas, are but phases of one and the same process of the growth of reasonableness. (CP 5.4, 1902)

For Peirce, "generalization, the spilling out of continuous systems, in thought, in sentiment, in deed, is the true end of life" (NEM 4.346).[83] In particular, our emotional life, whether it is esthetic, ethical or mystic, leans towards a melting of oneself with others. Peirce thinks that in "all his life long no son of Adam has ever fully manifested what there was in him" (CP 1.615), and that a human personality is an end, a continuity of an adaptive ideal, that is itself a continuum: ". . . each one of us is in his own real nature a continuum" (NEM 4.345).

Hence, human community is a continuum of a higher degree of generality than individuals, and in this continuum all individuals are in a reciprocal determination. Moreover, there is also a continuity between each one of us and the Creator:

. . . the barbaric conception of personal identity must be broadened. . . . All communication from mind to mind is through continuity of being. A man is capable of having assigned to him a rôle in the drama of creation, and so far as he loses himself in that rôle,—no matter how humble it may be,—so far he identifies himself with its Author. (CP 7.572, 1892–1893)[84]

One important aspect of the Supermultitudinous Period for Peirce, is that it gives an argument in favor of the agreement between his conception of continuity and his spiritual conception, that individuals should tend to loose their egoism and to become part of a continuum of a higher degree of generality.

5) Topological Period (1908–1913)[85]

The Supermultitudinous Period is characterized by the idea that for every continuum, there is room for the actualization of any multitude of points. But in his Topological Period, Peirce has doubts concerning this idea. In MS 204, written on May 24, 1908, Peirce wonders: ". . . whether continuity consists in the presence of just so many points or in something widely different". In the same text, Peirce writes that although there is no doubt for him that the Cantorian continuum "has what is called continuity in the calculus and theory of functions, it has not the continuity of a line." Once again, the question is not the mathematical legitimacy of the Cantorian continuum, but its adequacy to represent the "true" continuum.

Now, in his Topological Period, Peirce changes his mind about what is a "true" continuum. Although he maintains the idea of potentiality, the notion of continuity does not mainly rest on the notion of multiplicity anymore, but mainly on topological considerations and on the relations between the parts of a continuum. Such a change explains why I call this last period "topological".

As soon as Peirce gives up the idea that the main property of a continuum is being supermultitudinous, he has to find another way to explain why the parts form a continuous whole, and topology becomes essential for his theory of continuity. Indeed, Peirce defines topics or topology as the science which ". . . studies only the manner in which the parts of places are continuously connected . . . It is often called *topology*."[86]

The first occurrence of Peirce's doubts about the supermultitudinality of a continuum appears in a text published in 1908 but written in the summer of 1907, "A Note on Continuity"[87], in which he repositioned himself regarding the continuum that should be able in some way to be linearly arranged.

Should further investigation prove that a second-abnumeral multitude can in no way be linearly arranged, my former opinion that the common conception of a line implies that there is room upon it for

> any multitude of points whatsoever will need modification. (CP 4.639)

Nevertheless, such a doubt does not completely reject Peirce's previous conception of continuity, for even if it should be proved that no collection of higher multitude than the first multitude beyond the denumerable can be linearly arranged, this would not establish, from Peirce's point of view, the idea that a line consists of actual points. Indeed, the idea of potentiality remains essential for Peirce's conception of continuity.

> It will still remain true, after the supposed demonstration, that no collection of points, each distinct from every other, can make up a line, no matter what relation may subsist between them; and therefore whatever multitude of points be placed upon a line, they leave room for the same multitude that there was room for on the line before placing any points upon it. (CP 4.640)

Thus, although Peirce will reject the property of supermultitudinality for continuity, he will still hold that no collection of actual points could form a continuous collection.

However, Peirce's doubt is rather strange since, as remarked by Hourya Sinaceur, in order for the potential points of a circle to be linearly arranged, it requires that one actualized a point on the circle; namely, in Peirce's sense, it means breaking its continuity.

As we have seen, Peirce's conception of the meaning of "collection" changes around 1900. In 1898, Peirce seems to consider that a collection of potential points can be linearly arranged: "they might all have had that order of succession in the line and yet all have been at one point" (RLT, p. 160). But in 1908, Peirce seems to think that only actualized points can be linearly arranged: ". . . their units are inherently capable of being put into a linear arrangement in every order of succession" (NEM 3.881); and also: ". . . of which the units are in themselves capable of being put in a linear relationship" (CP 4.642).

