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PERCOLATION PROCESSES: LOWER BOUNDS FOR THE CRITICAL PROBABILITY

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1. Introduction. A percolation process is the spread of a fluid through a medium under the influence of a random mechanism associated with the medium. This contrasts with a diffusion process, where the random mechanism is associated with the fluid. Broadbent and Hammersley [1] gave examples illustrating the distinction.

Here we shall consider a *medium* consisting of an infinite set of *atoms* and *bonds*. A bond is a path between two atoms: it may be *undirected* (in which case it will allow passage from either atom to the other) or it may be *directed* (in which case it will allow passage from one atom to the other but not vice versa). Two atoms may be linked by several bonds, some directed and some undirected. Broadbent and Hammersley [1] dealt with *crystals*, i.e., media in which the atoms and bonds satisfied three postulates denoted by $P1$, $P2$, and $P3$. Here, however we shall dispense with $P1$ and a part of $P3$; and our surviving assumptions are:

$P2$. The number of bonds *from* (but not necessarily *to*) any atom is finite.

$P3$(a). Any finite subset of atoms contains an atom *from* which a bond leads to some atom not in the subset.

With this medium we associate the following *random mechanism*: each bond has an independent probability p of being *undammed* and $q = 1 - p$ of being *dammed*. *Fluid*, supplied to the medium at a set of *source atoms*, spreads along undammed bonds only (and in the permitted direction only for undammed directed bonds) and thereby *wets* the atoms it reaches. Associated with each atom A, there is a *critical probability* $p_d(A)$, defined as the supremum of all values of p such that, when A is the only source atom, A wets only finitely many atoms with probability one. We seek lower bounds for p_d.

An *n-stepped walk* is an ordered connected path along n bonds, each step being in a permitted direction along its bond and starting from the atom reached by the previous step. Walks (as opposed to fluid) may traverse dammed bonds: a walk is dammed or undammed according as it traverses at least one or no

Received October 29, 1956.

dammed bond. A walk is *self-avoiding* if it visits no atom more than once. The number of n-stepped self-avoiding walks starting from the atom A is denoted by $f_A(n)$. The *connective bound* of the medium is defined to be

$$\lambda = \sup_A \limsup_{n\to\infty} n^{-1} \log f_A(n). \tag{1}$$

Hammersley [3] showed that, under the fuller assumptions required by a crystal, there existed a *connective constant*

$$\kappa = \lim_{n\to\infty} n^{-1} \log f_A(n), \tag{2}$$

independent of A. Clearly $\lambda = \kappa$ when the latter exists.

The *principal n-neighbourhood* of an atom A, written $\mathbf{N}^n(A)$, is the atom A together with all atoms accessible from A by walks of n or fewer steps. The *principal n-boundary* of A is $\mathbf{B}^n(A) = \mathbf{N}^n(A) - \mathbf{N}^{n-1}(A)$, where $\mathbf{N}^0(A)$ is A alone. *A walk belongs to a set of atoms* $\mathbf{S}$ if every step of the walk *starts* from some atom of $\mathbf{S}$: notice that the final step may *terminate* at some atom not in $\mathbf{S}$.

2. Statement of results. Let $E_n(A, p)$ denote the expected number of atoms of $\mathbf{B}^n(A)$ which can be reached from A by at least one undammed walk belonging to $\mathbf{N}^{n-1}(A)$. Define $F_n = F_n(p) = \sup_A E_n(A, p)$.

THEOREM 1. $F_n(p_d + 0) \geqq 1$.

It is an easy matter to show from $P2$ and $P3(a)$ that $\mathbf{N}^n(A)$ has only finitely many atoms, and that every atom of $\mathbf{B}^n(A)$ can be reached from A by at least one (perhaps dammed) walk belonging to $\mathbf{N}^{n-1}(A)$, and that $\mathbf{B}^n(A)$ contains at least one atom. Therefore $E_n(A, p)$ is an increasing function of p, and $F_n(p)$ is a nondecreasing function of p, and $0 = E_n(A, 0) = F_n(0)$ and $1 \leqq E_n(A, 1) \leqq F_n(1)$. Thus Theorem 1 provides a lower bound for p_d, because p_d exceeds any solution of $F_n(p) < 1$.

