# Percolation of interdependent networks with intersimilarity

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Real data show that interdependent networks usually involve intersimilarity. Intersimilarity means that a pair of interdependent nodes have neighbors in both networks that are also interdependent [Parshani *et al*. Europhys. Lett. **92**, 68002 (2010)]. For example, the coupled worldwide port network and the global airport network are intersimilar since many pairs of linked nodes (neighboring cities), by direct flights and direct shipping lines, exist in both networks. Nodes in both networks in the same city are regarded as interdependent. If two neighboring nodes in one network depend on neighboring nodes in the other network, we call these links common links. The fraction of common links in the system is a measure of intersimilarity. Previous simulation results of Parshani *et al*. suggest that intersimilarity has considerable effects on reducing the cascading failures; however, a theoretical understanding of this effect on the cascading process is currently missing. Here we map the cascading process with intersimilarity to a percolation of networks composed of components of common links and noncommon links. This transforms the percolation of intersimilar system to a regular percolation on a series of subnetworks, which can be solved analytically. We apply our analysis to the case where the network of common links is an Erdős-Rényi (ER) network with the average degree $K$, and the two networks of noncommon links are also ER networks. We show for a fully coupled pair of ER networks, that for any $K \geqslant 0$, although the cascade is reduced with increasing $K$, the phase transition is still discontinuous. Our analysis can be generalized to any kind of interdependent random network systems.

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## I. INTRODUCTION

Single isolated networks have been extensively studied in the past decade [1–14]. Recently, much interest has been devoted to interdependent networks [15–27], which can model some real world catastrophic events, such as the electrical blackout in Italy on 28 September 2003 [28] and the US-Canada power system outage on 14 August 2003 [29]. Failures of a small number of power stations can cause further malfunction of nodes in their communication control network, which in turn leads to the shutdown of power stations [15,28]. This cascading process continues until no more nodes fail due to percolation or due to interdependence failures. In contrast to single networks where the percolation transition is continuous, in interdependent networks the transition is abrupt [15,16].

Real interdependent networks are sometimes coupled according to some intersimilarity features. Intersimilarity means the tendency of neighboring nodes in one network to be interdependent of neighboring nodes in the other network. Such coupled networks are more robust against cascading failures than randomly coupled interdependent networks. To quantify self-similarity, Parshani *et al*. introduced the interclustering coefficient (ICC), which measures the average number of common links per pair of interdependent nodes [30]. Common links are defined as follows: Given two coupled networks $A$ and $B$, and two nodes $a_k$ and $a_l$ which are linked in $A$, if their interdependent counterparts $b_k$ (corresponds to $a_k$) and $b_l$ (corresponds to $a_l$) in $B$ are also linked (in $B$), this pair of links is called a common link. Common links can be interpreted as follows: if two nodes in one network are linked, the tendency (probability) of their interdependent counterparts in the other network to be linked is a measure of the intersimilarity. Thus, the density of the common links reflects the intersimilarity of the two networks. In the extreme case where every pair of links is a common link, the two networks are identical. In this case no cascading failure will occur, since a failure in network $A$ will cause an identical failure in $B$ and there will be no cascading failure feedback to $A$. It is therefore expected that the more common links appear in the coupled networks system, the more robust it becomes. In the example of the coupled worldwide port network and the airline network, illustrated in Fig. 1, the fraction of common links is 0.12 for the port network and 0.18 for the airline network [30]. Therefore, developing a method to analyze cases where certain common topologies exist in the interdependent networks can help to understand the vulnerabilities of coupled complex systems in the real world as well as for designing robust infrastructures.

In this paper we introduce a method to analytically calculate the cascading process of failures in interdependent networks with common links. To analyze this problem, we consider the cluster components of the network composed of only common links after the initial attack. We will illustrate that all nodes in such a component will survive or fail simultaneously during the cascading process. Based on this fundamental feature, we divide the system into subnetworks according to the sizes of the components, and then contract all nodes in each component into a single node. After contraction, the system degenerates

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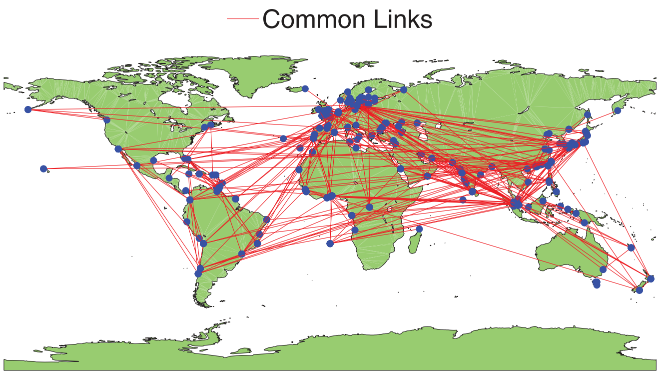


