Percolation in invariant Poisson graphs with i.i.d. degrees

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Abstract

Let each point of a homogeneous Poisson process in $\mathbb{R}^d$ independently be equipped with a random number of stubs (half-edges) according to a given probability distribution μ on the positive integers. We consider translation-invariant schemes for perfectly matching the stubs to obtain a simple graph with degree distribution μ. Leaving aside degenerate cases, we prove that for any μ there exist schemes that give only finite components as well as schemes that give infinite components. For a particular matching scheme that is a natural extension of Gale-Shapley stable marriage, we give sufficient conditions on μ for the absence and presence of infinite components.

1 Introduction

Let $\mathcal{P}$ be a homogeneous Poisson process with intensity 1 on $\mathbb{R}^d$. Furthermore, let μ be a probability measure on the strictly positive integers. We shall study translation-invariant simple random graphs whose vertices are the points of $\mathcal{P}$ and where the degrees of the vertices are i.i.d. with law μ. Deijfen [7] studied moment properties achievable for the edge lengths in such graphs. Here, we shall instead be interested in the percolation-theoretic question of whether the graph contains a component with infinitely many vertices.

We next formally describe the objects that we will work with. For any random point measure Λ we write $[\Lambda] := \{x \in \mathbb{R}^d : \Lambda(\{x\}) > 0\}$ for its support, or point-set. Let ξ be a random integer-valued measure on $\mathbb{R}^d$ with the same support as $\mathcal{P}$, and which, conditional on $\mathcal{P}$, assigns i.i.d. values with

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law μ to the elements of $[\mathcal{P}]$. The pair $(\mathcal{P}, \xi)$ is a marked point process with positive integer-valued marks. For $x \in [\mathcal{P}]$ we write D_x for $\xi(\{x\})$, which we interpret as the number of stubs at vertex x.

A **matching scheme** for a marked process $(\mathcal{P}, \xi)$ is a point process $\mathcal{M}$ on $(\mathbb{R}^d)^2$ with the property that almost surely for every pair $(x, y) \in [\mathcal{M}]$ we have $x, y \in [\mathcal{P}]$, and such that in the graph $G = G(\mathcal{P}, \mathcal{M})$ with vertex set $[\mathcal{P}]$ and edge set $[\mathcal{M}]$, each vertex x has degree D_x. The matching schemes under consideration will always be **simple**, meaning that G has no self-loops and no multiple edges, and **translation-invariant**, meaning that $\mathcal{M}$ is invariant in law under the action of all translations of $\mathbb{R}^d$. We say that a translation-invariant matching is a **factor** if it is a deterministic function of the Poisson process $\mathcal{P}$ and the mark process ξ, that is, if it does not involve any additional randomness. We write $\mathbb{P}$ and $\mathbb{E}$ for probability and expectation on the probability space supporting the random triplet $(\mathcal{P}, \xi, \mathcal{M})$. For later purposes, we note that the notion of a matching scheme generalizes from the Poisson case to general simple point processes.

Let $(\mathcal{P}^*, \xi^*, \mathcal{M}^*)$ be the Palm version of $(\mathcal{P}, \xi, \mathcal{M})$ with respect to $\mathcal{P}$, and write $\mathbb{P}^*$ and $\mathbb{E}^*$ for the associated probability law and expectation operator. Informally speaking, $\mathbb{P}^*$ describes the conditional law of $(\mathcal{P}, \xi, \mathcal{M})$ given that there is a point at the origin, with the mark process and the matching scheme taken as stationary background; see [16, Chapter 11] for more details. Note that since $\mathcal{P}$ is a Possion process, we have $[\mathcal{P}^*] \stackrel{d}{=} [\mathcal{P}] \cup \{0\}$. Let C denote the volume of the component of the origin vertex for $\mathcal{P}^*$, that is, C is the number of vertices that can be reached by a path in $G(\mathcal{P}^*, \mathcal{M}^*)$ from the origin. Our first result states that, excluding trivial cases, on one hand it is always possible to obtain configurations that contain only finite components in a translation-invariant way, while on the other hand infinite components can always be achieved. Furthermore, a connected graph is possible if and only if the expected degree is at least 2.

Theorem 1.1. *Let $\mathcal{P}$ be a Poisson process of intensity 1 in $\mathbb{R}^d$, for any $d \geq 1$, and let D be a random variable with law μ.*

(a) *For any degree distribution μ, there is a simple translation-invariant factor matching scheme such that $\mathbb{P}^*(C < \infty) = 1$.*

(b) *If $\mathbb{P}(D \geq 2) > 0$, then there is a simple translation-invariant factor matching scheme such that G has exactly one infinite component a.s., and furthermore $\mathbb{P}^*(C = \infty \mid D_0 \geq 2) = 1$.*

(c) *The following are equivalent.*

  (i) $\mathbb{E}[D] \geq 2$.

  (ii) *There exists a simple translation-invariant matching scheme for which the graph G is a.s. connected.*

  (iii) *There exists a simple* factor *matching scheme for which the graph G is a.s. connected.*

The implication (ii)⇒(i) of (c) is analogous to various results for percolation on lattices to the extent that the expected degree of vertices in infinite clusters must be at least 2; see, e.g. [11, Theorem 2] and [1, Theorem 6.1].

We move on to consider a particular natural type of matching scheme which in the special case where $\mu(\{1\}) = 1$ (i.e., deterministically one stub per vertex) is known as the **stable matching**. See, e.g., [14]. The natural extension to general degrees, called the **stable multi-matching**, is defined as follows; here and throughout, distance $|x - y|$ and edge length are defined in terms of Euclidean metric on $\mathbb{R}^d$.

