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PERFECT CATEGORIES II

(HEREDITARY CATEGORIES)

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We are familiar to study rings S with identity if we are interested in homological method on the ring theory. On the other hand, it seems for us that the theory of categories is some kind of generalization of the structure of S-modules. Especially, Grothendieck categories $\mathfrak{A}$ with generating sets of small projective are exactly generalizations of the category $\mathfrak{M}_S$ of S-modules.

Recentely, the author has pointed out in [13], by making use of [6] and Freyd's theorem (see [16]) that $\mathfrak{A}$ is equivalent to a full subcategory $\mathfrak{M}_R{}^+$ of $\mathfrak{M}_R$, where R is the induced ring from $\mathfrak{A}$ (see the definition in §1). In general, R does not contain the identity element, but R contains a set of mutually orthogonal idempotents $\{e_\alpha\}$ such that $R=\sum\oplus e_\alpha R=\sum\oplus Re_\alpha$.

It is natural from the reason of birth of R that $\mathfrak{M}_R{}^+$ has very similar properties to those in $\mathfrak{M}_R$. However, there are slightly different properties between them. For instance, let Δ be a division ring and T the ring of column finite matrices over Δ with degree α. Let $\{e_{ij}\}$ be the set of matrix units. Put R^l (resp. R^r)$=\sum_{i\geq j}\oplus e_{ij}\Delta$ (resp. $\sum_{i\leq j}\oplus e_{ij}\Delta$). If $|\alpha|$ is finite, then R^l, R^r have the same properties. If $|\alpha|$ is $\aleph_0$, then R^l and R^r do not have identities and R^l is semi-artinian and hereditary and R^r is perfect and hereditary., (see Theorem 3 in §5).

In this paper, we shall generalize above properties in a semi-perfect Grothendieck category and give types of hereditary and perfect or hereditary and semi-artinian categories in Theorems 3, 4, 5 and 6. They are generalizations of [3], Theorem 4.1, [14], Theorem 5 and [8], Theorem 5 in semi-primary hereditary rings. Finally, we shall show in Theorem 7 that a semi-perect Grothendieck category with bounded connected sequences (see §4) is a special type of subcategory of perfect and hereditary (or semi-artinian and hereditary) category and vice versa.

In this paper we do not assume that a ring R contains the identity element. We use the categorical terminology in [16]. By $\mathfrak{M}_R$ we denote the category of right R-modules and by Ab we denote the category of the abelian groups.

1. Preliminary results

Let $\mathfrak{A}$ be a Grothendieck category with a generating set of small projective, then $\mathfrak{A}$ is equivalent, by Freyd's theorem (see [16], Theorem 5.3) to a contravariant functor category ($\mathfrak{C}^\circ$, Ab) of an additive small category $\mathfrak{C}$. On the other hand, in this case P. Gabriel showed in [6] that ($\mathfrak{C}^\circ$, Ab) is equivalent to the full subcategory of modules over a ring R as follows:

Put $R=\sum_{C_\alpha\in\mathfrak{C}}\oplus[C_\alpha, C_\beta]$ as modules and we make R a ring by compositions of morphisms. We denote the identity morphism of $[C, C]$ by I_C. Then $\{I_C\}_{C\in\mathfrak{C}}$ is a set of mutually orthogonal idempotents and $R=\sum_C\oplus I_C R$. By $\mathfrak{M}_R{}^+$ we donte the full subcategory in the category $\mathfrak{M}_R$ of right R-modules, whose objects consist of all R-modules A such that $AR=A$. Then we note that $A=\sum\oplus AI_C$ and every R-submodule of A is in $\mathfrak{M}_R{}^+$. Similarly, we can define ${}_R\mathfrak{M}^+$. We know from [6], Proposition 2 in p. 347 that ($\mathfrak{C}$, Ab) (resp. ($\mathfrak{C}^\circ$, Ab)) is equivalent to ${}_R\mathfrak{M}^+$ (resp. $\mathfrak{M}_R{}^+$).

Conversely, let S be a ring, which is not necessarily to have the identity. We assume that S contains a set of mutually orthogonal idempotents $\{e_\alpha\}$ such that $S=\sum\oplus e_\alpha S=\sum\oplus Se_\alpha$. It is easily to check that $\{e_\alpha S\}$ is a generating set of small projective in $\mathfrak{M}_S{}^+$. Hence, $\mathfrak{M}_S{}^+$ is equivaient to ($\mathfrak{C}'^\circ$, Ab), where $\mathfrak{C}'$ is the pre-additive category $\{e_i S\}$ in $\mathfrak{M}_S{}^+$. Further $S\approx\sum\oplus[e_\alpha S, e_\beta S]$. Therefore, we call such a ring S an *induced ring* from a category and $\{e_\alpha\}$ is called a set of *generating idempotents.*

We shall use frequently some homological method over S in this paper. Hence, we shall give here some notes concerning with this method.

Let S be as above. We consider every things in $\mathfrak{M}_S{}^+$.

N.0. *Every sub or factor modules of A is in $\mathfrak{M}_S{}^+$.*

N.1. *P is projective if and only if P is a retract of a free S-module F. Sometimes we use a fact $F=\sum_u\oplus uS=\sum_{u,\alpha}\oplus ue_\alpha S$, where $\{u\}$ is a base.*

N.2. *For any elements x, y of A in $\mathfrak{M}_S{}^+$, there exist idempotents e_1, e_2 and e in S such that $xe_1=x$, $ye_2=y$ and $e_i=ee_i=e_ie$.*

N.3. $A\otimes_S S\approx A$, (use N.2). *However,* $\mathrm{Hom}_S(S, A)\approx\Pi Ae_\alpha$.

N.3′. $A\otimes_S A/\mathfrak{l}\approx A/A\mathfrak{l}$ *for any left ideal* $\mathfrak{l}$ *in* S.

N.4. $\mathrm{Tor}_S^n(\ ,\ )$ *commuts with direct limit*, (cf. [2]).

Let A be an object in $\mathfrak{A}$. By S_A we denote $[A, A]$ and J( ) meanes the Jacobson radical.

2. Perfect categories

Recentely, we have defined perfect (resp. semi-perfect) categories in [13]. We shall reproduce perfect categories as a form of induced ring from $\mathfrak{A}$.

Following to Mares [15], a projective object P is called *semi-perfect* if every factor object of P has a projective cover. If any coproduct of copies of P is semi-perfect, P is called *perfect*. A Grothendieck category $\mathfrak{A}$ is called *perfect* (resp *semi-perfect*) if every (resp. finitely generated) object has a projective cover. If a ring S has the identity, then the fact that S is semi-perfect is equivalent to a fact that $\mathfrak{M}_S$ is semi-perfect. However, if S does not contain the identity then the above statement is false (seeTheorem 1 below).