FIG. 1. (Color online) Common links in the interdependent geographic worldwide port network and the global airport network system. In the coupled network system, we identify common links if neighboring nodes (cities with direct transportation lines) in one network are also neighbors in the other network. For clarity, in the figure, 200 nodes are randomly chosen, and only common links attached to these nodes are plotted as solid (red) lines. Cities are shown as (blue) dots. Because two types of nodes (ports and airports) approximately correspond to the same cities, we just use single dots to denote them in the worldwide map.

into two randomly coupled networks without common links, which will be solved analytically. Here we find the exact solution for the case where $A$ and $B$ are fully interdependent Erdős-Rényi (ER) networks each of average degree $k$ and the network of common links is also an ER network with the average degree $K$. In this case we show that the interdependent networks system undergoes a first order transition for all $K \geqslant 0$.

The paper is organized as follows. In Sec. II we introduce the model of cascading failures in interdependent networks with common links. In Sec. III we analyze the cascading process and the final state. In Sec. IV we derive the theoretical solutions using multivariable generating functions. In Sec. V we show the first order transition for both the general case and the special case where common links form an ER network.

## II. THE MODEL

For simplicity and without loss of generality we analyze the percolation process in a system of two fully interdependent networks $A$ and $B$ of the same size $N$ with no-feedback condition [21] in the presence of common links. The no-feedback means that each $A$ node $a_k$ has one and only one dependency counterpart $b_k$ in network $B$, and $b_k$ must depend only on $a_k$. Initially, a fraction $1 - p$ of $A$ nodes are removed randomly. Due to interdependency, a corresponding fraction of $B$ nodes also fail. We denote by $A_0$ and $B_0$ the remaining networks of size $N_0 = pN$. Nodes of $A_0$ and $B_0$ are represented by $a_i$ and $b_i$, respectively, $i = 1, \ldots, N_0$, and $a_i$ interdepends on $b_i$ for all $i = 1, 2, \ldots, N$. The pair $(i,j)$ is a common link if $a_i$ is linked to $a_j$ and $b_i$ is also linked to $b_j$. We introduce a network $C_0$ that includes all $N_0$ nodes but only links that are common links. This means that the nodes $c_i$ and $c_j$ of $C_0$ will be linked if and only if both $a_i, a_j$ and $b_i, b_j$ are linked. Analogously, we define a network $C$ which is the collection of common links in the original networks $A$ and $B$. Thus network $C$ reduces to $C_0$ due to the initial attack. As shown in Fig. 2, we denote $\tilde{A}_0$ and $\tilde{B}_0$ to be the networks that are composed of the same $N_0$ nodes in $A_0$ and $B_0$ but only those links which are not common links. Therefore, networks $A_0$ and $B_0$ can be written as matrix summations:

$$A_0 = \tilde{A}_0 + C_0, \quad B_0 = \tilde{B}_0 + C_0. \tag{1}$$

We will investigate the robustness of such a system after the initial attack. Notice that when $C$ has no links since there is no common links in this system. This is the case of random coupling studied by Buldyrev *et al.* [15], since the probability to have a common link in random coupling approach to zero for large $N$. We will provide a method for analyzing the case when network $C_0$ has a given topological structure. Let $R_0(m), m = 1, 2, \ldots, M$ be the component size distribution of $C_0$. That is to say, if we randomly choose a node in network $C_0$, the probability that it belongs to a component of size $m$ in network $C_0$ is $R_0(m)$. This distribution is a characterization of both the degree and the structure of intersimilarity of the network. In the simulations, this distribution is used to construct the adjacency matrix of $C_0$ artificially. We also generate the matrices $\tilde{A}_0$ and $\tilde{B}_0$ directly and obtain networks $A_0$ and $B_0$ using the matrix addition.

The initial attack leads to failures of some other nodes in $A_0$ since those nodes will lose connectivity with the giant component $A_1$ of $A_0$ (a percolation failure). Consequently, in $B_0$, all nodes that depend on those nodes that have been removed in $A_0$ will fail due to interdependency relations (a dependency failure). We use $B_1$ to denote the remaining nodes