Definition 1.1. *A matching scheme $\mathcal{M}$ is said to be a* **stable multi-matching** *if a.s., for any two distinct points $x, y \in [\mathcal{P}]$, either they are linked by an edge or at least one of x and y has no incident edges longer than $|x-y|$.*

We remark that the concept of a stable multi-matching can be defined analogously for general point sets. Here however we will use the term restricted to the specific situation described above. We will see in Proposition 2.2 in Section 2 below that, for any dimension $d \geq 1$ and any μ, there is then a unique stable multi-matching. Our main result on the stable multi-matching is the following, giving sufficient conditions for existence and non-existence of an infinite cluster. It may be noted that the gap between the conditions in (a) and (b) is quite large; see Section 6 for some discussion on this point.

Theorem 1.2. *Consider the stable multi-matching.*

(a) *For any $d \geq 2$, there exists a $k = k(d)$ such that if $\mathbb{P}(D \geq k) = 1$, then $\mathbb{P}^*(C = \infty) > 0$.*

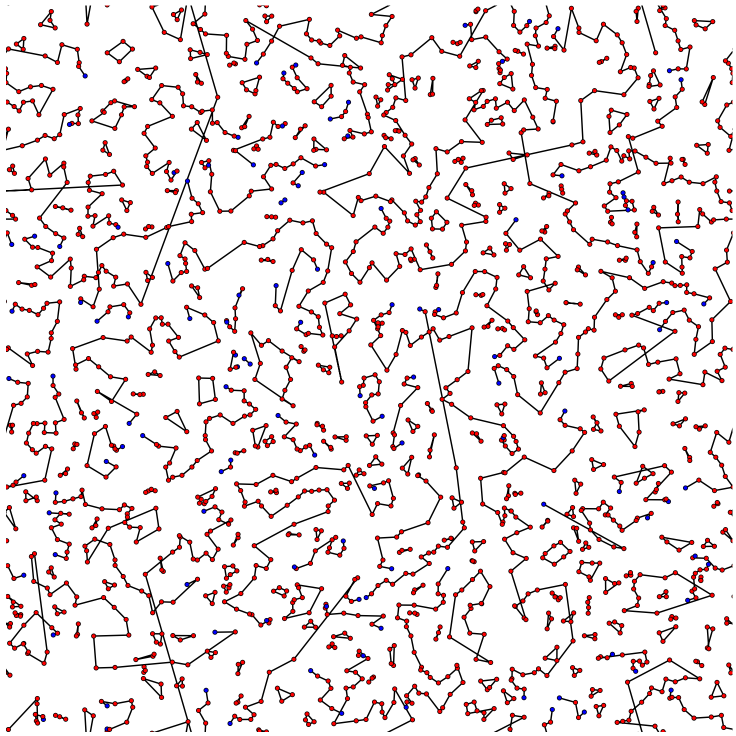
(b) *For any $d \geq 1$ we have that if $\mathbb{P}(D \leq 2) = 1$ and $\mathbb{P}(D = 1) > 0$, then $\mathbb{P}^*(C = \infty) = 0$.*

The rest of this paper is organized as follows. In Section 2 we offer some further background on the model considered here. In Section 3 we prove Theorem 1.1, while in Sections 4 and 5 we prove parts (a) and (b), respectively, of Theorem 1.2. Finally, in Section 6 we briefly mention some open problems and scope for further work.

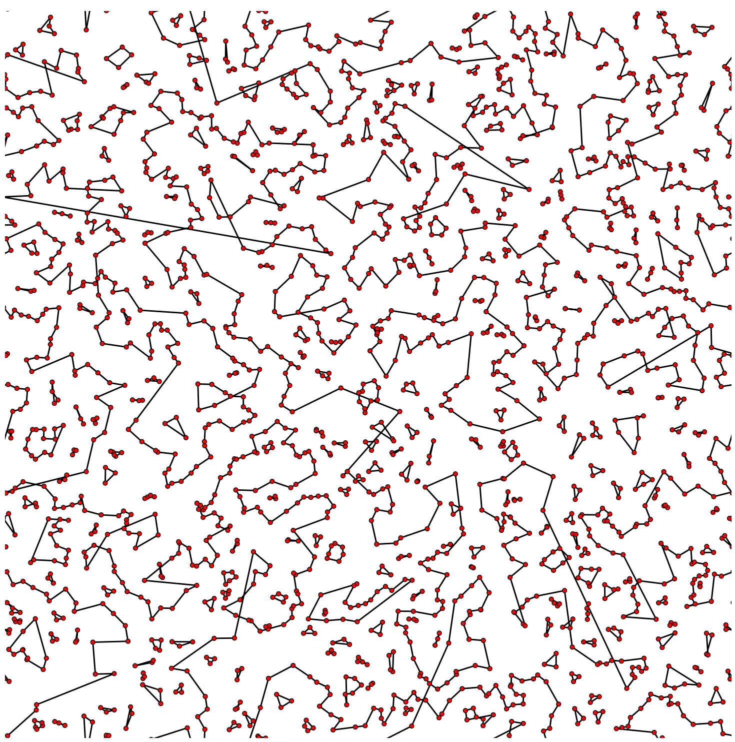
2 Background and preliminaries

2.1 Random graph models with i.i.d. degrees

Random graphs with prescribed degree distribution have been extensively studied in non-spatial settings; see e.g. [2], [3], [4], [5], [19] and [20]. Deijfen and Meester [9] studied the problem of constructing translation-invariant graphs with $\mathbb{Z}$ as vertex set and i.i.d. degrees assigned to the vertices. They focussed on the question of what moment properties on edge lengths are achievable. Deijfen and Jonasson [8] obtained further results in this direction, which