Let e be an idempotent in S. Following to [17] we call e *local* if eSe is a local ring or equivalently if Se (or eS) is completely indecomposable.

We have immediately from [11] and [15]

Theorem 1. *Let R be an induced ring from a category. Then the following are equivalent.*

1) *R is semi-perfect as an R-module in $\mathfrak{M}_R{}^+$.*

2) *$R=\sum\oplus f_\alpha R$, where $\{f_\alpha\}$ is a set of mutually orthogonal local idempotents and $\{f_\alpha R\}$ is right semi-T-nilpotent with respect to* $\mathrm{J}(R)$.

3) *$R/\mathrm{J}(R)$ is semi-simple as an R-module in $\mathfrak{M}_R{}^+$ and idempotents can be lifted modulo* $\mathrm{J}(R)$ *and* $\mathrm{J}(R)$ *is small in R.*

Proof. We note that $P\neq \mathrm{J}(P)$ for every non-zero projective module P by [13], Proposition 2 or [1], Proposition 2.7. Hence, 1) $\leftrightarrow$ 3) is obtained from [15], Theorem 4.3 and 5.1. 1) $\rightarrow$ 2). Let $R=\sum\oplus e_\alpha R$. Since $e_\alpha R$ is also semi-perfect, $e_\alpha R=\sum_{i+1}^{n_\alpha}\oplus f_{\alpha_i}R$ by [15], Corollary 4.4, where $\{f_{\alpha_i}\}$ is a set of mutually orthogonal and local idempotents. Furthermore, $\{f_{\alpha_i}R\}_{\alpha_i}$ is right semi-T-nilpotent by [11], Theorem 7. 2) $\rightarrow$ 1) is clear from [11], Theorem 7.

On the other hand, for $\mathfrak{M}_R{}^+$ we have immediately from [13], Proposition 5 and its corollary

Theorem 2. *Let R be an induced ring from a category. Then the following are equivalent.*

1) *$\mathfrak{M}^+{}_R$ is semi-perfect.*

1′) *${}_R\mathfrak{M}^+$ is semi-perfect.*

2) *$R=\sum\oplus f_\alpha R$.*

2′) *$R=\sum\oplus Rf_\alpha'$, where $\{f_\alpha\}$ and $\{f_\alpha'\}$ are sets of mutually orthogonal and local indempotents* (cf. [1], Theorem 2.1).

Let S be the ring of upper tri-angular matrices with infinite degree over a division ring and $\{e_{ij}\}_{i\leq j}$ be the complete set of matrix units. Put $R=\sum\oplus e_{ii}S$. Then R is semi-perfect as a right R-module but not as a left R-module. On the other hand, ${}_R\mathfrak{M}^+$ is semi-perfect.

We have already noted in [13], Remark that Theorem P in [1] are valid for an induced ring R. If we use N.0$\sim$N.4 and the idea given in [13], we can show that

Theorem P is true for R. We state here only its some parts, which we shall use later.

Theorem 2' (Theorem P in [1]). *Let R be as above. The following are equivalent.*

1) *$\mathfrak{M}_R^+$ is perfect.*

2) *$R=\sum\oplus f_\alpha R$, where $\{f_\alpha\}$ is a set of mutually orthogonal and local idempotents and $\{f_\alpha R\}$ is a right T-nilpotent system with respect to $\mathrm{J}(R)$, (the last condition is equivalent to $\mathrm{J}(R)$ being T-nilpotent).*

3) *Every right R-module in $\mathfrak{M}_R^+$ has the same weak as projective dimension.*

3 Categories of commutative diagrams

We recall, in this section, the concept of categories of (generalized) commutative diagrams in [9] and give relations between it and rings of (generalized) tri-angular matix rings in [8].

Let I be a linearly ordered set $(1, 2, \cdots, n)$ and $\{\mathfrak{A}_i\}_{i\in I}$ be a set of abelian categories. We assume that there exist functors T_{ij}: $\mathfrak{A}_j\to\mathfrak{A}_i$ for $i<j$ such that 1) T_{ij} is cokernel preserving, 2) there exist natural transformations ψ_{ijk}: $T_{ij}T_{jk}\to T_{ik}$ such that $\psi_{ijl}T_{kj}(\psi_{jkl})=\psi_{ikl}\psi_{ijl}$ for $i<j<k<l$. We define a category $\mathfrak{A}=[I, \mathfrak{A}_i]^l$ of commutative diagrams as follows: The objects A in $\mathfrak{A}$ consist of all n-tuple $(A_1, A_2, \cdots, A_n)$; $A_i\in\mathfrak{A}_i$ with arrows $d'_{ij}=d_{ij}T_{ij}$ such that $d_{ij}T_{ik}(d_{jk})=d_{ik}\psi_{ijk}$ for $i<j<k$. The morphisms $[A, B]_{\mathfrak{A}}$ consist of all n-tuple $(f_1, f_2, \cdots, f_n)$; $f_i\in[A_i, B_i]_{\mathfrak{A}_i}$ such that $f_id_{ij}{}^AT_{ij}=d_{ij}{}^BT_{ij}(f_j)$ for $i<j$, (see [9], p. 245). $\mathfrak{A}$ is an abelian category from [9], Proposition 1.1. We assume that $\mathfrak{A}_i$ has a projective class $\mathcal{E}_i$ (see [16], p. 136). We define adjoint functors S_i, T_i betweem $\mathfrak{A}_i$ and $\mathfrak{A}$ as follows: $S_i(A_i)=(T_{1i}(A_i), \cdots, T_{i-1i}(A_i), A_i, 0, \cdots, 0)$ with arrow $d_{hi}=I_{T_{hi}(A_i)}$ for $h<i$ and $d_{ef}=\psi_{efi}$ for $e<f<i$, and $T_i(A)=A_i$, where $A=(A_1, A_2, \cdots, A_n)$. Thus, $\mathfrak{A}$ has a projective class $\cap T_i^{-1}(\mathcal{E}_i)$ whose projectives are of the form $\sum\oplus S_i(P_i)$ and their retracts, where P_i is $\mathcal{E}_i$-projective by [9], Proposition 1.2'. We note that if we take $\mathcal{E}_i$ as the class of all epimorphisms in $\mathfrak{A}_i$ then $\cap T_i^{-1}(\mathcal{E}_i)$ is also the class of all epimorphisms in $\mathfrak{A}$ by [9], Proposition 1.1. Therefore, in this case $\mathcal{E}$-projective means usual projective.

