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PERFECT CLOAKING OF ELLIPSOIDAL REGIONS

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Abstract. Most of the existing explicit forms of cloaking devices concern spherical regions which exhibit the following behavior. If the cloaking region comes from blowing up a point we achieve perfect cloaking, but the conductivity tensor becomes singular and therefore hard to realize. A nonsingular conductivity tensor can be achieved by blowing up a small sphere, but then the cloaking is not perfect, since the interior field is controlled by the square of the radius of the initial small sphere. This behavior reflects the highly focusing effect of the spherical system where the 2-D manifold of a sphere degenerates to the 0-D manifold of its center as the radius diminishes to zero. In the present work, we demonstrate a cloaking region in the shape of an ellipsoid which achieves perfect interior cloaking and at the same time preserves the regularity of the conductivity tensor. This is possible since the confocal ellipsoidal system does not exhibit any loss of dimensionality as the ellipsoid collapses down to its focal ellipse. This is another example where high symmetry has a high price to pay. Furthermore, from the practical point of view, the most "economical" cloaking structure, for three dimensional objects, is provided by the ellipsoid which has three degrees of freedom and therefore can best fit any reasonable geometrical object. Cloaks in the shape of a prolate or an oblate spheroid, an almost disk, an almost needle, as well as a sphere are all cases of ellipsoidal degeneracies.

1. Introduction. The existence of hidden primary or secondary (such as scatterers) sources goes back to Helmholtz [17], where he demonstrated that it is possible to have currents within a conductive medium which generate a null field in its exterior. Electroencephalography, magnetoencephalography and electric impedance tomography are intimately related to this observation. Devaney and Wolf [8], Friedlander [9], Kerker [19] and Bleistein an Cohen [1] have investigated further this idea of hidden primary sources. Nevertheless, it is the last decay in which the idea of cloaking, as a technique of making objects "invisible", has gained a lot of attention [16], [14], [15], [10], [11], [12], [13], [6], [5], [22], [20], [2], [4], [23], [21], [3]. There are actually two ways to prove the existence of cloaked regions. One is based on spectral methods, where one proves that it is possible to

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choose the coefficients of an appropriate eigenfunction expansion in such a way that the excitation field does not enter the region in which we want to hide [5]. The other one utilizes the invariants of the governing equations with respect to coordinate transformations in such a way that a single point singularity in the domain of the transformation blows up to a three dimensional cloaked region in its range [11]. A fairly extended literature on the subject is reported in [12].

The present work is focused on the construction of a particular three dimensional cloak which can be adapted to almost any shape. In contrast to the isotropic behavior of a spherical cloak, the cloak we propose here has complete anisotropy which can be chosen at will. This report is organized as follows. In Section 2 we provide a short introduction to the ellipsoidal coordinate system which makes the paper self-readable. Then the transformation and its inverse is introduced and discussed in Section 3. Finally the material tensor, which guides the field to avoid the cloaking region, is calculated in Section 4.

2. The ellipsoidal system. We start with the essentials of the ellipsoidal system in order to fix the notation [24], [18]. The definition of an ellipsoidal system demands the determination of a reference ellipsoid

$$\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2} = 1, \tag{1}$$

where $0 < a_3 < a_2 < a_1 < +\infty$, which fixes the foci of the system and establishes the standards of every spatial direction. The reference ellipsoid (1) plays the role of the unit sphere in the case of the spherical system. The six foci's of the ellipsoidal system are located at the points $(\pm h_2, 0, 0)$, $(\pm h_3, 0, 0)$ and $(0, \pm h_1, 0)$, where