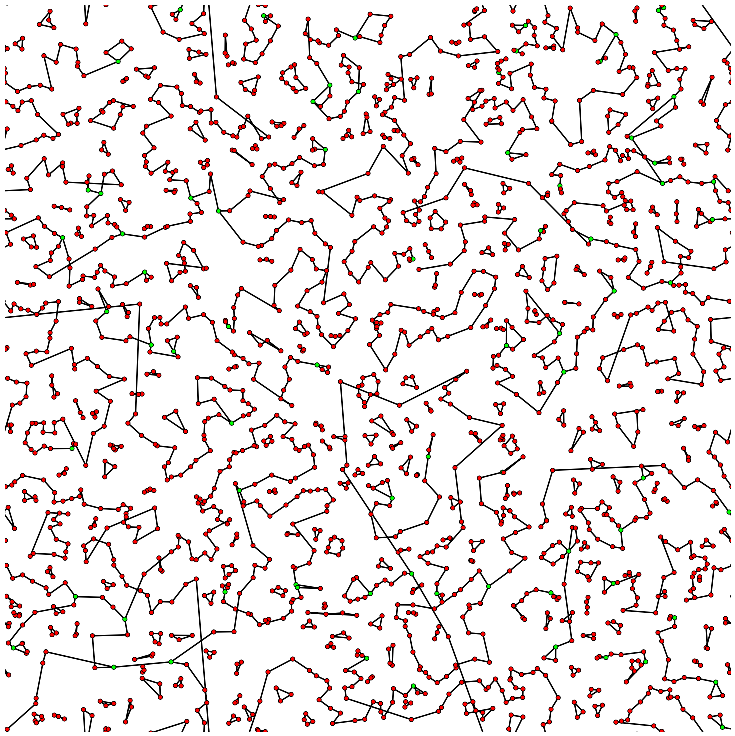


(a) $\mathbb{P}(D = 1) = 1 - \mathbb{P}(D = 2) = 0.05$.

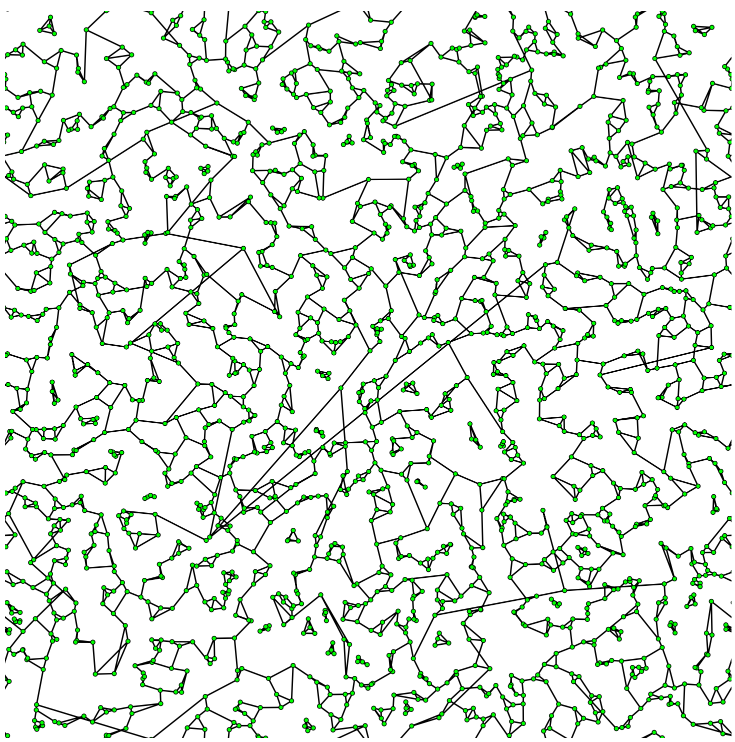


(b) $\mathbb{P}(D = 2) = 1$

Figure 1: Stable multi-matchings on the torus, with given degree distributions.



(a) $\mathbb{P}(D = 3) = 1 - \mathbb{P}(D = 2) = 0.05.$



(b) $\mathbb{P}(D = 3) = 1$

Figure 2: Stable multi-matchings on the torus, with given degree distributions.

Jonasson [15] extended to more general deterministic lattices. Finally Deijfen [7] considered the same problem for the Poisson process $\mathcal{P}$ on $\mathbb{R}^d$, which is the setting we are concerned with here.

2.2 Stable matchings and stable multi-matchings

The concept of stable matchings goes back to Gale and Shapely [10], and has been extensively studied ever since. Holroyd and Peres [14] considered the case of matching Poisson points in $\mathbb{R}^d$, while Holroyd et al. [13] went on to consider bipartite matching of two independent Poisson processes. These last two references provide constructions that will be useful to us in later sections. We isolate the relevant findings in the following result concerning the existence of matchings schemes with constant degree 1 in translation-invariant point processes. The **intensity** of a translation-invariant point process is the expected number of points in a fixed set of unit volume. A set $U \subset \mathbb{R}^d$ is said to be **non-equidistant** if there are no distinct points $x, y, u, v \in U$ with $|x - y| = |u - v|$ or $|x - y| = |y - v|$, while a **descending chain** is an infinite sequence $\{x_i\} \subset U$ such that $|x_i - x_{i-1}|$ is strictly decreasing.

Proposition 2.1 (Existence of matchings)**.**

(a) *Let $\mathcal{R}$ be translation-invariant point processes on $\mathbb{R}^d$ with finite intensity, and assume that a.s. $[\mathcal{R}]$ is non-equidistant and has no descending chains. Then a factor matching scheme for $\mathcal{R}$ with constant degree 1 exists.*

(b) *Let $\mathcal{R}$ and $\mathcal{S}$ be point processes on $\mathbb{R}^d$, jointly ergodic under translations, and with equal finite intensities. Assume that $[\mathcal{R}] \cup [\mathcal{S}]$ is almost surely non-equidistant and has no descending chain. Then there exists a factor matching scheme with constant degree 1 for $[\mathcal{R}] \cup [\mathcal{S}]$, having the property that every point in $[\mathcal{R}]$ is linked to a point in $[\mathcal{S}]$ and vice versa.*

Proof. As an example that proves (a), we can take the stable matching, whose existence and uniqueness is established in [14, pp. 10-11]. For (b) we can take the stable bipartite matching of $\mathcal{R}$ and $\mathcal{S}$ (i.e. the stable matching where two points that are either both in $\mathcal{R}$ or both in $\mathcal{S}$ are postulated to have distance ∞, while the distance between other pairs of points is the usual Euclidean one), whose existence and uniqueness is established in of [13, Proposition 9]. $\square$

We remark at this point that the Poisson process $\mathcal{P}$ satisfies the assumptions of Proposition 2.1 (a), because it satisfies the no descending chains property as first noted in [12]; see also [6].