We shall generalize the above category. Let I be a well orded set and $\{\mathfrak{A}_\alpha\}_I$ a family of Grothendieck categories. We assume that functors T_{ij} for $i<j$ (resp. $\tilde{T}_{ij}$ for $i>j$) are coproduct and cokernel preserving. We shall define $\mathfrak{A}=[I, \mathfrak{A}_i]^l$ as above, namely objects of $\mathfrak{A}$ are of forms $(A_1, \cdots, A_\alpha, \cdots)$ with arrows $d_{ij}T_{ij}$ for $i<j$, $i, j\in I$. Similarly, we define $\tilde{\mathfrak{A}}=[I, \mathfrak{A}_i]^r$ with $\tilde{T}_{ij}$. For any ordinal number α in I, we put $\mathfrak{A}^\alpha=[I^\alpha, \mathfrak{A}_i]^l$, where $I^\alpha=(1, \cdots, \alpha)$. Now we assume that all $\mathfrak{A}_i$ have generating sets of completely indecomposable and small projective $\{P_{i\alpha}\}$. Then $S_i(P_{i\alpha})=(T_{1i}(P_{i\alpha}), \cdots, T_{i-1i}(P_{i\alpha}), P_{i\alpha}, 0, \cdots, 0)=\boldsymbol{P}_\alpha$ is a member of gene-

rating set of small projective in $\mathfrak{A}$ by [16], p. 121, Proposition 1.5. Similarly, $\tilde{S}_i(P_{i\alpha})=(0, \cdots, P_{i\alpha}, \tilde{T}_{i+1i}(P_{i\alpha}), \cdots)=\tilde{\boldsymbol{P}}_\alpha$ is one in $\tilde{\mathfrak{A}}$. We have natural imbedding functors $\psi_\alpha: \mathfrak{A}^\alpha \to \tilde{\mathfrak{A}}, (\tilde{\varphi}_\alpha: \tilde{\mathfrak{A}}^\alpha \to \mathfrak{A})$ such that $\psi_\alpha(\boldsymbol{A}^{(\alpha)})=(A_1, \cdots, A_\alpha, 0, \cdots, 0)$, where $\boldsymbol{A}^{(\alpha)}=(A_1, \cdots, A_\alpha)$, $A_i \in \mathfrak{A}_i$. Making use of ψ_α, we may sassume that $\mathfrak{A}$ is the colimit of $\mathfrak{A}^\alpha$. We note that $\psi_\alpha(\boldsymbol{P}^{(\alpha)})$ is projective in $\tilde{\mathfrak{A}}$ if so is $\boldsymbol{P}^{(\alpha)}$ in $\mathfrak{A}^\alpha$, however $\tilde{\varphi}_\alpha(\boldsymbol{P}^{(\alpha)})$ is not projective. It is clear that $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ are Grothendieck categories by [9], Proposition 1.1.

Let R and $R^{(i)}$ be induced rings from $\mathfrak{A}$ and $\mathfrak{A}^i$, respectively. Put $\boldsymbol{P}_\alpha^{(i)}= S_i(P_{i\alpha})$. Since $[\boldsymbol{P}_\alpha^{(n)}, \boldsymbol{P}_\beta^{(m)}]=0$ if $n>m$, $R^{(n)}=\sum_{i\leq j\leq n}\sum_{\alpha,\beta}\oplus[\boldsymbol{P}_\alpha^{(i)}, \boldsymbol{P}_\beta^{(j)}]$ is a ring of generalized lower tri-angular matrices over rings $R_i=\sum_{\alpha,\beta}\oplus[\boldsymbol{P}_\alpha^{(i)}, \boldsymbol{P}_\beta^{(i)}]\approx\sum\oplus [P_{i\alpha}, P_{i\beta}]$. The natural immbedding ψ_i induces the natural imbedding: $R^{(i)} \to R=\sum_{s\leq t}\sum_{\alpha,\beta}\oplus[\boldsymbol{P}_\alpha^{(s)}, \boldsymbol{P}_\beta^{(t)}]$. Similarly, the induced ring R from $\tilde{\mathfrak{A}}$ is the ring of upper tri-angular matrices over R_i.

Conversely, let $S=\sum_I\oplus e_\alpha S$ be the induced ring with generating idempotents. We assume S is lower tri-angular, namely $\{e_\alpha\}$ is ranged as $\{e_\alpha^{(n)}\}$ such that $e_\alpha^{(n)}Se_\beta^{(m)}=0$ if $n<m$. Put $S_n=\sum_{\alpha,\beta}\oplus e_\alpha^{(n)}Se_\beta^{(n)}$ and $M_{nm}=\sum_{\alpha,\beta}\oplus e_\alpha^{(n)}Se_\beta^{(m)}$ for $n>m$. Let $\mathfrak{A}=[I, \mathfrak{M}_{S_n}{}^+]^l$ with functor $T_{ij}(-)=(-)\otimes_{S_j}M_{ji}$. Then $\mathfrak{M}_S{}^+$ is equivalent to $\mathfrak{A}$ and S and S_i are induced rings from $\mathfrak{A}$ and $\mathfrak{M}_{S_i}{}^+$, respectively.

From the above, we have

Lemma 1. *Let $\mathfrak{A}$ and $\mathfrak{A}^n$ be as above and R, $R^{(n)}$ be the induced rings from $\mathfrak{A}$ and $\mathfrak{A}^n$, respectively. Then $\mathfrak{A}=\varinjlim \mathfrak{A}^n$ and $R=\varinjlim R^{(n)}$.*

Let $\mathfrak{A}$ be a Grothendieck category. We call $\mathfrak{A}$ *semi-simple* if every object is a coproduct of minimal objects. $\mathfrak{A}$ is called *hereditary* (resp. *semi-hereditary*) if every sub-(resp. finitely generated) object of projective is also projective. Finally, $\mathfrak{A}$ is called *semi-artinian* if every non-zero object has the non-zero socle. It is clear that $\mathfrak{A}$ has a generating set and is semi-simple if and only if the induced ring R from $\mathfrak{A}$ is a directsum of minimal right ideals.
Therefore, we have nothing to study for semi-simple categories.

Proposition 1. *Let I be a well ordered set and $\{\mathfrak{A}_i\}_{i\in I}$ a set of semi-simple categories with generating sets. Then $\mathfrak{A}=[I, \mathfrak{A}_i]^l$ is semi-artinian and semi-perfect and $\tilde{\mathfrak{A}}=[I, \mathfrak{A}_i]^r$ is perfect.*

Proof. Let $\{P_{i\alpha}\}$ be a generating set of minimal projective in $\mathfrak{A}_i$. Since $[S_i(P_{i\alpha}), S_i(P_{i\alpha})]_{\mathfrak{A}}\approx[P_{i\alpha}, P_{i\alpha}]_{\mathfrak{A}_i}$, $\boldsymbol{P}_\alpha^{(i)}=S_i(P_{i\alpha})$ is small projective by [13], Corollary 1 to Lemma 2. Furthermore, $[P_\alpha^{(i)}, P_\beta^{(j)}]=0$ for $i>j$ and $[P_\alpha^{(i)}, P_\beta^{(j)}]= [P_\alpha^{(i)}, \mathrm{J}(P_\beta^{(j)})]$ for $i<j$. Hence, $\{\boldsymbol{P}_\alpha^{(i)}\}$ is a left T-nilpotent system with respect

to J($\mathfrak{A}$). Therefore, $\mathfrak{A}$ is semi-artinian and semi-prefect by [13], Proposition 3 and Corollary to Theorem 4. Similarly, we know by [13], Corollary to Theorem 4 that $\mathfrak{A}$ is perfect.