Let $P_n = P_n(A)$ be the probability that the single source atom A wets at least one atom of $\mathbf{B}^n(A)$.

THEOREM 2. *If* $F_n < 1$ *for some particular* n, *then* $P_N \leqq F_n^{[N/n]}$ *for all* N, *where* $[N/n]$ *denotes the integer part of* N/n.

Theorem 2 is a rather more precise form of Theorem 1 and will be required elsewhere [5]. Here we shall deduce Theorem 1 from Theorem 2.

THEOREM 3. $p_d \geqq e^{-\lambda}$.

Theorem 3 is a straightforward generalization of a previous result ([1], Theorem 7).

I have not yet succeeded in proving or disproving

CONJECTURE 1. *For each fixed* p, F_n *is a subexponential function of* n, *that is to say,* $F_{l+m} \leqq F_l F_m$.

Example 1 below shows that Theorem 1 is sometimes stronger than Theorem 3. Theoretically, Theorem 1 is never weaker than Theorem 3. However, when the medium is a crystal it is not hard to estimate κ by Monte Carlo methods, and Theorem 3 may prove more useful than Theorem 1 in cases in which F_n is hard to calculate for large n. Similarly, it may occasionally happen that Theorem

1, although weakened theoretically thereby, may yet be strengthened practically by redefining E_n as the expected number of wet atoms in $\mathbf{B}^n(A)$.

EXAMPLE 1. Let the atoms of the medium lie in the Euclidean plane at the points (x, y), where $x, y = 0, 1, 2, \cdots$. Suppose that from each (x, y) there is a directed bond to $(x + 1, y)$ and a directed bond to $(x, y + 1)$. Then $f_A(n) = 2^n$, $\lambda = \log 2$, and Theorem 3 gives $p_d \geqq \frac{1}{2}$. However, $F_2(p) = 4p^2 - p^4$; so that Theorem 1 gives $p_d > \sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}} = 0.518 \cdots$. F_3 gives a slightly sharper result, but is much more tedious to calculate. We may also notice that $F_1(p) = 2p$; so that $F_2 < F_1^2$, and inequality is certainly required in Conjecture 1.

EXAMPLE 2. Consider the familiar branching process in which each individual has 0, 1, or 2 descendants with independent probabilities q^2, $2pq$, p^2. This, like all branching processes, is a special form of percolation process (see [1] for further details of this question). We consider the atoms of the medium to be the actual or potential individuals of the branching process. Each atom has just two directed bonds from it, and the fluid is life transforming a potential into an actual individual. We have $F_n = (2p)^n$ and $p_d = \frac{1}{2}$, agreeing with well-known results. Also $\lambda = \log 2$. Thus Theorems 1 and 3 are equally strong and best possible. Also Conjecture 1 holds with equality. This and other branching processes suggest

PROBLEM 1. *Under what conditions is* $\liminf_{n\to\infty} F_n(p_d) = 1$ *valid?*

We cannot in general use E in the role of F nor omit the $+0$ in Theorem 1, though perhaps the exceptional cases are rare. Example 3 provides one such exception.

EXAMPLE 3. Suppose that the atoms $A_1, A_2, \cdots$ are connected by $2j$ directed bonds from A_j to A_{j+1}. Then

$$E_n(A_m, p) = \prod_{j=m}^{m+n-1} (1 - q^{2j}) < 1$$

for $p < 1$. However $p_d = 0$, because $E_\infty(A_m, p)$ is also the probability that A_m wets infinitely many atoms and the infinite product converges for $q < 1$. As a matter of passing interest, $E_\infty(A_1, p) = (\vartheta_1'/2q^{1/4})^{1/3}$, where q plays the usual theta-function role of Jacobi's nome [6], p. 473; see also [2], Sec. 21.7: indeed, $2j$ rather than j bonds from A_j preserved the nomenclature.