It is striking that the doubt Peirce had formulated in "A Note on Continuity" about the property of supermultitudinality for every continuum, seems to have vanished on May 24, 1908:

> I still think . . . that there is room on a line for a collection of points of *any* multitude whatsoever, and not merely for a multitude equal to that of the different irrational values, which is, excepting one, the smallest of all infinite multitudes, while there is a denumeral multitude of distinctly greater multitudes, as is now, on all hands, admitted. (NEM 3.880–881, MS 203)

Peirce's hesitations are probably connected with debates within the mathematical community. Although at the end of his life Peirce was isolated, he may have heard of a controversy against Zermelo's proof in 1904 that every set can be well-ordered. On May 24, 1908, Peirce apparently thinks that there is a proof of Borel supporting his idea that there is room on a line for a collection of points of any multitude, but he complains that he has never been able to get a copy of Borel's paper. However, Peirce is wrong in assuming that for Borel, Cantorian transfinite numbers are legitimate mathematical constructions.

It is on May 26, 1908, that Peirce finally gave up his idea that in every continuum there is room for whatever collection of any multitude. From now on, there are different kinds of continua, which have different properties. Some continua have a supermultitudinous multiplicity, but some do not. Since for Peirce "all the parts of a perfect continuum have the same dimensionality as the whole" (CP 4.642), the properties of time as a continuum, for example, are not exactly the same as those of our physical space as a continuum.

> Thus, I show that my true continuum might have room only for a denumeral multitude of points, or it might have room for just any abnumeral multitude of which the units are in themselves capable of being put in a linear relationship, or there might be room for all multitudes, supposing no multitude is contrary to a linear arrangement. (CP 4.642, 1908)

On May 26, 1908, in his last remark on "A Note on Continuity", Peirce states that he made a huge step forward to solve the question of continuity[88]. He left aside his previous distinction between the pseudo-continuum and the true continuum, for a new distinction between a perfect continuum and an imperfect continuum, this last being a continuum: "having topical singularities" (CP 4.642, 1908). According to his concept of continuity, "a top[olog]ical singularity . . . is a breach of continuity". But if the continuum has no topological singularity, then it is a perfect continuum whose essential character is:

> the absolute generality with which two rules hold good, first, that every part has parts; and second, that every sufficiently small part has the same mode of immediate connection with others as every other has. This manifestly vague statement will more clearly convey my idea (though less distinctly) than the elaborate full explication of it could. (CP 4.642, 1908)

The new aspect in this definition of a perfect continuum is: "that every sufficiently small part has the same mode of immediate connection with others as every other has"; but Peirce thinks that the theory of collections cannot correctly analyze this property. Indeed, since at least 1897, the idea that the mathematical theory of set is not enough to investigate the continuum was growing in Peirce's mind: "the development of projective geometry and of geometrical topics has shown that there are at least two large mathematical theories of continuity into which the idea of continuous quantity, in the usual sense of that word, does not enter at all" (CP 3.526). Moreover, we have seen that since 1900 Peirce has rejected the idea that, strictly speaking, a collection can be continuous.

Projective geometry is important for the study of continuity, since a projective line has no discontinuities at its extremities, unlike the Euclidean line. According to Peirce, it is a theorem that: "every continuum without singularities returns into itself" (NEM 2.184, MS 165, c.1895), and this is likely the reason why when discussing the linear continuum, Peirce usually does not refer to the straight line, as most mathematicians do, but to the circle. The reason for such a choice is likely that for projective geometry, and contrary to what occurs within Euclidean geometry, an infinite line is intuitively like a circle of an infinite radius.

But for the study of continuity, topology is even more important, for it is the only abstract geometry which purely deals with properties of continuity and discontinuity. Topology is for Peirce: "the full account of all forms of Continuity" (NEM 2.626, MS 145). Topology is essential for continuity because it is: "the study of the continuous connections and defects of continuity" (CP 4.219).

One can distinguish two ways in which topology helps understanding continuity.

First, with his Census-theorem (which corresponds roughly to the Euler-Poincaré characteristic) and his notion of "shape-class", Peirce tries to establish a classification of several kinds of continua according to their various dimensions and topological singularities[89]. This is what I propose to call external continuity, for it deals with properties shared by objects belonging to the same class, which is itself defined according to a homeomorphism. Although Listing was close to this idea, Peirce is perhaps, in the history of topology, the first to have understood that one could classify spatial complexes according to the value of their Census number[90].

