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PENALTY/FINITE-ELEMENT APPROXIMATIONS OF A CLASS OF UNILATERAL PROBLEMS IN LINEAR ELASTICITY*

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Abstract. The present paper is concerned with a development of a penalty/finite-element approximation of a class of unilateral problems in linear elasticity. A penalty method is applied to resolve the inequality constraint due to contact, and convergence with respect to the penalty parameter is discussed. Then finite-element approximations are introduced to the penalized formulation with a priori error estimates in terms of the penalty and mesh parameters. Several numerical examples are also given in the end of the paper.

1. Introduction. The present study is concerned with the development of a penalty/finite-element approximation to a class of contact problems which involve a deformation of a linearly elastic body supported unilaterally on a frictionless foundation. The kinematical restriction due to the rigid foundation is resolved by penalty methods, and the variational formulation of the penalized problem is discretized by finite-element methods.

The contact problem which will be discussed in this paper is called the Signorini problem and was solved by Fichera [1] in 1963. Details of the mathematical analysis such as existence, uniqueness, and regularity of the solution can be found in the monograph by Duvaut and Lions [2] or the paper by Kalker [3], for example. Finite-element analysis of the Signorini problem formulated by variational inequalities are studied by Hlavaček and Lovišek [4] and Kikuchi and Oden [5]. Since every admissible displacement is restricted by a constraint given by an inequality, the form of inequality must be solved directly in order to get a solution. This situation is not favorable for computations, despite the fact that the form of the inequality can be solved by various gradient methods with projection maps specially designed for constraints, as shown in Glowinski, Lions, and Tremolieres [6].

In order to avoid the constraint on admissible sets, one of the methods commonly used is the Lagrangian multiplier method with Uzawa's iterative algorithm. Formulations of contact problems in elasticity by this method are given in, e.g., Paczell [7] and Kikuchi and Song [8] together with finite-element approximations without convergence analysis. This method has been used by many authors without explicit mention in order

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to solve contact problems by finite-element methods (see Chan and Tuba [9] or Hughes et al. [10]). However, the Lagrangian multiplier method leads to slow convergence of its Uzawa iterative algorithm because of the restriction on the multiplier, while the displacement field is free from any constraints.

Another candidate for the resolution of the inequality constraints is the exterior penalty method introduced by Courant, Friedrichs, and Lewy [11] and extended by Zangwill [13]. As Courant [12] indicated, the penalty method makes physical sense. For example, the idea of penalty methods resolving the Dirichlet boundary condition $u = 0$ is that very stiff springs are set along the boundary (instead of the fixed condition). This means mathematically that the Dirichlet boundary condition is approximated by the third type of boundary condition $\partial u/\partial n = -u/\varepsilon$ for a small enough penalty parameter $\varepsilon > 0$. Physically ε^{-1} is the stiffness of the springs. We will apply this method to resolve the constraint due to the rigid foundation in contact problems.

Following the abstract mathematical analysis of penalty methods given by Lions [14] and Aubin [15], we will obtain (i) convergence of the penalty method for the Signorini problem with explicit estimates in terms of the penalty parameter ε, (ii) error estimates of the penalty/finite-element approximation with respect to ε and the mesh parameter h of the finite-element model, and (iii) several numerical examples which demonstrate the applicability of the present method to other practical contact problems.

The penalty resolution discussed in this paper seems to have a very close relation to the methods of interface (or bond, film) finite elements for solving two-body contact problems applied by Tsuta and Yamaji [16] and Yamada et al. [17]. The relationship of these two methods for a class of contact problems is discussed in Okabe and Kikuchi [18].

2. A penalization of an unilateral problem. Suppose that a body Ω is supported unilaterally on a rigid foundation and that Γ_C is a part of the boundary Γ of the body in which the true contact surface is included. Let s be the normal distance between the rigid foundation and the body. Then the unilateral contact condition

$$u_n - s \leq 0, \qquad \sigma \leq 0, \qquad (u_n - s)\sigma = 0 \quad \text{on} \quad \Gamma_C \tag{2.1}$$

must be satisfied. Here u_n and σ are the normal displacement and stress (contact pressure) on the boundary defined by

$$u_n = u_i n_i \quad \text{and} \quad \sigma = \sigma_{ij} n_j n_i, \tag{2.2}$$

respectively, where n is the unit vector normal to the boundary. The condition (2.1) means that the body cannot penetrate the rigid foundation and that the contact pressure exists at the point which the body touches the foundation.

It is a rather simple idea in physics that the rigid foundation can be approximated by continuously distributed springs the stiffness of which is given by $1/\varepsilon$ for sufficiently small $\varepsilon > 0$. If $\varepsilon \to 0$, the spring foundation becomes rigid. Thus condition (2.1) may be replaced by the relation

$$\sigma_\varepsilon = -(1/\varepsilon)(u_{\varepsilon n} - s)^+, \qquad a^+ = \max\{a, 0\} \tag{2.3}$$

which represents the boundary condition for the unilateral spring support. The replacement of (2.1) by (2.3) means that intense stress is produced on the boundary as a penalty, when the constraint $u_n - s \leq 0$ due to the rigid foundation is violated. The replacement

(2.3) of the rigid foundation by the stiff spring is called *penalization* of the constraint (2.1), and the parameter ε is called the *penalty parameter*.

If no frictional effect is assumed on the contact surface, the contact condition additional to (2.1) is

$$\sigma_i^T = 0, \qquad \sigma_i^T = \sigma_{ij} n_j - \sigma n_i \quad \text{on} \quad \Gamma_C. \tag{2.4}$$

That is, there are no stresses in the direction tangential to Γ_C.

Suppose that the body is fixed on a part Γ_D of the boundary, and that the body is subjected to the body force f and the traction t on the boundary $\Gamma_F = \Gamma - \bar{\Gamma}_D - \bar{\Gamma}_C$. Then the equilibrium equations are

$$-\sigma_{ij,j} = f_i \quad \text{in} \quad \Omega, \qquad u_i = 0 \quad \text{on} \quad \Gamma_D, \qquad \sigma_{ij} n_j = t_i \quad \text{on} \quad \Gamma_F. \tag{2.5}$$

Suppose that the body is linearly elastic:

$$\sigma_{ij} = E_{ijkh} \varepsilon_{kh}(u). \tag{2.6}$$

Here E_{ijkh} is a piecewise constant fourth-order tensor such that

$$E_{ijkh} = E_{khij} = E_{ijhk},$$

$$m > 0: \qquad E_{ijkh} X_{kh} X_{ij} \geq m X_{mn} X_{mn}, \qquad X_{ij} = X_{ji}, \tag{2.7}$$

and ε_{kh} is the linearized strain tensor defined by

$$\varepsilon_{kh}(u) = (u_{k,h} + u_{h,k})/2. \tag{2.8}$$

We have used the summation convention and $u_{k,h} = \partial u_k / \partial x_h$, etc.

A variational principle for the problem penalized as above is then given by

$$u_\varepsilon \in V: \qquad B(u_\varepsilon, v) + (1/\varepsilon)((u_{\varepsilon n} - s)^+, v_n) = f(v), \qquad v \in V \tag{2.9}$$

where

$$V = \{v \in H^1(\Omega): \qquad v = 0 \quad \text{on} \quad \Gamma_D\}, \tag{2.10}$$

$$B(u, v) = \int_\Omega E_{ijkh} \varepsilon_{kh}(u) \varepsilon_{ij}(v)\, dx, \tag{2.11}$$

$$f(v) = \int_\Omega f_i v_i\, dx + \int_{\Gamma_F} t_i v_i\, ds, \tag{2.12}$$

$$H^1(\Omega) = \{v: \quad v_i \in L^2(\Omega) \quad \text{and} \quad v_{i,j} \in L^2(\Omega)\}. \tag{2.13}$$

Let the Sobolev norm $\|\cdot\|_1$ be defined by

$$\|v\|_1 = \left| \int_\Omega (v_i v_i + v_{i,j} v_{i,j})\, dx \right|^{1/2} \tag{2.14}$$

The function $(\cdot, \cdot)$ is the $L^2(\Gamma_C)$ inner product defined by

$$(\tau, v_n) = \int_{\Gamma_C} \tau v_n\, ds \tag{2.14}$$

with the norm $|||\cdot|||$:

$$|||\tau||| = (\tau, \tau)^{1/2}.$$

We will assume that the boundary of the domain Ω is smooth enough so that $v_i \in H^{1/2}(\Gamma)$ and $v_n = v_i n_i \in H^{1/2}(\Gamma)$ for $v_i \in H^1(\Omega)$, etc. As shown in Nečas [19, p. 88], it suffices to assume

$$\Omega \in C^{1,1}. \tag{2.15}$$

In Secs. 6 and 7 we also use the Sobolev spaces $H^s(\Omega)$ and $H^r(\Gamma_C)$ with norms $\|\cdot\|_s$ and $\|\cdot\|_{r,\Gamma_C}$, respectively, for arbitrary real numbers s and r. Details of such spaces can be found in, e.g., Aubin [15, Chapter 6] and Nečas [19, Chapter 2].

