

Perfect fluid sources in 2+1 dimensions*

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Received 20 December 1993, in final form 17 August 1994

Abstract. We show that a class of metrics of Einstein theory with perfect fluid sources are also the solutions of the Deser–Jackiw–Templeton (DJT) theory with a cosmological constant. Because of this analogy we interpret a recently found black hole solution of the DJT equations as a rotating fluid solution of the Einstein theory in three dimensions.

PACS numbers: 0420J, 0450, 0460K, 1125

Recently [1] it has been shown that topologically massive gravity (TMG) [2] can be obtained from three-dimensional string theory with constant dilaton and vanishing gauge fields. Hence the black hole solutions of TMG might bring some new insights to three-dimensional gravitational theories [3] and also to the string theory in three dimensions [4].

In this paper we first connect the solutions of the Einstein equations with perfect fluid source to the solutions of TMG and then interpret the recently found black hole solution [5] as a rotating fluid in three dimensions.

TMG equations with a cosmological constant are given as follows.

$$G_{\nu}^{\mu} + \frac{1}{\mu} \eta^{\mu\beta\alpha} \nabla_{\alpha} (R_{\nu\beta} - \frac{1}{4} R g_{\nu\beta}) = \lambda \delta_{\nu}^{\mu}. \quad (1)$$

Here $G_{\mu\nu}$ and $R_{\mu\nu}$ are the Einstein and Ricci tensors respectively, $\eta_{\mu\alpha\beta} = \sqrt{-g} \epsilon_{\mu\alpha\beta}$ is the Levi-Civita tensor in three dimensions, μ and λ are respectively the DJT parameter and the cosmological constant.

Einstein's equations with a perfect fluid source are given as follows

$$G_{\mu\nu} = T_{\mu\nu}, \quad (2)$$

where

$$T_{\mu\nu} = (p + \rho) u_{\mu} u_{\nu} + p g_{\mu\nu}, \quad (3)$$

and where the fluid equations are obtained through the conservation equation $\nabla_{\mu} T^{\mu\nu} = 0$, p and ρ are respectively the pressure and mass density of the fluid and u_{μ} is the fluid's timelike unit 4-velocity, i.e. $u^{\mu} u_{\mu} = -1$.

We have the following:

Lemma. If p, ρ are constants and

$$\nabla_{\mu} u_{\nu} = \frac{\mu}{3} \eta_{\mu\nu\alpha} u^{\alpha} \quad (4)$$

* This work was partially supported by the Scientific and Technical Research Council of Turkey (TUBITAK).

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then any solution of the Einstein equations $G_{\mu\nu} = T_{\mu\nu}$ is also a solution of the TMG (1) with a cosmological constant $\lambda = (2p - \rho)/3$.

Proof of the lemma is straightforward. The Einstein and Ricci tensors in (1) are first expressed in terms of the perfect fluid energy-momentum tensor through (2) and (3). Then DJT equations (1) reduce to coupled equations for p , ρ and u^μ . These equations are identically satisfied for the conditions given by the lemma. Here we assumed that the 4-velocity vector is timelike. One has a similar lemma for the null case.

As an application of the above lemma let us consider the following spacetime metric

$$ds^2 = -\phi dt^2 + 2q dt d\theta + \frac{-q^2 + h^2 \psi}{\phi} d\theta^2 + \frac{1}{\psi} dr^2, \tag{5}$$

where

$$\begin{aligned} \phi &= a_0 & \psi &= b_0 + \frac{b_1}{r^2} + \frac{3\lambda_0}{4} r^2 \\ q &= c_0 + \frac{e_0 \mu}{3} r^2 & h &= e_0 r & \lambda_0 &= \lambda + \frac{\mu^2}{27} \end{aligned}$$

and where a_0, b_0, b_1, c_0 and e_0 are arbitrary constants. This metric solves the Einstein equations (2) with the following fluid parameters

$$\begin{aligned} p &= \frac{\mu^2}{9} & \rho &= \frac{\mu^2 - 9\lambda_0}{3} \\ u_\mu &= \sqrt{a_0}(1, 0, \Omega) & \Omega &= -\frac{\mu e_0 r^2 + 3c_0}{3a_0}. \end{aligned}$$

To have a physically reasonable solution we must have $\mu^2 > 9\lambda_0$ or $\lambda > \frac{2}{27}\mu^2$. When $p + \rho = 0$ the solution reduces to the Banados-Teitelboim-Zanelli (BTZ) metric. This limit is $\mu^2 = 9\lambda$. After performing the transformation $\theta = -\theta' + \omega t$, the parameters a_0, b_0, b_1, c_0, e_0 and ω can be arranged in such a way that the above metric (5) becomes

$$\begin{aligned} ds^2 = & - \left[-M + \frac{\mu^2}{9} r^2 + \frac{\mu^2}{18} \frac{\mu^2 - 9\lambda}{(2J\mu - 6M)} r^4 \right] dt^2 \\ & + \left(J - \frac{\mu}{3} \frac{\mu^2 - 9\lambda}{2J\mu - 6M} r^4 \right) dt d\theta + r^2 \left[1 - \frac{\mu^2 - 9\lambda}{2(2J\mu - 6M)} r^2 \right] d\theta^2 \\ & + \frac{dr^2}{-M + (J^2/4 r^2) + [(\mu^2 + 27\lambda) r^2/36]} \end{aligned} \tag{6}$$

with $-\infty < t < \infty, 0 < r < \infty$, and $0 \leq \theta \leq 2\pi$. Here the constants are chosen as

$$\begin{aligned} a_0 &= (J\mu - 3M)/3, & b_0 &= -M, & c_0 &= -J/2, \\ e_0 &= 1, & b_1 &= J^2/4, & \omega &= -\mu/3. \end{aligned} \tag{7}$$

The metric given in equation (6) describes a black hole with horizons r_+ and r_- defined through the following relations

$$M = \frac{r_+^2 + r_-^2}{l^2}, \quad J = \frac{2r_+ r_-}{l} \tag{8}$$

where

$$\frac{1}{l^2} = \frac{\mu^2 + 27\lambda}{36}$$

and r_{\pm} are given by

$$r_{\pm}^2 = \frac{1}{2}l(Ml \pm \sqrt{M^2 l^2 - J^2}). \quad (9)$$

Here we assume $\lambda_0 \neq 0$ ($l \neq \infty$) and $M^2 l^2 - J^2 \geq 0$. Two horizons coincide when $J^2 = M^2 l^2$, which is the extremal limit. There is also another extremal limit which corresponds to $l \rightarrow \infty$ or $\lambda_0 = 0$. In this case the horizon is located at $r^2 = J^2/4M$. The curvature invariants $R = -6\lambda$ and $R_{\mu\nu}R^{\mu\nu} = \frac{2}{27}(\mu^4 - 18\mu^2\lambda + 243\lambda^2)$ are regular everywhere. Further properties of the solution (6) will be discussed elsewhere.

Fluid parameters p and ρ remain the same but the velocity 4-vector becomes

$$u_{\mu} = \sqrt{a_0}(1 + \frac{1}{3}\mu\Omega', 0, \Omega') \quad (10)$$

where Ω' becomes

$$\Omega' = \frac{2\mu r^2 - 3J}{2\mu J - 6M}$$

and a_0 is given in (7). In the limit $\mu^2 = 9\lambda$ the above metric reduces exactly to the form given by BTZ. In that form M and J are respectively interpreted as the mass and angular momentum of the solution.

References

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