> Every place has a "Census-value" which consists of the Census number of its points minus that of its lines plus that of its surfaces minus that of its solids. The Census number of any homogeneous space is equal to its Chorisy minus its Cyclosy plus its Periphraxy minus its Apeiry. The "Census-Theorem" is that the Census Value of any place is unaffected by cutting it up by boundaries of lower dimensionality. (NEM 3.487)

Second is what I propose to call internal continuity, for it deals with the mode of immediate connection of the parts of a continuum. For Peirce, the nature of the differences between continua "depends on the manner in which they are connected. This connection does not spring from the nature of the individual units, but constitutes the mode of existence of the whole" (CP 4.219, "Multitude and Number").

The term "synesis" could be used for such a purpose. In order to define "synesis", Peirce says that it "cannot be defined in terms of Riemann's connectivity" (NEM 3.471), nor can it be defined "in terms of Listing's cyclosis and periphraxis, notwithstanding the value of those somewhat artificial conception" (NEM 3.471). In a nutshell, I think that Peirce's notion of synesis is a failed attempt to define a topological concept that would give an account for the mode of immediate connection of the parts of a continuum.

Peirce has struggled to give a satisfactory account of the mode of immediate connection of the parts of a continuum. In 1906, Peirce thought that "Whatever is continuous has material parts."[91] As Potter and Shields have pointed out, not everything that has material parts is continuous; it depends on the mode of connection between the parts[92].

Two years later, while trying to define a continuum according to its parts, Peirce distinguishes different modes of being for the whole and its parts. A part can have the same mode of being than its whole, but this is not mandatory, they can belong to different universes of experience. What are these universes?

> First, the Universes of Ideas, [or] arbitrary possibilities, second . . . the Universe of Singulars, comprising physical Things and single Facts, or actualisations of ideas in singulars, . . . third . . . the Universe of Minds with their Feelings, their Sensations of physical facts, . . . Esthetic, Moral, . . . Instincts, . . . Self-control, Habit-taking, Judgments, Conjectures, . . . Logical analysis, and Testings". (MS 204, 1908, May 24)

Peirce gives two examples of a physical reality that can be a part of a mental reality: "the existence of the Campanile of San Marco is a part of the last sight I had of it; though the part was a thing . . . the whole is a mental experience" MS 204; "the Human Body is perhaps a part of the human Mind" MS 204. Now, what are the definitions of the different kind of parts?

> By a "*material*" part, I mean one which belongs to the same principal division(s?) of the same universe; . . . I often use "*material part*" in a still narrower sense. . . By a *coexistential* part, I mean a part that exists in same one of "the three Universes of experience" as its whole. . . By a *copredicamental* part, I mean a part which belongs to the same "*predicament*", that is, to the same *summum genus* of the same universe as its whole does . . . A *homogeneous* part is a part which possesses all its real and non-partitional characters . . . that belong to its

> whole and that a part can possess. . . A *homogeneous* whole is a whole entirely composed of homogeneous parts . . . (*ibid*)

These concepts constitute for Peirce a previous step in his endeavor to elaborate a satisfactory definition of a perfect continuum (and also, as a consequence, of an imperfect continuum). But while studying the relations between a perfect continuum as a whole and its parts, Peirce is forced to presuppose time. Hence, unfortunately, his general definition of a perfect continuum presupposes time, which is a special continuum.

Two days later, on May 26, 1908, Peirce has found a better strategy which rests on topology. In order to define an imperfect continuum, he uses the notion of topological singularity, and he defines a perfect continuum by its having no topological singularity. However, he still faces the difficulty that: "In endeavoring to explicate 'immediate connection', I seem driven to introduce the idea of time" (CP 4.642, 1908, May 26).

In another text of 1908, Peirce states that what is homogeneous in all the parts of a perfect continuum, is the regularity of a certain kind of relation of each part to all the parts as a continuous whole[93].

> A perfect continuum belongs to the genus, of a whole all whose parts without any exception whatsoever conform to one general law to which same law conform likewise all the parts of each single part. Continuity is thus a special kind of generality, or conformity to one Idea. More specifically, it is a homogeneity, or generality among all of a certain kind of parts of one whole. Still more specifically, the characters which are the same in all the parts are a certain kind of relationship of each part to all the coördinate parts; that is, it is a regularity. (CP 7.535, note 6, 1908)

What is homogeneous in all the parts of a perfect continuum, is the regularity of a certain kind of relation of each part to all the parts as a continuous whole. But what is this kind of relation which makes this regularity a continuity? Because continuity is unbrokenness, Peirce's answer is that it is the relation or relations of contiguity, in which the passage from one part to a contiguous part is a continuous one. Now the question has become the following: What is this "passage"? For Peirce, "this passage seems to be an act of turning the attention from one part to another part", but such a definition involves time. Now, Peirce is not satisfied for: "time is a continuum; so that the prospect is that we shall rise from our analysis with a definition of continuity in general in terms of a special continuity" (CP 7.535, note 6, 1908).