3. Existence of the solution of the penalization. Let the operator $A: V \to V'$ be defined by

$$\langle A(u), v\rangle = B(u, v) + (1/\varepsilon)((u_n - s)^+, v_n) \tag{3.1}$$

where V' is the dual of V. By applying Korn's inequality [19, p. 192],

$$\int_\Omega \varepsilon_{ij}(v)\varepsilon_{ij}(v)\, dx \geq cm\|v\|_1^2, \qquad v \in V$$

with the assumption $\text{meas}(\Gamma_D) > 0$, the convexity assumption $(2.7)_2$ on the strain energy function yields the strong monotonicity of B:

$$B(u - v, u - v) \geq cm\|u - v\|_1^2.$$

Since

$$((a - c)^+ - (b - c)^+)(a - b) \geq ((a - c)^+ - (b - c)^+)^2,$$

for every real number a, b, c, we obtain

$$((u_n - s)^+, u_n - v_n) - ((v_n - s)^+, u_n - v_n) \geq |||(u_n - s)^+ - (v_n - s)^+|||^2 \geq 0.$$

Thus for some constant $m > 0$,

$$\langle A(u) - A(v), u - v\rangle \geq m\|u - v\|_1^2 \tag{3.2}$$

for every u, $v \in V$. That is, the operator A is *strongly monotone* on the space V.

On the other hand, the inequality

$$((a - c)^+ - (b - c)^+)\, d \leq |a - b|\, |d|$$

for every real numbers a, b, c, and d implies

$$\langle A(u) - A(v), w\rangle \leq M\|u - v\|_1 \|w\|_1 + (1/\varepsilon)|||u_n - v_n|||\, |||w_n|||$$

for every u, v, $w \in V$. Here

$$M = \left(\max_{i,j,k,h} \max_{x \in \Omega} |E_{ijkh}(x)|\right)\text{meas}(\Omega). \tag{3.3}$$

Applying the trace theorem [19, p. 99], we have

$$|||w_n||| \leq C\|w\|_1.$$

Then

$$\langle A(u) - A(v), w\rangle \leq (M + C/\varepsilon)\|u - v\|_1 \|w\|_1. \tag{3.4}$$

That is, the operator $A : V \to V'$ is *strongly continuous.*

Thus, the existence theorem for the penalized problem (2.9) follows from the theory of nonlinear monotone operators [14, p. 171]:

THEOREM 3.1. Suppose that (2.7) and (3.3) hold. Then there exists a unique solution u_ε to the penalized problem (2.9) provided that

$$f \in L^2(\Omega), \qquad t \in L^2(\Gamma_F), \quad \text{and} \quad s \in H^{1/2}(\Gamma_C). \tag{3.5}$$

Moreover, u_ε is uniformly bounded in ε:

$$\|u_\varepsilon\|_1 \le \|f\|_*/m \tag{3.6}$$

where

$$\|f\|_* = \|f\|_0 + \left\{\int_{\Gamma_F} t_i t_i \, ds\right\}^{1/2}.$$

Because of the reflexivity of the Sobolev space $H^1(\Omega)$, there exists a subsequence of $\{u_\varepsilon\} \in V$ which converges weakly to an element $u \in V$ as $\varepsilon \to 0$. Let such a subsequence be denoted again by $\{u_\varepsilon\}$. We shall show that the limit u belongs to the set

$$K = \{v \in V: \quad v_n - s \le 0 \quad \text{on} \quad \Gamma_C\} \tag{3.7}$$

and is the solution of the variational inequality

$$u \in K: \quad B(u, v-u) \ge f(v-u), \qquad v \in K. \tag{3.8}$$

The inequality (3.8) is the variational principle for the unilateral problem with the contact condition (2.1).

THEOREM 3.2. Suppose that conditions in Theorem 3.1 hold. Then the sequence $\{u_\varepsilon\} \in V$ of the solution of the penalized problem (2.9) converges to a unique solution of the variational inequality (3.8) as $\varepsilon \to 0$, provided that $\Omega \in C^{1,1}$.

Proof. From (2.9),

$$B(u_\varepsilon, v - u_\varepsilon) + (1/\varepsilon)((u_{\varepsilon n} - s)^+, v_n - u_{\varepsilon n}) = f(v - u_\varepsilon) \tag{3.9}$$

for every $v \in V$. For any $v \in K$, $(v_n - s)^+ = 0$. Then

$$B(u_\varepsilon, v - u_\varepsilon) \ge B(u_\varepsilon, v - u_\varepsilon) - (1/\varepsilon)((v_n - s)^+ - (u_{\varepsilon n} - s)^+, v_n - u_{\varepsilon n}) = f(v - u_\varepsilon),$$

i.e.

$$B(u_\varepsilon, v - u_\varepsilon) \ge f(v - u_\varepsilon), \qquad v \in K. \tag{3.10}$$

Since $v \to B(v, v)$ is strictly convex and Gâteaux-differentiable on V,

$$\liminf_{\varepsilon \to 0} B(u_\varepsilon, u_\varepsilon) \ge B(u, u).$$

Since $v \to f(v)$ is linear and continuous, we can take the limit of (3.10) as $\varepsilon \to 0$:

$$B(u, v - u) \ge f(v - u), \qquad v \in K.$$

We next show that $u \in K$. Putting v satisfying $v_n = s$ on Γ_C ($v_n \in H^{1/2}(\Gamma)$ since $\Omega \in C^{1,1}$) into (2.9) yields

$$\begin{aligned}((u_{\varepsilon n} - s)^+, (u_{\varepsilon n} - s)^+) &= \varepsilon\{B(u_\varepsilon, v - u_\varepsilon) - f(v - u_\varepsilon)\} \\ &\leq \varepsilon\{B(u_\varepsilon, v) - f(v - u_\varepsilon)\} \\ &\leq \varepsilon\{M\|u_\varepsilon\|_1 \|v\|_1 + C\|f\|_*(\|v\|_1 + \|u_\varepsilon\|_1)\}.\end{aligned}$$

Passing to the limit $\varepsilon \to 0$ yields

$$|||(u_n - s)^+||| \leq \lim_{\varepsilon \to 0} |||(u_{\varepsilon n} - s)^+||| = 0.$$

That is, $u_n - s \leq 0$ in the sense of $L^2(\Gamma_C)$. However, $u_n \in H^{1/2}(\Gamma_C)$ and $s \in H^{1/2}(\Gamma_C)$. Then

$$u_n - s \leq 0 \quad \text{in the sense of} \quad H^{1/2}(\Gamma_C);$$

that is, $u \in K$.

Since uniqueness of the solution of the variational inequality (3.10) is clear from the inequality

$$B(u - v, u - v) \geq m\|u - v\|_1^2,$$

every convergent subsequence of $\{u_\varepsilon\}$ has the same limit $u \in K$, which is the solution of (3.10). Therefore, the original sequence $\{u_\varepsilon\}$ itself must converge to the limit $u \in K$ as $\varepsilon \to 0$ because of the uniform boundedness of the sequence $\{u_\varepsilon\}$.

4. Existence of the contact pressure. It has been shown that the penalty solution u_ε converges to the solution u of the variational inequality as $\varepsilon \to 0$. It will be shown in this section that the approximation

$$\sigma_\varepsilon = -(1/\varepsilon)(u_{\varepsilon n} - s)^+ \tag{4.1}$$

to the contact pressure converges to that of the unilateral problem as $\varepsilon \to 0$.

THEOREM 4.1. Suppose that the conditions of Theorem 3.2 hold. Then the sequence $\{\|\sigma_\varepsilon\|^*_{1/2, \Gamma_C}\}$ is uniformly bounded in ε and $\{\sigma_\varepsilon\}$ converges to $\sigma \in N$ as $\varepsilon \to 0$. Moreover, the pair $\{u, \sigma\} \in K \times N$ satisfies the mixed (or saddle point) formulation

$$\begin{aligned} B(u, v) - [\sigma, v_n] &= f(v), \qquad v \in V, \\ [\tau - \sigma, u_n - s] &\geq 0, \qquad \tau \in N, \end{aligned} \tag{4.2}$$

where $[\cdot, \cdot]$ is the duality pairing on $(H^{1/2}(\Gamma_C))' \times H^{1/2}(\Gamma_C)$, and

$$N = \{\tau \in (H^{1/2}(\Gamma_C))': \qquad \tau \leq 0\}. \tag{4.3}$$

Proof. Substitution of (4.1) into (2.9) yields

$$B(u_\varepsilon, v) - [\sigma_\varepsilon, v_n] = f(v), \qquad v \in V,$$

since $[\tau, v_n] = (\tau, v_n)$ for $\tau \in L^2(\Gamma_C)$. By applying the trace theorem that the trace map is surjective from $H^1(\Omega)$ into $H^{1/2}(\Gamma)$, there is a positive constant α such that

$$\alpha\|\tau\|^*_{1/2, \Gamma_C} \leq \sup_{v \in H^1(\Omega)} ([\tau, v_n]/\|v\|_1).$$

Then

$$\|\sigma_\varepsilon\|^*_{1/2, \Gamma_C} \leq (M\|u_\varepsilon\|_1 + \|f\|_*)/\alpha.$$

This means that $\{\|\sigma_\varepsilon\|^*_{1/2,\Gamma_C}\}$ is uniformly bounded in ε, and thus there exists a subsequence of $\{\sigma_\varepsilon\}$ which converges to σ weakly in $(H^{1/2}(\Gamma_C))'$.