3. Proof of theorems. Let A_1 be a fixed atom and n a fixed positive integer. In studying the spread of fluid from the single source atom A_1, we shall suppose that the spreading occurs in consecutive recursively defined stages. Immediately before the jth stage takes place, we shall know two sets of atoms, denoted by $\mathbf{W}(j)$ and $\mathbf{S}(j)$ respectively. Here $\mathbf{W}(j)$ is the set of atoms already wet up to but excluding the jth stage, and $\mathbf{S}(j)$ is the set of atoms about to serve as sources in the jth stage. The process starts from $\mathbf{W}(1) = \mathbf{S}(1) = A_1$. If $\mathbf{S}(j)$ is empty, then $\mathbf{S}(j + 1)$ is empty, and $\mathbf{W}(j + 1) = \mathbf{W}(j)$. If $\mathbf{S}(j)$ is not empty, let A be any atom of $\mathbf{S}(j)$ and proceed as follows. Define $\mathbf{X}(A)$ to be the set of all atoms in $\mathbf{N}^n(A) - \mathbf{W}(j)$, which can be reached from A by at least one undammed

self-avoiding walk belonging to $[\mathbf{N}^{n-1}(A) - \mathbf{W}(j)] + A$. Define $\mathbf{Y}(A)$ as the intersection of $\mathbf{X}(A)$ and $\mathbf{B}^n(A)$. Lastly define

$$\mathbf{S}(j+1) = \sum_A \mathbf{Y}(A); \qquad \mathbf{W}(j+1) = \mathbf{W}(j) + \sum_A \mathbf{X}(A), \tag{3}$$

where $\sum_A$ denotes summation over all atoms A belonging to $\mathbf{S}(j)$. We shall also require sets of atoms $\mathbf{T}(1)$, $\mathbf{T}(2)$, $\cdots$ defined recursively by

$$\mathbf{T}(1) = A_1; \qquad \mathbf{T}(j+1) = \sum_{A \in \mathbf{T}(j)} \mathbf{B}^n(A), \qquad j = 1, 2, \cdots. \tag{4}$$

Let A be an atom in $\mathbf{S}(j)$, supposed not empty, and let B be an atom of $\mathbf{T}(j+1)$. We write $A \rightarrow B$ to denote the existence of at least one undammed self-avoiding walk from A to B belonging to $[\mathbf{N}^{n-1}(A) - \mathbf{W}(j)] + A$. By the foregoing definitions, $A \rightarrow B$ implies that $B \in \mathbf{S}(j+1)$; and conversely there exists an atom $A_j \in \mathbf{S}(j)$ if and only if we can find atoms $A_1, A_2, \cdots, A_{j-1}$ belonging to $\mathbf{S}(1), \mathbf{S}(2), \cdots, \mathbf{S}(j-1)$ such that $A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_{j-1} \rightarrow A_j$. Notice also that the sets $\mathbf{S}(1), \mathbf{S}(2), \cdots$ are mutually disjoint; and that $\mathbf{S}(1), \mathbf{S}(2), \cdots, \mathbf{S}(j)$ are all subsets of $\mathbf{W}(j)$. Finally $\mathbf{S}(j)$ is a subset of $\mathbf{T}(j)$. It may also be a subset of $\mathbf{T}(k)$ for $k \neq j$; but this will not affect our argument.