$$h_1^2 = a_2^2 - a_3^2, \tag{2}$$

$$h_2^2 = a_1^2 - a_3^2, \tag{3}$$

$$h_3^2 = a_1^2 - a_2^2, \tag{4}$$

and they are related by the equation

$$h_1^2 - h_2^2 + h_3^2 = 0. \tag{5}$$

The backbone of the ellipsoidal system is given by the focal ellipse

$$\frac{x_1^2}{h_2^2}+\frac{x_2^2}{h_1^2} = 1, x_3 = 0 \tag{6}$$

and the focal hyperbola

$$\frac{x_1^2}{h_3^2}-\frac{x_3^2}{h_1^2} = 1, x_2 = 0. \tag{7}$$

In the first octant, the ellipsoidal coordinates (ρ, μ, ν) are related to the Cartesian coordinates by

$$x_1 = \frac{\rho\mu\nu}{h_2 h_3}, \tag{8}$$

$$x_2 = \frac{\sqrt{\rho^2 - h_3^2}\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2}}{h_1 h_3}, \tag{9}$$

$$x_3 = \frac{\sqrt{\rho^2 - h_2^2}\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2}}{h_1 h_2}, \tag{10}$$

where $h_2 < \rho < +\infty$, $h_3 < \mu < h_2$, and $0 < \nu < h_3$, while the other seven octants are specified by considering the appropriate signs of the x_i's. The variable $\rho =$ constant specifies an ellipsoid, and therefore it corresponds to the radial variable of the spherical system. In particular, the focal ellipse (6) corresponds to the value $\rho = h_2$. The pair (μ, ν) identifies a point on the ellipsoid $\rho =$ constant and therefore it can be considered as the angular part of the spherical system.

By varying the variable ρ we obtain a family of ellipsoids,

$$\frac{x_1^2}{\rho^2} + \frac{x_2^2}{\rho^2 - h_3^2} + \frac{x_3^2}{\rho^2 - h_2^2} = 1, \quad \rho^2 \in (h_2^2, +\infty). \tag{11}$$

Similarly, the variation of the variable μ defines the family of hyperboloids of one sheet,

$$\frac{x_1^2}{\mu^2} + \frac{x_2^2}{\mu^2 - h_3^2} + \frac{x_3^2}{\mu^2 - h_2^2} = 1, \quad \mu^2 \in (h_3^2, h_2^2), \tag{12}$$

and the variation of the variable ν defines the family of hyperboloids of two sheets,

$$\frac{x_1^2}{\nu^2} + \frac{x_2^2}{\nu^2 - h_3^2} + \frac{x_3^2}{\nu^2 - h_2^2} = 1, \quad \nu^2 \in (0, h_3^2). \tag{13}$$

For a detailed analysis of the ellipsoidal system we refer to [7].

3. The ellipsoidal transformation. Suppose we want to cloak an ellipsoidal region which is bounded by (1) or, in view of (11), by $\rho = a_1$. Then we choose an ellipsoidal system (ρ, μ, ν) having (1) as a reference ellipsoid and define the transformation

$$\begin{aligned} \boldsymbol{f}(\rho, \mu, \nu) &= R(\rho)\frac{\mu\nu}{h_2 h_3}\hat{\boldsymbol{x}}_1 + \sqrt{R^2(\rho) - h_3^2}\frac{\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2}}{h_1 h_3}\hat{\boldsymbol{x}}_2 \\ &+ \sqrt{R^2(\rho) - h_2^2}\frac{\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2}}{h_1 h_2}\hat{\boldsymbol{x}}_3, \end{aligned} \tag{14}$$

where

$$R(\rho) = \sqrt{\frac{\rho_1^2 - a_1^2}{\rho_1^2 - h_2^2}(\rho^2 - h_2^2) + a_1^2} \tag{15}$$

with $\rho_1 > a_1$ and $0 \leqslant \nu^2 \leqslant h_3^2 \leqslant \mu^2 \leqslant h_2^2 \leqslant \rho^2$. Transformation (14) maps the focal ellipse $\rho = h_2$ to the reference ellipsoid $\rho = a_1$ and leaves the outer ellipsoid $\rho = \rho_1$ invariant. Hence, $\boldsymbol{f}$ restricts to a one-to-one map from the interior of the ellipsoid $\rho = \rho_1$ except the focal ellipse, to the ellipsoidal shell $a_1 < \rho < \rho_1$. Note that $\boldsymbol{f}$ is singular on the focal ellipse. This mapping connects the two points ρ and R on the coordinate curve