Next we shall show that the limit σ of a convergent subsequence of $\{\sigma_\varepsilon\}$ belongs to the set N. This follows from $\sigma_\varepsilon \le 0$ and closedness of the set N.

The first equation of (4.2) is clear. Indeed, (2.9) and (4.1) imply

$$B(u_\varepsilon, v) - (\sigma_\varepsilon, v_n) = f(v), \qquad v \in V.$$

Passing to the limit $\varepsilon \to 0$ yields

$$B(u, v) - [\sigma, v_n] = f(v), \qquad v \in V.$$

On the other hand,

$$0 \ge (\sigma_\varepsilon, u_{\varepsilon n} - s) = B(u_\varepsilon, u_\varepsilon) - f(u_\varepsilon) - (\sigma_\varepsilon, s).$$

Taking the limit as $\varepsilon \to 0$, we have

$$0 \ge B(u, u) - f(u) - [\sigma, s] = [\sigma, u_n - s] \ge 0;$$

That is,

$$[\sigma, u_n - s] = 0$$

The second inequality of (4.2) then follows from the fact that $\sigma \in N$ and $u \in K$.

It should be noted that the limit σ of the sequence $\{\sigma_\varepsilon\}$ may not belong to the space $L^2(\Gamma_C)$, even though each entry of the sequence belongs to $L^2(\Gamma_C)$. Indeed, suppose that a flat punch is indented into the semi-infinite linearly elastic foundation in R^2. Then the contact pressure is given by Garlin [20, p. 45] as

$$\sigma = C/\sqrt{a^2 - x^2} \quad \text{in} \quad (-a, a)$$

for some proper constant C. The x-axis is assumed to be the surface of the foundation, and the flat punch is located within the interval $[-a, a]$ on the x-axis. It is clear that $\sigma \notin L^2(-a, a)$.

In the subsequent analysis of finite-element approximations of the contact problem, however, we will assume that the contact pressure belongs to $L^2(\Gamma_C)$, although this assumption may not be true in all contact problems. Under this assumption, the mixed formulation (4.2) will be written as

$$B(u, v) - (\sigma, v_n) = f(v), \qquad v \in V, \qquad (\tau - \sigma, u_n - s) \ge 0, \qquad \tau \in N \cap L^2(\Gamma_C). \quad (4.4)$$

5. Finite-element approximations. Let V_h be the finite-element approximation of the space V defined by (2.10). Let $I(\cdot)$ be the operation of numerical integration given by

$$I(\tau^h, v_n^h) = \sum_{e=1}^{E'} \sum_{j=1}^{G} w_j^e \tau^h(\xi_j^e) v_n^h(\xi_j^e) \tag{5.1}$$

where E' is the number of elements related to the boundary Γ_C, G the number of points, w_j^e the weight, and ξ_j^e the local coordinates of points of numerical integration respectively. Here v^h and τ^h denote finite-element approximations of v and τ. The function v_n^h is the normal displacement of v^h defined by

$$v_n^h(\xi_j^e) = v^h(\xi_j^e) \cdot n(\xi_j^e) \qquad \text{(no summation)},$$

where j is the nodal point of the finite-element model. Moreover, v_n^h is the polynomial the order of which is same as that of v^h. Under these conventions, the approximation of (2.9) is given by

$$u_\varepsilon^h \in V_h: \qquad B(u_\varepsilon^h, v^h) + (1/\varepsilon)I((u_{\varepsilon n}^h - s)^+, v_n^h) = f(v^h), \qquad v^h \in V_h. \tag{5.2}$$

If the trapezoid rule is used for $I(\cdot)$, then $G = 2$, $w_1^e = w_2^e = h/2$, $\xi_1^e = -1$, and $\xi_2^e = 1$, where h is the length of the finite element. If Simpson's rule is applied, then $G = 3$, $w_1^e = w_3^e = h/3$, $w_2^e = 4h/3$, $\xi_1^e = -1$, $\xi_2^e = 0$, and $\xi_3^e = 1$.

We now define the finite-dimensional approximation σ_ε^h of the contact pressure such as (4.1). Let W_h be the space spanned by Lagrangian-type interpolation functions given on points of numerical integration (5.1). If the points of integration do not coincide with nodal points of the finite-element model on Γ_C, W_h is nonconforming; that is, every element of W_h need not to be continuous. If V_h is approximated by four-node isoparametric quadrilateral finite elements (Q_1 elements), and if the trapezoidal rule is applied for numerical integration I within an element, the space W_h is the set of piecewise linear polynomials which are traces of elements of V_h on the boundary. If the one-point Gaussian rule is used for the same elements, the space W_h is the set of all piecewise constant functions which are nonconforming. Let the approximation σ_ε^h be defined by

$$\sigma_\varepsilon^h(\xi_j^e) = -(1/\varepsilon)(u_{\varepsilon n}^h - s)(\xi_j^e)^+ \tag{5.3}$$

and $\sigma_\varepsilon^h \in W_h$, and let

$$N_h = \{\tau^h \in W_h: \qquad \tau^h(\xi_j^e) \le 0, \qquad 1 \le j \le G, \qquad 1 \le e \le E'\}. \tag{5.4}$$

Applying arguments similar to those in Theorem 3.1, we obtain the following results:

THEOREM 5.1. Suppose that same conditions in Theorem 3.1 hold. Then the sequence $\{u_\varepsilon^h\} \in V_h$ converges to the solution $u^h \in K_h$ of the variational inequality

$$u^h \in K_h: \qquad B(u^h, v^h - u^h) \ge f(v^h - u^h), \qquad v^h \in K_h \tag{5.5}$$

as $\varepsilon \to 0$, where

$$K_h = \{v^h \in V_h: \qquad (v_n^h - s)(\xi_j^e) \le 0, \qquad 1 \le e \le E', \qquad 1 \le j \le G\}. \tag{5.5}$$

The nonlinear equation (5.2) is solved by the method of successive iterations. Let ${}^t u_\varepsilon^h$ be the tth approximation of u_ε^h, and let

$$\begin{aligned}({}^t u_{\varepsilon n}^h - s)(\xi_j^e)^+ &= ({}^t u_{\varepsilon n}^h - s)(\xi_j^e) \qquad \text{if} \quad ({}^{t-1}u_{\varepsilon n}^h - s)(\xi_j^e) > 0,\\ &= \qquad 0 \qquad\qquad\quad \text{if} \quad ({}^{t-1}u_{\varepsilon n}^h - s)(\xi_j^e) \le 0.\end{aligned} \tag{5.6}$$

Then the variational equation in V_h

$${}^t u_\varepsilon^h \in V_h: \qquad B({}^t u_\varepsilon^h, v^h) + (1/\varepsilon)I(({}^t u_{\varepsilon n}^h - s)^+, v_n^h) = f(v^h), \qquad v^h \in V_h \tag{5.7}$$

becomes linear and may be solved by customary methods for the solution of systems of linear equations. We will repeat the above procedure until a small enough tolerance $\|{}^{t+1}u_\varepsilon^h - {}^t u_\varepsilon^h\|_1 / \|{}^{t+1}u_\varepsilon^h\|_1$ is obtained.

It can be easily verified that the operator $A_t: V_h \to (V_h)'$, defined by

$$\langle A_t({}^t u^h), v^h\rangle = B({}^t u_\varepsilon^h, v^h) + (1/\varepsilon)I(({}^t u_{\varepsilon n}^h - s)^+, v_n^h),$$

is monotone continuous and coercive on V_h independently of the parameter t, $t = 1, 2, \ldots, M$. Thus, existence of a unique solution ${}^t u_\varepsilon^h$ of the problem (5.7) is assured, and furthermore $\|{}^t u_\varepsilon^h\|_1$ is uniformly bounded in t. Therefore there exists a subsequence of $\{{}^t u_\varepsilon^h\}$ which converges to the limit u_ε^h as $t \to \infty$. Because of the uniqueness of solutions to the problem (5.2), the sequence $\{{}^t u_\varepsilon^h\}$ itself converges to the solution u_ε^h as $t \to \infty$.

As shown in Sec. 8, convergence of (5.7) is sometimes obtained with zero tolerance within a few iterations.

6. Convergence with respect to ε and h. In this section we obtain the a priori estimates of the form $B(u - u_\varepsilon^h, u - u_\varepsilon^h)$ and $|||\sigma - \sigma_\varepsilon^h|||$ in terms of the penalty parameter ε and the mesh size h of the finite-element model.