Lemma 2 ([19]). *Let S be an induced ring from a category and e an idempotent. If SeS is projective in* $\mathfrak{M}_S{}^+$, *then Se is projective in* $\mathfrak{M}_{eSe}$.

Proof. $\mathrm{Hom}_S(eS, S)=Se$ by N. 3 and the trace ideal $\tau_S(eS)=SeS$. We quote here Silver's proof in [19], Theorem 2.5. Let $0\to K\to Se\otimes_T eS\xrightarrow{\tau} SeS\to 0$ be exact, where $T=\mathrm{Hom}_S(eS, eS)=eSe$. A diagram;

$$\begin{array}{ccc} Se\otimes_T eS\otimes_S Se & \xrightarrow{\tau\otimes 1} & SeS\otimes_S Se \\ \downarrow 1\otimes\mu & & \downarrow \mu \\ Se & \longleftarrow & SeSe \end{array}$$

is commutative, where μ is the multiplication. Since Se is S-projective by N. 1, μ is isomorphic by N. 3. On the other hand, $Se=SeSe$. Hence, $\tau\otimes 1$ is isomorphic. Therefore, $K\otimes_S Se=0$, which implies $K\otimes_S SeS=0$ by τ. Since $(eS)SeS=eS$, $eS\otimes_S SeS\approx eS$. Hence, $Se\otimes_T eS\approx Se\otimes_T eS\otimes_S SeS\approx SeS\otimes_S SeS$ is S-projective by the assumption, (which is obtained from the first exact sequence by taking $\otimes_S SeS$). Noting $Se\otimes_S Se\approx eSe$, we can prove from the proof of [19], Lemma 2.8 that Se is projective in $\mathfrak{M}_{eSe}$.

Corollary ([18], [8]). *Let S and e as above. If S is hereditary in* $\mathfrak{M}_S{}^+$, *then eSe is hereditary in* $\mathfrak{M}_{eSe}$.

Proof. Let $\mathfrak{r}$ be a right ideal in eSe. Since $\mathfrak{r}S$ is S-projective, $\mathfrak{r}=\mathfrak{r}eSe$ is a coretract of copies of Se. Hence, $\mathfrak{r}$ is eSe-projective by the lemma.

4. J-nilpotent and connected sequence

In the structure theorems of semi-primary and hereditary rings the nilpotency of the radical is very important. (cf. [4], [8] and [14]). We define the nilpotency of projective object in a catrgory.

Let $\mathfrak{A}$ be a semi-perfect Grothendieck category with a generating set of (completely indecomposable and) small projective $\{P_\alpha\}$. For an object A in $\mathfrak{A}$ we put $\mathrm{J}^n(A)=\mathrm{J}(\mathrm{J}^{n-1}(A))$. If $\mathrm{J}^m(A)=0$ for some m, we call A *J-niplotent*. If $\mathrm{J}^{n-1}(A)\neq 0$, $\mathrm{J}^n(A)=0$, n is called the *index* of A. Next, we generalize the notion of a connected sequence of idempotents in [14]. A sequence $(P_1, P_2\cdots, P_n)$ is called a *left connected sequence* if $[P_{i+1}, \mathrm{J}(P_i)]\neq 0$ for $i=1, \cdots, n-1$ and n is called the *length* of the sequence. Similarly, a sequence $(P_1, P_2, \cdots, P_n)$ is called a *right connected sequence*, if $[P_i, \mathrm{J}(P_{i+1})]\neq 0$ for $i=1, \cdots, n-1$. By $lC(P_\alpha)$ (resp. $rC(P_\alpha)$) we denote all left (resp. right) connected sequences such thar $P_1=P_\alpha$. A sequence in $lC(P_\alpha)$ with maximal lenght is called a *maximal sequence*. By $\mathrm{lL}(P_\alpha)$ (resp. $\mathrm{rL}(P_\alpha)$) we denote the lenght of maximal sequences of P_α, (if $lC(P_\alpha)$ has non maximal sequences, $\mathrm{lL}(P_\alpha)=\infty$).

We note that if P_α's are completey indecomposable and projective, $[P_\alpha, P_\beta]=[P_\alpha, \mathrm{J}(P_\beta)]$ if $P_\alpha \not\approx P_\beta$ by [13]., Corollary to Lemma 2. From now on when we consider connected sequences, we take completely indecomposable projective objects, unless otherwise stated.

Proposition 2. *Let A be a* J*-nilpotent object of index n. Take $\{A_i\}_{i=1}^{n+1}$ $(A_1=A)$ and $f_i\in[A_i, \mathrm{J}(A_{i-1})]$. Then $f_1\cdots f_n=0$. Especially, if A is projective, then $\mathrm{J}(S_A)^n=0$.*

Proof. We assume $f_i\cdots f_n(A_{n+1})\subset \mathrm{J}^{n-i+1}(A_i)$. Then $f_{i-1}f_i\cdots f_n(A_{n+1})\subset \mathrm{J}^{n-i+1}(f_{i-1}(A_i))\subset \mathrm{J}^{n-i+1}(\mathrm{J}(A_{i-1}))=\mathrm{J}^{n-i+2}(A_{i-1})$. Hence, $f_1\cdots f_n=0$. If A is projective, $[A, \mathrm{J}(A)]\supset \mathrm{J}(S_A)$ by [20], Lemma. Therefore, $\mathrm{J}(S_A)^n=0$.

Corollary. *Let $\mathfrak{A}$ be a semi-perfect Grothendieck category with a generating set of small projective P_α. If all P_α are* J*-nilpotent, $\mathfrak{A}$ is semi-artinian. Moreover, if the indeces are bounded, $\mathfrak{A}$ is perfect.*

Proof. We may assume that P_α's are completely indecomposable by [13], Corollary 1 to Theorem 4. Hence, $\mathfrak{A}$ is semi-artinian by Proposition 2 and [13], Proposition 3. If the indeces are bounded, $\mathfrak{A}$ is perfect by [13], Lemma 6.