Moving on to stable multi-matchings, consider the following procedure for matching the stubs of $(\mathcal{P}, \xi)$. In a set of points $S \subset \mathbb{R}^d$, call a distinct pair $x, y \in S$ **mutually closest** if x is the closest point to y in $S \setminus \{y\}$, and vice-versa.

Step 1. Consider the set $[\mathcal{P}]$ of all points. An edge is created between each mutually closest pair in this set, and one stub is removed from each of these points.

Step 2. Consider the set of all points that still have at least one stub after step 1. Two such points are called compatible if no edge was created between them in step 1. An edge is created between each compatible mutually closest pair in this set, and one stub is removed from each of these points.

$$\vdots$$

Step n. Consider the set of all points that still have at least one stub. Two such points are called compatible if no edge has been created between them. An edge is created between each compatible mutually closest pair in this set, and one stub is removed from each of these points.

$$\vdots$$

It is immediate that this procedure will not produce self-loops or multiple edges, and the resulting process is clearly translation-invariant. We will show that a.s. all stubs are eventually matched, and moreover that the resulting graph is the unique stable multi-matching of $(\mathcal{P}, \xi)$.

Proposition 2.2. *Let $(\mathcal{P}, \xi)$ be a marked Poisson process as before. Almost surely, the iterative multi-matching procedure described above exhausts the set of stubs, and the limiting graph (after an infinite number of iterations) is a stable multi-matching. No other stable multi-matching of $(\mathcal{P}, \xi)$ exists.*

Proof. For the case where $\mu(\{1\}) = 1$ this is an application the result from [14] mentioned in the proof of Proposition 2.1 (a). The general case is a straightforward adaptation of their argument, as follows.

Let $\mathcal{P}'$ be the process of points with at least one unmatched stub on them after the above matching procedure is completed. Then $\mathcal{P}'$ is an ergodic point process and hence has either a.s. infinitely many points or a.s. no points. To rule out the former case, call two points in $[\mathcal{P}']$ compatible if they do not have an edge between them in the configuration obtained from the matching procedure. Then create a directed graph G' with $[\mathcal{P}']$ as vertex set by drawing a directed edge from each point to its nearest compatible point (which exists because the initial numbers of stubs were finite). Some thought reveals that G' cannot contain cycles of length more than two, and that each finite component must contain precisely one cycle of length two. However, a cycle of length two is also impossible, since it corresponds to two mutually closest points with no edge between them and an unmatched stub at each point, and between such points an edge would indeed have been created at some stage in the matching procedure. Hence G' has no finite components, and no cycles. This implies that if $[\mathcal{P}']$ is non-empty, then following the outgoing edges starting at any $x \in [\mathcal{P}']$ yields a descending chain. Since descending chains do not exist $\mathcal{P}$ (and hence not in $\mathcal{P}'$), we conclude that $[\mathcal{P}']$ is a.s. empty, as desired.

That the resulting multi-matching is stable follows from the definition: an unstable pair of points would have had an edge created between them at some stage of the matching procedure. That it is the unique matching with this property follows by induction over the stages in the algorithm to show that each edge that is present in the resulting configuration must be present in any stable matching of the stubs. □

Remark 2.1. Note that the given procedure works and proves Proposition 2.2 in the greater generality where the Poisson process $\mathcal{P}$ is replaced by any point process satisfying the requirements of Proposition 2.1.

Remark 2.2. We will later want to apply the given procedure in situations where some pairs of vertices already have an edge between them and additional connections between such vertices are prohibited. Provided the existing edges form a translation-invariant process, the proof of Proposition 2.2 shows that the process still exhausts all remaining stubs, and results in a translation-invariant process.

2.3 Mass transport

The so-called mass transport method in percolation theory was originally developed for the setting of nonamenable lattice (see [1] for background) but has turned out to be a convenient tool also for the more familiar setting of processes living on $\mathbb{Z}^d$ or $\mathbb{R}^d$. Here we formulate a special case adapted to the particular needs of the present paper. We define a **mass transport** to be a random measure T on $(\mathbb{R}^d)^2$ that is invariant in law under translations of $\mathbb{R}^d$, that is, $T(A+x, B+x) \stackrel{d}{=} T(A,B)$ for all Borel $A, B \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$, where we write $T(A,B) := T(A \times B)$. We interpret $T(A,B)$ as the amount of mass transported from A to B. Let Q denote the unit cube $[0,1)^d$.

Lemma 2.1 (Mass Transport Principle)**.** *Let T be a mass transport. Then*

$$\mathbb{E}\, T(Q, \mathbb{R}^d) = \mathbb{E}\, T(\mathbb{R}^d, Q)\,.$$

Proof.

$$\mathbb{E}\, T(Q, \mathbb{R}^d) = \sum_{z \in \mathbb{Z}^d} \mathbb{E}\, T(Q, Q+z) = \sum_{z \in \mathbb{Z}^d} \mathbb{E}\, T(Q-z, Q) = \mathbb{E}\, T(\mathbb{R}^d, Q)\,. \quad \square$$

3 Anything is possible

The task in this section is to prove Theorem 1.1, and we begin with part (a).

Proof of Theorem 1.1 (a). We need to describe a factor matching scheme that gives only finite components. To this end, let $\mathcal{P}_n$ denote the process of points $x \in [\mathcal{P}]$ with $D_x = n$ (recall that D_x denotes the number of stubs attached to x). We will partition $[\mathcal{P}_n]$ into groups of size $n+1$. The configuration is then taken to consist of complete graphs on each of these groups.