As we have seen, in his cosmology, Peirce considers that time and space are belated realities in the evolution of the Universe, because from a topological point of view, time has just one dimension, whereas the original continuum must have many more dimensions, and so, originally, there was no time. Moreover, because time is not cyclical, it is a continuum of one dimension with topological singularities, namely its beginning and its end.

There is here a tension in Peirce's thought on continuity over the question of the significance of the continuity of time for other kinds of continua. On the one hand, Peirce thinks that time is a just a special continuity and that topology is: "the full account of all forms of Continuity" (NEM 2.626, MS 145). However, on the other hand, Peirce thinks that topology "presupposes the doctrine of time, because it considers motions." (NEM 2.481, MS 137, 1904) Moreover, Peirce claims in around 1911 that because pure mathematics deals exclusively with the consequences deducible from hypotheses arbitrarily posited, the question of the best theory of continuity ". . . is beyond the jurisdiction of Pure Mathematics" (CP 6.182)[94]. So it seems that even at the end of his life he considers that topology cannot itself gives a satisfactory account of the continuity of time.

On December 26, 1913, less than four months before he died of cancer on April 19, 1914, Peirce still had the ambition to offer a mathematical alternative to Cantor and Dedekind's theory of continuity. This last text is just a short attempt, but it is surprising as it is much more algebraical than topological. The reason is likely that Peirce was desperately aiming to get an important mathematical result on continuity[95].

Thus, I agree with Potter and Shields that there is a last period beginning in 1908. However, it is wrong to assume that the specificity of this period is to emphasize the continuity of time, for this question is important in all of Peirce's periods (with maybe the exception of the Cantorian Period). For example, in a text written at the end of his Supermultitudinous Period, Peirce refers to his 1868 article "Some Consequences of Four Incapacities" to explain that time is continuous.

> The argument which seems to me to prove, not only that there is such a conception of continuity as I contend for, but that it is realized in the universe, is that if it were not so, nobody could have any memory. If time, as many have thought, consists of discrete instants, all but the feeling of the present instant would be utterly non-existent. But I have argued this elsewhere[96]. (CP 4.641, 1907)

That time is continuous is defended in texts of different periods, for example: "time is a continuum" (CP 1.499, c.1896). However, it is true that a specificity of Peirce's last period is that, in his attempts to explicate 'immediate connection', he is "driven to introduce the idea of time".

Although I put emphasis on modifications, some important aspects of Peirce's philosophy of continuity are not modified in his Topological Period, like the idea of potentiality. Another example is when Peirce states that:

> We directly perceive the continuity of consciousness; and if anybody objects, that which is not really continuous may seem so, I reply, "Aye, but it could not seem so, if there were not some consciousness that is so." I should like to see a good criticism of that reply. (CP 6.182, c.1911)

Another example is Peirce's claim in 1893 that: "Once you have embraced the principle of continuity no kind of explanation of things will satisfy you except that they [the laws of nature] grew" (CP 1.175)

As we will see below, even in his last period, Peirce did not fully succeed in developing a mathematical framework for the dynamical aspects of continuity. One implicit but important aspect of Peirce's Topological Period is that the mathematical theory of collection is not enough to investigate the continuum. Peirce's mature conception of the continuum is incompatible with modern point-set topology which defines the continuum in a Cantorian spirit. In particular, Arnold Johanson remarks that Peirce's continuum "is not a Hausdorff space, and hence does not have the nice separation properties" so useful in mathematical analysis[97]. But as pointed out by Johanson, "topology without points" could be used in order to formalize Peirce's conceptualization of continuity. Johanson claims that "though many of Peirce's ideas about continua are in conflict with modern point-set topology, they are in substantial agreement with many of the conceptions of topology without points"[98].

What is topology without points? It is a topological theory in which points as ultimate parts do not exist. Hence in topology without points the connectedness must have a different definition than in point-set topology[99]. The main philosophical interest of a definition of continuity within point-set topology is that it is a satisfactory topological framework to deal with an Aristotelian continuum.