Proposition 3. *Let $\mathfrak{A}$ be a Grothendieck category with a generating set of samll objects. We assume that $\mathfrak{A}$ is semi-hereditary. Then for any completely indecomposable projective, P_j,*

1) *Any non-zero element in $[P_1, P_2]$ is monomorphic.*

2) *If P_i is* J*-nilpotent of index n_i, then $[P_1, P_2]=0$ if $n_1>n_2$ or $n_1=n_2$, $P_1\not\approx P_2$ and moreover $[P_1, \mathrm{J}(P_1)]=0$.*

3) *If P_1 is* J*-nilpotent, $lL(P_1)<$ the index of P_1. If $\mathfrak{A}$ is hereditaty and perfect, then $[P_\alpha, P_\beta]\neq 0$ implies $[P_\beta, P_\alpha]=0$ for any non-isomorphic completely indecomposable projecitves P_α, P_β and $[P_\alpha, \mathrm{J}(P_\alpha)]=0$.*

Proof. 1). P_i is finitely generated by [13], Corollary to Lemma 2. Hence $\mathrm{Im}\, f$ is projective by the assumption for $f\in[P_1, P_2]$. Therefore, $f=0$ or f is monomorphic, since P_1 is indecomposable. 2). Since $\mathrm{J}(P)\supset \mathrm{J}(Q)$ for $P\supset Q$, $[P_1, \mathrm{J}(P_1)]=0$. Similarly. $[P_1, P_2]=0$ if $n_1>n_2$ or $P_1\not\approx P_2$, since $\mathrm{J}(P_2)$ is uniquc maximal in P_2. 3). It is clear from 1) and Proposition 2. The last statement is clear from Proposition 2 and [13], Lemma 6.

For the connected sequences we obtain similarly from the definition

Proposition 4. *Let $\mathfrak{A}$ and P_i be as above. We assume $lL(P_i)=n_i$ (resp. $rL(P_i)=m_i$). If $n_1>n_2$ (resp. $m_1<m_2$) or $n_1=n_2$* (resp. $m_1=m_2$) *$P_1\not\approx P_2$, then $[P_1, P_2]=0$ and $[P_1, J(P_1)]=0$.*

5. Perfect and hereditary categories

Let R be a ring with identity. We showed in [3], [8] and [14] that every hereditary semi-primary ring is a ring of lower triangular matrices over semi-simple artinian rings. We also studied hereditary categories of commutative diagrams in [9] (cf. Lemma 2). On the other hand, we defined perfect categories in the pervious section. Using them, we shall study, in this section, perfect categories with some assumptions, which is a generalization of [8].

First, we give an example. Let Δ be a division ring and S the ring of column finite matrices with countably infinite degree over Δ. Let $\{e_{ij}\}$ be the completely set of matrix units. Put $R=\sum_{i\leq j}\oplus e_{ij}S=\sum_{i\leq j}\oplus e_{ij}R$, (resp. $\tilde{R}=\sum_{i\geqslant j}\oplus e_{ij}S$). Then $R=\cup R_n$ (resp. $\tilde{R}=\cup\tilde{R}_n$), where $R_n=\sum_{i\leq n}\oplus e_{ij}R$ (resp. $\tilde{R}_n=\sum_{i\leq n}\oplus e_{ij}R$). R_n and $\tilde{R}_n$ are hereditary by [5] or [8], Theorem 1. Moreover, $e_{ij}R$ is J-nilpotent of index i. We shall show from Theorem 3 below that R and $\tilde{R}$ are hereditary in $\mathfrak{M}_R{}^+$ and $\mathfrak{M}_{\tilde{R}}{}^+$, respectively. We note that $\mathrm{lL}(e_{ii}R)=i$, but $\mathrm{rL}(e_{ii}R)=\infty$, (resp. $\mathrm{rL}(e_{ii}\tilde{R})=i$, but $\mathrm{lL}(e_{ii}\tilde{R})=\infty$).

Lemma 3 ([18], Proposition 1). *Let $\{\Delta_i\}_I$ be a set of division rings and R the induced ring from $[I, \mathfrak{M}_{\Delta_i}]^l$. If the radical N of R is projective in $\mathfrak{M}_R{}^+$, then R is hereditary.*

Proof. Let $R=\sum_I\oplus e_\alpha R$. Then $N=\sum_{\alpha>\beta}\oplus e_\alpha Re_\beta$ and $\mathfrak{r}_\alpha=\sum_{\beta;<\alpha}\oplus e_\alpha Re_\beta$ is projective by the assumption and [13], Lemma 7. It is clear that every minimal object in $\mathfrak{M}_R{}^+$ is isomorphic to some $e_\alpha R/\mathfrak{r}_\alpha=\Delta_\alpha$. From the assumption hd. $e_\alpha R/\mathfrak{r}_\alpha\leqslant 1$. We shall show by the standard argument that R is hereditary. Let M be an object in $\mathfrak{M}_R{}^+$ and $0\to M\to Q_0\to Q_1\to 0$ exact with Q_0 injective. Then $0=\mathrm{Ext}^2(\Delta_i, M)=\mathrm{Ext}^1(\Delta_i, Q_1)$. We shall show that Q_1 is injective in $\mathfrak{M}_R{}^+$. Let

$$\begin{array}{ccccc} 0 & \to & A & \overset{i}{\to} & B \\ & & \downarrow f & & \\ & & Q_1 & & \end{array}$$

be an exact sequence. We take a maximal extension f_0: $A_0\supset A\to Q_1$. If $A_0\neq B$, there exists b in B such that $(bR+A_0)/A_0$ is minimal, since R is semi-artinian by Proposition 1. Hence, $\mathfrak{r}=\{r\mid\in R, br\in A_0\}$ is a maximal right ideal. We define $g:\mathfrak{r}\to Q_1$ by $g(r)=f_0(br)$. Since $\mathrm{Ext}^1(R/\mathfrak{r}, Q_1)=0$, $0\leftarrow[\mathfrak{r}, Q_1]\leftarrow[R, Q_1]$ is exact and we have $g' \in [R, Q_1]$ such that $g'|\mathfrak{r}=g$. Since $B \in \mathfrak{M}_R{}^+$, there exists an idempotent e in R such that $be=b$. Put $q_1=g'(e)$ and $f_1(a_0+br)=f_0(a_0)+q_1r$. If $br \in A_0$, $f_1(br)=f_0(ber)=g(er)=g'(er)=q_1r$. Hence, we have an extension of A_0. Therefore, $A_0=B$ and Q_1 is injective. Thus, R is hereditary.