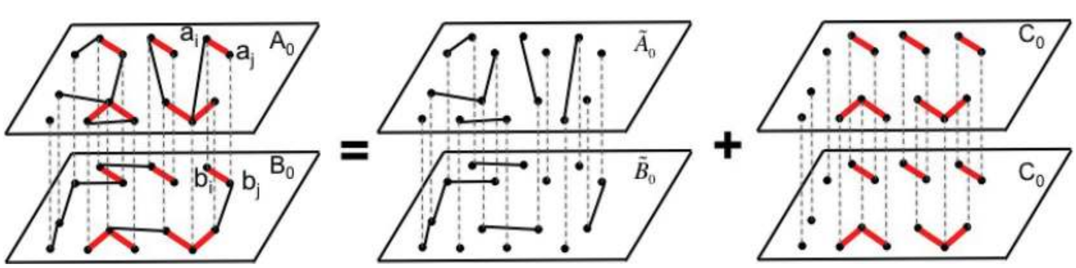


FIG. 2. (Color online) Decomposition of networks according to the common links. Dots in the upper plane are the nodes from network $A_0$ and those in the lower plane are the nodes from network $B_0$. Interdependent nodes are connected by dashed lines. Solid thick (red) lines are the common links, and other links are shown in solid thin lines. Network $C_0$ is composed of common links. $\tilde{A}_0$ and $\tilde{B}_0$ are the complementary networks with respect to $C_0$. $a_i$ and $a_j$ are linked nodes in network $A_0$; their interdependent counterparts $b_i$ and $b_j$ are also linked in network $B_0$. Thus the link is a common link and appears in network $C_0$.

in $B_0$. Then, similarly, a percolation failure will occur in $B_1$. This will induce an iterative process of percolation failures and dependency failures in the system [31]. Finally, if no further failure occurs, this cascading process will end with a total collapse or two remaining giant components of the same size. We are interested here in the relationship between $p$ and the size of the final mutual giant component.

Notice that during the cascading process, if a node in a component of $C_0$ survives, the whole component will survive, and if a node in a component fails, the whole component will fail. Inspired by this basic fact, as shown in Fig. 3, we divide networks $A_0$ and $B_0$ respectively according to the component sizes in network $C_0$. That is to say, in network $A_0$, all those nodes that belong to components of size $m = 1,2,\ldots,M$ in $C_0$ compose a subnetwork denoted by $A_0^{(m)}$. Here $M$ is the largest component size of network $C_0$. Network $B_0$ can be divided into $B_0^{(m)}$ analogously. In this way the size of each subnetwork $A_0^{(m)}$ or $B_0^{(m)}$ is $N_0^{(m)} = R_0(m)N_0$, where $m = 1,2,\ldots,M$.

As depicted in Fig. 3, we also contract $A_0$ and $B_0$ according to the components of network $C_0$. In other words, in network $A_0$, all $m$ nodes that belong to each component of size $m$ in $C_0$ are merged into a single new node, and all links connected to at least one of these $m$ nodes are also merged on the new node. All of these merged nodes and links form a contracted network $A_0'$ with subnetworks

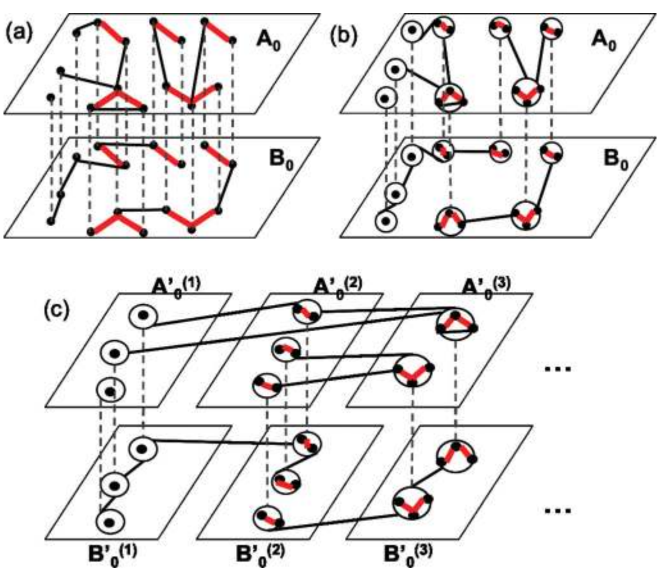


FIG. 3. (Color online) A sketch of the method of dividing and contracting the system after the initial attack. (a) The remaining interdependent networks $A_0$ and $B_0$ right after the initial attack. Thick (red) lines are the common links. Other links are shown in (black) solid thin lines. Interdependent nodes are connected by dashed lines. (b) Nodes in the same component of $C_0$ are contracted. Nodes that are within one circle belong to the same component in network $C_0$. Because of the first principle we propose, all the nodes from one component either survive together or die together. Thus the component can be regarded as a super node (if survived). The links between nodes from different components are now links between the two components. (c) The contracted network is further decomposed into subnetworks according to the component size in $C_0$. For example, $A_0'^{(2)}$ is the collection of super nodes of component size 2 in $A_0$, etc.