Take n such that $\mathcal{P}_n$ is non-empty. To partition $[\mathcal{P}_n]$, we assign each point in $[\mathcal{P}_n]$ a type $i \in \{1, \dots, n+1\}$ as follows. Let R^* be the distance from the origin to the closest other point in the Palm version of $\mathcal{P}_n$ and let $0 = r_0, r_1, \dots, r_n, r_{n+1} = \infty$ be such that

$$\mathbb{P}^*(r_{i-1} < R^* \leq r_i) = \frac{1}{n+1}, \quad i = 1, \dots, n+1. \tag{1}$$

For $x \in [\mathcal{P}_n]$, let R_x denote the distance to the nearest other point in $[\mathcal{P}_n]$. We assign $x \in [\mathcal{P}_n]$ type i if $r_{i-1} < R_x \leq r_i$, and let $\mathcal{P}_n^i$ be the process of points of $\mathcal{P}_n$ of type i. Note that this assignment involves no randomness beyond the Poisson process itself, and that for each given n, the processes $\mathcal{P}_n^1, \dots, \mathcal{P}_n^{n+1}$ have equal intensities and are jointly ergodic under translations. By Proposition 2.1 (b), this means that for each $i = 1, \dots, n$ we can find a matching scheme that matches each type i point to a unique type $i+1$ point and vice versa. The components of the graph obtained by taking the union of these matchings partition $[\mathcal{P}_n]$ into groups of size $n+1$ with one point of each type in each group. □

For the proofs of parts (b) and (c) of Theorem 1.1 the following lemma will be useful.

Lemma 3.1. *For a Poisson process with exactly* 2 *stubs on each point, there exists a factor matching scheme in which G has a single component consisting of a doubly infinite path.*

Proof. This is contained in the proof of [14, Theorem 1]. For expository purposes, let us nevertheless say a few words about how it is proved. The main step is to construct, in a translation-invariant way, a one-ended tree whose vertex set is $[\mathcal{P}]$. Once that is done, the single doubly infinite path is easily constructed from the tree by first ordering the children of each vertex according to distance from the parent, then ordering all vertices according to depth-first search, and finally linking each pair vertices that fall next to each other in this ordering by an edge. □

Remark 3.1. If we relax the requirement in Lemma 3.1 to ask for a union of doubly infinite paths rather than a single infinite path (this will be enough for our proof of Theorem 1.1 (b) but not for the proof of Theorem 1.1 (c)), then the tree construction of [14] can be replaced by the following construction: Define the cone $V = \{y \in \mathbb{R}^d : y_1 \geq |(y_2, \dots, y_d)|\}$, where $y = (y_1, \dots, y_d)$, and, for $x \in [\mathcal{P}]$, put a directed edge to the (almost surely unique) point in

$(x+V)\cap[\mathcal{P}]$ whose first coordinate is minimal among all points in $(x+V)\cap[\mathcal{P}]$. The resulting graph is clearly a forest and it is shown in [14, pp. 10-11] that the trees are indeed one-ended. Directed infinite paths can then be created from each tree as in the proof of Lemma 3.1.

Proof of Theorem 1.1 (b). It is sufficient to provide a factor matching scheme where all vertices of degree at least 2 belong to a single infinite component.

To match the stubs in the Poisson configuration in such a way that an infinite component is obtained we proceed as follows. First consider all vertices of degree at least 2 and create in a translation-invariant and deterministic way a directed infinite path that contains all of them; this is possible by Lemma 3.1 (or, if we opt for a union of infinite paths which is sufficient for the existence but not for the uniqueness of the infinite component, by the more elementary result in Remark 3.1). When this is done we are left with a reduced stub configuration. This is then matched up using the stable multi-matching described prior to Proposition 2.2 with the obvious modification that we do not allow connections between points that already have an edge between them arising from the connections along the paths. Proposition 2.2 along with Remark 2.2 guarantee that this indeed leads to a multi-matching. $\square$

For the proof of Theorem 1.1 (c), one more lemma – a generalization of Proposition 2.1 (b) – will be convenient.

Lemma 3.2. *Let ν be a probability measure on the strictly positive integers and let X be a random variable with law ν. Let $\mathcal{R}$ and $\mathcal{S}$ be translation-invariant point processes on $\mathbb{R}^d$, jointly ergodic under translations, and with finite intensities $\lambda_{\mathcal{R}}$ and $\lambda_{\mathcal{S}}$ satisfying*

$$\lambda_{\mathcal{R}} \le \mathbb{E}[X]\lambda_{\mathcal{S}}\,. \tag{2}$$

Assign degree 1 to each point in $\mathcal{R}$ and assign i.i.d. degrees with law ν to the points in $\mathcal{S}$. If $[\mathcal{R}]\cup[\mathcal{S}]$ is almost surely non-equidistant and has no descending chains, then there exists a translation-invariant partial matching scheme, a deterministic function of $(\mathcal{R},\mathcal{S})$, that matches each point in $[\mathcal{R}]$ to a stub in $[\mathcal{S}]$. If (2) holds with equality, then the procedure also exhausts all stubs in $[\mathcal{S}]$.