The above proof suggests us

Corollary. *Let R be an induced ring from a category and M an object in $\mathfrak{M}_R{}^+$. Then M is injective if and only if any element in $[\mathfrak{r}, M]$ is extended to $[R, M]$ for any right ideal $\mathfrak{r}$ of R.*

In the first step, we shall generalize the conditions in [9], Theorem 3.12. For $[I, \mathfrak{A}_i]$ with functors $T_{\alpha\beta}$

$(*)^r$ 1) *$\psi_{\alpha\beta\gamma}$: $T_{\alpha\beta}T_{\beta\gamma} \to T_{\alpha\gamma}$ is monomorphic for all $\alpha>\beta>\gamma$,*
2) *For any given numbers $\alpha=\alpha_1<\alpha_2<\cdots<\alpha_n=\beta$*

$$T_{\beta\alpha}(P_\alpha) = T_{\beta\alpha_2}T_{\alpha_2\alpha_1}(P_\alpha) \oplus T_{\beta\alpha_3}(K^{\alpha_3}(P_\alpha)) \oplus \cdots \oplus T_{\beta\alpha_{i-1}}(K^{\alpha_{n-1}}(P_\alpha)) \oplus K^{\alpha}(P_\alpha),$$

where P_α is any object in $\mathfrak{A}_\alpha$ and $K_h^{\alpha_i}(P_\alpha)$ are defined inductively from the above equality and this equality is given through $\psi_{\alpha\beta\gamma}$.

(resp. $(*)^l$ *replacing $\alpha>\beta>\gamma$ and $\alpha_1<\alpha_2<\cdots<\alpha_n$ in $(*)^r$ by $\alpha<\beta<\gamma$ and $\alpha_1>\alpha_2>\cdots>\alpha_n$).*

Theorem 3. ([9]). *Let I be a well ordered set and $I^{\aleph_0}=(1, 2, \cdots, n, \cdots)$ the set of natural numbers. Let $\{\mathfrak{A}_i\}_I$ be a set of semi-simple Grothendieck categories with generating sets. If $\tilde{\mathfrak{A}}=[I, \mathfrak{A}]^r$ (resp. $[I, \mathfrak{A}_i]^l$) is hereditary, then functors $T_{\alpha\beta}$ satisfy $(*)^r$ (resp. $(*)^l$). Conversely, $\tilde{\mathfrak{A}}=[I, \mathfrak{A}_i]^r$ (resp. $\mathfrak{A}=[I^{\aleph_0}; \mathfrak{A}_i]^l$)[1] satisfies the condition $(*)$, then $\tilde{\mathfrak{A}}$ and $\mathfrak{A}$ are hereditary.*

Proof. Let $\{P_{i\alpha}\}$ be a generating set of minimal objects in $\mathfrak{A}_i$ and $R=\sum_{i\leq j}\oplus[P_\alpha^{(i)}, P_\beta^{(j)}]$ be the induced ring from $\mathfrak{A}$ with functors T_{ij}, where $P^{(i)}=\tilde{S}_i(P_{i\alpha})$. We assume that $\mathfrak{A}$ and hence, R are hereditary. Since $\mathfrak{A}_i$ is semi-simple with generating set, $\mathfrak{A}_i$ is a coproduct of simple categories $\mathfrak{A}_{i\alpha}$. We may assume that $P_{i\alpha_i}$ is a generator in $\mathfrak{A}_{i\alpha_i}$. Furthermore, $\mathfrak{A}=[I, \mathfrak{A}_i]^r \approx [I', \mathfrak{A}_{i\alpha_i}]^r$ with functors $T'_{i\alpha_i, j\beta_j}$ such that $T'_{i\alpha_i i\beta_i}=0$, $T'_{m\beta_m n\alpha_n}=p_m\beta_m T_{mn} i_{n\alpha_n}$ for $n<m$ and $T_{ij}=\sum\oplus T'_{i\alpha_i, j\beta_j}$, where i is the inclusion $\mathfrak{A}_{n\alpha_n}$ to $\mathfrak{A}_n$ and p is the projection of $\mathfrak{A}_m$ to $\mathfrak{A}_{m\beta_m}$. Let $n_1<n_2<\cdots<n_m$ be given numbers of I and $\mathfrak{A}^m=[(n_1, \cdots n_m), \mathfrak{A}_i]^r$. Put $e(n_i, \alpha_i)=\sum_{i=1}^t 1_{P_{n_i\alpha_i}}$ in $\tilde{R}$ for any finite number of $P_{n_i\alpha_i}$. Let $\tilde{R}^{(m)}$ be the induced subring of $\tilde{R}$ from $\mathfrak{A}^m$. Then $\tilde{R}(n_i, \alpha_i)=e(n_i, \alpha_i)\tilde{R}e(n_i, \alpha_i)=e(n_i, \alpha_i)\tilde{R}^m e(n_i, \alpha_i)$. Furthermore, $\tilde{R}^m=\bigcup e(n_i, \alpha_i)\tilde{R}e(n_i, \alpha_i)$, where (n_i, α_i) runs over all (n_i, α_i) and n_i may be overlaped, and $M=\bigcup Me(n_i, \alpha_i)$ for $M\in \mathfrak{M}_R{}^+$ and $Me(n_i, \alpha_i)\in \mathfrak{M}_R{}^+(n_i, \alpha_i)$. On the other hand, $\tilde{R}(n_i, \alpha_i)$ is hereditary

1) If $|I|>\aleph_0$ or $|I|=\aleph_0$ and I contains the last element, $[I, \mathfrak{A}_i]^l$ is, in general, not hereditary by Lemma 8 in the forth comming paper of the same title III.

by Corollary to Lemma 2. Hence, w.gl.dim $\tilde{R}(n_i, \alpha_i) \leq 1$. Therefore, w.gl.dim $\tilde{R}^m \leq 1$ by [7], Proposition 1. It is clear that $\mathfrak{A}^m$ is perfect. Thus, $\tilde{R}^m$ is hereditary by Theorem 2′. Therefore, the condition $(*)^r$ is obtained from [9], Theorem 3.12. Similarly, we obtain $(*)^l$ for hereditary categories $[I, \mathfrak{A}_i]^l$. Conversely, we assume $\mathfrak{A}=[I, \mathfrak{A}_1]^r$ satisfies $(*)^r$. Then from the above $\tilde{R}=\cup \tilde{R}(n_i)$, where $\tilde{R}(n_i)=[(n_1, \cdots, n_t), \mathfrak{A}_i]^r$ for any finite subset $(n_1, \cdots, n_t)$ of I. $\tilde{R}(n_i)$ is hereditary by [9], Theorem 3.12. Therefore, $\mathfrak{A}$ is hereditary from the above arguments and a fact that $\tilde{\mathfrak{A}}$ is perfect. Next, we consider $\mathfrak{A}$. Let $R=\sum \oplus e_{n\alpha_n}R$ be the induced ring and $R_n=\sum_{i\leq n} \oplus e_{i\alpha_i}R$. Then $\mathrm{J}(R)=\sum_{m<n} \oplus e_{n\alpha_n}Re_{m\alpha_m}$ and $\mathfrak{r}_n=\sum_{m,\alpha_n,\alpha_m} \oplus e_{n\alpha_n}Re_{m\alpha_m}$ is projective, in $\mathfrak{M}_{R_n}{}^+$ by [9], Theorem 2.13 and hence, projective in $\mathfrak{M}_R{}^+$ by the structure of $\mathfrak{A}$. Therefore, $\mathrm{J}(R)=\sum \oplus \mathfrak{r}_i$ is projective. Thus, $\mathfrak{A}$ is hereditary by Lemma 3.