$A_0'^{(m)}$, $m = 1,2,\ldots,M$. Network $B_0$ is contracted similarly. In this way the size of each subnetwork $A_0'^{(m)}$ or $B_0'^{(m)}$ is just the number of components of size $m$ in network $C_0$, that is, $N_0'^{(m)} = \frac{R_0(m)N_0}{m}, m = 1,2,\ldots,M$. And the size of $A_0'$ or $B_0'$ is $N_0' = \sum_{m=1}^{M} N_0^{(m)} = \sum_{m=1}^{M} \frac{R_0(m)N_0}{m} = N_0 \sum_{m=1}^{M} \frac{R_0(m)}{m}$. We denote $\langle m \rangle = (\sum_{m=1}^{M} \frac{R_0(m)}{m})^{-1}$ as the average component size in $C_0$. Thus, we have $N_0' = \frac{N_0}{\langle m \rangle}$.

As shown in Fig. 3(c), common links do not exist any more in the contracted system, because each common link always lies inside a component of network $C_0$. In fact, after the initial attack, the cascading process in the contracted system is equivalent to the cascading on the original system. Therefore, we only need to focus on the cascade process in the contracted system.

## III. THEORETICAL APPROACH

Here we exhibit, step by step, the theoretical analysis for the cascading process starting from $A_0'$ and $B_0'$. In the first stage, the size of the remaining functional giant component $A_1'$ can be obtained using the method proposed by Leicht and D'Souza [32]. We regard $A_0'$ as a system of $M$ coupled subnetworks $A_0'^{(m)}$, $m = 1,2,\ldots,M$. If the degree distribution $p_{A_0'}^{mm'}$ from a randomly chosen node in $A_0'^{(m)}$ to all nodes in $A_0'^{(m')}$ (called $m \to m'$ degrees) can be exactly evaluated, then the whole system can be described using multivariable generating functions. Usually, we analyze the cascading process by three steps to obtain the recursive system [15].

We use $g_{A_0'}^{(m)}(p_1,p_2,\ldots,p_M)$ to denote the fraction of nodes in the giant component of $A_0'^{(m)}$ after randomly removing a fraction $1-p_{m'}$ of nodes in each subnetwork $A_0'^{(m')}$, $m' = 1,2,\ldots,M$. Here we contract the two coupled networks after the initial attacking. It implies that $p_{m'} = 1$ at the beginning of the cascading process on the contracted two coupled networks. Thus, the remaining functional part in each subnetwork is $\psi_1^{(m)} = g_{A_0'}^{(m)}(1,1,\ldots,1)$ in the first stage. Note that at each time step $n$ during the cascading process, the accumulative failures in networks $A_0'^{(m)}$ (or $B_0'^{(m)}$) can be equivalently regarded as randomly attacking certain fractions $1-\psi_n'^{(m)}$ (or $1-\phi_n'^{(m)}$) of nodes in networks $A_0'^{(m)}$ (or $B_0'^{(m)}$), and $\psi_n^{(m)}$ (or $\phi_n^{(m)}$) are used to denote the resulting giant component sizes in $A_0'^{(m)}$ (or $B_0'^{(m)}$).

The second stage [15] is equivalent to randomly attacking a fraction $1-\psi_1^{(m)}$ of nodes in each subnetwork $B_0'^{(m)}$, $m = 1,2,\ldots,M$. We let $\phi_1'^{(m)} = \psi_1^{(m)}$. Therefore, the remaining giant component of $B_0'$ is $\phi_1^{(m)} = \phi_1'^{(m)} g_{B_0'}^{(m)}(\phi_1'^{(1)},\phi_1'^{(2)},\ldots,\phi_1'^{(M)})$.

The third stage is equivalent to randomly removing a fraction $1-g_{B_0'}^{(m)}(\phi_1^{(1)},\phi_1^{(2)},\ldots,\phi_1^{(M)})$ of nodes in $A_0'^{(m)}$. We let $\psi_2'^{(m)} = g_{B_0'}^{(m)}(\phi_1^{(1)},\phi_1^{(2)},\ldots,\phi_1^{(M)})$, $m = 1,2,\ldots,M$. Thus, the remaining fraction in $A_0'$ is $\psi_2^{(m)} = \psi_2'^{(m)} g_{A_0'}^{(m)}(\psi_2'^{(1)},\psi_2'^{(2)},\ldots,\psi_2'^{(M)})$.