THEOREM 6.1. Suppose that conditions of Theorems 4.1 and 5.1 hold. Then

$$B(u - u_\varepsilon^h, u - u_\varepsilon^h) \le B(u - u_\varepsilon^h, u - v^h) + (\sigma, v_n^h - u_n) + (\sigma - \tau^h, u_n - u_{\varepsilon n}^h) + (\tau^h - \sigma, u_n - s) + E_I(\tau^h, u_{\varepsilon n}^h - s) + I(\tau^h - \sigma_\varepsilon^h, \tau^h)\varepsilon \tag{6.1}$$

for every $v^h \in K_h$ and $\tau^h \in N_h$, where

$$E_I(f, g) = (f, g) - I(f, g), \qquad \forall f, g \in C(\bar{\Omega}).$$

Proof. From (4.4) and (5.2),

$$\begin{aligned} B(u - u_\varepsilon^h, u - u_\varepsilon^h) &= B(u - u_\varepsilon^h, u - v^h) + B(u - u_\varepsilon^h, v^h - u_\varepsilon^h) \\ &= B(u - u_\varepsilon^h, u - v^h) + (\sigma, v_n^h - u_{\varepsilon n}^h) - I(\sigma_\varepsilon^h, v_n^h - u_{\varepsilon n}^h). \end{aligned}$$

The last two terms are then estimated by

$$\begin{aligned} (\sigma, v_n^h - u_{\varepsilon n}^h) - I(\sigma_\varepsilon^h, v_n^h - u_{\varepsilon n}^h) &= (\sigma, v_n^h - u_n) + (\sigma - \tau^h, u_n - u_{\varepsilon n}^h) + (\tau^h, u_n - s) \\ &\quad - (\tau^h, u_{\varepsilon n}^h - s) - I(\sigma_\varepsilon^h, v_n^h - s) + I(\sigma_\varepsilon^h, u_{\varepsilon n}^h - s) \\ &\le (\sigma, v_n^h - u_n) + (\sigma - \tau^h, u_n - u_{\varepsilon n}^h) + (\tau^h - \sigma, u_n - s) \\ &\quad - E_I(\tau^h, u_{\varepsilon n}^h - s) - I(\tau^h, u_{\varepsilon n}^h - s) + I(\sigma_\varepsilon^h, u_{\varepsilon n}^h - s) \\ &\le (\sigma, v_n^h - u_n) + (\sigma - \tau^h, u_n - u_{\varepsilon n}^h) + (\tau^h - \sigma, u_n - s) \\ &\quad - E_I(\tau^h, u_{\varepsilon n}^h - s) + I(\sigma_\varepsilon^h - \tau^h, (u_{\varepsilon h}^h - s)^+). \end{aligned}$$

Then the estimate (6.1) follows from

$$I(\sigma_\varepsilon^h - \tau^h, (u_{\varepsilon n}^h - s)^+) = I(\sigma_\varepsilon^h - \tau^h, -\varepsilon\sigma_\varepsilon^h) \le -I(\sigma_\varepsilon^h - \tau^h, \tau^h)\varepsilon.$$

The first four terms of the right-hand side of (6.1) are the error of the interpolations by finite-element methods. The fifth and sixth terms are errors of the numerical integration method I. The last one is part of the penalty approximation.

Let the approximation Λ_h of the trace operator $v \to v_n$ be defined by

$$(\tau^h, \Lambda_h(v^h)) = I(\tau^h, v_n^h). \tag{6.2}$$

THEOREM 6.2. Suppose that conditions of Theorem 6.1 hold. Further suppose that there exist an element $v^h \in V_h$ and positive number α_h such that

$$\Lambda_h(v^h) = \tau^h, \qquad \alpha_h \|v^h\|_1 \le |||\tau^h||| \tag{6.3}$$

for every $\tau^h \in \mathrm{Rg}(\Lambda_h)$. Then

$$I(\tau^h - \sigma_\varepsilon^h, \tau^h - \sigma_\varepsilon^h) \le M\|u - u_\varepsilon^h\|_1 |||\tau^h - \sigma_\varepsilon^h|||/\alpha_h + |||\tau^h - \sigma||| \, |||\tau^h - \sigma_\varepsilon^h||| - E_I(\tau^h, \tau^h - \sigma_\varepsilon^h) \tag{6.4}$$

for every $\tau^h \in N_h$.

Proof. From (4.4) and (5.2),

$$\begin{aligned} I(\tau^h - \sigma_\varepsilon^h, v_n^h) &= I(\sigma - \sigma_\varepsilon^h, v_n^h) + I(\tau^h - \sigma, v_n^h) \\ &= (\sigma, v_n^h) - I(\sigma_\varepsilon^h, v_n^h) + I(\sigma, v_n^h) - (\sigma, v_n^h) + I(\tau^h - \sigma, v_n^h) \\ &= B(u - u_\varepsilon^h, v^h) + I(\tau^h - \sigma, v_n^h) + I(\sigma, v_n^h) - (\sigma, v_n^h) \\ &= B(u - u_\varepsilon^h, v^h) + (\tau^h - \sigma, v_n^h) + E_I(\tau^h, v_n^h). \end{aligned}$$

Applying (6.3) yields

$$I(\tau^h - \sigma_\varepsilon^h, \tau^h - \sigma_\varepsilon^h) \le M\|u - u_\varepsilon^h\|_1 |||\tau^h - \sigma_\varepsilon^h|||/\alpha_h + (\tau^h - \sigma, \tau^h - \sigma_\varepsilon^h) + E_I(\tau^h, v_n^h).$$

Then the estimate (6.4) follows by applying Schwartz' inequality to the second term of the right-hand side of the above inequality.

Remark 6.1. Similar estimates have been derived by using a different methodology for obtaining errors of the finite-element approximations in Oden, Kikuchi, and Song [21]. There, errors of $\|u - u_\varepsilon^h\|_1$ and $|||\sigma - \sigma_\varepsilon^h|||$ are obtained through a mixed finite-element method.

It is an easy task to obtain a priori error estimates for $\|u - u_\varepsilon^h\|_1$ and $|||\sigma - \sigma_\varepsilon^h|||$ from (6.1) and (6.4), using results of numerical integration, interpolation properties of finite-element methods, and assumptions on the regularity of the solution. We will give an example of this.

Example 6.1. For the numerical integration I, we suppose that

$$I(v_n^h, v_n^h) \ge C_1 |||v_n^h|||^2, \qquad I(\tau^h, v_n^h) \le C_2 |||\tau^h||| \, |||v_n^h||| \tag{6.5}$$

and

$$|E_I(\tau^h, v_n^h)| \le C_3 h^{\lambda_1}, \qquad |E_I(\tau^h, \hat{\tau}^h)| \le C_4 h^{\lambda_2} \tag{6.6}$$

for every $v^h \in V_h$, $\tau^h \in W_h$, and $\hat{\tau}^h \in W_h$. Let the finite-element model satisfy the following interpolation properties:

$$v^I \in K_h\colon \|v - v^I\|_r \le C_5 h^{\mu_1} \|v\|_s, \qquad \tau^I \in N_h\colon \|\tau - \tau^I\|_{g,\Gamma_C} \le C_6 h^{\mu_2} \|\tau\|_{m,\Gamma_C} \tag{6.7}$$

for every $v \in H^s(\Omega)$ and $\tau \in H^m(\Gamma_C)$, where

$$\begin{aligned} \mu_1 &= \min\{k + 1 - r, s - r\}, \qquad & r \le s, \\ \mu_2 &= \min\{\hat{k} + 0.5 - g, m - g\}, \qquad & g \le m, \end{aligned} \tag{6.8}$$

and $(k, \hat{k})$ is the order of complete polynomials included in interpolations of v and τ, respectively. The numbers r and g may be negative, as shown in Babuska and Aziz [22, p. 95].

Suppose that

$$u \in H^s(\Omega), \qquad \sigma \in H^{s-1.5}(\Gamma_C) \tag{6.9}$$

under the assumption that

$$s \in H^{s-0.5}(\Gamma_C). \tag{6.10}$$

Further suppose that the constant α_h in condition (6.3) is represented by

$$\alpha_h = \alpha h^{\delta}, \qquad \delta \in R. \tag{6.11}$$

Under the above assumptions, the estimate (6.1) becomes

$$\begin{aligned} m\|u - u_\varepsilon^h\|_1^2 \le{}& M\|u - u_\varepsilon^h\|_1 \|u - v^h\|_1 + \|\sigma\|_{s-1.5,\Gamma_C} \|v_n^h - u_u\|_{1.5-s,\Gamma_C} \\ &+ \|\sigma - \tau^h\|_{-0.5,\Gamma_C} \|u_n - u_{\varepsilon n}^h\|_{0.5,\Gamma_C} + \|\tau^h - \sigma\|_{0.5-s,\Gamma_C} \|u_n - s\|_{s-0.5,\Gamma_C} \\ &+ C_3 h^{\lambda_1} + |||\tau^h - \sigma_\varepsilon^h|||\, |||\tau^h|||\varepsilon. \end{aligned}$$

Applying Young's inequality, (6.6) and (6.7), we can obtain that

$$\|u - u_\varepsilon^h\|_1^2 \le C(\|u\|_s, \|\sigma\|_{s-1.5,\Gamma_C}) h^{2\gamma_1} + |||\tau^h - \sigma_\varepsilon^h|||\varepsilon + C_3 h^{\lambda_1}, \tag{6.12}$$

where $\gamma_1 = \min\{k, \hat{k} + 1, s - 1\}$. Similarly, from (6.4),

$$C_1 |||\tau^h - \sigma_\varepsilon^h|||^2 \le M\|u - u_\varepsilon^h\|_1 |||\tau^h - \sigma_\varepsilon^h|||/\alpha_h + |||\tau^h - \sigma|||\, |||\tau^h - \sigma_\varepsilon^h||| + C_4 h^{\lambda_2}.$$

