Perfect GMV-Algebras

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Dedicated to W. Charles Holland on the occasion of his 70^{th} birthday

Abstract

The focus of this paper is the class of perfect GMV-algebras, which includes all non-commutative analogues of perfect MV-algebras. As in the commutative case, we show that each perfect GMV-algebra possesses a single negation, it is generated by its infinitesimal elements, and can be realized as an interval in a lexicographical product of the lattice-ordered group of integers and an arbitrary lattice-ordered

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group. Further, we establish that the category of perfect GMV-algebras is equivalent to the category of all lattice-ordered groups. The variety of GMV-algebras generated by the class of perfect GMV-algebras plays a key role in our considerations. Among other results, we describe a finite equational basis for this variety and prove that it fails to satisfy the amalgamation property. In fact, we show that uncountably many of its subvarieties fail this property.

1 Introduction

In this section we provide an overview of the contents and aims of the paper. The reader should refer to the next section or the explicitly cited references in the literature for definitions of undefined concepts.

MV-algebras are the algebraic counterparts of infinite-valued sentential calculus of Łukasiewicz logic, as Boolean algebras are the counterparts of classical propositional logic. By Mundici's fundamental result, [Mun], every MV-algebra can be realized as the MV-algebra $\Gamma(G, u)$ based on the interval $[0, u]$ of a unital Abelian lattice-ordered group (henceforth, ℓ-group) (G, u). Because of the non-idempotency of the MV-algebraic conjunction, unlike Boolean algebras, the class of MV-algebras contains subdirectly irreducible members that are not simple, as well as members that are not semisimple, i.e., the intersection of their maximal ideals is non-trivial. We refer to this intersection as the radical of the MV-algebra in question and to the non-zero elements of the radical as infinitesimals. Perfect MV-algebras, introduced by Belluce, Di Nola and Lettieri in [BDL], may be viewed as the most compelling examples of non-archimedean MV-algebras, in the sense that they are generated by their infinitesimals. In a perfect MV-algebra, every element belongs to either its radical or its coradical (set-theoretic complement). It is shown in [DiLe1] that given a perfect MV-algebra M, there exists an abelian ℓ-group G, such that

$$M \cong \Gamma(\mathbb{Z} \times_{lex} G, (1, 0)), \tag{1.1}$$

where $\mathbb{Z}$ is the ℓ-group of integers, $\mathbb{Z} \times_{lex} G$ is the lexicographic product of $\mathbb{Z}$ and G, and Γ is the aforementioned Mundici functor.

GMV-algebras, called also pseudo MV-algebras [GeIo] or non-commutative MV-algebras [Rac], are non-commutative versions of MV-algebras and the algebraic counterparts of noncommutative many valued logic. The key representation theorem for GMV-algebras is Dvurečenskij's [Dvu1] non-commutative generalization of Mundici's result, which states that every GMV-algebra can be realized as the GMV-algebra $\Gamma(G, u)$ based on the interval $[0, u]$ of a unital ℓ-group (G, u). This result provides a new bridge between different research areas, including GMV-algebras, unital ℓ-groups, noncommutative many valued logic, soft computing and quantum structures (see [DvPu]). Further, it motivates a new set of questions in the study of ℓ-groups (see, for example, [DvHo]). A generalization of Dvurečenskij's result for a wider class of residuated structures was established by Galatos and Tsinakis in [GaTs]. The reader should be cautioned that, in the latter paper, the term GMV-algebra is used for possibly unbounded structures and the algebras of the present paper are referred to as bounded GMV-algebras.

This paper initiates the study of perfect GMV-algebras. Unlike arbitrary GMV-algebras, which admit two negations, the class of perfect GMV-algebras admits a single negation. However, the existence of a single negation does not imply that perfect GMV-algebras are close to being commutative. For example, given an arbitrary ℓ-group, Condition (1.1) above yields a GMV-algebra with a unique negation. In fact, every perfect GMV-algebra can be described by (1.1). We remark that Leuştean [Leu] established an analogous result. Her approach to the study of strong perfect GMV-algebras is somewhat different than ours.

We prove that the category of perfect GMV-algebras is categorically equivalent to the category of all ℓ-groups. The variety of GMV-algebras generated by the class of perfect GMV-algebras plays a key role in our considerations. Among other results, we describe a finite equational basis for this variety and prove that it is generated by any algebra described by (1.1), as long as G is a doubly transitive ℓ-group. Further, we show that, unlike the situation with perfect MV-algebras, this variety and the class of perfect GMV-algebras fail the amalgamation property, and exhibit uncountably many subvarieties of the former that fail the same property. The basic technical tools we employ to establish these results are the representation of GMV-algebras via unital ℓ-groups, the notion of a top variety introduced in [DvHo], and McCleary's Trichotomy Classification Theorem of primitive ℓ-groups.

The paper is organized as follows. The basic properties of GMV-algebras are reviewed in Section 2. Symmetric GMV-algebras are studied in Section 3, where also one of the results in [DDJ] is corrected. Perfect GMV-algebras are introduced in Section 4, and their categorical equivalence with the class of all ℓ-groups is established in Section 5. The variety generated by all perfect GMV-algebras is studied in Section 6. Section 7 provides an alternative short discussion of GMV-algebras within the framework of residuated lattices. This alternative perspective is used to demonstrate that the subvariety lattice of the variety of symmetric GMV-algebras has the cardinality of the continuum. Finally the amalgamation property is investigated in the last section of the paper.

2 Basic Notions

GMV-algebras were independently introduced in [GeIo] and [Rac] under the names pseudo MV-algebras and generalized MV-algebras, respectively. According to [GeIo], a *GMV-algebra* is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$, where the derived operation $\odot$ appearing in the axioms (A6) and (A7) is defined by

$$y \odot x = (x^- \oplus y^-)^\sim.$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = 0 \oplus x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$;

(A4) $1^\sim = 0$; $1^- = 0$;

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;

(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$;[2]

(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;

(A8) $(x^-)^\sim = x$.

Given a GMV-algebra M, the stipulation $x \leq y$ iff $x^- \oplus y = 1$ induces a partial order on M. M actually becomes a distributive lattice with respect to this order, with joins and meets being defined by $x \vee y = x \oplus (x^\sim \odot y)$ and $x \wedge y = x \odot (x^- \oplus y)$. A GMV-algebra M is an MV-algebra iff $x \oplus y = y \oplus x$ for all $x, y \in M$. We refer the reader to [GeIo] for the basic properties of GMV-algebras.

[2] $\odot$ has a higher priority than $\oplus$.

As a means of an example, if u is an arbitrary positive element of a – not necessarily Abelian – ℓ-group G,

$$\Gamma(G,u) := [0,u]$$

and

$$\begin{array}{rcl} x \oplus y & := & (x+y) \wedge u, \\ x^- & := & u - x, \\ x^\sim & := & -x + u, \\ x \odot y & := & (x - u + y) \vee 0, \end{array}$$

then $(\Gamma(G,u); \oplus, ^-, ^\sim, 0, u)$ is a GMV-algebra [GeIo].

Let now $(M; \oplus, ^-, ^\sim, 0, 1)$ be a GMV-algebra. Define a partial binary operation $+$ on M as follows: $x+y$ is defined iff $x \le y^-$, and in this case

$$x + y := x \oplus y.$$

It is clear that $x+y$ is defined iff $y \le x^\sim$. In addition $a \le b$ iff there exists $c \in M$ such that $a + c = b$ iff there exists $d \in M$ such that $d + a = b$.

Moreover, for $a \le b$, we define two subtractions, $\div$and $\dot{-}$by $a \div b := a^\sim \odot b$ and $b \dot{-} a = b \odot a^-$. Then $a + a \div b = b = b \dot{-} a + a$.

As was noted earlier, the GMV-algebras $(\Gamma(G,u); \oplus, ^-, ^\sim, 0, u)$ comprise, up to isomorphism, all GMV-algebras due to the following basic representation result of Dvurečenskij [Dvu1, Thm 3.9, Thm 6.4].

Theorem 2.1 *For any GMV-algebra M, there exists a unique (up to isomorphism) unital ℓ-group G with a strong unit u such that $M \cong \Gamma(G,u)$. The functor Γ defines a categorical equivalence between the category of GMV-algebras and the category of unital ℓ-groups. In addition, if h is an isomorphism of M with $\Gamma(G,u)$, (G,h) is the universal group for M: that is, for any partially ordered group $(K; +, \le, 0)$ and any order and $+$ preserving mapping $g : M \to K$, there exists a homomorphism of partially ordered groups $g' : G \to K$ such that $g = g' \circ h$.*

An *ideal* of a GMV-algebra M is a subset I of M satisfying the following conditions for all elements $x, y \in M$: (i) $0 \in I$; (ii) if $x, y \in I$, then $x \oplus y \in I$ and $y \oplus x \in I$; and (iii) if $x \in I$ and $y \le x$, then $y \in I$. It can be shown that a non-void subset I of M is an ideal of M iff (i) $x, y \in I$ and $x+y \in M$ imply $x+y \in I$, and (ii) $a \leq b \in I$ implies $a \in I$.

Throughout the remainder of this paper, the algebraic closure family of ideals of M will be denoted by $\mathcal{I}(M)$.

An ideal I of M is said to be (i) *normal* if $x \oplus I = I \oplus x$ for all $x \in M$; [3] (ii) *maximal* if it is a proper ideal of M and is not included into any proper ideal of M. We will denote by $\mathcal{M}(M)$ the set of maximal ideals of M. It can be shown that an ideal I is normal iff $x + I = I + x$ for any $x \in M$.

Theorem 2.1 gives a one-to-one correspondence between the ideals, normal ideals, maximal ideals of $M = \Gamma(G, u)$, and convex ℓ-subgroups, $\mathcal{C}(G)$, ℓ-ideals, $\mathcal{L}(G)$, and maximal convex ℓ-subgroups, $\mathcal{M}(G)$, of (G, u), see [Dvu2]; the one-to-one mapping $\phi: \ \mathcal{I}(M) \to \mathcal{C}(G)$ is defined by

$$\phi(I) = \{x \in G : \ \exists\, x_i, y_j \in I, \quad x = x_1 + \cdots + x_n - y_1 - \cdots - y_m\}. \quad (2.1)$$

A *state* on a GMV-algebra M is a mapping $m: \ M \to [0,1]$ such that (i) $m(1) = 1$, and (ii) $m(a+b) = m(a) + m(b)$ whenever $a+b$ is defined in M. If m is a state on M, the set

$$\mathrm{Ker}(m) := \{a \in M : \ m(a) = 0\}$$

is a normal ideal of M.

Let $\mathcal{S}(M)$ denote the set of all states of a GMV-algebra M. Then $\mathcal{S}(M)$ is a convex set. Let $\partial_e \mathcal{S}(M)$ be the set of all extremal states on M. A net $\{m_\alpha\}$ of states on M *converges weakly* to a state m iff $m_\alpha(a) \to m(a)$ for every $a \in M$. Therefore, in view of the Krein-Millman theorem, M has at least one state iff M has an extremal state. It is well-known that every MV-algebra possesses at least one state. More generally, every normal-valued GMV-algebra admits a state, but there exist noncommutative stateless GMV-algebras (see [Dvu2]).

A *state-morphism* on a GMV-algebra M is a mapping m from M into the standard MV-algebra $[0,1]$ such that, for all $a, b \in M$, $m(a \oplus b) = m(a) \oplus_{\mathbb{R}} m(b)$; (ii) $m(a^-) = m(a^\sim) = 1 - m(a)$; and (iii) $m(1) = 1$.

The class of extremal states on a GMV-algebra M was characterized in [Dvu2] by the following equivalent statements:

[3] $x \oplus I := \{x \oplus y : \ y \in I\}$ and likewise $I \oplus x$. The sets $x + I$ and $I + x$ are defined analogously.

(i) m is an extremal state on M.

(ii) m is a state-morphism on M.

(iii) $m(x \wedge y) = \min\{m(x), m(y)\},\ x, y \in M$.

(iv) $\mathrm{Ker}(m)$ is a maximal ideal.

Moreover, there is a one-to one correspondence between the set, $\partial_e \mathcal{S}(M)$, of extremal states on M and the set, $\mathcal{N}\mathcal{M}(M)$, of normal and maximal ideals of M given by

$$m \leftrightarrow \mathrm{Ker}(m). \tag{2.2}$$

Let $a \in M$ and an integer $n \geq 0$ be given. We define $0a := a$ and $na = (n-1)a + a$ for $n \geq 1$ whenever the right-hand side exists in M.