If A is an atom of the nonempty set $\mathbf{S}(j)$, we define its score $\theta(A) = \sum_B \theta(B)$, where $\sum_B$ denotes summation over all atoms $B \in \mathbf{S}(j-1)$ such that $B \rightarrow A$. We begin this recursive definition from $\theta(A_1) = 1$ when $j = 1$. To the set $\mathbf{T}(j)$ we attach the score

$$\phi_j = \begin{cases} \sum_{A \in \mathbf{S}(j)} \theta(A) & \text{if } \mathbf{S}(j) \text{ is not empty,} \\ 0 & \text{if } \mathbf{S}(j) \text{ is empty.} \end{cases} \tag{5}$$

Suppose that $\mathbf{W}(j-1)$ is given, and that $\mathbf{S}(j-1)$ is not empty. This means that $\mathbf{S}(1), \mathbf{S}(2), \cdots, \mathbf{S}(j-1)$ are all given and not empty. Hence $\theta(B)$ is given for each $B \in \mathbf{S}(j-1)$. Consider the conditional expectation of ϕ_j given $\mathbf{W}(j-1)$ with nonempty $\mathbf{S}(j-1)$. We have

$$\begin{aligned} &E[\phi_j \mid \mathbf{W}(j-1), \quad \mathbf{S}(j-1) \neq 0] \\ &\qquad = \sum_{A \in \mathbf{T}(j), B \in \mathbf{S}(j-1)} \theta(B) \text{ Prob } [B \rightarrow A \mid \mathbf{W}(j-1)] \\ &\qquad = \sum_{B \in \mathbf{S}(j-1)} \theta(B) \sum_{A \in \mathbf{T}(j)} \text{Prob } [B \rightarrow A \mid \mathbf{W}(j-1)]. \end{aligned} \tag{6}$$

Since $B \rightarrow A$ involves the existence of at least one undammed self-avoiding walk from B to A belonging to $[\mathbf{N}^{n-1}(B) - \mathbf{W}(j-1)] + B$, this event depends only upon the condition of bonds whose condition does not affect $\mathbf{W}(j-1)$. Hence,

$$\text{Prob } [B \rightarrow A \mid \mathbf{W}(j \leqq 1)] \leqq \text{Prob } [B \sim A], \tag{7}$$

where $B \sim A$, in the unconditional probability on the right of (7), means that there is at least one undammed walk from B to A belonging to $\mathbf{N}^{n-1}(B)$. Then, by definition of $E_n(B, p)$ and $F_n(p)$, we have

$$\sum_{A \in \mathbf{T}(j)} \text{Prob } [B \sim A] = E_n(B, p) \leqq F_n(p). \tag{8}$$

Combination of (6), (7), and (8) yields

$$E[\phi_j \mid \mathbf{W}(j-1), \quad \mathbf{S}(j-1) \neq \mathbf{0}] \leqq F_n(p) \sum_{B \in \mathbf{S}(j-1)} \theta(B). \tag{9}$$

In (9), we can remove the condition $S(j-1) \neq 0$, provided we interpret the right-hand empty sum as zero when $\mathbf{S}(j-1) = \mathbf{0}$. Hence,

$$\begin{aligned} E[\phi_j] &= \sum E[\phi_j \mid \mathbf{W}(j-1)] \text{ Prob } [\mathbf{W}(j-1)] \\ &\leqq F_n(p) \sum_{B \in \mathbf{S}(j-1)} \sum \theta(B) \text{ Prob } [\mathbf{W}(j-1)] = F_n(p) E[\phi_{j-1}]. \end{aligned} \tag{10}$$

Since *every* walk from A_1 to $\mathbf{B}^N(A_1)$ contains at least N steps, A_1 cannot wet any atom of $\mathbf{B}^N(A_1)$ unless none of $\mathbf{S}(1)$, $\mathbf{S}(2)$, $\cdots$, $\mathbf{S}(\nu+1)$ are empty, where $\nu = [N/\nu]$. If $\mathbf{S}(\nu+1)$ is not empty, $\phi_{\nu+1} \geqq 1$. Thus, by (10),

$$P_N \leqq \text{Prob } [\phi_{\nu+1} \geqq 1] \leqq E[\phi_{\nu+1}] \leqq F_n^{\nu} = F_n^{[N/n]}, \tag{11}$$

which is Theorem 2. The relation (11) is true but useless if $F_n \geqq 1$.