Applying Young's inequality and others, we obtain

$$|||\tau^h - \sigma_\varepsilon^h|||^2 \le C(\|u - u_\varepsilon^h\|_1^2/\alpha_h^2 + |||\tau^h - \sigma|||^2) + C_4 h^{\lambda_2},$$

i.e.

$$|||\tau^h - \sigma_\varepsilon^h||| \le C(\|u - u_\varepsilon^h\|_1/\alpha_h + |||\tau^h - \sigma|||) + (C_4 \lambda^{\lambda_2})^{1/2}. \tag{6.13}$$

Combining (6.12) and (6.13) yields

$$\|u - u_\varepsilon^h\|_1^2 \le C(\|u\|_s, \|\sigma\|_{s-1.5,\Gamma_C})(h^{2\gamma_1} + h^{\gamma_2}\varepsilon) + C_3 h^{\lambda_1} + C_4^{1/2} h^{\lambda_2/2}\varepsilon, \tag{6.14}$$

where $\gamma_2 = \min\{-\delta, s - 1.5\}$. Substitution of (6.14) into (6.13) implies

$$\begin{aligned} |||\tau^h - \sigma_\varepsilon^h||| \le{}& \hat{C}(\|u\|_s, \|\sigma\|_{s-1.5,\Gamma_C})(h^{\gamma_3} + h^{\gamma/2-\delta}\varepsilon^{1/2}) \\ &+ C_3 h^{\lambda_1/2-\delta} + C_4^{1/4} h^{\lambda_2/4-\delta}\varepsilon^{1/2} \end{aligned}$$

where $\gamma_3 = \min\{k - \delta, \hat{k} + 1 - \delta, s - 1 - \delta, s - 1.5\}$. Then it follows from the triangle inequality that

$$\begin{aligned} |||\sigma - \sigma_\varepsilon^h||| \le{}& \hat{C}(\|u\|_s, \|\sigma\|_{s-1.5,\Gamma_C})(h^{\gamma_3} + h^{\gamma_2/2-\delta}\varepsilon^{1/2}) \\ &+ C_3 h^{\lambda_1/2-\delta} + C_4^{1/4} h^{\lambda_2/4-\delta}\varepsilon^{1/2}. \end{aligned} \tag{6.15}$$

The properties of interpolation of finite-element methods (6.7) are defined in restricted sets K_h and N_h instead of V_h and W_h. Thus, we must recall results by Falk [23] as well as general interpolation theories given by Ciarlet [24]. Let Σ_C be a set of nodal points on $\bar{\Gamma}_C$ and let U_h be a finite-element approximation, containing piecewise complete kth-order polynomials, of the Sobolev space $H^s(\Omega)$. Then, following Falk [23], we may assume the following interpolation properties: if $v \in H^s(\Omega)$ satisfies $v - s \le 0$ in the sense of $H^s(\Omega)$, then there is an element $v^h \in U_h$ such that

$$\begin{aligned} &\|v^h - v\|_m \le Ch^{\mu}\|v\|_s, \qquad \mu = \min\{s - m, k + 1 - m\}, \\ &(v - s)(\Sigma_C) \le 0. \end{aligned} \tag{6.7)*}$$

Under this interpolation property, we need to consider conditions (6.5), (6.6) and (6.11) for specific finite elements and numerical integration.

7. Constant α_h for nine-node elements. We shall consider nine-node isoparametric quadrilateral ($Q_2 -$) elements, specifically, in this section and shall obtain the constant α_h, (6.3), for the case when Simpson's rule is used to integrate the penalty term numerically.

If nine-node elements are used for the finite-element approximation, the boundary value of the function $v^h \in V_h$ is a piecewise quadratic polynomials. If Simpson's rule is applied for the numerical integration I, the approximation τ^h of the contact pressure becomes a piecewise quadratic polynomial as well as the boundary value of v^h. Thus the space W_h is spanned by basis functions which are same as boundary traces of basis functions for V_h. Within a finite element related to the boundary Γ_C, the functions v_n^h and $\tau^h \in W_h$ are spanned by

$$\begin{aligned} &N_1(\xi) = \xi(\xi - 1)/2, \\ &N_2(\xi) = \xi(\xi + 1)/2, \qquad \xi \in [-1, 1], \\ &N_3(\xi) = 1 - \xi^2. \end{aligned} \tag{7.1}$$

We first note properties of Simpson's rule for the numerical integration I.

THEOREM 7.1. Let I be Simpson's rule of numerical integration. Then

$$I(v_n^h, v_n^h) \geq |||v_n^h|||^2, \tag{7.2}$$

$$I(\tau^h, v_n^h) \leq 2|||\tau^h|||\,|||v_n^h|||, \tag{7.3}$$

$$|E_I(\tau^h, v_n^h)| \leq C_3 h^4 \|\tau^h\|_{1\cdot 5, \Gamma_C} \|v_n^h\|_{2\cdot 5, \Gamma_C}, \tag{7.4}$$

$$|E_I(\tau^h, \hat{\tau}^h)| \leq C_4 h^3 \|\tau^h\|_{1.5, \Gamma_C} \|\hat{\tau}^h\|_{1.5, \Gamma_C} \tag{7.5}$$

for every $v^h \in V_h$ and $\tau^h, \hat{\tau}^h \in W_h$, where

$$\|\tau^h\|_{s, \Gamma_C} = \left\{ \sum_{e=1}^{E'} \|\tau^h\|_{s, \Gamma_e}^2 \right\}^{1/2},$$

etc., for $s > 1$.

Proof. Note that v_n^h and τ^h are all quadratic polynomials within an element. Let

$$v_n^h = v_1 + v_2\xi + v_3\xi^2,$$
$$\tau^h = \tau_1 + \tau_2\xi + \tau_3\xi^2, \qquad \text{etc.}$$

Then

$$(\tau^h, \tau^h) = \sum_{e=1}^{E'} h(6\tau_1^2 + 2\tau_2^2 + 6\tau_3^2/5 + 4\tau_1\tau_3)/3,$$

$$I(\tau^h, \tau^h) = \sum_{e=1}^{E'} h(6\tau_1^2 + 2\tau_2^2 + 2\tau_3^2 + 4\tau_1\tau_3)/3.$$

Since τ^h and v_n^h have same form, we can easily conclude (7.2) and (7.3). Note that

$$I(\tau^h, v_n^h) \leq \{I(\tau^h, \tau^h)\}^{1/2}\{I(v_n^h, v_n^h)\}^{1/2}.$$

Direct evaluation of the term E_I provides the estimate (7.4). In fact,

$$|E_I(\tau^h, v_n^h)| = \sum_{e=1}^{E'} 4h|\tau_3 v_3|/15 = \sum_{e=1}^{E'} 4h^5|(\tau^h|_{\Gamma_e})''(v_n^h|_{\Gamma_e})''|/15$$
$$\leq (4h^4/15)\|(\tau^h)''\|_{-0.5,\Gamma_C}\|(v_n^h)''\|_{0.5,\Gamma_C} \leq C_3 h^4\|\tau^h\|_{1.5,\Gamma_C}\|v_n^h\|_{2.5,\Gamma_C}$$

where g'' means the second derivative of g in the global coordinates, i.e. d^2g/dx^2. Similarly, we have

$$|E_I(\tau^h, \hat{\tau}^h)| = \sum_{e=1}^{E'} 4h|\tau_3\hat{\tau}_3|/15 \leq (4h^4/15)\|(\tau^h)''\|_{0,\Gamma_C}\|(\hat{\tau}^h)''\|_{0,\Gamma_C}.$$

Applying the inequality

$$\|(\tau^h)''\|_{0,\Gamma_C} \leq \|\tau^h\|_{2,\Gamma_C}$$

and applying the *inverse property* of finite-element approximations

$$\|\tau^h\|_{2,\Gamma_C} \leq Ch^{-1/2}\|\tau^h\|_{1.5,\Gamma_C},$$

we can obtain the estimate (7.5):

$$|E_I(\tau^h, \hat{\tau}^h)| \leq C_4 h^3\|\tau^h\|_{1.5,\Gamma_C}\|\hat{\tau}^h\|_{1.5,\Gamma_C}.$$

Our next objective is to find the "constant" α_h which is essential to the estimates of finite-element approximations.

LEMMA 7.1. If Simpson's rule is applied for numerical integration, there exists an v_n^h, $v^h \in V_h$ and $v_n^h = v^h \cdot n$, such that

$$I(\tau^h, v_n^h) \geq |||\tau^h|||\,|||v_n^h||| \tag{7.6}$$

for every $\tau^h \in W_h$.

Proof. Taking

$$v_n^h(\xi_j^e) = \tau^h(\xi_j^e), \qquad 1 \leq j \leq G, \qquad 1 \leq e \leq E' \tag{7.7}$$

in (5.1) yields

$$I(\tau^h, v_n^h) = \{I(\tau^h, \tau^h)\}^{1/2}\{I(v_n^h, v_n^h)\}^{1/2}.$$

Then (7.6) follows from the results in Theorem 7.1.