An element a of a GMV-algebra M is said to be an *infinitesimal* if $na \in M$ for any integer $n \geq 1$, or equivalently, $n \odot a := a \oplus \cdots \oplus a \leq a^-$ for any $n \geq 1$. We denote by $\mathrm{Infinit}(M)$ the set of all infinitesimal elements of M. It is clear that (i) $0 \in \mathrm{Infinit}(M)$, (ii) if $a \leq b \in \mathrm{Infinit}(M)$, then $a \in \mathrm{Infinit}(M)$, (iii) $1 \notin \mathrm{Infinit}(M)$.

If a GMV-algebra M has only one infinitesimal, 0, then M is commutative, i.e., an MV-algebra.

We define (i) the *radical* of a GMV-algebra M, $\mathrm{Rad}(M)$, as the set

$$\mathrm{Rad}(M) = \bigcap \{I :\ I \in \mathcal{M}(M)\},$$

and (ii) the *normal radical* of M, $\mathrm{Rad}_n(M)$, as the set

$$\mathrm{Rad}_n(M) = \bigcap \{I :\ I \in \mathcal{N}(M) \cap \mathcal{M}(M)\}.$$

We have, see [DDJ, Prop. 4.1, Thm 4.2], that

$$\mathrm{Rad}(M) \subseteq \mathrm{Infinit}(M) \subseteq \mathrm{Rad}_n(M). \tag{2.3}$$

If M is an MV-algebra, the inclusions in (2.3) are equalities. An example in [DDJ, Ex. 4.10] demonstrates that the last inclusion of (2.3) may be proper for arbitrary GMV-algebras. However, if every maximal ideal is normal, then all inclusions in (2.3) are equalities.

3 Symmetric GMV-algebras

GMV-algebras have, in general, two negations, $^-$ and $^\sim$. A GMV-algebra M is said to be *symmetric* (or, more precisely, have *symmetric negations*) if $a^- = a^\sim$, for all $a \in M$. The class of all symmetric GMV-algebras forms a variety, $\mathcal{SPMV}$, which contains as a proper subvariety the variety of all MV-algebras.

For example, if G is an arbitrary noncommutative ℓ-group, then $M = \Gamma(\mathbb{Z} \times_{lex} G, (1,0))$ gives a noncommutative symmetric GMV-algebra; if G is linear, then so is $M = \Gamma(\mathbb{Z} \times_{lex} G, (1,0))$. Moreover, M admits a unique state.

We recall that if $M = \Gamma(G,u)$, then $a^- = a^\sim$ holds for all $a \in M$ iff u is in the center of G, i.e., $u + g = g + u$, for all $g \in G$.

The following proposition was proved in [DDJ, Prop. 6.5].

Proposition 3.1 *The following hold in a symmetric GMV-algebra M.*

(i) *For all $x, y \in M$, $x + y - x,\ -x + y + x \in M$.*

(ii) *If $x \in \mathrm{Infinit}(M)$, then $-y + x + y,\ y + x - y \in \mathrm{Infinit}(M)$, for all $y \in M$.*

The following example, due to A.M.W. Glass (oral communication), provides a negative answer to a problem posed in [DDJ, Prob. 6.7] as to whether any symmetric GMV-algebra admits a state. In our discussion below we use Holland's fundamental representation of an ℓ-group as a group of automorphisms of a linear set [Hol].

Example 3.2 *There exists a symmetric stateless GMV-algebra M such that* $\mathrm{Rad}(M) = \{0\} \neq \mathrm{Infinit}(M)$ *and* $\mathrm{Infinit}(M)$ *is not an ideal of M.*

Proof. Let $\mathrm{Aut}(\mathbb{R})$ denote the group of automorphisms $g:\ \mathbb{R} \to \mathbb{R}$, and set

$$G = \{g \in \mathrm{Aut}(\mathbb{R}) :\ (t+1)g = tg + 1,\ \forall\, t \in \mathbb{R}\}.$$

Then G is an ℓ-group with a strong unit $u:\ t \mapsto t+1$ ($t \in \mathbb{R}$) that is central. Indeed, if $g \in G$, and $0g = s > 0$, let $n \in \mathbb{Z}_+$ be an integer such that $n > s$. For all $t \in \mathbb{R}$, let $m \in \mathbb{Z}$ with $m < t < m+1$. Then $tu^{-(m+1)} < 0$ and $0u^m < t$. So $tu^{-(m+1)}g < 0g = s < n = 0u^n$, which gives

$tg = tu^{-(m+1)}gu^{(m+1)} < 0u^{n+m+1} < tu^{n+1}$ since u is central. Thus $tg < tu^{n+1}$ for all $t \in \mathbb{R}$ so $g < u^{n+1}$.

Similarly, if $0g \leq 0$, then $u^{-(n'+1)} < g$ (since $g^{-1} < u^{n'+1}$ for some $n' \in \mathbb{Z}^+$), which proves u is a strong unit for G.

Let $t_0 \in \mathbb{R}$ and $G_{t_0} = \{g \in G : \ t_0 g = t_0\}$ be the stabilizer of t_0. Then $u \notin G_{t_0}$ and if H is any subgroup of G strictly containing G_{t_0}, there is $h \in H$ with $t_0 h \neq t_0$. By replacing h by h^{-1}, we may assume that $t_0 h > t_0$.

Now if $t_0 h > t_0 + 1$, then $h \geq u$, and if $t_0 h < t_0 + 1$, there exists $g \in G_{t_0} \subset H$ with $t_0 + 1/2 < t_0 hg < t_0 + 1$ and $t_0 hg > (t_0 + 1)h^{-1}$. This follows from the fact that $G_{t_0} = G_{t_0+1}$ and $h \notin G_{t_0}$, and there is an open unit interval $I \subseteq \mathbb{R}$ with $t_0 \in I$ and $I \subseteq \mathrm{supp}(h) = \{t \in \mathbb{R} : \ th \neq h\}$. Since $h \in G$, we have $(I + 1) := \{t + 1 : \ t \in I\} \subseteq \mathrm{supp}(h)$. So $t_0 hgh > t_0 + 1$.

It follows that $t_0(hgh)^2 > t_0 + 2$ and hence $u < (hgh)^2 \in H$. Thus if H is a convex subgroup, then $u \in H$. Consequently, G_{t_0} is a maximal convex ℓ-subgroup of G not containing u. If $f \in G$ with $t_0 f = t_0 + 1/4$, then $f^{-1}G_{t_0}f = G_{t_0+1/4} \neq G_{t_0}$ so G_{t_0} is not normal in G.

On the other hand, $(G, \mathbb{R})$ is a transitive and primitive periodic ℓ-group [Gla1, Ex.1.9.2, p. 93], and G has only trivial normal convex ℓ-subgroup, i.e., no maximal convex ideal of G is normal.

Consequently, $M = \Gamma(G, u)$ is a symmetric stateless GMV-algebra. Since every stabilizer G_t is a convex maximal ℓ-subgroup of G, then $G_t \cap M$ is a maximal ideal of M, and $\mathrm{Ker}(M) = \{0\}$.

Let $g \in G_{t_0} \cap M$. Then for $t_0 < t < t_0 + 1$ we have $t_0 < tg^n < t_0 + 1$ for all $n \geq 1$, so that $g \in \mathrm{Infinit}(M)$ and $G_{t_0} \subseteq \mathrm{Infinit}(M)$ for all $t_0 \in \mathbb{R}$.

If $\mathrm{Infinit}(M)$ were an ideal of M, by Proposition 3.1, $\mathrm{Infinit}(M)$ would be a non-trivial normal ideal, which contradicts the fact that G has no non-trivial ℓ-ideals. $\square$

According to [DDJ, Thm 5.5], if $\mathrm{Infinit}(M)$ is an ideal of a symmetric GMV-algebra M, M admits a state. The converse statement is not valid in general as the following example shows.

Example 3.3 $\mathrm{Infinit}(M)$ *is not always an ideal in a symmetric GMV-algebra, M, with a state.*

Proof. Let M_1 be the symmetric stateless GMV-algebra of Example 3.2 and let $M_2 = \Gamma(\mathbb{R}, 1)$. Then $M = M_1 \times M_2$ is a symmetric GMV-algebra with a unique state, s, namely $s(g, t) = t$ for any $(g, t) \in M_1 \times M_2$. Since

$\mathrm{Infinit}(M) = \mathrm{Infinit}(M_1) \times \mathrm{Infinit}(M_2)$, [DDJ, Prop. 4.9], we have that $\mathrm{Infinit}(M)$ is not an ideal of M since $\mathrm{Infinit}(M_1)$ is not an ideal of M_1. $\square$

Example 3.3 shows that, in [DDJ, Thm 4.3], the implication "the existence of a state on M implies $\mathrm{Infinit}(M)$ is an ideal" is incorrect, consequently, all statements using this implication are not necessarily valid.

Remark 3.4 *It is worth mentioning that we can replace M_2 in Example 3.3 by any MV-algebra, and thereby obtain infinitely many nonisomorphic symmetric GMV-algebras M for which $\mathrm{Infinit}(M)$ is not an ideal of M.*

Let $a, b \in G$, we say that a is *infinitarily small with respect* to b (in symbols, $a \ll b$) if $na \leq b$ for any $n \in \mathbb{Z}$.

We set

$$\mathrm{Infinit}(G, u) := \{g \in G : \; g \ll u\}.$$

If $\mathrm{Infinit}(M)$ is an ideal of M, then by [DDJ, Thm 5.2],

$$\phi(\mathrm{Infinit}(M)) = \{x_0 - y_0 : \; x_0, y_0 \in \mathrm{Infinit}(M)\} = \mathrm{Infinit}(G, u). \tag{3.1}$$

As shown in [DDJ, Cor. 7.5], the inequality $\mathrm{Infinit}(G, u) \neq \mathrm{Infinit}(G, 2u)$ can occur. However, if $M = \Gamma(G, u)$ is symmetric, then, by [DDJ, Thm 5.5],

$$\mathrm{Infinit}(G, u) = \mathrm{Infinit}(G, ku), \tag{3.2}$$

for every integer $k \geq 1$.

Proposition 3.5 *In a symmetric GMV-algebra M, $\mathrm{Infinit}(M)$ is an ideal of M if and only if $\mathrm{Rad}(M) = \mathrm{Infinit}(M)$. Moreover, in this case, M possesses at least one state.*

Proof. Let $\mathrm{Infinit}(M)$ be an ideal of M. By Proposition 3.1, $\mathrm{Infinit}(M)$ is a normal ideal. According to (2.3), it is necessary to verify that $\mathrm{Infinit}(M) \subseteq \mathrm{Rad}(M)$. Let $a \in \mathrm{Infinit}(M)$ and $a \notin \mathrm{Rad}(M)$. There exists a maximal ideal I of M such that $a \notin I$. Hence, the ideal $I_0(I, a)$ – generated by I and the element a – must coincide with M. Therefore, $1 = a_1 + x_1 + \cdots + a_k + x_k$ for some $a_i \leq a$ and $x_i \in I$ $(i = 1, \ldots, k)$. Then $1 = a_1 + a_2' + \cdots + a_k' + x_1 + \cdots + x_k$; by Proposition 3.1, all a_i' are infinitesimal, so that $a_0 := a_1 + a_2' + \cdots + a_k' \in \mathrm{Infinit}(M)$, since $\mathrm{Infinit}(M)$ is an ideal of M and $x_0 := x_1 + \cdots + x_k \in I$.

Then $1 = a_0 + x_0 = a_0^- + a_0$. But $a_0 \leq a_0^-$, which entails $x_0 \geq a_0$, so that $a_0 \in I$, and $1 = a_0 + x_0 \in I$, a contradiction. Hence, $\mathrm{Rad}(M) = \mathrm{Infinit}(M)$.