But point-set topology is not the only interesting analogy with Peirce's mathematical conception of continuity. Within the (mathematical) theory of category, *Smooth Infinitesimal Analysis* (SIA) is also a promising analogy. However, some Peircean commentators have, even recently, wrongly put forward analogies between Peirce's continuum and various versions of Non Standard Analysis (NSA), like those of Robinson or Conway, although these systems are extensions of the Cantorian continuum, very far from the Aristotelian continuum[100].

It is worth noting that for Peirce's Existential Graphs (EG), there is a deep link between logic, topology, and continuity. For example, Peirce writes:

> I ask you to imagine all the true propositions to have been formulated; and since facts blend into one another, it can only be in a continuum that we can conceive this to be done. This continuum must clearly have more dimensions than a surface or even than a solid; and we will suppose it to be plastic, so that it can be deformed in all sorts

> of ways without the continuity and connection of parts being ever ruptured. Of this continuum the blank sheet of assertion may be imagined to be a photograph. (CP 4.512)[101]

Some commentators have pointed out the relevance of Peirce's EG for contemporary works in logic:

> There is a scattering of topological methods in logic today. . . Perhaps there is a general field of the topology of logic, of which Peirce's existential graphs are our as yet most comprehensive portion, but which are a precursor of developments yet to come. (Dusek, 1993, p. 58)

Indeed, Pietarinen has shown that Peirce's EG are very interesting to provide a philosophical foundation for game semantics[102]. Moreover, Fernando Zalamea has shown that Lawvere's "Geometrizing Logic" program could be seen as a modern development of Peirce's Existential Graphs whose logic rests on his theory of continuity[103].

Since the perceptive continuum cannot be adequately formalized by the Cantorian continuum, one philosophical interest of Peirce's Topological Period could be the study of the perceptive continuum. With his Catastrophe Theory (CT), René Thom has elaborated a logic of the tension between continuity and discontinuity that could shed light on the perceptive continuum. However, the problem with this dynamic logic is that discontinuities are mathematically predetermined, whereas it sounds as if within the perceptive continuum, the discontinuities occur according to our evolutive aims[104].

Now, Peirce's theory of evolution can, more adequately than Thom's CT, account for such an evolutive and dynamic logic:

> To say that mental phenomena are governed by law does not mean merely that they are describable by a general formula; but that there is a living idea, a conscious continuum of feeling, which pervades them, and to which they are docile. (CP 6.152, *The Law of Mind*)

Thus, Peirce is aware that a mathematical formalization that would restrain evolution to predetermined phenomena is irrelevant for the study of the dynamical aspects of his synechism. But unfortunately, in Peirce's time, mathematics was unable to correctly grasp such an issue. One century later, we know how important and how difficult is such a project. Although Peirce was a forerunner in this research, Peirce's Topological Period fails to account adequately for the dynamical aspects of continuity.

Conclusion

What is now called the mathematical theory of sets plays an important part in Peirce's theory of continuity, from 1884 until 1907. However, after 1900, this is gradually replaced by his considerations on topology. I think that the contemporary theory called: *Smooth Infinitesimal Analysis* (SIA) has deep similarities with Peirce's conceptions, much more than the *Non-Standard Analysis* à la Robinson. Nevertheless, it is important to understand that although Peirce's considerations on continuity bear deep mathematical insights, they are not purely mathematical; there is always a tension between mathematical, philosophical and logical considerations. For Peirce, in 1893, "the reality of continuity appears most clearly in reference to mental phenomena"[105], such as thought, memory and perception; and in 1904 Peirce, states something similar: "true continuity is confusedly apprehended in the continuity of common sense"[106], by which he means mainly the continuity of time for "the connection of past and future seems truly continuous".

In reference to Charles Hartshorne's paper: "Continuity, the Form of Forms", the importance of continuity for Peirce's philosophy can be understood through his conception of philosophy, which can be summarized as the descriptive science of the most general characters of experience and the explanatory science of the total universe of possibility, actuality and generality[107]. For Peirce, everything that has an aspect of generality is a kind of continuity, and in every experience there is some regularity, some generality, some continuity, if only because every experience involves time, which is a continuum. Moreover, predictions in science imply continuity between mind and Nature or between ideas.