Theorem 4. *Let $\mathfrak{A}$ be a semi-perfect Grothendieck category with a generating set of small projectives. If $\mathfrak{A}$ is perfect and hereditary, then $\mathfrak{A}$ is equivalent to $[I, \mathfrak{A}_i]^r$ with functors T_{ij}, which satisfy the condition $(*)^r$. If $\mathfrak{A}$ is semi-artinian and hereditary, then $\mathfrak{A}$ is equivalent to $[I, \mathfrak{A}_i]^l$ with functors T_{ij}, which satisfy the condition $(*)^l$, where I is a well ordered set and $\mathfrak{A}_i$'s are semi-simple categories with generating sets.*

Proof. We assume that $\mathfrak{A}$ is perfect and hereditary. Let $\{P_\alpha\}$ be a generating set of indecomposable projective objects in $\mathfrak{A}$. Since $\mathfrak{A}$ is perfect, there exists P_α such that $[P_\alpha, P_\beta]=0$ for all $P_\alpha \not\approx P_\beta$ and $[P_\gamma, \mathrm{J}(P_\gamma)]=0$ for all P_γ by Proposition 3. We denote all of such a type P_α by $P_\alpha^{(1)}$. If we take out all of $\{P_\alpha^{(1)}\}$ from $\{P_\gamma\}$, we can find projectives P_β such that $[P_\beta, P_\gamma]=0$ if $P_\beta \not\approx P_\gamma$ and $P_\gamma \in \{P_\alpha\}-\{P_\beta^{(1)}\}$. We denote such P_β by $P_\beta^{(2)}$. We can define $P^{(\alpha)}$ inductively. Then the induced ring R from $\mathfrak{A}$ is a ring of tri-angular matrices: $R=\sum_{i\leq j} \oplus [P_\alpha^{(j)}, P_\beta^{(i)}]$. Hence, $\mathfrak{A}$ is equivalent to $\tilde{A}=[I, \mathfrak{M}_S{}^+]^r$ with functors $T_{ij}(-)=(-)\otimes_{S_j}(\sum [P_\gamma^{(i)}, P_\gamma^{(j)}]$, where $S_i=\sum \oplus [P_\gamma^{(i)}, P_\gamma^{(j)}]$ is semi-simple. On the other hand T_{ij} is coproduct and cokernel preserving. Hence, T_{ij}'s satisfy $(*)^r$ by Theorem 3. The remaining part is proved similarly to the above.

The above proof suggests us

Proposition 5. *Let $\mathfrak{A}$ be a Grothendieck category with a generating set of completely indecomposable and samll projective $\{P_\alpha\}_I$. Then $\mathrm{lL}(P_\alpha)$ (resp. $\mathrm{rL}(P_\alpha)$) is bounded for any $\alpha \in I$ if and only if $\mathfrak{A}$ is equivalent to $[I^{\aleph_0}, \mathfrak{A}_i]^l$ (resp. $[I^{\aleph_0}, \mathfrak{A}_i]^r$), where $\mathfrak{A}_i$'s are semi-simple categories with generating sets. Therefore, if $\mathrm{lL}(P_\alpha)$ (resp. $\mathrm{rL}(P_\alpha)$ is bounded for any α, then $\mathfrak{A}$ is semi-artinian (resp. perfect) and P_α is J-nilpotent for all α.*

Theorem 5. *Let $\mathfrak{A}$ be a semi-perfect Grothendieck category with a generating set of projective and samll objects P_α. Then $\mathfrak{A}$ is semi-hereditary and all P_α are J-*

nilpotent if and only if $\mathfrak{A}$ *is equivalent to* $[I^{\aleph_0}, \mathfrak{A}_i]^l$ *with functors* T_{ij}, *which satisfy the condition* $(*)l$ *in Theorem* 3, *where* $\mathfrak{A}_i$*'s are semi-simple categories with generating sets.*

Proof. It is clear from the defintion of $[I^{\aleph_0}, \mathfrak{A}_i]^l$, Proposition 3,3, Theorem 4 and Proposition 5.

Theorem 6. *Let* $\mathfrak{A}$ *and* $\{P_\alpha\}$ *be as in Theorem* 5. *If* $\mathfrak{A}$ *is semi-hereditary, then the following are equivalent.*

1) *All* P_α *are* J*-nilpotent.*

2) $lL(P_\alpha)<\infty$ *for any* α, *(*P_α *may not be indecomposable).*

Furthermore the following are equivalent.

1) *All* P_α *are* J*-nilpotent and* $\mathfrak{A}$ *is perfect.*

2) $lL(P_\alpha)$ *and* $rL(P_\alpha)$ *are bounded for any* α, *(*P_α *may not be indecomposable).*

Proof. It is clear from Theorems 4 annd 5, and König Graph theorem and Krull-Remak-Schmidt's theorem, since $\boldsymbol{P}_\alpha$'s are small.

Theorem 7. *Let* $\mathfrak{A}$ *be a semi-perfect Grothendieck category with a generating set of completely indecomposable projective and small objects* P_α. *Then the following are equivalent.*

1) $\mathfrak{A}$ *is equivalent to a category of commutative diagrams* $[I^{\aleph_0}, \mathfrak{A}_i]l$ *(resp.* $[I^{\aleph_0}, \mathfrak{A}_i]^r$*) over semi-simple categories* $\mathfrak{A}_i$ *with generating sets.*

2) $lL(P_\alpha)<\infty$ *(resp.* $rL(P_\alpha)<\infty$*) for all* α.

3) *There exists a fully imbedding functor* φ *of* $\mathfrak{A}$ *to a hereditary category of commutative diagrams* $\mathfrak{B}=[I^{\aleph_0}, \mathfrak{A}_i']^l$ *with functors* T_{ij}' *over semi-simple categories* $\mathfrak{A}_i'$ *such that* $[P_\alpha/\mathrm{J}(P_\alpha), P_\alpha/\mathrm{J}(P_\alpha)]_{\mathfrak{A}}\approx[\varphi(P_\alpha)/\varphi(\mathrm{J}(P_\alpha)), \varphi(P_\alpha)/\varphi(\mathrm{J}(P_\alpha))]_{\mathfrak{B}}=[(\boldsymbol{P}_\alpha/\mathrm{J}(\boldsymbol{P}_\alpha), \boldsymbol{P}_\alpha/\mathrm{J}(\boldsymbol{P}_\alpha)]_{\mathfrak{B}}$, *(resp. changing* $[\ \ ]^l$ *by* $[\ \ ]^r$*), where* $\boldsymbol{P}_\alpha$ *is a projective cover of* $\varphi(P_\alpha)$ *in* $\mathfrak{B}$ *and* $\{\boldsymbol{P}_\alpha\}$ *is a generatinga set.*