Conversely, let $\mathrm{Rad}(M) = \mathrm{Infinit}(M)$. Hence, $\mathrm{Infinit}(M)$ is an ideal and, by Proposition 3.1, $J := \mathrm{Infinit}(M)$ is a normal ideal of M, so that, $\phi(J)$ is an ℓ-ideal of G. Suppose $n(a/\phi(J)) \leq u/\phi(J)$ for any integer $n \geq 1$. Therefore, there exists $u_n \in \phi(J)$ such that $na \leq u + u_n \leq u + u_n^+ \leq 2u$, when we have used (5.3). By (3.2), $a \ll 2u$, i.e., $a \in \mathrm{Infinit}(M)$, which proves that $a/\phi(J) = 0$ and that $G/\phi(J)$ is an Abelian ℓ-group such that $M/\mathrm{Infinit}(M)$ is an MV-algebra having a state. This implies that M has a state, too. $\square$

Let M be a GMV-algebra and $A \subseteq M$. We set $A^- := \{a^- : a \in A\}$, $A^\sim := \{a^\sim : a \in A\}$, and $A^* := A^- \cup A^\sim$. If M is symmetric, it is clear that $A^* = A^- = A^\sim$.

Proposition 3.6 *Let M be a symmetric GMV-algebra. If a, b are infinitesimals and $a + b = b + a$, then $a + b \in \mathrm{Infinit}(M)$. Moreover,* $\mathrm{Infinit}(M) \cap \mathrm{Infinit}(M)^* = \emptyset$.

Proof. Let $a, b \in \mathrm{Infinit}(M)$ and $a + b = b + a$. Assume $M = \Gamma(G, u)$ and calculate addition in G: $n(a + b) = na + nb \leq 2u$ which by (3.2) yields $n(a + b) \in M$ for all $n \geq 1$, that is, $a + b \in \mathrm{Infinit}(M)$. Assume now that $a \in \mathrm{Infinit}(M) \cap \mathrm{Infinit}(M)^*$. Then $a, a^- \in \mathrm{Infinit}(M)$, but $a + a^- = 1 = a^- + a$. Thus $1 = a + a^- \in \mathrm{Infinit}(M)$, which is a contradiction. $\square$

4 Perfect GMV-algebras

Perfect MV-algebras were introduced and studied in [BDL]. In this section, we generalize this notion to the wider class of GMV-algebras.

We have seen that if G is an ℓ-group, then $M = \Gamma(\mathbb{Z} \times_{lex} G, (1, 0))$ is a symmetric GMV-algebra, which is commutative iff G is. Moreover, it satisfies the following properties.

(i) $M = \{(0, g) : g \in G^+\} \cup \{(1, -g) : g \in G^+\}$,

(ii) $\{(0, g) : g \in G^+\} \cap \{(1, -g) : g \in G^+\} = \emptyset$,

(iii) $\{(0, g) : g \in G^+\}^* = \{(1, -g) : g \in G^+\}$ and $\{(1, -g) : g \in G^+\}^* = \{(0, g) : g \in G^+\}$,

(iv) $\mathrm{Infinit}(M) = \{(0, g) : g \in G^+\} = \mathrm{Rad}(M)$,

(v) if $g, h \in G^+$, $(1, -g) + (1, -h)$ is not defined in M,

(vi) $(0, g) \leq (1, -h)$ for all $g, h \in G^+$, and

(vii) M admits a unique state, s; namely $s(0, g) = 0$ and $s(1, -g) = 1$ for all $g \in G^+$.

We will show in Section 5 that if M is a symmetric GMV-algebra satisfying $M = \mathrm{Rad}(M) \cup \mathrm{Rad}(M)^*$, then there exists an ℓ-group G such that $G \cong \Gamma(\mathbb{Z} \times_{lex} G, (1, 0))$.

Proposition 4.1 *If M is a symmetric GMV-algebra such that* $\mathrm{Rad}(M) = \mathrm{Infinit}(M)$, *then* $\mathrm{Rad}(M)$ *is a semigroup with respect to + satisfying the following properties.*

(i) 0 *is the neutral element* $\mathrm{Rad}(M)$,

(ii) *the cancelation law holds in* $\mathrm{Rad}(M)$,

(iii) *if* $a + b = 0$ *for* $a, b \in \mathrm{Rad}(M)$, *then* $a = b = 0$,

(iv) $\mathrm{Rad}(M) \cap \mathrm{Rad}(M)^* = \emptyset$,

(v) *if* $a, b \in \mathrm{Rad}(M)^*$, *then* $a + b$ *is not defined in* M, *and*

(vi) *if* $a \in \mathrm{Rad}(M)$ *and* $b \in \mathrm{Rad}(M)^*$, *then* $a \leq b$.

Proof. Assume that $a, b \in \mathrm{Rad}(M)$. By the Riesz decomposition property, there exist $a_1, b_1, c \in M$ such that $a = a_1 + c$, $b = b_1 + c$, and $a_1 + b_1 + c = b_1 + a_1 + c \in M$. Note that $a_1, b_1, c \in \mathrm{Rad}(M)$. The assumptions yield $a_1 + b_1 + c \in \mathrm{Infinit}(M) = \mathrm{Rad}(M)$, therefore, $(a_1 + b_1 + c) + (a_1 + b_1 + c) \in M$ which gives $a + b$ is defined in M, consequently, $a + b \in \mathrm{Rad}(M)$. Conditions (i)–(iii) are now evident.

(iv) Assume $a \in \mathrm{Rad}(M) \cap \mathrm{Rad}(M)^*$. Then $a \in \mathrm{Rad}(M)^*$, so that $a^- \in \mathrm{Rad}(M)$. This gives $1 = a + a^- \in \mathrm{Rad}(M)$, a contradiction.

(v) Assume $a + b \in M$. Then $a \leq b^- \in \mathrm{Rad}(M)$, which implies that $a \in \mathrm{Rad}(M)$, a contradiction.

(vi) If $b \in \mathrm{Rad}(M)^*$, then $b^- \in \mathrm{Rad}(M)$, which implies that $a + b^-$ is defined in $\mathrm{Rad}(M)$. Hence $a \leq (b^-)^- = b$. $\square$

Proposition 4.2 *For a symmetric GMV-algebra M, the following statements are equivalent:*

(i) $\mathrm{Rad}(M) \cup \mathrm{Rad}(M)^* = M$.

(ii) $\mathrm{Infinit}(M)$ *is an ideal of* M *and* $\mathrm{Infinit}(M) \cup \mathrm{Infinit}(M)^* = M$.

(iii) M *admits a state and* $\mathrm{Infinit}(M) \cup \mathrm{Infinit}(M)^* = M$.

(iv) *There exists a proper ideal* A *of* M, *such that* $A \cup A^* = M$, $A \cap A^* = \emptyset$, *and* $a \leq b$ *whenever* $a \in A$ *and* $b \in A^*$. *In such a case,* A *is the unique maximal ideal of* M.

(v) $\mathrm{Infinit}(M) \cup \mathrm{Infinit}(M)^* = M$ *and* $a \leq b$ *whenever* $a \in \mathrm{Infinit}(M)$ *and* $b \in \mathrm{Infinit}(M)^*$.

If the preceding equivalent conditions are satisfied, then $\mathrm{Rad}(M) = \mathrm{Infinit}(M)$, $\mathrm{Infinit}(M)$ *is the unique maximal ideal of* M, *and* M *admits a unique state,* s *– defined by* $s(a) = 0$ *if* $a \in \mathrm{Infinit}(M)$ *and* $s(a) = 1$ *if* $a \in \mathrm{Infinit}(M)^*$.

In addition, conditions (vi)–(viii) *below are equivalent, and are implied by* (i)–(v).

(vi) $\mathrm{Rad}_n(M) \neq M$ *and* $\mathrm{Rad}_n(M) \cup \mathrm{Rad}_n(M)^* = M$.

(vii) M *admits only a two-valued state.*

(viii) *If* I *is a maximal and normal ideal of* M, *then* $M/I = \Gamma(\mathbb{Z}, 1)$.

Proof. We can assume $M = \Gamma(G, u)$, for some ℓ-group G.

(i) $\Rightarrow$ (ii). Assume that the inclusion $\mathrm{Rad}(M) \subseteq \mathrm{Infinit}(M)$ is proper, and let $a \in \mathrm{Infinit}(M)$ but $a \notin \mathrm{Rad}(M)$. Then $a^- \in \mathrm{Rad}(M)$, so that $a^- \in \mathrm{Infinit}(M)$. Therefore, for any integer $n \geq 1$, we have $(n+1)a^- \leq u$, i.e. $(n+1)(u-a) \leq u$. Since u is central, we have $(n+1)u - (n+1)a \leq u$ and $nu \leq (n+1)a \leq u$, which is impossible. (An alternative proof can be obtained with the use of Proposition 3.6.) Therefore $a \in \mathrm{Rad}(M)$, i.e., $\mathrm{Infinit}(M) = \mathrm{Rad}(M)$, and $\mathrm{Infinit}(M)$ is an ideal.

(ii) $\Rightarrow$ (iii). We have that $\mathrm{Infinit}(M) = \mathrm{Rad}(M)$ which, by Proposition 3.5, implies that M admits a state, say s. The state s is unique, since if $a \in \mathrm{Infinit}(M)$ then $s(a) = 0$ and if $a \in \mathrm{Infinit}(M)^*$, then $s(a) = 1$.

(iii) $\Rightarrow$ (iv). It is clear that M admits a unique state, s. Then $\mathrm{Rad}_n(M) = \mathrm{Ker}(s) \supseteq \mathrm{Infinit}(M)$. If $a \in \mathrm{Rad}_n(M) \setminus \mathrm{Infinit}(M)$, then $a^- \in \mathrm{Infinit}(M)$, $s(a^-) = 0$ and $a^- \in \mathrm{Rad}_n(M)$, which is a contradiction. Hence $\mathrm{Ker}(s) = \mathrm{Rad}_n(M) = \mathrm{Infinit}(M) = \mathrm{Rad}(M)$. If we set $A = \mathrm{Rad}(M)$, we obtain (iv) in light of Proposition 4.1.

(iv) $\Rightarrow$ (i). If $a, b \in A$, then $a \leq b^-$. Hence, $a + b$ is defined in M and $a + b \in A$. In particular, $A \subseteq \mathrm{Infinit}(M)$. Choose $z \in \mathrm{Infinit}(M) \setminus A$. Then $z \in A^*$, i.e., $z^- \in A$ and $z^- \in \mathrm{Infinit}(M)$. Similarly, as in the proof of the implication (i) $\Rightarrow$ (ii), we obtain a contradiction. Since A is an ideal and $A = \mathrm{Infinit}(M), \mathrm{Rad}(M) = A$ and (i) holds by Proposition 3.5.

(ii) $\Rightarrow$ (v). It is clear that (ii) $\Rightarrow$ (v). Suppose now (v). By Proposition 3.6, $\mathrm{Infinit}(M) \cap \mathrm{Infinit}(M)^* = \emptyset$. Let now $a, b \in \mathrm{Infinit}(M)$. Then $a \leq b^-$ and so $a+b \in M$. We claim that $a+b \in \mathrm{Infinit}(M)$. If not then for any integer n $na, nb \leq a+b$; in particular, $2a \leq a+b$ and $a \leq b$. Likewise, $2b \leq a+b$ which implies $b \leq a$ and $a = b$. Therefore, $2a = a+b \in \mathrm{Infinit}(M)^*$, which is a contradiction. We have shown that $\mathrm{Infinit}(M)$ is an ideal of M.

Let us assume now that the equivalent conditions (i)–(v) hold and let I be a maximal ideal of M. Then $I \supseteq \mathrm{Rad}(M) = \mathrm{Ker}(s)$ and s is a state-morphism. Thus the maximality of I and $\mathrm{Ker}(s)$ imply that $I = \mathrm{Infinit}(M)$ and that $\mathrm{Infinit}(M)$ is the unique maximal ideal of M.

Next, it is clear that (i)–(v) imply (vi). We complete the proof by establishing the equivalence of (vi)–(viii).

(vi) $\Rightarrow$ (vii). It is clear that M admits a unique state, s, namely, $s(a) = 0$ if $a \in \mathrm{Rad}_n(M)$ and $s(a) = 1$ if $a \in \mathrm{Rad}_n(M)^*$. Hence $\mathrm{Rad}_n(M) = \mathrm{Ker}(s)$.