If

$$F_n = F_n(p) < 1, \tag{12}$$

then

$$\lim_{N \to \infty} P_n = 0, \tag{13}$$

by (11). Since $\mathbf{N}^N(A_1)$ contains only finitely many atoms, (13) implies that A_1 wets infinitely many atoms with probability zero. Therefore, by definition of $p_d = p_d(A_1)$,

$$p \leqq p_d . \tag{14}$$

Since (14) is a consequence of (12), we deduce Theorem 1.

Theorem 3 is easy; for A_1 does not wet $\mathbf{B}^N(A_1)$ unless there is at least one N-stepped self-avoiding walk from A_1. The probability of this event is less than or equal to the expected number of such walks, namely $p^N f_A(N)$, because all bonds of a self-avoiding walk are distinct. If $p < e^{-\lambda}$, $\lim_{n\to\infty} p^N f_A(N) = 0$ by (1), and Theorem 3 follows.

To see that Theorem 1 is always as strong as Theorem 3, notice that, given $\epsilon > 0$, there exists n such that $f_A(m) \leqq e^{(\lambda+\epsilon)m}$ for $m \geqq n$; and hence $E_n(A, p) \leqq \sum_{m \geqq n} (pe^{\lambda+\epsilon})^m$. The right-hand side does not depend on A, so that we may write $F_n(p)$ for $E_n(A, p)$. Then, if $p < e^{-(\lambda+\epsilon)}$, $F_n \to 0$ as $n \to \infty$; and the result follows because ϵ is arbitrary.

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NON-PARAMETRIC UP-AND-DOWN EXPERIMENTATION[1]

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1. Introduction. Let $Y(x)$ be a random variable such that $P(Y(x) = 1) = F(x)$ and $P(Y(x) = 0) = 1 - F(x)$ where $F(x)$ is a distribution function. It is sometimes of interest, as in sensitivity experiments, to estimate a given quantile of $F(x)$ with observations distributed like $Y(x)$ where the choice of x is under control. A procedure for estimating the median was suggested by Dixon and Mood [2]. The validity of their procedure depends on the assumption that $F(x)$ is normal. Robbins and Monro [6] suggested a general scheme which can be used for estimating any quantile and which imposes no parametric assumptions on $F(x)$. Their method does assume, however, that the range of possible experimental values of x is the real line. In practice, this will not be the case. Limitations on the precision of measuring instruments, or natural limitations such as when x is obtained by a counting procedure, will usually restrict the experimental range of x to a set of numbers of the form

$$a + hn (-\infty < a < \infty, h > 0, n = 0, \pm 1, \cdots).$$

In this note we suggest a non-parametric procedure for estimating any quantile of $F(x)$ on the basis of quantal response data when, experimentally, x is restricted to the form $a + hn$.

For convenience we assume $a = 0$, $h = 1$. Suppose we wish to estimate that value of $x = \theta$ such that $F(\theta - 0) \leqq \alpha \leqq F(\theta)$, $\frac{1}{2} \leqq \alpha < 1$. If $0 < \alpha \leqq \frac{1}{2}$ or $a \neq 0$ or $h \neq 1$ the necessary modifications will be apparent. The experimental procedure is as follows: choose x_1 arbitrarily. Recursively, let

$$\begin{aligned} x_n &= x_{n-1} - 1, && \text{with probability } \frac{1}{2\alpha} \text{ if } y_{n-1} = 1, \\ &= x_{n-1} + 1, && \text{with probability } 1 - \frac{1}{2\alpha} \text{ if } y_{n-1} = 1, \\ &= x_{n-1} + 1, && \text{with probability 1 if } y_{n-1} = 0. \end{aligned} \tag{1}$$

Received May 28, 1956; Revised January 29, 1957.
[1] Research supported by the United States Air Force through the Office of Scientific Research of the Air Research and Development Command.