Generally, we have $\psi_n'^{(m)} = g_{B_0'}^{(m)}(\phi_{n-1}'^{(1)},\phi_{n-1}'^{(2)},\ldots,\phi_{n-1}'^{(M)})$, and $\psi_n^{(m)} = \psi_n'^{(m)} g_{A_0'}^{(m)}(\psi_n'^{(1)},\psi_n'^{(2)},\ldots,\psi_n'^{(M)})$; $\phi_n'^{(m)} = g_{A_0'}^{(m)}(\psi_n'^{(1)},\psi_n'^{(2)},\ldots,\psi_n'^{(M)})$, and $\phi_n^{(m)} = \phi_n'^{(m)} g_{B_0'}^{(m)}(\phi_n'^{(1)},\phi_n'^{(2)},\ldots,$

$\phi_n^{\prime(M)}$), $m = 1,2,\ldots,M$. From these we can calculate the giant component sizes in the whole original networks $A$ and $B$. For example, the fraction of nodes in the giant component of network $A$ is

$$\mu_{An} = \frac{\sum_{m=1}^{M} \psi_n^{(m)} N_0^{\prime(m)} m}{N} = p \sum_{m=1}^{M} \psi_n^{(m)} R_0(m). \tag{2}$$

In the final stage, where the process of cascading failures ceases, we have $\psi_n^{\prime(m)} = \psi_{n-1}^{\prime(m)} = \phi_n^{\prime(m)} = \phi_{n-1}^{\prime(m)}$ for all $m$. Let $x_m = \psi_n^{\prime(m)}$, and $y_m = \phi_n^{\prime(m)}$. We arrive a system of $x_m$ and $y_m$:

$$x_m = g_{A_0'}^{(m)}(y_1,y_2,\ldots,y_M), \quad y_m = g_{B_0'}^{(m)}(x_1,x_2,\ldots,x_M), \tag{3}$$

where $m = 1,2,\ldots,M$.

## IV. ANALYTICAL SOLUTION

This system can be analytically solved using $M$-variant generating functions. Similar to Ref. [32], for a system of $M$ interconnected subnetworks $A_0^{\prime(m)}$, we define the generating function for the degree distributions for each subnetwork as

$$G_{A_0'}^{(m)}(\xi_1,\xi_2,\ldots,\xi_M) = \sum_{k_1,k_2,\ldots,k_M} p_{k_1,k_2,\ldots,k_M}^{(m)} \xi_1^{k_1} \xi_2^{k_2} \ldots \xi_M^{k_M}, \tag{4}$$

where $p_{k_1,k_2,\ldots,k_M}^{(m)}$ is the probability that a randomly chosen node in $A_0^{\prime(m)}$ has $k_{m'}$ $m \to m'$ degrees. Moreover, the generating function for the underlying branching processes for each subnetwork is

$$G_{A_0'}^{(mm')}(\xi_1,\xi_2,\ldots,\xi_M) = \frac{\frac{\partial}{\partial \xi_{m'}} G_{A_0}^{(m)}(\xi_1,\xi_2,\ldots,\xi_M)}{\frac{\partial}{\partial \xi_{m'}} G_{A_0}^{(m)}(\xi_1,\xi_2,\ldots,\xi_M)\,|_{\xi_1=\xi_2=\ldots=\xi_M=1}}, \tag{5}$$

where $m' = 1,2,\ldots,M$. Then the fraction of nodes in the giant component after randomly removing a fraction $1-p_m$ of nodes in each subnetwork $A_0^{\prime(m)}$ is

$$g_{A_0'}^{(m)}(p_1,p_2,\ldots,p_M) = 1 - G_{A_0'}^{(m)}(1-p_1(1-u_{1m}),1-p_2(1-u_{2m}),\ldots,1-p_M(1-u_{Mm})). \tag{6}$$

Here $u_{m'm}$ satisfies

$$u_{m'm} = G_{A_0'}^{m'm}(1-p_1(1-u_{1m'}),1-p_2(1-u_{2m'}),\ldots,1-p_M(1-u_{Mm'})), \tag{7}$$

where $m,m' = 1,2,\ldots,M$. For network $B_0'$, we can define the analogous generating functions and obtain similarly the giant component size.

For simplicity, we assume that all $m \to m'$ degree distributions in $A_0'$ and $B_0'$ are Poisson distributions, whose average degrees are $k_{mm'}^{A_0'}$ and $k_{mm'}^{B_0'}$, respectively. For example, if both $\tilde{A}$ and $\tilde{B}$ are Erdős-Rényi (ER) networks with average degrees $a$ and $b$, respectively, and the initial attack on network $A$ is random, then these $m \to m'$ degrees in $A_0$ have Poisson distributions for all $m$ and $m'$. Then, according to the result in Ref. [32], we have

$$G_{A_0'}^{m} = G_{A_0'}^{mm'} = \exp\left(-\sum_{m'=1}^{M} k_{mm'}^{A_0'}(1-\xi_{m'})\right), \quad G_{B_0'}^{m} = G_{B_0'}^{mm'} = \exp\left(-\sum_{m'=1}^{M} k_{mm'}^{B_0'}(1-\xi_{m'})\right). \tag{8}$$