LEMMA 7.2. For every $v^h \in V_h$, there exists a constant β such that

$$\beta h^{1/2}\|v^h\|_1 \leq |||v_n^h|||, \tag{7.8}$$

where h is the mesh size of finite elements.

Proof. We recall the trace theorem on $H^1(\Omega)$ and the inverse property of the finite-element approximation:

$$\|v^h\|_1 \leq C\|v_n^h\|_{1/2,\Gamma_C} \tag{7.9}$$

for $v^h \notin \ker(\gamma_n^C)$, and

$$\|v_n^h\|_{1/2,\Gamma_C} \leq \hat{C}h^{-1/2}|||v_n^h|||, \tag{7.10}$$

where $\gamma_n^C: H^1(\Omega) \to H^{1/2}(\Gamma_C)$ is the trace operator defined by

$$\gamma_n^C(v)(\equiv v_n) = v \cdot n \quad \text{on} \quad \Gamma_C$$

if $v \cdot n \in C^\infty(\Gamma_C)$. Then (7.8) follows from (7.9) and (7.10).

Combining these two results, we can obtain the constant α_h

THEOREM 7.2. Let Simpson's rule be applied for the numerical integration I. Then there exist an element $v^h \in V_h$ and a positive constant β such that

$$\Lambda_h(v^h) = \tau^h, \qquad \beta h^{1/2}\|v^h\|_1 \leq |||\tau^h||| \tag{7.11}$$

for every $\tau^h \in Rg(\Lambda_h)$, i.e., $\alpha_h = \beta h^{1/2}$.

Proof. By using the results in Lemma 7.1 and 7.2, it can be proved that

$$v^h \in V_h: (\tau^h, \Lambda_h(v^h)) = I(\tau^h, v_n^h) \geq \beta h^{1/2} |||\tau^h||| \, \|v^h\|_1.$$

On the other hand, (7.3) implies

$$(\tau^h, \Lambda_h(v^h)) = I(\tau^h, \tau^h) \leq 2|||\tau^h|||^2.$$

Thus, we have

$$\beta h^{1/2} |||\tau^h||| \, \|v^h\|_1 \leq |||\tau^h|||^2$$

Let us now apply the result $\alpha_h = \beta h^{1/2}$ to the estimate obtained in Sec. 6.

Example 7.1 (Continuation of Example 6.1). If the following regularity

$$u \in H^3(\Omega) \quad \text{and} \quad \sigma \in H^{3/2}(\Gamma_C) \tag{7.12}$$

is assumed, interpolations v^h and τ^h of u and σ satisfy

$$\|v_n^h\|_{2.5,\Gamma_C} \leq M_1, \qquad \|\tau^h\|_{1.5,\Gamma_C} \leq M_2$$

for positive constants M_1 and M_2 independent of the mesh size h of the finite-element model. Thus, the estimates (6.14) and (6.15) yield

$$\|u - u_\varepsilon^h\|_1 \leq C(h^2 + h^{-1/4}\varepsilon^{1/2}), \qquad |||\sigma - \sigma_\varepsilon^h||| \leq C(h^{3/2} + h^{-3/4}\varepsilon^{1/2}). \tag{7.13}$$

Furthermore, if $\varepsilon = h^{9/2}$, then

$$\|u - u_\varepsilon^h\|_1 \leq Ch^2, \qquad |||\sigma - \sigma_\varepsilon^h||| \leq Ch^{3/2} \tag{7.14}$$

are obtained.

Remark 7.1. As shown in (7.13) and (7.14), the rate of convergence of $\sigma - \sigma_\varepsilon^h$ differs from that of $u - u_\varepsilon^h$ by quantities of the order of magnitude $O(h^{1/2})$. However, it is possible to recover $O(h^{1/2})$-rate of convergence using duality arguments. Indeed,

$$\|\sigma - \sigma_\varepsilon^h\|_{-1/2,\Gamma_C} = \sup_{v \in H^1(\Omega)} \frac{\langle \sigma - \sigma_\varepsilon^h, v_n \rangle}{\|v\|_1},$$

$$\langle \sigma - \sigma_\varepsilon^h, v_n \rangle = \langle \sigma - \sigma_\varepsilon^h, v_n - v_n^h \rangle + B(u - u^h, v^h) - E_I(\sigma_\varepsilon^h, v_n^h), \forall v^h \in V_h.$$

If we assume the interpolation v^I of v in V_h for v^h, then we have

$$\|\sigma - \sigma_\varepsilon^h\|_{-1/2,\Gamma_C} \leq C_1 |||\sigma - \sigma_\varepsilon^h||| h^{1/2} + C_2\|u - u^h\|_1 + C_3 h^2.$$

Here we have used (6.7) and Theorem 7.1. Thus, we have

$$\|\sigma - \sigma_\varepsilon^h\|_{-1/2,\Gamma_C} \leq C(h^2 + h^{-1/4}\varepsilon^{1/2})$$

and if $\varepsilon = h^{9/2}$,

$$\|\sigma - \sigma_\varepsilon^h\|_{-1/2,\Gamma_C} \leq Ch^2.$$

Thus the final estimate of errors of the penalty finite-element approximation for Q_2-elements with Simpson's rule has been obtained as

$$\|u - u_\varepsilon^h\|_1 + \|\sigma - \sigma_\varepsilon^h\|_{-1/2,\Gamma_C} \leq C(h^2 + h^{-1/4}\varepsilon^{1/2})$$

which indicates $O(h^2)$-convergence in the mesh parameter of the finite element model and $O(\varepsilon^{1/2})$-convergence in the penalty. This estimate shows that the approximation (5.2) is of good quality for both penalty and finite-element methods. Note that the penalty part of the estimate include the negative power of the mesh parameter. This means that the penalty parameter cannot be arbitrary in the mesh size h. In order to keep $O(h^2)$-convergence, it suffices to take $\varepsilon = h^{9/2}$.

8. Numerical experiments. Three numerical examples will be solved by the penalty/finite-element approximation discussed in above. The first example is one of Hertz contact problems, and the numerical results will be compared with the Hertz solutions given by Goldsmith [25]. The second example is a rigid punch problem and is designed to check the estimates obtained in Secs. 6 and 7. The last example shows how a two-body contact problem is solved.

Example 8.1. Let an infinitely long circular cylinder rest unilaterally on a flat foundation, and let it be subjected to a uniform line load along the top of the cylinder. Under the condition that the material of cylinder is homogeneous and isotropic, this can be

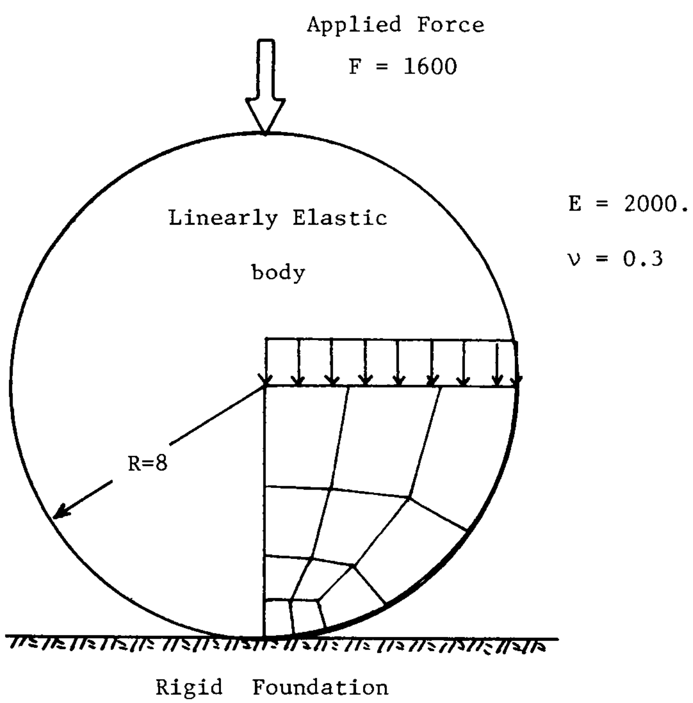

FIG. 8-1(a). A finite-element model of a Hertz problem.