(vii) $\Rightarrow$ (viii). Let s be only a two-valued state. Then $\mathrm{Rad}_n(M) = \mathrm{Ker}(s)$ and $I = \mathrm{Ker}(s)$ is a normal ideal such that $M/I = \Gamma(\mathbb{Z}, 1)$. Assume that J is a normal ideal of M such that $M/J = \Gamma(\mathbb{Z}, 1)$. Since $\Gamma(\mathbb{Z}, 1)$ is a two-element Boolean algebra, it has only a two-valued state, say μ. The mapping $\hat{\mu}$ on M defined by $\hat{\mu}(a) = \mu(a/J)$, $a \in M$, is an extremal state on M. It follows that $s = \hat{\mu}$ and $I = \mathrm{Ker}(s) = J$.

(viii) $\Rightarrow$ (vi). As in the preceding implication, M admits a state, s. Then $\mathrm{Ker}(s)$ is a normal ideal such that $M/\mathrm{Ker}(s) = \Gamma(\mathbb{Z}, 1)$, which yields that s is the unique state on M and, moreover, s is two-valued. Whence $\mathrm{Ker}(s) = \mathrm{Rad}_n(M)$ and (vi) holds. $\square$

Remark 4.3 *We note that all statements* (i)–(viii) *of Proposition* 4.2 *are equivalent in the setting of MV-algebras. However, this is no longer the case for GMV-algebras. Indeed, take* M_1 *from Example* 3.2 *and let* $M_2 = \Gamma(\mathbb{Z}, 1)$. *Then* $M = M_1 \times M_2$ *is a symmetric GMV-algebra having only one two-valued state, but* (ii) *of Proposition* 4.2 *does not hold.*

We say a non-trivial symmetric GMV-algebra M is *perfect* if $\mathrm{Rad}(M) \cup \mathrm{Rad}(M)^* = M$.

It is clear that the class of perfect GMV-algebras is not closed under direct products, but, as the next proposition shows, it is closed under subalgebras and non-trivial homomorphic images.

Proposition 4.4 *If M is a perfect GMV-algebra, then so is any non-trivial homomorphic image of M and any subalgebra of M.*

Proof. Let $h: M \to N$ be a surjective homomorphism of GMV-algebras, and assume that N is non-trivial. In light of of Proposition 4.2(v), if $x \in \mathrm{Infinit}(M)$ then $h(x) \in h(\mathrm{Infinit}(M))$; and if $x^- \in \mathrm{Infinit}(M)$ then $h(x^-) \in h(\mathrm{Infinit}(M))$ and $h(x) \in (h(\mathrm{Infinit}(M)))^*$. Thus, $h(\mathrm{Infinit}(M)) = \mathrm{Infinit}(h(M))$ and $h(\mathrm{Infinit}(M)^*) = (\mathrm{Infinit}(h(M)))^*$, showing that $h(M)$ is perfect.

Similarly, if N is a GMV-subalgebra of M, then N is perfect by Proposition 4.2(v). $\square$

5 The Categorical Equivalence of Perfect GMV-algebras

In the present section, we establish that the category of perfect GMV-algebras is categorically equivalent to the category of ℓ-groups. An analogous result was proved also by Leuştean [Leu] in a different way.

Let $\mathcal{PGMV}$ be the category whose objects are perfect GMV-algebras and morphisms are homomorphisms of GMV-algebras. Let $\mathcal{L}$ be the category of ℓ-groups and ℓ-group homomorphisms.

We define a functor $\mathcal{E}: \mathcal{L} \to \mathcal{PGMV}$ as follows: If G is an ℓ-group, let

$$\mathcal{E}(G) := \Gamma(\mathbb{Z} \times_{lex} G, (1,0)), \tag{5.1}$$

and if h is an ℓ-group homomorphism with domain G, we set

$$\mathcal{E}(h)(x) = \begin{cases} (0, h(g)) & \text{if } x = (0,g), \\ (1, -h(g)) & \text{if } x = (1,-g), \end{cases} \tag{5.2}$$

where $g \in G^+$.

Proposition 5.1 *$\mathcal{E}$ is a faithful and full functor from the category $\mathcal{L}$ of ℓ-groups into the category $\mathcal{PGMV}$ of perfect GMV-algebras.*

Proof. Let h_1 and h_2 be two morphisms from G into G' such that $\mathcal{E}(h_1) = \mathcal{E}(h_2)$. Then $(0, h_1(g)) = (0, h_2(g))$ for all $g \in G^+$, and hence $h_1 = h_2$.

To prove that the functor $\mathcal{E}$ is full, consider a GMV-algebra homomorphism $f : \Gamma(\mathbb{Z} \times_{lex} G, (1,0)) \to \Gamma(\mathbb{Z} \times_{lex} G', (1,0))$. Then for each $g \in G^+$ there exists a unique $g' \in G'^+$ such that $f(0,g) = (0,g')$. Define the mapping $h : G^+ \to G'^+$ by $h(g) = g'$ iff $f(0,g) = (0,g')$. Note that $h(g_1 + g_2) = h(g_1) + h(g_2)$ for all $g_1, g_2 \in G^+$, and note further that h preserves finite meets and joins. Assume now that $g \in G$ is arbitrary. If $g = g_1 - g_2 = g_1' - g_2'$, for $g_1, g_2, g_1', g_2' \in G^+$, then $g_1 + g_2' = g_1' + g_2$. This shows that the assignment $h(g) = h(g_1) - h(g_2)$ is a well-defined extension of h to the whole ℓ-group G.

We assert that the aforementioned map h preserves binary meets in G, i.e., $h(a \wedge b) = h(a) \wedge h(b)$, whenever $a, b \in G$. Let $a, b \in G$. We have $a = a^+ - a^-$, $b = b^+ - b^-$, $a = -a^- + a^+$, and $b = -b^- + b^+$. Since $h((a^+ + b^-) \wedge (a^- + b^+)) = h(a^+ + b^-) \wedge h(a^- + b^+)$, subtracting $h(b^-)$ from the right hand side and $h(a^-)$ from the left hand side, we obtain the assertion.

In conclusion, we have proved that h is a homomorphism of ℓ-groups, and that $\mathcal{E}(h) = f$. $\square$

We recall that by a *universal group* for a GMV-algebra M we mean a pair (G, γ) consisting of an ℓ-group G and a G-valued measure $\gamma : M \to G^+$ (i.e., $\gamma(a + b) = \gamma(a) + \gamma(b)$ whenever $a + b$ is defined in M) such that the following conditions hold: (i) $\gamma(M)$ generates G; and (ii) if H is a group and $\phi : M \to H$ is an H-valued measure, then there exists a group homomorphism $\phi^* : G \to H$ such that $\phi = \phi^* \circ \gamma$.

Due to [Dvu1], every GMV-algebra admits a unique, up to isomorphism, universal group, and the group homomorphism ϕ^* is unique. The universal group for $M = \Gamma(G, u)$ is (G, id) where id is the embedding of M into G.

Proposition 5.2 *Let M be a perfect GMV-algebra. Then there is a unique (up to isomorphism) ℓ-group G such that $M \cong \Gamma(\mathbb{Z} \times_{lex} G, (1,0))$.*

Proof. Let M be a perfect GMV-algebra. Without loss of generality, we can assume $M = \Gamma(G_0, u)$ for some unital ℓ-group (G_0, u). By Proposition 4.1, $\mathrm{Rad}(M)$ is a cancellative semigroup satisfying Birkhoff's conditions [Bir, Thm XIV.2.1; Fuc, Theorem II.4] for being the positive cone of a unique (up to isomorphism) directed po-group G. Since $\mathrm{Rad}(M)$ is a lattice, we have that G is an ℓ-group.

$\Gamma(\mathbb{Z} \times_{lex} G, (1,0))$ is a perfect GMV-algebra. Let us define a mapping $f : M \to \Gamma(\mathbb{Z} \times_{lex} G, (1,0))$ by $f(a) = (0,a)$ and $f(a^-) = (1,-a)$ if $a \in \mathrm{Rad}(M)$.

Then f is injective, onto, and preserves $^-$, $+$, $\vee$ and $\wedge$. Consequently, f can be extended to an injective group homomorphism $\hat{f}:\ G_0 \to \mathbb{Z} \times_{lex} G$.

We show that $\hat{f}$ is an ℓ-group homomorphism. The proof will proceed in several steps.

Step 1. Let $a, b, u_0 \in G_0^+$. If $\hat{f}(a \wedge b) = \hat{f}(a) \wedge \hat{f}(b)$ and $\hat{f}(u_0 \wedge (b - (a \wedge b))) = \hat{f}(u_0) \wedge \hat{f}(b - (a \wedge b))$, then

$$\hat{f}((a + u_0) \wedge b) = \hat{f}(a + u_0) \wedge \hat{f}(b).$$

Indeed, we have

$$u_0 \wedge (b - (a \wedge b)) + (a \wedge b) = (u_0 + a \wedge b) \wedge b = (u_0 + a) \wedge (u_0 + b) \wedge b = (u_0 + a) \wedge b,$$

which yields

$$\begin{aligned} \hat{f}((a + u_0) \wedge b) &= \hat{f}(u_0 \wedge (b - (a \wedge b))) + \hat{f}(a \wedge b) \\ &= [\hat{f}(u_0) \wedge (\hat{f}(b) - (\hat{f}(a) \wedge \hat{f}(b)))] + (\hat{f}(a) \wedge \hat{f}(b)) \\ &= [(\hat{f}(u_0) + (\hat{f}(a) \wedge \hat{f}(b)))] \wedge \hat{f}(b) \\ &= (\hat{f}(u_0) + \hat{f}(a)) \wedge (\hat{f}(u_0) + \hat{f}(b)) \wedge \hat{f}(b) = \hat{f}(a + u_0) \wedge \hat{f}(b). \end{aligned}$$

Step 2. $\hat{f}(a \wedge b) = \hat{f}(a) \wedge \hat{f}(b)$, whenever $a \in G_0^+$ and $b \in \Gamma(G_0, u)$.

Let $a \in G_0^+$ and $b \in \Gamma(G_0, u)$. Since $\Gamma(G_0, u)$ generates G_0^+ as a semigroup, a can be written as $a = a_1 + \cdots + a_n$, for some $a_1, \ldots, a_n \in \Gamma(G_0, u)$. The proof will proceed by mathematical induction on n.

If $n = 1$, the statement is trivial. Suppose now that the statement holds for all $a' = a_1 + \cdots + a_i$, with $1 \le i \le n$. Set $a = a_1 + \cdots + a_n$, $u_0 = a_{n+1}$. Then there exist $v_1, \ldots, v_k \in \Gamma(G_0, u)$ such that $b = (v_1 + \cdots + v_k) + (a \wedge b)$. Since $v := v_1 + \cdots + v_k \le b \in \Gamma(G_0, u)$, $v \in \Gamma(G_0, u)$. Hence $v = b - (a \wedge b)$. Since $\hat{f}$ preserves meets in $\Gamma(G_0, u)$, we have $\hat{f}(u_0 \wedge v) = \hat{f}(u_0) \wedge \hat{f}(v)$. Thus, by the induction hypothesis for a and b, we get $\hat{f}(u_0 \wedge (b - (a \wedge b))) = \hat{f}(u_0) \wedge \hat{f}(b - (a \wedge b)) = \hat{f}(u_0) \wedge (\hat{f}(b) - (\hat{f}(a) \wedge \hat{f}(b)))$. By Step 1, $\hat{f}((a + u_0) \wedge b) = \hat{f}(a + u_0) \wedge \hat{f}(b)$, that is, $\hat{f}((a_1 + \cdots + a_{n+1}) \wedge b) = \hat{f}(a_1 + \cdots + a_{n+1}) \wedge \hat{f}(b)$ for all n.

Step 3. $\hat{f}(a \wedge b) = \hat{f}(a) \wedge \hat{f}(b)$ whenever $a, b \in G_0^+$.

Let $a = a_1 + \cdots + a_n$, $b = b_1 + \cdots + b_k$. The proof will follow complete induction on k.

If $k = 1$, we apply Step 2. Suppose now that the assertion holds for all j with $1 \le j \le k$. Let $B = a$, $A = b_1 + \cdots + b_k$ and $u_0 = b_{k+1}$. By Step 2,

$\hat{f}(u_0 \wedge (B-(A\wedge B))) = \hat{f}(u_0)\wedge \hat{f}(B-(A\wedge B))$ and $\hat{f}(A\wedge B) = \hat{f}(A)\wedge \hat{f}(B)$. Therefore the conditions of Step 1 are satisfied, so that $\hat{f}((A+u_0)\wedge B) = \hat{f}(A+u_0)\wedge \hat{f}(B)$, which proves $\hat{f}((a_1+\cdots+a_n)\wedge(b_1+\cdots+b_{k+1})) = \hat{f}(a_1+\cdots+a_n)\wedge \hat{f}(b_1+\cdots+b_{k+1})$, for each n and each k.