Proof. Define

$$m = \inf\left\{ j : \sum_{i=1}^{j} \mathbb{P}(X \ge i)\lambda_{\mathcal{S}} \ge \lambda_{\mathcal{R}} \right\}$$

(with $m=\infty$ if (2) holds with equality) and, if $m \ge 2$, let

$$p_i = \frac{\mathbb{P}(X \ge i)\lambda_{\mathcal{S}}}{\lambda_{\mathcal{R}}}, \quad i = 1,\dots,m-1,$$

and if $m<\infty$ let $p_m = 1-\sum_{i=1}^{m-1} p_i$. If $m=1$, just let $p_1=1$. As in the proof of Theorem 1.1 (a), we let R^* be the distance from the origin to the closest other point in the Palm version of $\mathcal{R}$ and, analogously to (1), we define real numbers $0 < r_1 < \cdots < r_{m-1}$ such that

$$\begin{array}{lcl} \mathbb{P}^*(R^* \leq r_1) & = & p_1 \\ \mathbb{P}^*(r_1 < R^* \leq r_2) & = & p_2 \\ & \vdots & \\ \mathbb{P}^*(r_{m-2} < R^* \leq r_{m-1}) & = & p_{m-1} \\ \mathbb{P}^*(R^* > r_{m-1}) & = & p_m. \end{array}$$

For $x \in [\mathcal{R}]$, say that x is of type $i \in \{1, \ldots, m-1\}$ if r_i is the first number in the ordered sequence $r_1 < \cdots < r_{m-1}$ that exceeds the distance from x to the nearest other point in $[\mathcal{R}]$, and of type m if the distance from x to the nearest other point in $[\mathcal{R}]$ is larger than r_{m-1}. This divides the process $\mathcal{R}$ into processes $\mathcal{R}_i$ ($i = 1, \ldots, m$) with intensities $\lambda_{\mathcal{R}} p_i$. Write $\mathcal{S}_i$ for the process of vertices in $\mathcal{S}$ with degree at least i. By the choice of p_i, for $i = 1, \ldots, m-1$, the intensity of $\mathcal{R}_i$ coincides with the intensity of $\mathcal{S}_i$, and condition (2) implies that the intensity of $\mathcal{R}_m$ is at most by the intensity of $\mathcal{S}_m$ (indeed, the intensity of $\mathcal{R}_m$ is $\lambda_{\mathcal{R}} - \sum_{i=1}^{m-1} \mathbb{P}(X \geq i)\lambda_{\mathcal{S}}$ which does not exceed $P(X \geq m)\lambda_{\mathcal{S}}$ by the choice of m).

Now, for each $i = 1, \ldots, m$, match the points of $\mathcal{R}_i$ to the points of $\mathcal{S}_i$ using Proposition 2.1 (b). For $i = 1, \ldots, m-1$ we get a perfect matching of the points (since the intensities of the processes coincide) and, for $i = m$, it is not difficult to see that all points in $\mathcal{R}_m$ get matched up (while some points in $\mathcal{S}_m$ may not be used). $\square$

Proof of Theorem 1.1 (c). To show that conditions (i), (ii) and (iii) are equivalent, it suffices to show that (iii)⇒(ii)⇒(i)⇒(iii). Since (iii)⇒(ii) is trivial, we only need to show that (i)⇒(iii) and that (ii)⇒(i).

To prove (i)⇒(iii), first note that the matching scheme described in the proof of Theorem 1.1 (b) gives a connected graph as soon as all vertices have degree at least 2 (provided we use the construction in Lemma 3.1 rather than the one in Remark 3.1). To handle the case where $\mathbb{P}(D = 1) > 0$, let X be a random variable distributed as $D - 2$ conditional on that $D \geq 3$ and note that $\mathbb{E}[D] \geq 2$ implies that

$$\mathbb{P}(D = 1) \leq \mathbb{E}[X]\mathbb{P}(D \geq 3) \tag{3}$$

with equality if and only if $\mathbb{E}[D] = 2$.

Consider first the case $\mathbb{E}[D] = 2$. For a matching scheme here, first employ the scheme in the proof of Theorem 1.1 (b) in order to connect up all points in $[\mathcal{R}]$ that are assigned degree 2 or more into an infinite path. This leaves the points that are assigned degree 1, plus the points initially assigned degree 3 or more, each having two of their stubs already matched. Since (3) holds with equality, Lemma 3.2 is exactly tailored to produce a factor matching of the degree 1 points to the unmatched stubs of the degree ≥ 3 points. This gives a connected graph, so the case $\mathbb{E}[D] = 2$ is settled.

For the case $\mathbb{E}[D] > 2$ we proceed as with $\mathbb{E}[D] = 2$ by first constructing the infinite path and then connecting up degree 1 points to it by the scheme offered in the proof of Lemma 3.2. This time, however, the latter scheme, although resulting in a connected graph, fails to use up all the stubs of the degree ≥ 3 points. These remaining stubs can be hooked up to each other by the stable multi matching scheme described prior to Proposition 2.2 with the restriction that we do not allow connections between points that already have an edge between them on the infinite path. Using Remark 2.2, this completes the proof of the (i)$\Rightarrow$(iii) implication.

To prove (ii)$\Rightarrow$(i) we employ a mass-transport argument. Assume (ii), and let $\mathcal{M}$ be a matching scheme that produces a connected graph. Consider the mass transport where each point $x \in [\mathcal{P}]$ sends a unit mass to $y \in [\mathcal{P}]$ if and only if x and y are connected by an edge, and removing that edge would leave x in a finite component. Note that the mass M_x^{out} sent from x cannot exceed 1. We claim that, for any vertex x,

$$D_x - 2 \geq M_x^{\text{in}} - M_x^{\text{out}} . \tag{4}$$

This follows by considering separately the two cases $M_x^{\text{out}} = 1$ and $M_x^{\text{out}} = 0$. When $M_x^{\text{out}} = 1$ we get $M_x^{\text{in}} = D_x - 1$ and (4) holds with equality. When $M_x^{\text{out}} = 0$ we have that x is connected to infinity via at least two edges adjacent to it, which implies that $M_x^{\text{in}} \leq D_x - 2$, and again (4) holds.