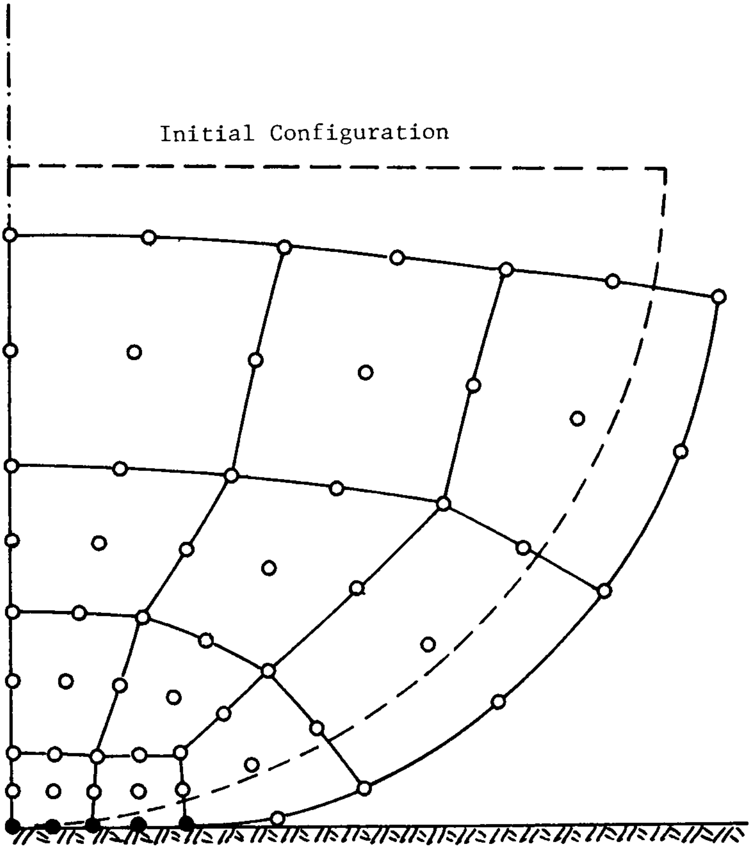

FIG. 8-1(b). Deformed configuration by Q_2-elements and Simpson's rule.

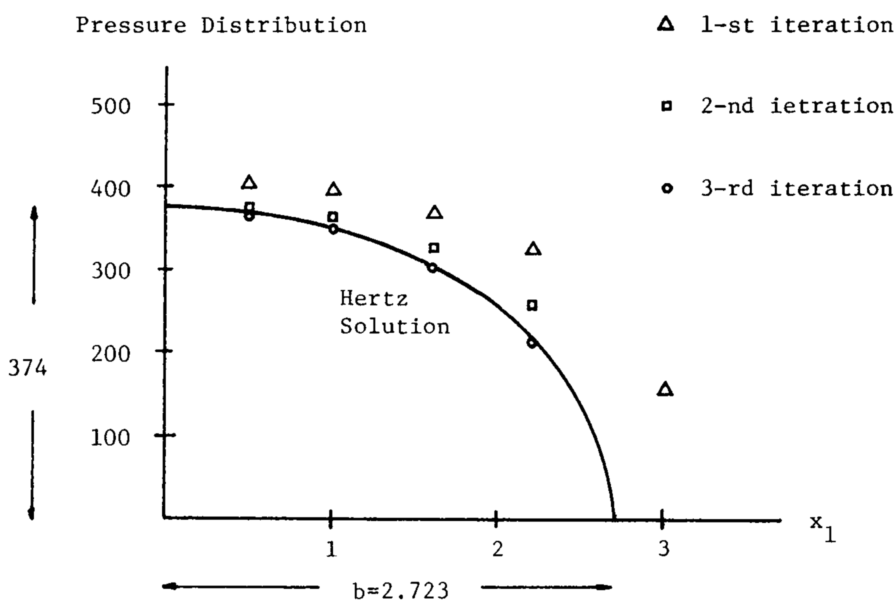

FIG. 8-1(c). Pressure distributions with the Hertz solution.

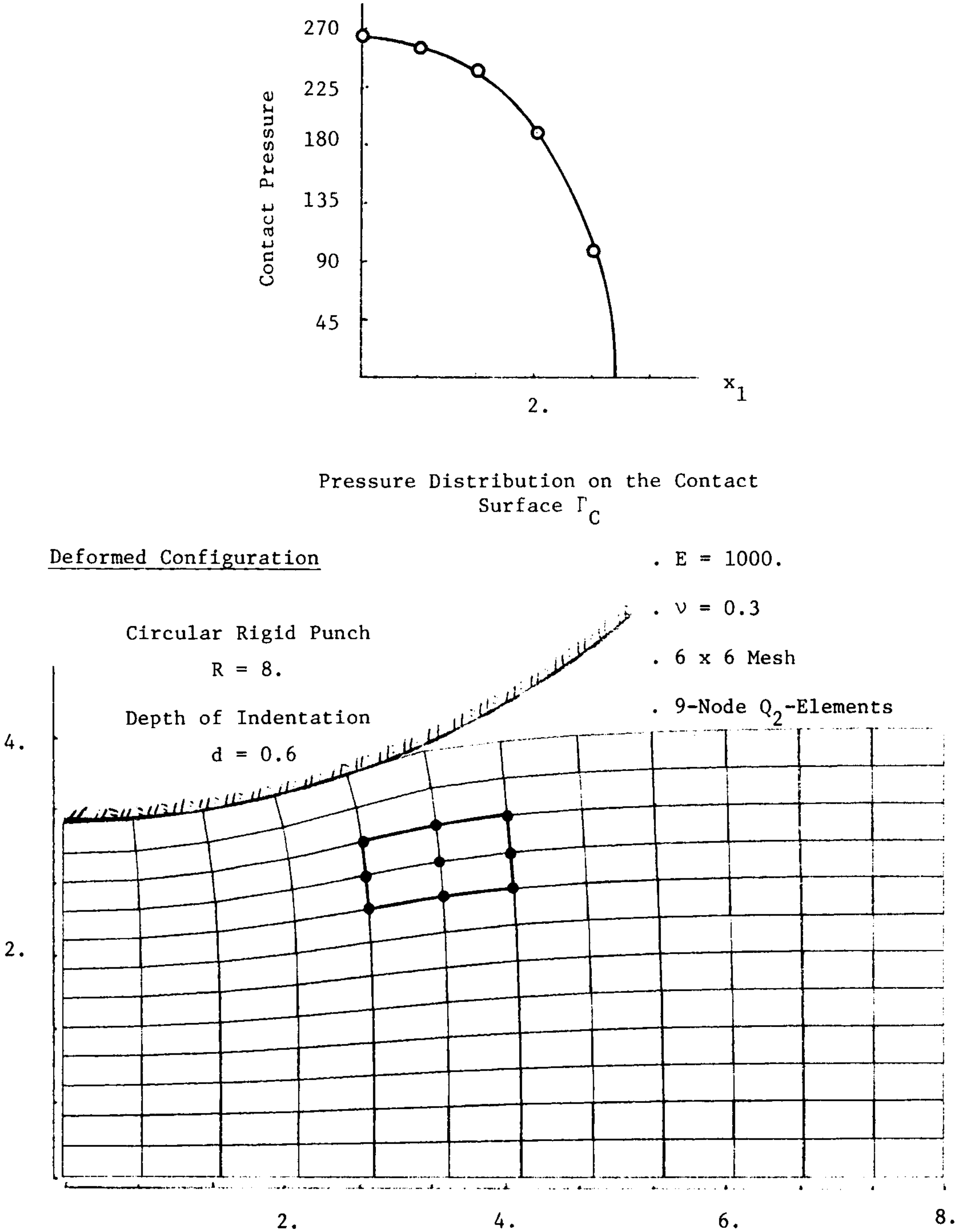

FIG. 8-2(a). A rigid punch problem by Q_2-element and Simpson's rule of numerical integration.

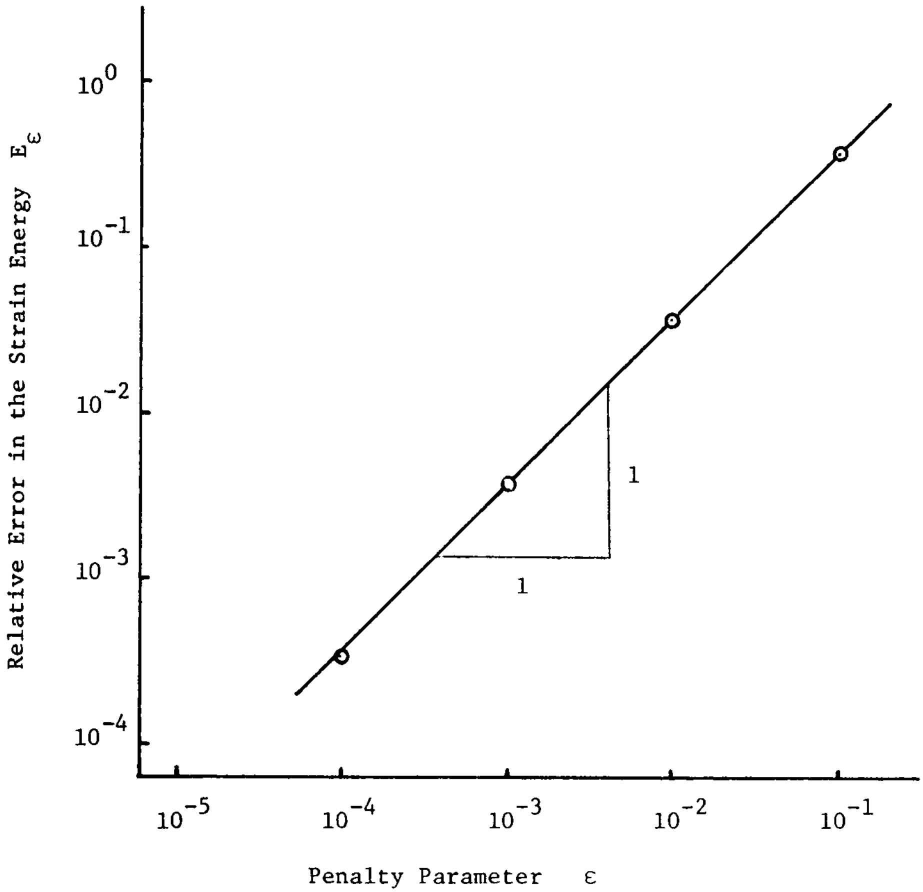

FIG. 8-2(b). Convergence of the penalty method ($\varepsilon \to 0$).

considered as a problem of plane strain. Let the cylinder be characterized by a Young's modulus of $E = 1000$ and a Poisson's ratio of $\nu = 0.3$. Fig. 8-1(a) shows a finite-element model, with its physical dimensions. Here the St. Venant principle is applied so that a quarter of the cross-section of the cylinder is solved.