Step 4. $\hat{f}(a\wedge b) = \hat{f}(a)\wedge \hat{f}(b)$ whenever $a, b \in G_0$. Then $a = a^+ - a^-$, $b = b^+ - b^-$, $a = -a^- + a^+$ and $b = -b^- + b^+$. Invoking Step 3, we get $\hat{f}((a^+ + b^-)\wedge(a^- + b^+)) = \hat{f}(a^+ + b^-)\wedge \hat{f}(a^- + b^+)$. Subtracting $\hat{f}(b^-)$ from the right hand side and $\hat{f}(a^-)$ from the left hand side, we obtain the claim.

We have verified that f is an MV-isomorphism of the GMV-algebras M and $\Gamma(\mathbb{Z}\times_{lex} G, (1,0))$. $\square$

Proposition 5.3 *The functor $\mathcal{E} : \mathcal{L} \to \mathcal{PGMV}$ has a left-adjoint.*

Proof. We show that given a perfect GMV-algebra M there is a universal arrow (G, f). This means that f is a homomorphism from M into $\mathcal{E}(G)$ such that if G' is an object from $\mathcal{L}$ and f' is a homomorphism from M into $\mathcal{E}(G')$, then there exists a unique morphism $f^* : G \to G'$ such that $\mathcal{E}(f^*)\circ f = f'$.

We have, from Proposition 5.2 and its proof, that $(\mathbb{Z}\times_{lex} G, f)$ is a universal group for M, where $f : M \to \Gamma(\mathbb{Z}\times_{lex} G, (1,0))$ is defined by $f(a) = (0, a)$ and $f(a^-) = (1, -a)$, for $a \in \mathrm{Rad}(M)$. Then it is straightforward to verify that (G, f) is a universal arrow for M. $\square$

We define a functor $\mathcal{P} : \mathcal{PGMV} \to \mathcal{L}$ as follows. If M is a GMV-algebra, let $\mathcal{P}(M) := G$ where $(\mathbb{Z}\times_{lex} G, f)$ is the universal group for M. It is clear that if $f_0 : M \to N$ is a morphism of GMV-algebras, then it can be uniquely extended to a homomorphism $\mathcal{P}(f_0)$ from G into G_1, where $(\mathbb{Z}\times_{lex} G_1, f_1)$ is the universal group for the perfect GMV-algebra N.

Proposition 5.4 *The functor $\mathcal{P}$ is the left-adjoint of the functor $\mathcal{E}$.*

Proof. It follows from the construction of the universal group. $\square$

We present now the main result of this section.

Theorem 5.5 *The pair $(\mathcal{P}, \mathcal{E})$ of functors constitutes a categorical equivalence between the category, $\mathcal{PGMV}$, of perfect GMV-algebras and the category, $\mathcal{L}$, of ℓ-groups.*

In addition, if $h : \mathcal{E}(G) \to \mathcal{E}(H)$ is a homomorphism of GMV-algebras, then there exists a unique homomorphism $f : G \to H$ of unital ℓ-groups such that $h = \mathcal{E}(f)$, and f is surjective (respectively, injective) if so is h.

Proof. According to [MaL, Thm IV.4.1], it is necessary to show that, for a perfect GMV-algebra M, there is an object G in $\mathcal{L}$ such that $\mathcal{E}(G)$ is isomorphic to M. To show that, we take a universal group $(\mathbb{Z} \times_{lex} G, f)$. Then $\mathcal{E}(G)$ and M are isomorphic. $\square$

Let M be a GMV-algebra. For any integer $n \geq 0$ and any element $x \in M$ we set $n \odot x = x_1 \oplus \cdots \oplus x_n$ and $x^n = x_1 \odot \cdots \odot x_n$, where $x_1 = \cdots = x_n := x$.

Remark 5.6 *In view of Theorem* 5.5, *any perfect GMV-algebra satisfies the identity*

$$2 \odot x^2 = (2 \odot x)^2. \tag{5.3}$$

6 Variety of Perfect GMV-algebras

We have noted above that the class, $\mathcal{PGMV}$, of perfect GMV-algebras is not a variety. Let $\mathcal{V}(\mathcal{PGMV})$ denote the variety generated by $\mathcal{PGMV}$. In this section we present an equational basis for $\mathcal{V}(\mathcal{PGMV})$ and describe a single algebra that generates it.

We introduce the following important example from [DvHo].

Let $u \in \mathrm{Aut}(\mathbb{R})$ be the translation $tu = t + 1$, $t \in \mathbb{R}$, and let

$$\mathrm{BAut}(\mathbb{R}) = \{g \in \mathrm{Aut}(\mathbb{R}) : \ \exists\, n \in \mathbb{N}, \ \ u^{-n} \leq g \leq u^n\}.$$

Then $(\mathrm{BAut}(\mathbb{R}), u)$ is a doubly transitive unital ℓ-permutation group, and according to [Gla2, Lem 10.3.1], the variety of GMV-algebras generated by $\Gamma(\mathrm{BAut}(\mathbb{R}), u)$ is the variety of all GMV-algebras.

It is worth mentioning, that if (G, u) is an arbitrary doubly transitive unital ℓ-group, then by [DvHo, Thm 4.11], the variety of GMV-algebras generated by $\Gamma(G, u)$ is the variety of all GMV-algebras.

We show below that $\mathcal{V}(\mathcal{PGMV})$ is generated by a perfect GMV-algebra, thereby generalizing the result in [DiLe3, Thm 3.8] which states that $\Gamma(\mathbb{Z} \times_{lex} \mathbb{Z}, (1, 0))$ is a generator for the variety generated by all perfect MV-algebras.

Given any ℓ-group G, we set $\mathcal{V}(\mathcal{E}(G))$ for the variety generated by the perfect GMV-algebra $\mathcal{E}(G)$.

Theorem 6.1 *If G is a doubly transitive ℓ-group, then $\mathcal{V}(\mathcal{PGMV}) = \mathcal{V}(\mathcal{E}(G))$. In particular, an identity holds in every perfect GMV-algebra if and only if it holds in $\mathcal{E}(G)$.*

Proof. Let G be a doubly transitive ℓ-groups, and define $\mathcal{E}(G)$ via (5.1). Let M be any perfect MV-algebra. In view of Proposition 5.2, there exists a unique ℓ-group G_M such that $M = \mathcal{E}(G_M)$. Since every doubly transitive ℓ-group generates the variety $\mathcal{L}$ of ℓ-groups, [Gla2, Lem. 10.3.1], and hence all free ℓ-groups are ℓ-subalgebras of powers G^I of G. Thus, there exist an ℓ-subgroup K of a power G^I of G and a surjective ℓ-homomorphism $f : K \to G_M$. By Theorem 5.5 and condition (5.2), $M = \mathcal{E}(G_M) = \mathcal{E}(f)(\mathcal{E}(K))$.

Define a mapping $\rho : \mathcal{E}(G^I) \to (\mathcal{E}(G))^I$ by $\rho(0,(g_i)_{i\in I}) = \{(0,g_i)\}_{i\in I}$ and $\rho(1,(-g_i)_{i\in I}) = \{(1,-g_i)\}_{i\in I}$, for $g_i \in G^+$ and $i \in I$. Then ρ is an embedding, and $\mathcal{E}(G^I) \in \mathcal{V}(\mathcal{E}(G))$. Since $\mathcal{E}(K)$ is a subalgebra of $\mathcal{E}(G^I)$, we have $\mathcal{E}(K) \in \mathcal{V}(\mathcal{E}(G))$ and $M \in \mathcal{V}(\mathcal{E}(G))$, since it is a homomorphic image of $\mathcal{E}(K) \in \mathcal{V}(\mathcal{E}(G))$. $\square$

We mention in passing that $\mathrm{Aut}(\mathbb{R})$ is a doubly transitive ℓ-group that satisfies the conditions of Theorem 6.1.

Proposition 6.2 *If $\mathcal{K}$ is a nonempty family of GMV-algebras that admit at least one state, then every non-trivial GMV-algebra in the variety, $\mathcal{V}(\mathcal{K})$, generated by $\mathcal{K}$ admits a state. In addition, the class $\mathcal{SGMV}$ – consisting of all GMV-algebras that admit a state or are trivial – is a variety.*

Proof. If $\{0\} \neq M \in \mathcal{V}(\mathcal{K})$, then there exist an algebra F, which is a subdirect product of a family $(M_i : i \in I)$ of algebras in $\mathcal{K}$, and a surjective GMV-homomorphism $h : F \to M$. It is clear that each of $\prod_{i\in I} M_i$ and F admits a state. Because there is a bijective correspondence between the set of extremal states and the set of maximal ideals that are also normal, F has a maximal filter that is also normal. It follows that $h(I)$ is a maximal filter of M that is also normal, and hence M admits a state. $\square$

An immediate consequence of Proposition 6.2 is that every non-trivial GMV-algebra in $\mathcal{V}(\mathcal{PGMV})$ admits a state.

Motivated by the results of [DiLe1], we introduce the class, $\mathcal{BP}$, of GMV-algebras. A GMV-algebra M is a member of $\mathcal{BP}$ if M is trivial or each maximal ideal I of it is normal and satisfies the equalities $I^- = I^\sim$ and $I \cup I^* = M$. We will show that $\mathcal{BP}$ is a variety, thereby generalizing the corresponding result for MV-algebras [AmLe]. Our proof makes use of ideas developed in [DvHo].

Let $M = \Gamma(G, u)$ be a GMV-algebra, where (G, u) is a unital ℓ-group. By a *value* of u in (G, u) we mean a convex ℓ-subgroup V of (G, u) that is maximal with respect to not containing u. Hence, $\phi^{-1}(V)$ is a maximal ideal of M, where ϕ is defined by (2.1), and, conversely, if I is a maximal ideal of M, then $\phi(I)$ is a value of u in (G, u).

For any value V of (G, u), we set

$$K(V) = \bigcap_{g \in G} g^{-1} V g$$

(we momentarily employ multiplicative notation for (G, u)). Then $K(V)$ is a normal convex ℓ-subgroup of (G, u) contained in V, and $(G/K(V), G/V)$ is a primitive transitive ℓ-permutation group, called a *component* of G.

Let $\mathcal{V}$ be a variety of GMV-algebras and let $\Gamma^{-1}(\mathcal{V}) = \{(G, u) : \Gamma(G, u) \in \mathcal{V}\}$. Then, [DvHo, Thm 3.1], $\Gamma^{-1}(\mathcal{V})$ is an equational class of unital ℓ-groups. We point out here, as a word of caution, that $\Gamma^{-1}(\mathcal{V})$ is not a variety in the usual sense of universal algebra, but rather a class of unital ℓ-groups described by equations in the language of unital ℓ-groups.

Let

$$\mathcal{T}(\mathcal{V}) = \{\Gamma(G, u) : \Gamma(G/K(V), u/K(V)) \in \mathcal{V}\} \cup \{\{0\}\}.$$

By [DvHo, Cor. 4.5], $\mathcal{T}(\mathcal{V})$ is a variety, referred to as a *top variety* of $\mathcal{V}$.

Let $\mathcal{V}_P$ be the variety of symmetric GMV-algebras satisfying the identity (5.3). Then

$$\mathcal{V}(\Gamma(\mathbb{Z}, 1)) \subseteq \mathcal{V}(\mathcal{PGMV}) \subseteq \mathcal{V}_P.$$

Theorem 6.3 *$\mathcal{T}(\mathcal{V}_P) = \mathcal{BP}$, and hence $\mathcal{BP}$ is a variety. Moreover, it satisfies $\mathcal{T}(\mathcal{BP}) = \mathcal{BP} = \mathcal{T}(\mathcal{V}(\Gamma(\mathbb{Z}, 1))) = \mathcal{T}(\mathcal{V}(\mathcal{E}(\mathbb{Z})))$.*

Proof. Let $M = \Gamma(G, u) \in \mathcal{T}(\mathcal{V}_P)$ and let I be a maximal ideal of M. Then $V = \phi(I)$ is a value of (G, u) and $G/K(V) \in \Gamma^{-1}(\mathcal{V}_P)$. Now $G/K(V)$ is transitive and primitive and hence we can invoke McCleary's Trichotomy Classification Theorem of primitive ℓ-groups – [Gla2, Thm 7E], [Dar, Thm 33.10]. According to this theorem, $G/K(V)$ is either (1) Abelian, hence an ℓ-subgroup of the reals, (2) doubly transitive, or (3) periodic. Clearly $G/K(V)$ cannot be doubly transitive, since then $\mathcal{V}_P = \mathcal{GMV}$, which is impossible.