By the mass transport principle (Lemma 2.1), the expectation of the right-hand side of (4) summed over all Poisson points x in the unit cube Q is 0. But the expectation of $D_x - 2$ summed over all Poisson points in the unit cube Q is simply $\mathbb{E}[D] - 2$ (because the Poisson process has intensity 1), so (4) implies $\mathbb{E}[D] \geq 2$, as desired. $\square$

4 Percolation in the stable multi-matching

In this section we prove Theorem 1.2 (a), that is, we show that if each Poisson point has sufficiently many stubs attached to it, then the edge configuration resulting from the stable multi-matching percolates. The proof uses the result from [18] concerning domination of r**-dependent** random fields by product measures, where a random field $\{X_z\}_{z\in\mathbb{Z}^d}$ is said to be r-dependent if for any two sets $A, B \in \mathbb{Z}^d$ at l_∞-distance at least r from each other we have that $\{X_z\}_{z\in A}$ is independent of $\{X_z\}_{z\in B}$. The version we need is as follows.

Theorem 4.1 (Liggett, Schonmann & Stacey (1997))**.** *For each $d \geq 2$ and $r \geq 1$ there exists a $p_c = p_c(d, r) < 1$ such that the following holds. For any r-dependent random field $(X_z)_{z\in\mathbb{Z}^d}$ satisfying $\mathbb{P}(X_z = 1) = 1 - \mathbb{P}(X_z = 0) \geq p$ with $p > p_c$, the 1's in $(X_z)_{z\in\mathbb{Z}^d}$ percolate almost surely.*

Proof of Theorem 1.2 (a). The idea of the proof is a renormalization procedure. We partition $\mathbb{R}^d$ into cubes and declare a cube to be **good** if it contains at least one but not too many Poisson points and if the same holds for all cubes close to it, where "too many" and "close" will be specified below. We then use Theorem 4.1 to deduce that the good cubes can be made to percolate, and we observe that, if each Poisson point has sufficiently many stubs attached to it, then each point in a good cube must be connected to each point in its adjacent cubes in the stable multi-matching. This forces the existence of an infinite component in the stable multi-matching.

To make this reasoning more precise, for $a \in \mathbb{R}$, let $a\mathbb{Z}^d = \{az : z \in \mathbb{Z}^d\}$ and partition $\mathbb{R}^d$ into cubes $\{C_{az}\}_{z\in\mathbb{Z}^d}$ centered at the points of $a\mathbb{Z}^d$ and with side a. Two cubes C_{az} and C_{ay} are called adjacent if $|z-y| = 1$, and we write $m = m(d)$ for the smallest integer such that the maximal possible Euclidean distance between points in adjacent cubes does not exceed ma. For each cube C_{az} a super-cube S_{az} is defined, consisting of the cube itself along with all cubes C_{ay} with y at l_∞-distance at most $2m$ from z. A super-cube hence contains $(4m+1)^d$ cubes.

Now, call a cube C_{az} **acceptable** if it contains at least one and at most $n = n(d)$ Poisson points, where n will be specified below, and it is **good** if all cubes in S_{az} are acceptable. We have that

$$\mathbb{P}(C_{az} \text{ is acceptable}) = 1 - \mathbb{P}[\mathcal{P}(C_{az}) = 0] - \mathbb{P}[\mathcal{P}(C_{az}) > n].$$

The first probability on the right side can be made arbitrarily small by taking a large, and, for a fixed a, the second probability can be made arbitrarily small by taking n large. Hence, by choosing first a large and then n very large, we can make the probability that a cube is good come arbitrarily close to 1. In particular, by Theorem 4.1, we can make it large enough to guarantee that the good cubes percolate. Fix such values of a and n, let $k = n(4m+1)^d$ and assume that $\mathbb{P}(D \geq k) = 1$. We will show that then each Poisson point in a good cube is connected to all Poisson points in the adjacent cubes, which completes the proof.

Say that a point $x \in [\mathcal{P}]$ with D_x stubs **desires** a point $y \in [\mathcal{P}]$ if y is one of the D_x nearest points to x in $[\mathcal{P}]$. Then a Poisson point x in a good cube C_{az} desires all points in the adjacent cubes: For any point y in an adjacent cube, the distance between x and y is at most ma, and the Euclidean ball $B_{ma}(x)$ with radius ma centered at x is contained in the supercube S_{az}, which contains at most k Poisson points (indeed, all $(4m+3)^d$ cubes in S_{az} are acceptable, which means that each one of them contains at most n points). Since $D_x \geq k$, it follows that y desires all points in $B_{ma}(x)$, so in particular x desires y. Furthermore, each Poisson point y in a cube that is adjacent to a good cube C_{az} desires each point in the good cube: Since the distance between x and y is at most ma, we have that $B_{ma}(y) \subset B_{2ma}(x)$. Moreover, the ball $B_{2ma}(x)$ is contained in the supercube S_{az}, which contains at most k points.

Hence $B_{ma}(y)$ contains at most k points and, since $D_y \geq k$, it follows that y desires all points in $B_{ma}(y)$.

We have shown that each point in a good cube desires each point in the adjacent cubes and vice versa. All that remains is to note that two points that desire each other will indeed be matched. This follows from the definition of the stable multi-matching. Hence each point in a good cube is connected to each point in the adjacent cubes and since the good cubes percolate this proves part (a) of Theorem 1.2. $\square$

Remark 4.1. An easy modification of the above proof reveals that the following slightly stronger variant of Theorem 1.2 holds. For any $\varepsilon > 0$, there exists $k = k(d, \varepsilon)$ such that if $\mathbb{P}(D \geq k) > \varepsilon$, then $\mathbb{P}^*(C = \infty) > 0$.

5 Non-percolation in the stable multi-matching

In this section we prove Theorem 1.2 (b), that is, we show that if all points have degree at most 2 and there is a positive probability for degree 1, then almost surely the stable multi-matching gives configurations with only finite components. The proof is based on the following lemma.