Is everything continuous for Peirce? It is clear that there are discontinuities, for Peirce states against Hegel that both Firstness and Secondness are not reducible to Thirdness. But according to Peirce, nothing is absolutely isolated or separated; hence, there is no absolute discontinuity. Anderson has pointed out that according to Peirce's tychism, there can be no rational continuity between past events and spontaneity. But discontinuity is relative, not absolute, like a new curve on a blackboard which is a discontinuity, but which "derives its discontinuity from the continuity of the blackboard itself" (RLT, p. 263, 1898). Moreover, Peirce's theory of evolution proceeds from an original continuum towards a final continuum, which is final less as a result than as a principle. In other words, there are different kinds of discontinuities but within different kinds of continua. This can explain Peirce's enigmatic claim that: "the doctrine of continuity is that all things so swim in continua" (CP 1.171, Summer 1893).

Peirce's theory of continuity is especially powerful for it combines the Aristotelian notion of inexhaustible potentiality, the Cantorian notion of transfinite diversity and the Kantian-Hegelian notion of homogeneity of diversity in Unity.

In a nutshell, even if Peirce did not fully succeed in his various attempts to give mathematics a more satisfying definition of continuity than Cantor's, I hope to have shown that Peirce's clarifications of continuity abound in fruitful reasoning, mathematical and philosophical.

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ABBREVIATIONS

Some sources are referred to throughout this article by the following abbreviations.

Baldwin: *Baldwin Dictionary of Philosophy and Psychology*

CD: *Century Dictionary*

CP: *Collected Papers of Charles Sanders Peirce*

MS: Microfilm version of the Peirce manuscripts in Houghton Library, Harvard University; numbers indicate those from Richard Robin's: *Annotated Catalogue of the Papers of Charles S. Peirce*

NEM: *The New Elements of Mathematics*

RLT: *Reasoning and the Logic of Things: The Cambridge Conferences Lectures of 1898*

TCSPS: *Transactions of the Charles S. Peirce Society*

W: *Writings of Charles S. Peirce: A Chronological Edition*

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NOTES

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2. End of 1893–beginning of 1894.

3. November–December 1893.

4. Article "Synechism" written by Peirce. One can also notice this passage: "I like to call my theory Synechism, because it rests on the study of Continuity" (RLT, p. 261).

5. Russell, 1903, quoted in Hudry, 2004, note 1.

6. "A Sketch of Logical Critic", c. 1911.

7. §1 "Three Grades of Clearness" in "The Logic of Relatives" in *The Monist*, vol. 7, pp. 161–217.

8. MS 1597 is Peirce's interleaved copy of the *Century Dictionary*, on which he has modified many articles.

9. There has not been time to comment here about Moore's new paper "The Genesis of the Peircean Continuum". I agree with Moore on a lot of points, but I think that in Peirce's mature conception of continuity topology plays a prominent role, although Moore seems skeptical about this claim (see Moore, 2007, p. 462).

10. Max Fisch has put forward one way to consider Peirce's evolution toward the question of nominalism (Fisch, 1971, p. 2). He has come up with three stages:

1) a nominalistic stage (1851–1868) in which Peirce thinks that only seconds can be real.

2) an anti-nominalistic stage (1868–1889) in which Peirce thinks that both seconds and thirds can be real.

3) a realistic stage (1889–1914) in which Peirce thinks that firsts, seconds and thirds can be real.

In this paper, I propose a division of Peirce's intellectual life based on Peirce's evolution on the question of continuity and not the question of nominalism. This is the reason why my anti-nominalistic period (1868–1884) differs from the anti-nominalistic stage (1868–1889) characterized by Max Fisch.

11. Peirce add the following remark: "Accordingly, just as we say that a body is in motion, and not that motion is in a body we ought to say that we are in thought and not that thoughts are in us."

12. In a draft titled "Time and Thought", found in MS 376 (CP 7.346–353) and MS 377.

13. Moore, 2007, p. 426.

14. "The Conception of Time essential in Logic."

15. "The Doctrine of Chances".

16. Dedekind, 1963. The original article was first published in 1888.

17. W 4.299–309, "On the Logic of Number", in *The American Journal of Mathematics*, vol. 4, p. 85–95, 1881. Peirce has always claimed to be the first to have made this discovery, but there is evidence that even though Peirce made this discovery independently, Dedekind was the first. Hourya Sinaceur told me that the first draft of "Was sind und was sollen die Zahlen?",—which can be seen at the "Niedersächsische Staats- und Universitätsbibliothek" of Göttingen—, was written in 1872. She also told me that in May 1877, Dedekind informed Heinrich Weber about his idea about how to define the infinite. In 1848, Bolzano was close to the idea by discovering a relation which hold between two infinite sets; but Bolzano's definition is not a definition of an infinite set *per se*. In 1877 Cantor found something similar to Bolzano. In a nutshell, although Peirce was the first to publish it in 1881, the first modern definition of an infinite set was given by Dedekind in 1872.