Similarly the third possibility is excluded because otherwise, there would exist an ℓ-subgroup of $G/K(V)$ with a doubly transitive ℓ-homomorphic image, which again would imply the absurdity $\mathcal{V}_P = \mathcal{GMV}$.

Consequently, $(G/K(V), u/K(V))$ is an ℓ-subgroup of the reals, and since $\Gamma((G/K(V), u/K(V)))$ satisfies (5.3), this yields $(G/K(V), u/K(V)) \cong (\mathbb{Z}, 1)$. Hence, $\Gamma(G/K(V), u/K(V))$ admits a two-valued extremal state, say s, such that $\mathrm{Ker}(s) = \phi^{-1}(K(V))$. Since $\mathrm{Ker}(s)$ is a maximal ideal of M, $K(V)$ is a value of u in (G, u), so that $K(V) = V$ and V is normal. This implies that I is normal, $I^- = I^\sim$, and $I \cup I^* = M$, i.e., $M \in \mathcal{BP}$.

Conversely, let $M \in \mathcal{BP}$ and let $M = \Gamma(G, u)$. Then any value V of u in (G, u) is normal, and $V = K(V)$, so that $G/K(V)$ is a subgroup of the reals. Since $\phi^{-1}(K(V)) \cup \phi^{-1}(K(V))^* = M$, we have $(G/K(V), u/K(V)) \cong (\mathbb{Z}, 1)$. It is clear that $\Gamma(\mathbb{Z}, 1) \in \mathcal{V}_P$, i.e., $M = \Gamma(G, u) \in \mathcal{T}(\mathcal{V}_P)$.

The equality $\mathcal{T}(\mathcal{V}_P) = \mathcal{BP}$ implies that $\mathcal{BP}$ is a variety, since, as noted above, $\mathcal{T}(\mathcal{V}_P)$ is a variety.

It is clear that $\mathcal{BP} \subseteq \mathcal{T}(\mathcal{BP})$. Let now $M = \Gamma(G, u) \in \mathcal{T}(\mathcal{BP})$, and let I be a maximal ideal of M. Then $G/K(V) \in \Gamma^{-1}(\mathcal{BP})$, where $V = \phi(I)$. As in the first part of the proof, we can show that $G/K(V)$ is an ℓ-subgroup of the reals. Then M admits an extremal state s such that $\mathrm{Ker}(s) = \phi^{-1}(K(V)) \subseteq I$. The maximality of the normal ideal $\mathrm{Ker}(s)$ yields that $\mathrm{Ker}(s) = I$, so that $K(V) = V$. Since $G/K(V) = G/V \subseteq \mathbb{R}$, $\{0/V\}$ is a unique maximal ideal of $G/K(V)$, $(G/K(V), u/K(V)) \cong (G/K(V), u/K(V))/\{0/V\} \cong (\mathbb{Z}, 1)$, which entails $I^- = I^\sim$ and $I \cup I^* = M$, i.e., $M \in \mathcal{BP}$.

In a similar manner we prove $\mathcal{BP} = \mathcal{T}(\mathcal{V}(\Gamma(\mathbb{Z}, 1)))$. Let $M \in \mathcal{T}(\mathcal{V}(\mathcal{E}(\mathbb{Z})))$, then as above, we have $M/I \in \mathcal{V}(\mathcal{E}(\mathbb{Z}))$, and M/I is an MV-algebra for any maximal ideal I of M which is an MV-subalgebra of $\Gamma(\mathbb{R}, 1)$. But due to [DiLe2, Thm 18], this is equivalent to the statement $(M/I)/\mathrm{Rad}(M/I) \in \Gamma(Z, 1)$. Because $\mathrm{Rad}(M/I)$ is the zero ideal of M/I, we have $M/I \in \mathcal{V}(\mathcal{E}(\mathbb{Z}))$ iff $M/I \in \mathcal{V}(\Gamma(Z, 1))$. Therefore, $\mathcal{T}(\mathcal{V}(\mathcal{E}(\mathbb{Z}))) = \mathcal{T}(\mathcal{V}(\Gamma(\mathbb{Z}, 1)))$. $\square$

Let $\mathcal{M}$ be the set of GMV-algebras M such that either every maximal ideal of M is normal or M is trivial. Proceeding as in the proof of Theorem 6.3, we can show that $\mathcal{M}$ is a variety such that

$$\mathcal{M} = \mathcal{T}(\mathcal{MV}) = \mathcal{T}(\mathcal{N}) = \mathcal{T}(\mathcal{M}), \tag{6.1}$$

where $\mathcal{MV}$ is the variety of MV-algebras and $\mathcal{N}$ is the class of normal valued

GMV-algebras, which according to [Dvu2, Thm 6.8] is a variety. In addition,

$$\mathcal{M} \subsetneqq \mathcal{SGMV} = \mathcal{T}(\mathcal{SGMV}). \tag{6.2}$$

Combining Theorem 6.3 and (6.1), we have

$$\mathcal{V}(\mathcal{PGMV}) \subseteq \mathcal{V}_P \subseteq \mathcal{T}(\mathcal{V}_P) = \mathcal{BP} \subseteq \mathcal{M}. \tag{6.3}$$

Let $\mathcal{SYM}$ be the variety of symmetric GMV-algebras, and set $\mathcal{SBP} = \mathcal{SYM} \cap \mathcal{BP}$.

We recall for $M \in \mathcal{SYM} \cap \mathcal{M}$, all conditions (i)–(viii) of Proposition 4.2 are equivalent.

In what follows, we show that

$$\mathcal{SBP} = \mathcal{V}_P = \mathcal{V}(\mathcal{PGMV}).$$

Let M be a GMV-algebra. An element $e \in M$ is said to be an *idempotent* (or *Boolean*) if $e \odot e = e$. According to [GeIo, Lem. 4.1, Prop. 4.2], the following statements are equivalent:

(i) an element $e \in M$ is an idempotent;

(ii) $e \oplus e = e$;

(iii) $e \wedge e^- = 0$;

(iv) $e \wedge e^\sim = 0$;

(v) $e \vee e^- = 1$;

(vi) $e \vee e^\sim = 0$.

Let $B(M)$ be the set of idempotents of M. Then (i) $0, 1 \in B(M)$; (ii) $e^- = e^\sim$, if $e \in B(M)$; (iii) $x \oplus e = x \vee e = e \oplus x$, $x \odot e = x \wedge e = e \odot x$, for $x \in M$; (iv) $(B(M); \vee, \wedge, ^-, 0, 1)$ is a Boolean algebra: and (v) $B(M)$ is the greatest GMV-subalgebra of M that is also a Boolean algebra (see [GeIo, Cor. 4.4]). Moreover, if $e \in B(M)$, then the interval $[0, e]$ endowed with $\oplus_e, ^{-_e}, ^{\sim_e}$ and $0, e$ is a GMV-algebra, where $x \oplus_e y = x \oplus y$, $x^{-_e} = e \odot x^-$, and $x^{\sim_e} = e \odot x^\sim$, for $x, y \in [0, e]$.

We say that a GMV-algebra M is (i) *finitely subdirectly irreducible* if whenever M is a subdirect product of finitely many GMV-algebras $M_1, \ldots, M_n$, then $M \cong M_i$ for some $i = 1, \ldots, n$; (ii) *directly indecomposable* if M is non-trivial and whenever $E \cong M_1 \times M_2$, then either M_1 or M_2 is trivial.

It is clear that if M is subdirectly irreducible, then M is finitely subdirectly irreducible.

Proposition 6.4 *A GMV-algebra M is directly indecomposable if and only if $B(M) = \{0, 1\}$. If M is finitely subdirectly irreducible, then $B(M) = \{0, 1\}$.*

Proof. Let e be an idempotent of M. We define a mapping $p_e : M \to [0, e]$ by $p_e(x) := x \wedge e$, $x \in M$. According to [Dvu3, Prop. 4.1 (vii)] and [Dvu4, Cor. 4.3], p_e is a surjective homomorphism of GMV-algebras. Moreover, $M \cong [0, e] \times [0, e^-]$ under an isomorphism $x \mapsto (p_e(x), p_{e^-}(x))$, $x \in M$. (Note that $x = p_e(x) + p_{e^-}(x)$.)

If now M is directly indecomposable, then $M \cong [0, e]$ or $M \cong [0, e^-]$, that is, $e \in \{0, 1\}$.

Conversely, let $B(M) = \{0, 1\}$ and let $M \cong M_1 \times M_2$. If both M_1 and M_2 are non-trivial, then $(0_1, 1_2)$ and $(1_1, 0_2)$ are non-trivial idempotents of $M_1 \times M_2$, which is absurd.

Finally, if M is finitely subdirectly irreducible, the isomorphism $M \cong [0, e] \times [0, e^-]$, for any idempotent $e \in B(M)$, yields $B(M) = \{0, 1\}$. $\square$

Lemma 6.5 *If $M \in \mathcal{SBP}$ is subdirectly irreducible, then either M is trivial or $M = \langle \mathrm{Rad}(M) \rangle \in \mathcal{PGMV}$.*

Proof. Assume $M = \Gamma(G, u)$ for a non-trivial unital ℓ-group (G, u). In view of Theorem 2.1, M is subdirectly irreducible iff G is subdirectly irreducible. Hence, [Gla2, Cor. 7.1.3], G has a faithful transitive representation. By [Gla2, Cor. 7.1.1], this is possible iff there is a prime subgroup C of G such that $\bigcap_{g \in G} g^{-1}Cg = \{1\}$ (here, we use multiplicative notation for (G, u)). The partially ordered set $\Omega := \{Cg : g \in G\}$ of right cosets of C – with respect to the partial order defined by $Cg \leq Ch$ iff $g \leq ch$, for some $c \in C$ – is a totally ordered set in this case. Further, G has a faithful transitive representation on Ω, namely $\psi(f) = Cgf$, $f \in G$, with $\mathrm{Ker}(\psi) = \bigcap_{g \in G} g^{-1}Cg = \{1\}$.

Since the system of prime subgroups of G forms a root system, there is a unique maximal ideal I of M such that $C \subseteq \phi(I) =: \hat{I}$, where $\phi(I)$ is defined by (2.1).

It is clear that $x \wedge x^- \in \mathrm{Rad}(M)$ for any $x \in M$.

Claim. If $x \in I$, then $x = x \wedge x^-$.

There are two possibilities: (1) $Cg = Cg(x \wedge x^-)$ and (2) $Cg \neq Cg(x \wedge x^-)$.

(1) Let $Cg = Cg(x \wedge x^-)$. Then $x \wedge x^- \in g^{-1}Cg \subseteq g^{-1}\hat{I}g = \hat{I}$. Because $g^{-1}Cg$ is also prime, we have $x \in g^{-1}Cg$. Hence, $Cgx = Cg$, and $Cgx = Cg = Cg(x \wedge x^-) \leq Cgx^-$.

(2) Let $Cg \neq Cg(x \wedge x^-)$. The transitivity of G implies that there exists an $h \in G$ such that $Cgh = Cg(x \wedge x^-)$. Then $gh = cg(x \wedge x^-)$ for some $c \in C$, and $h = g^{-1}cg(x \wedge x^-) \in \hat{I}$. Hence, $Cgh = Cghh^{-1}(x \wedge x^-)$ and $h^{-1}(x \wedge x^-) = (h^{-1}x) \wedge (h^{-1}x^-) \in (gh)^{-1}C(gh)$. Since $(gh)^{-1}C(gh)$ is prime and $h \in \hat{I}$, we get $h^{-1}x \in (gh)^{-1}C(gh)$. Then $h^{-1}x = (gh)^{-1}cgh$ for some $c \in C$, and $gx = ghh^{-1}x = cgh$, i.e., $Cgx = Cgh$. But $Cgh = Cg(x \wedge x^-) \leq Cgx^-$.