Lemma 5.1. *In any translation-invariant matching scheme, a.s. G has no component consisting of a singly infinite path.*

Proof. Assume that components consisting of a singly infinite paths occur with positive probability, and consider the mass transport where each Poisson point that sits on a singly infinite path sends mass 1 to the endpoint of the path. With positive probability the unit cube Q contains such an endpoint, and hence the expected mass that is received by Q is infinite. But the expected mass that is sent out from Q is at most 1, because the expected number of Poisson points in Q is 1. This contradicts the mass transport principle (Lemma 2.1). $\square$

Proof of Theorem 1.2 (b). Let the degree distribution be such that $\mathbb{P}(D \leq 2) = 1$ and $\mathbb{P}(D = 1) > 0$. The only infinite components that can occur when $\mathbb{P}(D \leq 2) = 1$ are infinite paths of degree 2 vertices that are connected to each other and, by Lemma 5.1, any such path has to be bi-infinite. Assume for contradiction that such a bi-infinite path occurs with positive probability. We will describe a coupled configuration of vertex degrees, where, with positive probability, the edge configuration remains unchanged except that a doubly infinite path is cut apart and turned into two singly infinite paths. This conflicts with Lemma 5.1.

Given the Poisson process $\mathcal{P}$ with associated degrees $\{D_x\}_{x\in[\mathcal{P}]}$, we now introduce a modified degree process $\{\widetilde{D}_x\}_{x\in[\mathcal{P}]}$. Conditional on $\mathcal{P}$ and $\{D_x\}_{x\in[\mathcal{P}]}$, let $\{\widetilde{D}_x\}_{x\in[\mathcal{P}]}$ be independent random variables chosen as follows. With probability $1 - e^{-|x|}$, we set $\widetilde{D}_x = D_x$, where $|x|$ is the Euclidean distance from

x to the origin. With the remaining probability $e^{-|x|}$, the degree $\widetilde{D}_x$ is taken to be an independent random variable with law μ. The Poisson points that receive a newly generated degree in $\widetilde{D}_x$ are referred to as **re-randomized**. Note, crucially, that $\{\widetilde{D}_x\}_{x\in[\mathcal{P}]}$ has the same distribution as $\{D_x\}_{x\in[\mathcal{P}]}$. Note also that

$$\mathbb{E}\sum_{x\in[\mathcal{P}]}\mathbf{1}[x \text{ re-randomized}] = \int_{\mathbb{R}^d} e^{-|x|}dx < \infty\,.$$

Hence, by the Borel-Cantelli lemma, the number of re-randomized points is finite almost surely.

Now, take a configuration of Poisson points and associated degrees for which the graph G resulting from the stable multi-matching contains some bi-infinite path. Let $\widetilde{G}$ be the graph resulting from the modified degrees $\{\widetilde{D}_x\}_{x\in[\mathcal{P}]}$. Along each such path $(x_i)_{i=-\infty}^{\infty}$, there must be an edge that is locally maximal, that is, an edge (x_i, x_{i+1}) with $|x_{i+1}-x_i| > \max\{|x_i - x_{i-1}|, |x_{i+2}-x_{i+1}|\}$. To see this, note that if such an edge did not exist, the vertices of the path would either constitute a descending chain, or contain a single locally minimal edge (defined in obvious analogy with locally maximal). Descending chains do not occur in Poisson processes (as noted in Section 2.2) while chains with a single minimal edge are ruled out by a mass transport argument similar to the one on the proof of Lemma 5.1 (let each vertex on such a path send unit mass to the midpoint of the unique invariant edge). Let (x_m, x_{m+1}) be a locally maximal edge – say the one with a vertex closest to the origin. Write A for the event that x_m and x_{m+1} are the only two vertices that are re-randomized and that $\widetilde{D}_{x_m} = \widetilde{D}_{x_{m+1}} = 1$, that is, the degrees of x_m and x_{m+1} are changed to 1's while the rest of the degrees remain unchanged.

We claim that, on A, the modified graph $\widetilde{G}$ consists of the same edges as in the original G except that the edge between (x_m, x_{m+1}) is absent. Indeed, since $|x_{m+1}-x_m| \geq \max\{|x_{m+2}-x_{m+1}|, |x_m - x_{m-1}|\}$, the edge (x_m, x_{m+1}) is created at a later stage in the matching procedure than the edges (x_{m-1}, x_m) and (x_{m+1}, x_{m+2}). On the event A, no stubs have been added or removed in the modified configuration except that one stub is taken away from each of x_m and x_{m+1}. Hence, up until the stage when the edge (x_m, x_{m+1}) was created in the original process, precisely the same edges are created in the modified configuration. At this stage, the vertices x_m and x_{m+1} do not have a stub on them, and so the edge (x_m, x_{m+1}) is not created in the modified configuration. After this stage, the situation is as in the original configuration, and so the same edges are again created.

The above shows that, on A, the modified stable multi-matching for the modified configuration contains two singly infinite paths $(x_{m+1}, x_{m+2}, \ldots)$ and $(x_m, x_{m-1}, \ldots)$. All that remains it to note that, since the number of re-randomized vertices is finite almost surely, the event A has positive probability. We have hence derived a contradiction with Lemma 5.1. $\square$

6 Open problems

Closing the gap in Theorem 1.2. Theorem 1.2 provides conditions for when the stable multi-matching contains an infinite component and for when it consists only of finite components. These conditions however are quite far apart and it would be desirable to obtain a more precise understanding for when the stable multi-matching percolates. Consider for instance the case with exactly two stubs attached to each point, that is, $\mu(\{2\})$. Do infinite components occur in this case? Simulations appear to suggest a positive answer $d = 1$, but are less conclusive in $d = 2$.

Theorem 1.2 (b) states that percolation does not occur when there are only degree 1 and degree 2 vertices. Roughly speaking, this is because the degree 1 vertices serve as dead ends in the configuration. Does this phenomenon persist when a small proportion of degree 3 vertices is added? Does a sufficiently large proportion of degree 1 vertices always guarantee non-percolation?

Finally, if degree distribution μ results in an infinite cluster, and we replace μ by a distribution μ' that stochastically dominates μ, do we still get an infinite cluster?

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