18. Cf W3.105–106, "The Conception of Time Essential in Logic", July 1873: "A *continuum* (like time and space as they actually are) is defined as something any part of which however small itself has parts of the same kind. . . . And so nothing is true of a point which is not at least on the limit of what is true for spaces and times."

19. Cf W2.209–211; 1868.

20. See Fisch, 1971, p. 195.

21. André De Tienne informed me that this text was written at the same time as "One, Two, Three"; "One, Two, Three: An Evolutionist Speculation"; "First, Second, Third", W 5. 294–308.

22. "A Guess at the Riddle", Fall 1887–Winter 1888.

23. W 6.172, CP 1.359.
24. Ibid.
25. Ibid.
26. CP 6.121.
27. Potter and Shields, 1977, p. 24.
28. That cancellation took the form of the word ‘Stet’, which means: “Let it stand the way it was before my correction.”
29. Commentary on MS 1604.
30. CP 6.174; 1906.
31. CP 8 Bibliography General 1893 [G-1893–5].
32. CP 6.104; 1892.
33. Between Summer 1892 and early 1893
34. CP 1.163; Summer 1893.
35. CP 4.119; 1893.
36. CD, article “limit”, 1883–1888.
37. CD, article “Continuity”.
38. MS 950, p. 1; c.1893 according to Robin.
39. “On Continuous Series and the Infinitesimal”, MS 718.
40. NEM 3.95; MS 28.
41. Cantor wrote in 1882 that: “The hypothesis of the continuity of space is therefore nothing other than the assumption, arbitrary in itself, of a complete, one-to-one correspondence between the three dimensional purely arithmetical continuum (x, y, z) and the space underlying the world of phenomena”. Quoted in Dauben, 1979, p. 86.
42. Bell, 1998, p. 5.
43. CP 8.216; see also CP 6.168, CP 6.182, and NEM 2.531.
44. “The Law of Mind”.
45. MS 28; c.1897 according to Robin; NEM 3.94, “Multitude and Continuity”, 1895–1900 according to Carolyn Eisele.
46. “On Continuous Series and the Infinitesimal”.
47. NEM 2.483, MS 137, 1904.
48. MS 1597 in the article “Continuity”. CP 6.122, “The Law of Mind”, 1892. In “The Logic of Quantity”, CP 4.121, 1893. In CP 6.166, 1892–1893.
49. MS 1597, article “continuity”; (see also CP 6.166), Summer 1892–Early 1893.
50. In a letter which he wrote to William James in 1897, Peirce states that: “[Is] possibility a mode of being[?]. . . Precisely so . . . I reached this truth by studying the question of possible grades of multitude” (CP 8.308). See also Murphey, p. 394.
51. “Transfinite cardinal” does not belong to Peirce’s terminology. Indeed, Peirce states that: “G. Cantor has introduced among mathematicians a bias use of the term ‘cardinal number’ to signifiy multitude” (MS 1597, article “cardinal”). However, in MS 1597, Peirce complains that the CD has not defined ‘transfinite’. See also “As for Cantor’s cardinal transfinites, though called numbers by him, they are not properly so called but are *multitudes*, or maninesses of infinite collections” (NEM 3.989, 1906).
52. Cantor, 1999b.
53. “The Logic of Relatives”, in *The Monist*, vol. 7, pp. 161–217.
54. “The Logic of Quantity”.

55. “Multitude and Quantity”. Robin thinks this manuscript was written around 1895, but I think it is 1896–1897.

56. NEM 3, p. vii.

57. Matthew Moore, in a private communication.

58. In another text, where Peirce uses the term ‘abnumeral’ instead of ‘abnumerable’, he says that: “The first abnumerable multitude is that of the irrational numbers of the calculus . . .” (MS 1170 A, article “abnumerable”).