Here the average $m \to m'$ degrees in $A_0'$ and $B_0'$ are $k_{mm'}^{A_0'} = ampR_0(m')$ and $k_{mm'}^{B_0'} = bmpR_0(m')$, respectively. Notice that, in this case, $u_{m1} = u_{m2} = \cdots = u_{mM} \triangleq u_m$, $m = 1,2,\ldots,M$. Therefore,

$$g_{A_0'}^{(m)}(p_1,p_2,\ldots,p_M) = 1 - u_m(p_1,p_2,\ldots,p_M), \tag{9}$$

where $u_m(p_1,p_2,\ldots,p_M)$, $m = 1,2,\ldots,M$ is the solution of the following set of equations:

$$u_m = G_{A_0'}^{m}(1-p_1(1-u_1),1-p_2(1-u_2),\ldots,1-p_M(1-u_M)) = \exp\left(-\sum_{m'=1}^{M} k_{mm'}^{A_0'} p_{m'}(1-u_{m'})\right), \tag{10}$$

where $m = 1,2,\ldots,M$. Similarly, for network $B_0'$,

$$g_{B_0'}^{(m)}(p_1,p_2,\ldots,p_M) = 1 - v_m(p_1,p_2,\ldots,p_M), \tag{11}$$

where $v_m$ satisfies

$$v_m = G_{B_0'}^{m}(1-p_1(1-v_1),1-p_2(1-v_2),\ldots,1-p_M(1-v_M)) = \exp\left(-\sum_{m'=1}^{M} k_{mm'}^{B_0'} p_{m'}(1-v_{m'})\right) \tag{12}$$

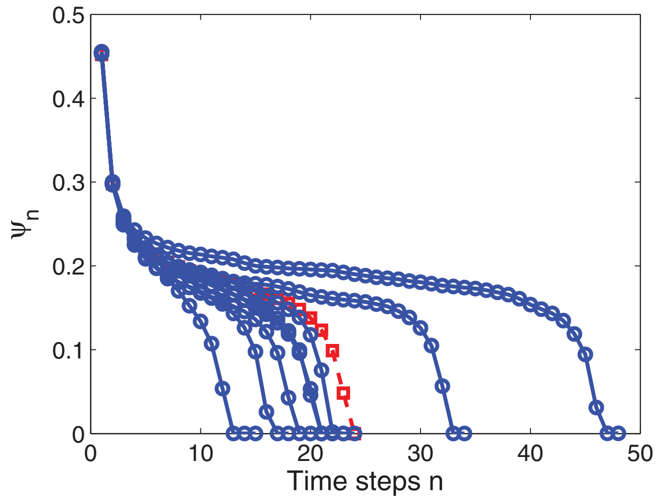


FIG. 4. (Color online) The evolution during cascading failures of the giant component size of network $A$, $\mu_{An}$, for both the theory and simulations. Here, $a = b = 3$, $K = 0.8$, $N = 100\,000$, and $M_{\max} = 50$. The (blue) full circles show eight realizations of simulations with total collapse at the critical threshold $p = p_c = 0.5528$. The (red) dashed rectangles show the theoretical prediction for $p_c = 0.551$.

where $m = 1,2,\ldots,M$. Thus, the giant component size $\mu_{An}$ at time step $n$ can be predicted theoretically according to Eq. (2). Figure 4 shows this cascading process for both the theory and simulations near criticality when $\tilde{A}$, $\tilde{B}$, and $C$ are all ER networks, and their average degrees are $a = 3$, $b = 3$, and $K = 0.8$.

The system of the final stage can be written as

$$x_m = 1 - u_m(y_1,y_2,\ldots,y_M), \quad y_m = 1 - v_m(x_1,x_2,\ldots,x_M). \tag{13}$$

By excluding $x_m$ and $y_m$, we finally obtain

$$u_m = \exp\left(-\sum_{m'=1}^{M} k_{mm'}^{A'_0}(1-u_{m'})(1-v_{m'})\right),$$
$$v_m = \exp\left(-\sum_{m'=1}^{M} k_{mm'}^{B'_0}(1-u_{m'})(1-v_{m'})\right). \tag{14}$$

Therefore,

$$u_m = \exp\left(-amp\sum_{m'=1}^{M} R_0(m')(1-u_{m'})(1-v_{m'})\right),$$
$$v_m = \exp\left(-bmp\sum_{m'=1}^{M} R_0(m')(1-u_{m'})(1-v_{m'})\right), \tag{15}$$

where $m = 1,2,\ldots,M$.