The deformed configuration of the body is shown in Fig. 8-1(b) with the distribution of the contact pressure and the Hertz solution. It is easily seen that the quality of the numerical solutions is quite nice, even though a rather coarse mesh is used. Convergence of the successive interation (5.7) is obtained at the third iteration. The path of the convergence is shown in Fig. 8-1(c) for the contact pressure.

Example 8.2. Let a rigid circular cylinder be indented into a thick linearly elastic plate which is homogeneous and isotropic (e.g., $E = 1000$ and $\nu = 0.3$). A finite element model, with physical dimensions, is given in Fig. 8-2(a). An objective of this example is to check the convergence of the method described in Sec. 5. Since the exact solutions u and σ are not available, the estimate $\|u - u_\varepsilon^h\|_1$ cannot in general be obtained. We will consider relative errors of the strain energy of the body E_ε and E_h defined by

$$E_\varepsilon = F(u_\varepsilon^h) - F(u_{\bar{\varepsilon}}^h), \qquad E_h = F(u_{\bar{\varepsilon}}^h) - F(u_{\bar{\varepsilon}}^{\bar{h}}),$$

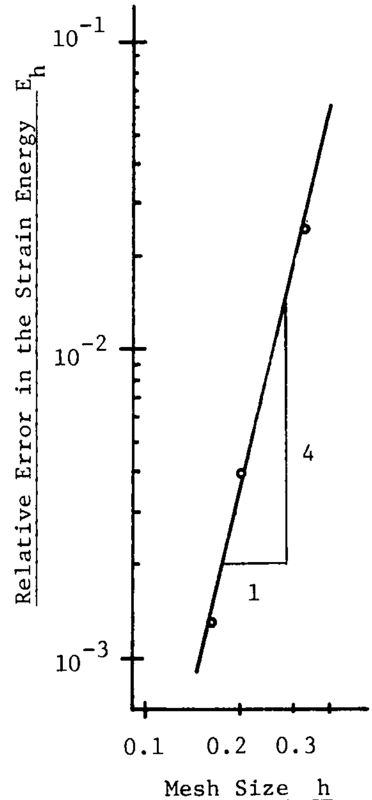

FIG. 8-2(c). Convergence of the finite-element method (Q_2-elements, $h \to 0$).

where " ^ " indicates fixed values, and $F(v) = 1/2B(v, v)$. It is easily shown that if the estimate (7.14) holds, the relative errors E_ε and E_h are asymptotically bounded by

$$E_\varepsilon \leq C_1 \varepsilon + C_2, \qquad E_h \leq C_3 h^4 + C_4 .$$

Numerical results are shown in Fig. 8-2(b) and Fig. 8-2(c). We again observe clear agreement between theoretical and numerical results.

Example 8.3. The method described above can be extended to two-body contact problems without any essential modifications of the theory. For the case of two-body contact problems, the kinematical constraint $(2.1)_1$ of the contact condition is replaced by

$$u_n^1 - u_n^2 \leq s$$

where u_n^i is the normal displacement of the ith body and s is the normal distance of two bodies.

A sample problem is solved numerically: the analysis of an elastic pin-joint problem shown in Fig. 8-3(a). The physical dimensions of the model are also shown in Fig. 8-3(a), and the problem is assumed to be plane strain. The successive iterative algorithm converges at the third iteration with the numerical results shown in Fig. 8-3(b).

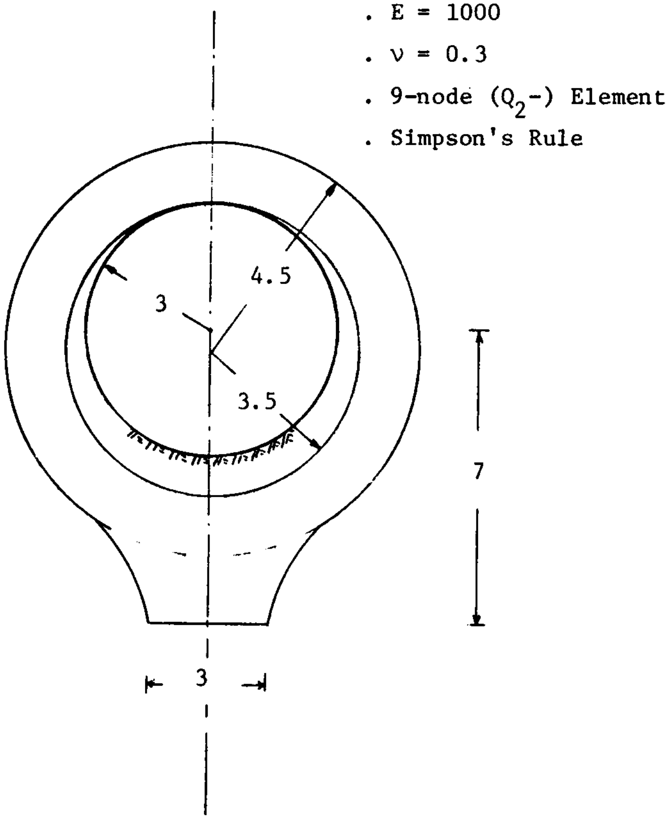

FIG. 8-3(a). A finite-element model of the pin-joint problem.

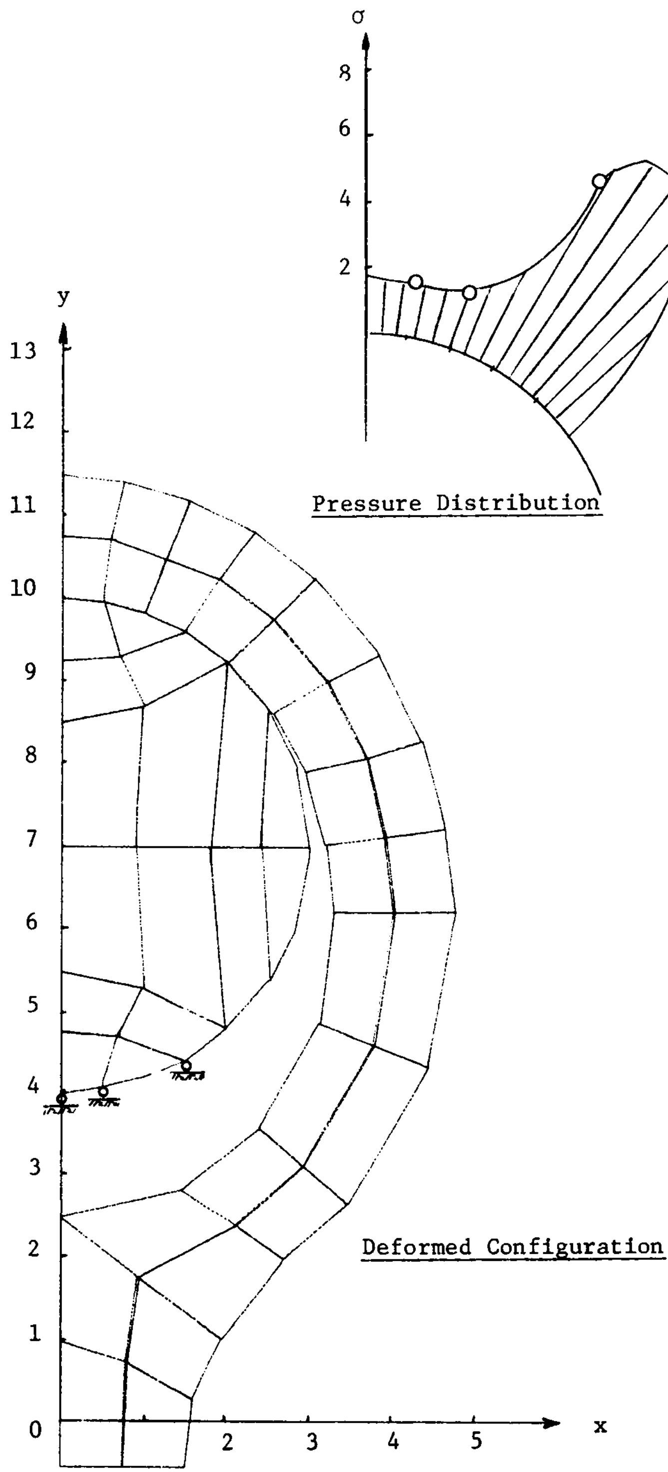

FIG. 8-3(b). Deformed configuration and contact pressure of the pin-joint problem.

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