Combining (1) and (2), we get $Cgx \leq Cgx^-$ for all $g \in G$, i.e., $x \leq x \wedge x^- \leq x$ and $x = x \wedge x^- \in \mathrm{Rad}(M)$. This completes the proof of the claim.

Let now $x^- \in I$. The claim above implies that $x^- = x \wedge x^- \in \mathrm{Rad}(M)$, that is, $M = \langle \mathrm{Rad}(M) \rangle$ and $M \in \mathcal{PGMV}$. $\square$

The next theorem shows that $\mathcal{V}(\mathcal{PGMV}) = \mathcal{V}_P = \mathcal{SBP}$. This result generalizes the corresponding result for MV-algebras, [DiLe1, Thm 5.11, Cor. 5.2], and implies, in particular, that $M \in \mathcal{V}(\mathcal{PGMV})$ iff it satisfies the identity identity (5.3).

Theorem 6.6 $\mathcal{V}(\mathcal{PGMV}) = \mathcal{V}_P = \mathcal{SBP}$.

Proof. Take $M \in \mathcal{SBP}$. Then M is a subdirect product of a family $(M_t : \ t \in T)$ of subdirectly irreducible algebras in $\mathcal{SBP}$. According to Lemma 6.5, every M_t is a perfect GMV-algebras. Hence, $M \in \mathcal{V}(\mathcal{PGMV})$, i.e., $\mathcal{SBP} \subseteq \mathcal{V}(\mathcal{PGMV})$. Thus, (6.3) completes the proof of the theorem. $\square$

As a direct consequence of Theorem 6.6, we have the following result.

Corollary 6.7 $\mathcal{T}(\mathcal{V}(\mathcal{PGMV})) \cap \mathcal{SYM} = \mathcal{V}(\mathcal{PGMV})$.

Proof. It follows from Theorems 6.3 and 6.6 $\square$

Corollary 6.8 *Every non-trivial member of $\mathcal{V}(\mathcal{PGMV})$ is a subdirect product of perfect GMV-algebras.*

Theorem 6.9 *If $M \in \mathcal{SBP}$ is non-trivial, then*

$$M = \langle \mathrm{Rad}(M) \cup B(M) \rangle.$$

Proof. Due to Corollary 6.8, M is a subdirect product of a family, $(M_t : t \in T)$, of perfect GMV-algebras. We can think of each $x \in M$ as an infinite vector $x = (x_t)_{t \in T}$, where $x_t \in M_t$ for all $t \in M_t$.

Claim 1. $2 \odot x^2 \in B(M)$.

Indeed, we have for all x_t, $2 \odot x_t^2 = 0_t$ if $x_t \in \mathrm{Rad}(M_t)$, and $2 \odot x_t^2 = 1_t$ if $x_t \in \mathrm{Rad}(M_t)^*$. Therefore, $2 \odot x^2 \in B(M)$.

Claim 2. $x \wedge x^- \in \mathrm{Rad}(M)$.

This is evident.

Claim 3. $x = (2 \odot x^2)^- \odot (x \wedge x^-) \oplus (2 \odot x^2) \odot (x \vee x^-)$ *for all* $x \in M$.

If $x_t \in \mathrm{Rad}(M_t)$, then $x_t = (2 \odot x_t^2)^- \odot (x_t \wedge x_t^-) \oplus (2 \odot x_t^2) \odot (x_t \vee x_t^-)$. Similarly, if $x_t \in \mathrm{Rad}(M_t)^*$, then $x_t = (2 \odot x_t^2)^- \odot (x_t \wedge x_t^-) \oplus (2 \odot x_t^2) \odot (x_t \vee x_t^-)$.

In conclusion, if $x \in M$, then $x \in \langle \mathrm{Rad}(M) \cup B(M) \rangle$. $\square$

Corollary 6.10 *A non-trivial GMV-algebra $M \in \mathcal{SBP}$ is perfect if and only if $B(M) = \{0, 1\}$. Moreover, the class of perfect GMV-algebras is first order definable in the variety $\mathcal{M}$ by*

$$[(\forall\, x)(x^- = x^\sim)],$$

$$[(\forall\, x)(2 \odot x^2 = (2 \odot x)^2)], \quad [(\forall\, x)((x^2 = x) \longrightarrow ((x = 0) \vee (x = 1))].$$

Proof. The first part follows from Theorem 6.9. $\square$

7 Cardinality of Symmetric GMV-algebra Varieties

The preceding sections demonstrate that symmetric GMV-algebras provide an ideal environment for extending many fundamental properties of MV-algebras. Thus the question arises as to how different symmetric GMV-algebras are from MV-algebras. One way to phrase this question in mathematical terms is to ask how large is the subvariety lattice of symmetric GMV-algebras. The primary purpose of this section is to show that this lattice has the cardinality of the continuum, thereby demonstrating that there exist uncountably many subvarieties of symmetric GMV-algebras exceeding the variety of MV-algebras. Our proof makes extensive use of our results on perfect GMV-algebras and uses the language of residuated lattices. As it is shown in [GaTs], there is an easy description of GMV-algebras within the framework of residuated lattices. We start with some definitions.

A *residuated lattice* (abbreviated, RL) is an algebra $L = (L; \wedge, \vee, \cdot, \backslash, /, e)$ such that (i) $(L; \wedge, \vee)$ is a lattice, (ii) $(L; \cdot, e)$ is a monoid, and (iii) the operation $\cdot$ is residuated with $\backslash$ and $/$ as its residuals. This means that for all $x, y, z \in L$, we have the equivalences

$$x \cdot y \leq z \ \Leftrightarrow \ x \leq z/y \ \Leftrightarrow \ y \leq x\backslash z.$$

Throughout the remainder of this section we use the convention that, in the absence of parentheses, $\cdot$ is performed first, followed by $\backslash, /$ and then $\vee, \wedge$. We will also write $xy = x \cdot y$. The class of RLs forms a variety, denoted by $\mathcal{RL}$. We refer the reader to [JiTs], and its predecessor [BlTs], for a systematic discussion of residuated lattices.

An algebra $(L; \wedge, \vee, \cdot, \backslash, /, e, v)$ is said to be a *pointed residuated lattice* (abbreviated, pointed RL) provided it satisfies the following two conditions (i) $(L; \wedge, \vee, \cdot, \backslash, /, e)$ is a residuated lattice, and (ii) v is a distinguished element of $(L; \wedge, \vee, \cdot, \backslash, /, e)$.

We denote by $p\mathcal{RL}$ the variety of pointed RLs. One can identify $\mathcal{RL}$ with the subvariety of $p\mathcal{RL}$ satisfying the law $e = v$.

For example, if G is an ℓ-group (written additively), then $(G; \wedge, \vee, \cdot, \backslash, /, 0)$ is a residuated lattice with respect to the operations $x \cdot y = x + y$, $x\backslash y = -x+y$, and $x/y = x-y$. Similarly, if G^- is the negative cone of an ℓ-group G, then $(G^-, \wedge, \vee, \cdot, \backslash, /, 0)$ is an RL with respect to $\cdot = +$, $x\backslash y = (-x+y) \wedge 0$, and $x/y = (x-y) \wedge 0$.

Let $(M; \oplus, ^-, ^\sim, 0, 1)$ be a GMV-algebra. Then $(M; \wedge, \vee, \cdot, \backslash, /, 1, 0)$ is a bounded pointed RL satisfying $y/(x\backslash y) = x \vee y = (y/x)\backslash y$, where $\cdot = \odot$, $x\backslash y = x^- \oplus y$, $x/y = x \oplus y^\sim$, and 0 and 1 are the bottom element and top element, respectively. Conversely, if $(M; \wedge, \vee, \cdot, \backslash, /, 1, 0)$ is a bounded pointed RL satisfying $y/(x\backslash y) = x \vee y = (y/x)\backslash y$, then $(M; \oplus, ^-, ^\sim, 0, 1)$ is a GMV-algebra, where $x \oplus y = 0/((y\backslash 0) \cdot (x\backslash 0))$ $(= ((0/y) \cdot (0/x))\backslash 0)$, $x^- = x\backslash 0$, $x^\sim = 0/x$, and $x \odot y = x \cdot y$.

We denote by $\mathcal{L} := \mathcal{L}_{\mathcal{RL}}$, $\mathcal{L}^- := \mathcal{L}^-_{\mathcal{RL}}$, and $\mathcal{GMV} := \mathcal{GMV}_{p\mathcal{RL}}$ the varieties of ℓ-groups, negative cones of ℓ-groups axiomatized relative to $\mathcal{RL}$, and the variety of GMV-algebras axiomatized relative to $p\mathcal{RL}$.

The following lemma can be found in [BCGJT] and [JiTs].

Lemma 7.1 (1) *The variety of ℓ-groups is term-equivalent to the subvariety, $\mathcal{L}$, of $\mathcal{RL}$ axiomatized, relative to $\mathcal{RL}$, by the identities $(e/x)x = e$.*

(2) *The variety $\mathcal{L}^-$ is axiomatized, relative to $\mathcal{RL}$, by the identities $x\backslash xy = y = yx/x$ and $y/(x\backslash y) = x \vee y = (y/x)\backslash y$.*

(3) *The variety of GMV-algebras is term-equivalent to the subvariety, $\mathcal{GMV}$, of $p\mathcal{RL}$ axiomatized, relative to $p\mathcal{RL}$, by the identities $y/(x\backslash y) = x \vee y = (y/x)\backslash y$ and $x \wedge v = v$.*

If $\mathcal{G}$ is a variety of ℓ-groups, according to [BCGJT, Thm 7.1], the mapping $\mathcal{G} \to \mathcal{G}^-$ is a lattice isomorphism between the subvariety lattices of $\mathcal{L}$ and $\mathcal{L}^-$.

Theorem 7.2 *The subvariety lattice of the variety of all symmetric GMV-algebras has the cardinality of the continuum.*

Proof. Given an ℓ-group G, we define the perfect GMV-algebra $\mathcal{E}(G)$ via (5.1). The GMV-algebra $\mathcal{E}(G)$ can be converted into a pointed RL $(\mathcal{E}(G); \wedge, \vee, \cdot, \backslash, /, 1, 0)$, with $\cdot$, $\backslash$ and $/$ defined as above, and $1 = (1, 0)$ and $0 = (0, 0)$.

The radical of $\mathcal{E}(G)$ is the set $\mathrm{Rad}(\mathcal{E}(G)) = \{(0, g) : \ g \in G^+\}$; $\mathrm{Rad}^*(\mathcal{E}(G)) = \{(1, g) : \ g \in G^-\}$ is a subalgebra of the the RL reduct of $\mathcal{E}(G)$ (we recall that $(0, g) < (1, -h)$ for all $g, h \in G^+$.) Moreover, it is isomorphic to the RL G^-, and $\mathrm{Rad}^*(\mathcal{E}(G)) = \{x \vee x\backslash 0 : \ x \in \mathcal{E}(G)\} = \{x \vee x/0 : \ x \in \mathcal{E}(G)\}$.

If $\mathcal{G}$ is a variety of ℓ-groups, then $\mathcal{E}(\mathcal{G}) = \{\mathcal{E}(G) : \ G \in \mathcal{G}\}$ is a category, and using the technique used in Section 5, Theorem 5.5, we can show that also $\mathcal{G}$ is categorically equivalent to $\mathcal{E}(\mathcal{G})$ under the restriction of the functor $\mathcal{E}$ to $\mathcal{E}(\mathcal{G})$. Let $\mathcal{V}(\mathcal{E}(\mathcal{G}))$ be the variety of symmetric GMV-algebras generated by $\mathcal{E}(\mathcal{G})$. This defines a mapping Φ that sends each $\mathcal{G}$ to $\mathcal{V}(\mathcal{E}(\mathcal{G}))$. Φ is an order preserving map from the subvariety lattice of $\mathcal{L}$ into the subvariety lattice of $\mathcal{GMV}$.