(μ, ν)= constant, and therefore it is completely determined by the scalar transformation (15). Consequently,

$$\rho(R) \quad = \quad \sqrt{\frac{\rho_1^2 - h_2^2}{\rho_1^2 - a_1^2}(R^2 - a_1^2) + h_2^2} \tag{16}$$

and the inverse of $\boldsymbol{f}$ is given by

$$\begin{aligned} \boldsymbol{f}^{-1}(R, \mu, \nu) \quad &= \quad \rho(R)\frac{\mu\nu}{h_2 h_3}\hat{\boldsymbol{x}}_1 + \sqrt{\rho^2(R) - h_3^2}\,\frac{\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2}}{h_1 h_3}\hat{\boldsymbol{x}}_2 \\ &+ \quad \sqrt{\rho^2(R) - h_2^2}\,\frac{\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2}}{h_1 h_2}\hat{\boldsymbol{x}}_3. \end{aligned} \tag{17}$$

The value of a_1 determines the size and the shape of the cloaked region, the value of ρ_1 determines the invariant exterior boundary and the ellipsoidal distance $\rho_1 - a_1$ controls the thickness of the cloak.

4. The material tensor. The Dirichlet problem for the Laplace equation in a domain V refers to finding a harmonic function u in V which takes the preassigned values h on the boundary ∂V. If the domain V is equipped with the Riemannian metric (g_{ij}), then

$$\Delta u(x_1, x_2, x_3) \quad = \quad \frac{1}{\sqrt{|g|}}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial}{\partial x_i}\left(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x_j}\right)u(x_1, x_2, x_3), \tag{18}$$

where (g^{ij}) is the inverse of (g_{ij}) and $|g|$ denotes the determinant of the metric tensor. The map

$$\Lambda_g(h) \quad = \quad \sum_{i=1}^{3}\sum_{j=1}^{3}\left(\sqrt{|g|}n_i g^{ij}\frac{\partial}{\partial x_j}\right)u, \quad \boldsymbol{r} \in \partial V, \tag{19}$$

known as the Dirichlet-to-Neumann map, remains invariant under any diffeomorphic transformation that reduces to the identity on the boundary [10]. This observation allows one to interpret the effect of the transformation as a change of the material tensor that characterizes the medium in V. In other words, the mathematical transformation is absorbed by the physical characteristics of the medium. The material properties of the medium are represented by the symmetric tensor

$$\sigma^{ij} \quad = \sqrt{|g|}g^{ij}. \tag{20}$$

Therefore, we have to calculate the metric that corresponds to the inverted transformation mapping $\boldsymbol{f}^{-1}$. This will lead us to the metric $\tilde{g}^{ij}$ and to the material tensor

$$\tilde{\sigma}^{ij}\rho \quad = \sqrt{|\tilde{g}|}\tilde{g}^{ij}. \tag{21}$$

After long and tedious calculations with the inverted ellipsoidal transformation $\boldsymbol{f}^{-1}$, we obtain the following expression for the inverted metric:

$$\widetilde{g}_{RR} = \frac{R^2}{\rho^2(R)}\left(\frac{\rho_1^2-h_2^2}{\rho_1^2-a_1^2}\right)^2\frac{(\rho^2(R)-\mu^2)(\rho^2(R)-\nu^2)}{(\rho^2(R)-h_3^2)(\rho^2(R)-h_2^2)}, \tag{22}$$

$$\widetilde{g}_{\mu\mu} = \frac{(\rho^2(R)-\mu^2)(\mu^2-\nu^2)}{(\mu^2-h_3^2)(h_2^2-\mu^2)}, \tag{23}$$