59. NEM 3.777, L 73, December 23, 1900.

60. L 148; in a letter sent to F.W. Frankland on May 08th, 1906.

61. Murphey, 1993, p.260–263.

62. Potter and Shields, 1977, p.27–29.

63. In another text, written afterwards, Peirce uses the term ‘abnumeral’ instead of ‘abnumerable’, and he says that: “There is no multitude greater than all finitely abnumerable multitudes, since if we attempt to conceive of such a collection it will be found that the individual members become indefinite and lose their distinct identities, so that there is no longer any collection and there consequently is no multitude” (MS 1170 S, article “abnumerable”).

64. MS 1147, Baldwin, “Mathematical Logic”, written in 1900 according to André De Tienne.

65. RLT, 1898, p. 247. CP 6.185, 1. Potential Aggregates, The Logic of Continuity.

66. CP 3.568, “Infinitesimals”, a letter to the Editor of *Science*, vol. 2, p. 430–433, March 16th, 1900.

67. Ibid.

68. MS 955 (or CP 1.164); and also CP 4.123.

69. “Peirce’s Continuum”, 1995, p. 8.

70. In other words, I think that what Moore call Peirce’s Cambridge Conferences (1898) theory of the continuum, or CC theory (see Moore, 2007, p. 425–426), becomes gradually obsolete after 1900 because of Peirce’s new conception of collection, and after 1908 because of Peirce’s concern that multiplicity might not be the best criteria to define continuity, and also because of Peirce’s mature conviction that topology, and not set theory, is the good mathematical tool to conceptualize continuity.

71. RLT, p. 50.

72. Ehrlich, 2006.

73. MS 28, “Multitude and Continuity”.

74. Bell, 1998, p. 5.

75. 1893–1895 according to Robin.

76. NEM 4.343, MS 439, 1898, “Detached Ideas continued and the Dispute between Nominalists and Realists”.

77. Quoted by Max Fisch, “Peirce’s Arisbe: The Greek Influence In His Later Philosophy”, *Transactions of the Charles S. Peirce Society*, 1971, p. 198.

78. *Ibid*, quoted by Max Fisch, 1971.

79. W 2.227, CP 5.289, 1868.

80. Merrell, 1991, p. 187.

81. MS 1147, Baldwin: “Mathematical Logic”.

82. Anderson, 1987.

83. MS 439, RLT, p. 163.

84. “Synechism and Immortality”.

85. In MS 1597, in the article "mathematics", Peirce put forward a division of mathematics, in which he claims that the theory of functions explores the mathematics of integers and irrational quantities, but that it is topology that explores the mathematics of continuity:

Mathematics of integers (theory of functions)

Mathematics of irrational quantity (theory of functions)

Higher quite undeveloped forms of mathematics

Mathematics of continuity (Topology or Geometrical Topic or Topical Geometry).

86. CD, Supplement, 1909; article "Topics". For other definitions of topology and its links with continuity, see Havenel, forthcoming.

87. CP 4. 639–641.

88. CP 4.642.

89. For a more detailed explanation, see my forthcoming paper: "Peirce's Topological Concepts", section: "Peirce on Topological Singularities and defects of Continuity".

90. Among the many subjects in which Peirce is surprisingly modern, I am currently working, with Marc Guastavino, on Peirce's version of the fixed point theorem.

91. CP 6.174, "Continuity Redefined", in "The Bedrock beneath Pragmaticism".

92. Potter and Shields, 1977, p. 30.

93. CP 7.535, note 6.

94. "A Sketch of Logical Critic", c.1911 according to the *Collected Papers*. In this text written at the end of his life, Peirce seems to use indifferently the terms 'continuity' and 'continuum'. This constitutes another reason against maintaining that throughout Peirce's work on the subject, there is always a clear distinction between the words 'continuity' and 'continuum'.

95. MS 140.

96. CP 5.289, "Some Consequences of Four Incapacities", 1868.

97. Johanson, 2001, p. 10.

98. Johanson, 2001, p. 10.

99. For a more detailed presentation of "topology without points", see Johanson, 1981 and Johnstone, 1983.

100. Havenel, 2006, Chapter 6.

101. One can also notice this passage: "The line of identity . . . represents Identity to belong to the genus Continuity and to the species Linear Continuity . . . The Phemic Sheet . . . is the most appropriate Icon possible of the continuity of the Universe of Discourse. . ." (CP 4.561, note 1).

102. Pietarinen, 2006.

103. Zalamea, 2003, p. 157.

104. Visetti, 2004.

105. CP 8 Bibliography General 1893 [G-1893–5].

106. NEM 2.483, 1904.

107. I have slightly modified Hartshorne's definition in: Hartshorne, 1929, p. 522.