Proof. 1) $\to$ 2) is clear from the observation in §3. 2) $\to$ 1). It is proved by Proposition 5. 1) $\to$ 3). Let $\mathfrak{A}\approx[I^{\aleph_0}, \mathfrak{A}_i]^l$ with functor $T_{ij}(i<j)$. Let $\mathfrak{A}_i$ be a minimal object in $\mathfrak{A}_i$. Then $P_i=S_i(A_i)=(T_{1i}(A_i), T_{2i}(A_i), \cdots, T_{i-1i}(A_i)$ $A_i, 0\cdots)$ is a member of a generating set in $\mathfrak{A}$. Let $\mathfrak{B}=[I^{\aleph_0}, \mathfrak{A}_i]$, with functor $T_{ij}'=\sum_{i_1<i_2\dots<i_t<j}\oplus T_{ii}T_{i_1,i_2}\cdots T_{i_{1_i}}\oplus T_{tj}$, changing arrows for $d_{ij}(\sum\phi_{i_1\cdots i_t}T_{ii_1}\cdots T_{i_tj}$ $\oplus 1T_{ij})$, where $\phi_{i_1\cdots i_t}$ are natural transformations $T_{ii_1}\cdots T_{i_tj}\to T_{ij}$. We have a faithful functor φ: $\mathfrak{A}\to\mathfrak{B}$ from [9], p 197. Put $\boldsymbol{P}_i=S_i'(\mathfrak{B}_i)$ in $\mathfrak{B}$. Since $\mathrm{J}(P_i)$ $=(T_{1i}(A_i), \cdots, T_{i-1j}(A_i), 0, \cdots)=\varphi(\mathrm{J}(P_i))$ except arrows and $\mathrm{J}(\boldsymbol{P}_i)=(T_{1i}'(A_i), \cdots,$ $T'_{i-1k}(A_i), 0, \cdots)$, $[P_i/\mathrm{J}(P_i), P_i/\mathrm{J}(P_i)]_{\mathfrak{A}}\approx[A_i, A_i]_{\mathfrak{A}_i}\approx[\boldsymbol{P}_i/\mathrm{J}(\boldsymbol{P}_i), \boldsymbol{P}_i/\mathrm{J}(\boldsymbol{P}_i)]_{\mathfrak{B}}$. If we take the natural morphism f: $\boldsymbol{P}_i\to\varphi(P_i)$, which is induced from 1_{A_j}, f is epimorphic amd $\boldsymbol{P}_i$ is a projective cover of $\varphi(P_i)$, since $\mathrm{J}(\varphi(P_1))$ is unique maximal in $\varphi(P_i)$. Fruthermore, $\mathfrak{B}$ is hereditary from Theorem 4 and [10], Corollary to Theorem 10. 3) $\to$ 2). We assume that there exists φ as in 3). If

$[P_\alpha, P_\beta]_{\mathfrak{A}} \neq 0$ for $P_\alpha \not\approx P_\beta$, then $[\boldsymbol{P}_\beta, \boldsymbol{P}_\beta] \neq 0$ and $\boldsymbol{P}_\alpha \not\approx P_\beta$, since $\boldsymbol{P}_\alpha$'s are projective covers of $\varphi(P_\alpha)$'s. Further, $[P_\alpha, \mathrm{J}(P_\alpha)]_{\mathfrak{B}}$ is isomorphic to a submodule of $[\varphi(P_\alpha), \varphi(\mathrm{J}(P_\alpha))]_{\mathfrak{B}} \subset [\varphi(P_\alpha), \mathrm{J}(\varphi(P_\alpha))]_{\mathfrak{A}}$, since $\varphi(\mathrm{J}(P_\alpha)) \neq \varphi(P_\alpha)$, $\boldsymbol{P}_\alpha$ is a projetive cover of $\varphi(P_\alpha)$ and $\boldsymbol{P}_\alpha$ is indecomposable. On the other hand, $[\varphi(P_\alpha), \mathrm{J}(\varphi(P_\alpha))]$ is induced from $[\boldsymbol{P}_\alpha, \mathrm{J}(\boldsymbol{P}_\alpha)]=0$ since $1\mathrm{L}(\boldsymbol{P}_\alpha)<\infty$. Hence, $1\mathrm{L}(P_\alpha) \leqslant 1\mathrm{L}(\boldsymbol{P}_\alpha)<\infty$. We have similar results for $[I^{\aleph_0}, \mathfrak{A}_i]^r$.

Finally, if we restruct ourselves to a ring with identity, we have immediately from Proposition 5.

Proposition 6. *Let R be semi-perfect ring with identity. and $\{e_i\}_{i=1}^n$ be a complete set of mutually orthogonal and local idempotents. Then the following are equivalent.*

1) *$rL(e_iR)$ (or $rL(Re_i)$)$<\infty$ for any i.*
2) *$lL(e_iR)$ (or $lL(Re_i)$)$<\infty$ for any i.*
3) *R is generalized tri-angular matrix ring over semi-smiple artinan rings.*

In such a case, R is semi-primary. Especially, if R is right (left) perfect and hereditary R is a semi-primary ring.

REMARK. It is clear that Theorems 3, 6 and 7 are generalizations of [3], Theorem 4.1, [14], Theorem 5 and [8], Theorem 5. However, we drop the assertion gl.dim $R/N^2<\infty$, because it seems to us that it does not contain a special categorical meaning. If we want this result, we may consider the ring induced from a category.

Let Δ be a divsion ring and T_n the ring of lower tri-angular matrices over Δ with degree n. Then $\mathfrak{A}=\pi\ \mathfrak{M}_{T_n}$ is a hereditary and perfect category and generators are J-nilpotent, whose indeces are not bounded. Let S be the ring of lower tri-angular matrices over Δ with countable infinite degree and $\{e_{ih}\}$ the set of matrix units. We condider a subset $e_{i'j'}$ as follows: if $i'=1 j'=1$, if $i'=2$, $j'=2$ and if $i'=3, j'=1, 2, 3$. We assume $i'<3$. If $i' \neq n(n+1)/2$ for any n, $j'=i'$. If $i'=n(n+1)/2$ for some n, j' are $\{(n-1)(n-2)/2\}+3$ nearest numbers from i, except $n(n-1)/2$, $(n-1)(n-2)/2, \cdots$, (for instance, if $i'=15$, $j'=15, 14, \cdots, 11, {}^{10}9, \cdots, 7, {}^{6}5$). Put $R=\sum \oplus e_{i'j'}\Delta$. Then we can easily check that $\mathfrak{M}_R{}^+$ is heredetary, perfect and all P_α are J-nilpotent without boundary and further, $\mathfrak{M}_R{}^+$ can not be expressed as a coproduct of two full subcategories.

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