$$\tilde{g}_{\nu\nu} = \frac{(\rho^2(R)-\nu^2)(\mu^2-\nu^2)}{(h_3^2-\nu^2)(h_2^2-\nu^2)}, \tag{24}$$

where, due to the orthogonality of the ellipsoidal system, every other component of the metric tensor vanishes. Furthermore,

$$\sqrt{|\tilde{g}|} = \frac{\rho_1^2-h_2^2}{\rho_1^2-a_1^2}\frac{R(\rho^2(R)-\mu^2)(\rho^2(R)-\nu^2)(\mu^2-\nu^2)}{\rho(R)\sqrt{\rho^2(R)-h_3^2}\sqrt{\rho^2(R)-h_2^2}\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}}$$

$$= \frac{\rho_1^2-h_2^2}{\rho_1^2-a_1^2}\frac{R\mu\nu}{x_1x_2x_3}\frac{(\rho^2(R)-\mu^2)(\rho^2(R)-\nu^2)(\mu^2-\nu^2)}{h_1^2h_2^2h_3^2}, \tag{25}$$

which finally implies the material tensor

$$\tilde{\sigma} = \begin{pmatrix} \tilde{\sigma}_{\rho\rho} & 0 & 0 \\ 0 & \tilde{\sigma}_{\mu\mu} & 0 \\ 0 & 0 & \tilde{\sigma}_{\nu\nu} \end{pmatrix}, \tag{26}$$

where

$$\tilde{\sigma}_{\rho\rho} = \frac{\rho_1^2-a_1^2}{\rho_1^2-h_2^2}\frac{\rho^2\mu\nu}{x_1x_2x_3}\frac{(\rho^2-h_3^2)(\rho^2-h_2^2)(\mu^2-\nu^2)}{h_1^2h_2^2h_3^2}\left[\frac{\rho_1^2-a_1^2}{\rho_1^2-h_2^2}(\rho^2-h_2^2)+a_1^2\right]^{-\frac{1}{2}}, \tag{27}$$

$$\tilde{\sigma}_{\mu\mu} = \frac{\rho_1^2-h_2^2}{\rho_1^2-a_1^2}\frac{\mu\nu}{x_1x_2x_3}\frac{(\mu^2-h_3^2)(h_2^2-\mu^2)(\rho^2-\nu^2)}{h_1^2h_2^2h_3^2}\left[\frac{\rho_1^2-a_1^2}{\rho_1^2-h_2^2}(\rho^2-h_2^2)+a_1^2\right]^{\frac{1}{2}}, \tag{28}$$

$$\tilde{\sigma}_{\nu\nu} = \frac{\rho_1^2-h_2^2}{\rho_1^2-a_1^2}\frac{\mu\nu}{x_1x_2x_3}\frac{(h_3^2-\nu^2)(h_2^2-\nu^2)(\rho^2-\mu^2)}{h_1^2h_2^2h_3^2}\left[\frac{\rho_1^2-a_1^2}{\rho_1^2-h_2^2}(\rho^2-h_2^2)+a_1^2\right]^{\frac{1}{2}}. \tag{29}$$

It is of interest to note that the material tensor (26)-(29) above remains bounded away from zero, as well as from infinity, on both boundaries of the cloak, i.e. on $\rho = a_1$ and on $\rho = \rho_1$. The relative problem for the case of a spherical cloak leads to a material tensor that vanishes on the inner boundary of the cloak [12]. It seems that the singular behavior of the spherical case is due to the fact that the transformation map blows up the origin to a full sphere, while in the ellipsoidal case a point on the focal ellipse is mapped just to two symmetric points on the inner boundary of the cloak. In other words, the inverse map exhibits a strong focusing effect in the spherical case, sending a two dimensional manifold to a single point, while in the ellipsoidal case, the inverse map sends just two points to one. On the other hand, in both the spherical and the ellipsoidal case, there are submanifolds in the interior of the cloak where the material tensor vanishes because the determinant of the metric vanishes there, but this is due to the particular system.

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