By solving this system, we can get $\mu_\infty^{(m)} = (1-u_m)(1-v_m)$, $m = 1,2,\ldots M$. This is the fraction of the mutual giant component in each subnetwork $A_0'^{(m)}$ or $B_0'^{(m)}$. The fraction of the mutual giant component in the original system of $A$ and $B$ is $\mu_\infty = p\sum_{m=1}^{M}\mu_\infty^{(m)}R_0(m)$.

Notice that in Eq. (15), $u_m = u_1^m$, and $v_m = u_1^{mb/a}$, $m = 1,2,\ldots,M$. Therefore, the system can be simplified to a single equation for $u_1$,

$$u_1 = \exp\left(-ap\sum_{m'=1}^{M} R_0(m')\left(1-u_1^{m'}\right)\left(1-u_1^{m'b/a}\right)\right). \tag{16}$$

Thus, the fraction of the mutual giant component becomes

$$\mu_\infty = p\sum_{m=1}^{M} R_0(m)\left(1-u_1^{m}\right)\left(1-u_1^{mb/a}\right) = \frac{-\log u_1}{a}. \tag{17}$$

## V. RESULTS

One trivial solution of Eq. (16) is $u_1 = 1$. In some cases, other nontrivial solutions exist in the interval $[0,1)$. The smallest solution $u_{1\min}$ corresponds to the size of the final mutual giant component $-\log u_{1\min}/a$. Consider $F_2(u_1) = \exp(-ap\sum_{m'=1}^{M} R_0(m')(1-u_1^{m'})(1-u_1^{m'b/a}))$ and $F_1(u_1) = u_1$. Then the critical point $u_{1c}$ and $p_c$ is where a nontrivial solution that satisfies $F_1(u_{1c}) = F_2(u_{1c})$ and $F_1'(u_{1c}) = F_2'(u_{1c})$ emerges. Note that all the analysis is done here on the contracted network system after the initial random removal, which means there is no initial attacking on the contracted system. If $M$ is finite, at the solution $u_1 = 1$, we have $F_1(1) = F_2(1) = 1$, but $F_1'(1) = 1 \neq F_2'(1) = 0$. This means these two curves cannot be tangent to each other at $u_1 = 1$. Therefore, $u_1 = 1$ cannot be a critical value for a second order transition, and only first order phase transitions at $u_{1c} < 1$ occur in systems with a finite $M$.

Here we further investigate the case where $C$ (network composed of common links) is an ER network with an average degree $K$. After the random initial attack, $C_0$ is also an ER network, whose average degree becomes $Kp$. The component

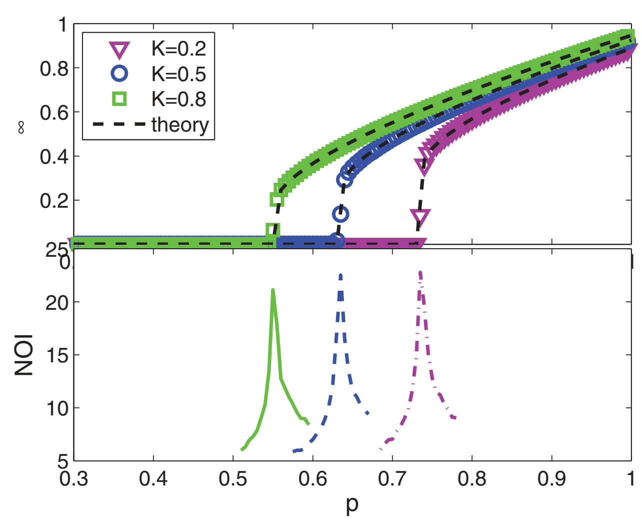


FIG. 5. (Color online) Comparison of the simulation results (symbols) for $\mu_\infty$ (top figure) against the theoretical results (curves) when network $C_0$ is an ER network. Here, we choose $a = b = 3$, $K = 0.2,0.5,0.8$, and $N = 100\,000$. We also show (bottom figure) the number of iterative failures (NOI) in the simulation for the same values of $K$. Usually the peak of NOI values indicates the value of the critical threshold at the first order phase transition [31]. We can see the excellent agreement between the theoretical and simulation results.

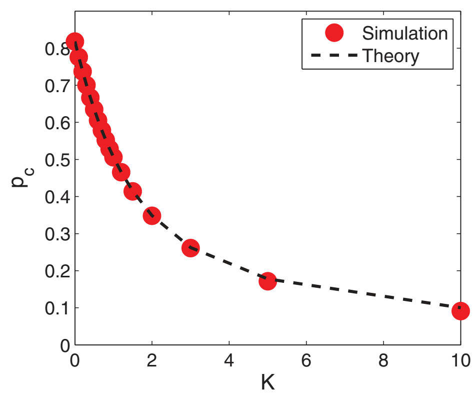


FIG. 6. (Color online) The dependence of the critical value $p_c$ on the average degree $K$ of common links for interdependent ER networks systems. The average degrees of networks $A_0$ and $B_0$ are $a = b = 3$, and that of $C_0$ is $K$. The network size is $N = 300\,000$, and the number of realizations in simulations is 100. The curve is the theoretical results (using $M_{\max} = 50$), and the (red) full circles are the corresponding simulation results.

size distribution of $C_0$ is $R_0(m) = \frac{1}{m!}(mKp)^{m-1}e^{-mKp}, m = 1,2,\ldots$. In this case, $M$ should be in principle infinite. However, when we substitute this distribution into Eq. (16), it is appropriate to use a truncation $m = 1,2,\ldots,M < \infty$ on the infinite sum, since $R_0(m)$ decays exponentially, and the largest cluster of network $C_0$ cannot be as large as $O(N)$ when $Kp < 1$. This means the transition point $p_c < 1/K$. Therefore, no matter for $K < 1$ or $K \geqslant 1$, we can use Eq. (16) directly to determine $p_c$, and the transition is always discontinuous. (Note that in the theory, $N \to \infty$, and $K$ is a finite constant according to the definition of ER networks.) The theoretical

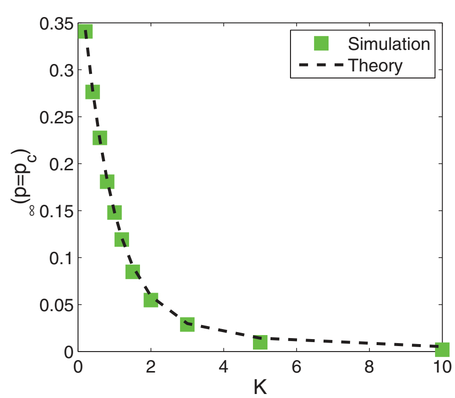


FIG. 7. (Color online) The dependence of the size of the jump at the critical threshold of the first order phase transitions on the average degree $K$ of network $C_0$. The average degrees of networks $A_0$ and $B_0$ are $a = b = 3$. The network size is $N = 100\,000$, and the number of realizations in simulations is 100. The dashed curve is the theoretical result ($M = 50$), and the (green) solid squares are the corresponding simulation results at the critical thresholds. From the shape of the curve, we can see that the size of the jump at the critical point does not reach 0 for a very large $K$, which strongly supports that if the two mutually depending networks are not identical, the phase transition is always discontinuous.

results are in excellent agreement with simulations, as shown in Figs. 5, 6, and 7. Figure 5 shows the phase transitions for three different $K$ values. Figure 6 exhibits the effect of $K$ on the transition point $p_c$. Note that when $K$ is much larger than $a$ and $b$, $p_c$ will be very close to the transition point of a single ER network $p_c^{II} = 1/K$, since the system has only very small differences from two identical networks. Surprisingly, our results indicate that as long as $a,b > 0$, which means two networks are not identical, the equation describing the system will become Eq. (16), and the transition will move suddenly from second order when $a = b = 0$ to first order. Moreover, Fig. 7 illustrates that when $K$ gets larger, the jump at the transition point will converge to 0 when $K \to +\infty$, since $0 \leqslant \mu_\infty(p = p_c) \leqslant p_c < 1/K \to 0$, but it will never reach 0 at any finite $K$.

## VI. SUMMARY

In this paper we provide an exact solution for interdependent networks with common links (representing intersimilarity in the system), which can be found in many real world network systems. We treat the components composed of intersimilar links as a new kind of nodes, and these new nodes form a new mutually interdependent network system with degree correlation, which comes from the correlation between component sizes. In order to deal with this kind of degree correlation, we decompose the new network system into a series of subnetworks according to their component sizes. That is, the new node corresponds to the same component size in each of subnetworks respectively. Then we employ a high dimensional generating function to describe this system and obtain the exact percolation equations, which can be solved numerically. If the two mutually interdependent networks are fully intersimilar or identical ($a = b = 0$), we know that the percolation is exactly the same with that on a single network and must be a second order phase transition. From the above analysis, we surprisingly find that when the two mutually interdependent networks are not identical ($a,b > 0$), the transition is totally different from single networks and is always of first order. Notice that our method can be simplified to a single equation only for ER networks. Moreover, the cutoff $M$ cannot be used if a giant component exists in $C_0$. Therefore, we will try to develop new methods to solve other cases, such as power-law networks, in future works.

After this paper was submitted, we learned of a related analytical work on multiplex networks (specific case of interdependent networks) submitted recently to arXiv [33].

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