Claim. Φ *is an order-embedding.*

Let now $\mathcal{G}$ be an ℓ-group variety, axiomatized relative to $\mathcal{RL}$ by a set Σ of identities. Let Σ^- denote the corresponding set of identities axiomatizing $\mathcal{G}^-$, see [BCGJT, Lem. 7.4, Cor. 7.8, Thm 7.9]. Lastly, let Σ^-_v be the set of identities obtained from Σ^- by replacing each variable x occurring in an identity in Σ^- by the term $x \vee v/x$ $(= x \vee x\backslash v)$, where v is a variable not occurring in Σ^-, and we identify e with 0. Then, for each ℓ-group G, $G \models \Sigma$ iff $G^- \models \Sigma^-$ iff $\mathrm{Rad}^*(\mathcal{E}(G)) \models \Sigma^-$ iff $\mathcal{E}(G) \models \Sigma^-_v$. Equivalently, $G \in \mathcal{G}$ iff $\mathcal{E}(G) \in \Phi(\mathcal{G}) = \mathcal{V}(\mathcal{E}(\mathcal{G}))$. Thus, Φ is an order-embedding.

Finally, since the lattice of ℓ-group varieties has the cardinality of the continuum, the claim implies that so does the subvariety lattice of symmetric GMV-algebras. This is, of course, the maximum possible, since the language is finite. $\square$

Using results of [BCGJT], starting with a variety, $\mathcal{G}$, of ℓ-groups defined by a set Σ of identities, we describe the set of identities which describes the variety $\Phi(\mathcal{G})$ and which is derived from Σ.

Given a group term $g(x_1, \ldots, x_m)$ (written multiplicatively) and a variable z distinct from $x_1, \ldots, x_m$, let

$$\bar{g}(z, x_1, \ldots, x_m) = g(z^{-1}x_1, \ldots, z^{-1}x_m).$$

Since ℓ-groups are distributive, and since $\cdot$ and $^{-1}$ distributes over $\vee$ and $\wedge$, every identity can be reduced to a finite conjunctions of inequalities of the form $e \leq g_1 \vee \cdots \vee g_n$. If G is an ℓ-group, then according to [BCGJT, Lem. 7.6], $G \models e \leq g(x_1, \ldots, x_m)$ iff $G \models x_1 \vee \cdots \vee x_m \vee z \leq e$ implies $e \leq \bar{g}(z, x_1, \ldots, x_m)$. In view of [BCGJT, Lem. 7.7], for any group term g, there is an RL term $\hat{g}$ such that $(g \wedge e)^G|_{G^-} = \hat{g}^{L^-}$. This can be done using conjugation: $xy^{-1} = y^{-1}(yxy^{-1}) = y^{-1}(yx/y)$. In general, any group term g can be written in the form $g = p_1q_1^{-1}p_2q_2^{-1} \cdots p_nq_n^{-1}$, where $p_i, q_i \in G^+$. It can be rewritten in the form

$$q_1^{-1}q_2^{-1} \cdots q_n^{-1}(q_n(\cdots(q_2(q_1p_1/q_1)p_2/q_2)\cdots)p_n/q_n).$$

Thus we can take $\hat{g} = s\backslash t$, where

$$s = q_n \cdots q_2q_1 \quad \text{and } t = q_n^{-1}(q_n(\cdots(q_2(q_1p_1/q_1)p_2/q_2)\cdots)p_n/q_n.$$

Therefore, [BCGJT, Thm 7.9], let $\mathcal{G}$ be a variety of ℓ-groups defined by a set Σ of identities, which we may assume are of the form $e \leq g_1 \vee \cdots \vee g_n$. If

$$\Sigma^- = \{e = \hat{\bar{g}}_1 \vee \cdots \vee \hat{\bar{g}}_n : \ e \leq g_1 \vee \cdots \vee g_n \text{ is in } \Sigma\},$$

then Σ^- is an equational basis for $\mathcal{G}^-$. Changing e to 0 and each variable x occurring in an identity in Σ^- to $x \vee x/0$, we obtain the equational base, Σ_v^-, for $\Phi(\mathcal{G})$.

For example, the variety, $\mathcal{R}$, of representable ℓ-groups is axiomatized, relative to the variety, $\mathcal{L}$, of ℓ-groups by the identity $e \leq x^{-1}yx \vee y^{-1}$. Applying the above procedure, we see that $\mathcal{R}^-$ is axiomatized, relative to $\mathcal{L}^-$, by $e = zx\backslash(zy/z)z \vee y\backslash z$. Lastly, $\Phi(\mathcal{R})$ is axiomatized, relative to $\mathcal{GMV}$, by the identity $0 = [(z \vee z/0) \odot (x \vee x/0)\backslash((z \vee z/0) \odot (y \vee y/0)/(z \vee z/0)) \odot (z \vee z/0)] \vee [(y \vee y/0)\backslash(z \vee z/0)]$.

Or if $\mathcal{N}$ is the variety of normal-valued ℓ-groups, then is axiomatized, relative to $\mathcal{L}$, by the identity $(x\wedge e)^2(y\wedge e)^2 \leq (y\wedge e)(x\wedge e)$. The corresponding identity for $\mathcal{N}^-$ is $x^2y^2 \leq yx$, and the identity for $\Phi(\mathcal{N})$ is $(x \vee x/0) \odot (x \vee x/0) \odot (y \vee y/0) \odot (y \vee y/0) \leq (y \vee y/0) \odot (x \vee x/0)$.

It is worth recalling that if an ℓ-group G is a generator of a variety $\mathcal{G}$ of ℓ-groups, then $\Phi(\mathcal{G}) = \mathcal{V}(\mathcal{E}(G))$, compare with Theorem 6.1. For example, the variety of Abelian ℓ-groups, $\mathcal{A}$, has two important generators $G = \mathbb{R}$ and $G = \mathbb{Z}$, so that $\Phi(\mathcal{A}) = \mathcal{V}(\mathcal{E}(\mathbb{R})) = \mathcal{V}(\mathcal{E}(\mathbb{Z}))$, and the variety $\mathcal{G} = \mathcal{L}$ is generated by any doubly transitive G, so that $\Phi(\mathcal{L}) = \mathcal{V}(\mathcal{E}(G))$.

8 Coproducts and Amalgamation of Perfect GMV-algebras

In the final section of the paper, we study the amalgamation property for the category of perfect GMV-algebras and for related varieties. In particular, we prove that the variety $\mathcal{SBP}$ fails the amalgamation property. In fact, we show much more, namely, that there exist uncountably many varieties of symmetric GMV-algebras that fail the amalgamation property. En route, we also consider the related concept of a coproduct.

If I is an ideal of $M \in \mathcal{SYM} \cap \mathcal{M}$, then the GMV-subalgebra $\langle I\rangle$ of M generated by I has underlying set

$$\langle I\rangle = I \cup I^*.$$

Let M_1 be a GMV-subalgebra of $M_2 \in \mathcal{SYM} \cap \mathcal{M}$. We first claim that if J is a maximal ideal of M_2, then $I = J \cap M_1$ is a maximal ideal of M_1. Indeed, $I = J \cap M_1$ is an ideal of M_1 and it is normal and proper, since $1 \notin J$. Now a normal ideal I is maximal iff given $z \notin I$ there is an integer $n \geq 1$ such that $(n \odot z)^- \in I$. Thus, if $z \in M_1 \setminus I$, then $z \in M_2 \setminus J$, J is normal, and hence $(n \odot z)^- \in J$ and $(n \odot z)^- \in I$. Consequently, I is a maximal ideal of M_1.

Conversely, if I is a maximal ideal of M_1, there exists a maximal ideal J of M_2 such that $I = J \cap M_1$. Indeed, let $\hat{I} = \{x \in M_2 :\ \exists\, a \in I, x \leq a\}$. $\hat{I}$ is a proper ideal of M_2 $(1 \notin \hat{I})$. There exists a maximal ideal J of M_2 such that $\hat{I} \subseteq J$. By the discussion of the previous paragraph, $J \cap M_1$ is a maximal ideal of M_1, it contains I, and the maximality of I yields that $I = J \cap M_1$.

The preceding observations yield that $\mathrm{Rad}(M_1) = \mathrm{Rad}(M_2) \cap M_1$. Therefore, $\langle \mathrm{Rad}(M) \rangle$ is the greatest perfect subalgebra of $M \in \mathcal{SYM} \cap \mathcal{M}$.

Proposition 8.1 *For any $M \in \mathcal{SYM} \cap \mathcal{M}$, set*

$$M_{\mathcal{SBP}} := \bigcap \{\langle I \rangle : \ I \in \mathcal{M}(M)\}. \tag{8.1}$$

Then $M_{\mathcal{SBP}} \in \mathcal{SBP}$.

Proof. It is clear that $M_{\mathcal{SBP}} \subseteq M \in \mathcal{SYM} \cap \mathcal{M}$. Let $x \in M_{\mathcal{SBP}}$ and let J be a maximal ideal of $M_{\mathcal{SBP}}$. There exists a maximal ideal K of M such that $J = M_{\mathcal{SBP}} \cap K$. It is clear that $x \in \langle K \rangle$, therefore, $x \in (K \cup K^*) \cap M_{\mathcal{SBP}} = (K \cap M_{\mathcal{SBP}}) \cup (K^* \cap M_{\mathcal{SBP}}) = J \cup J^*$, i.e., $M_{\mathcal{SBP}} \in \mathcal{SBP}$. $\square$

Proposition 8.2 *Let $M \in \mathcal{SYM} \cap \mathcal{M}$ and let N be a GMV-subalgebra of M such that $N \in \mathcal{SBP}$. Then $N \subseteq M_{\mathcal{SBP}}$.*

Proof. It is clear that $N \in \mathcal{SYM} \cap \mathcal{M}$. If I is any maximal ideal of M, then $J := I \cap N$ is a maximal ideal of N. Since $N \in \mathcal{SBP}$, we have $N = J \cup J^* = (I \cap N) \cup (I^* \cup N) = \langle I \rangle \cap N$, which is true for any $I \in \mathcal{M}(M)$. Hence, $N = M_{\mathcal{SBP}} \cap N \subseteq M_{\mathcal{SBP}}$, as stated. $\square$

Theorem 6.9 can be generalized as follows giving a GMV-analogue of [DiLe1, Thm 5.8].

Theorem 8.3 *If $M \in \mathcal{SYM} \cap \mathcal{M}$, then*

$$M_{\mathcal{SBP}} = \langle \mathrm{Rad}(M) \cup B(M) \rangle.$$

Proof. It is clear that $B(M) \subseteq M_{\mathcal{SBP}}$ and $\mathrm{Rad}(M) \subseteq M_{\mathcal{SBP}}$. According to Proposition 8.2, $M_{\mathcal{SBP}}$ is the largest subalgebra of M belonging to $\mathcal{SBP}$. Due to Theorem 6.9, $\langle \mathrm{Rad}(M_{\mathcal{SBP}}) \cup B(M_{\mathcal{SBP}}) \rangle = M_{\mathcal{SBP}}$. On the other hand, $B(M_{\mathcal{SBP}}) \subseteq B(M)$ and $\mathrm{Rad}(M_{\mathcal{SBP}}) \subseteq \mathrm{Rad}(M)$, proving that $M_{\mathcal{SBP}} = \langle \mathrm{Rad}(M) \cup B(M) \rangle$. $\square$

Let $\mathcal{V}$ be a class of GMV-algebras and let $(M_t : \ t \in T)$ be a family of algebras in $\mathcal{V}$. A *$\mathcal{V}$-coproduct* of this family is a GMV-algebra $M \in \mathcal{V}$, denoted by $\bigsqcup_{t \in T}^{\mathcal{V}} M_t$, together with a family of GMV-homomorphisms $(f_t : M_t \to M : \ t \in T)$ such that

(i) $\bigcup_{t\in T} M_t$ generates M; and

(ii) If $N \in \mathcal{V}$ and $(g_t : M_t \to N : t \in T)$ is a family of GMV-homomorphisms, then there exists a (necessarily) unique GMV-homomorphisms $h : M \to N$ such that $g_t = hf_t$, for all $t \in T$.

It is clear that if a $\mathcal{V}$-coproduct of a family exists, then it is unique up to isomorphism. Further one can easily show with the use of universal algebraic techniques that $\mathcal{V}$-coproduct of any family exists whenever $\mathcal{V}$ is a variety.

If in the preceding definition we require that all the homomorphisms f_t $(t \in T)$ be injective, then we refer to this coproduct as the *$\mathcal{V}$-free product* of the family in question. It is clear that if a free product exists, so does the corresponding coproduct, and are equal. The converse is not true. For example, let M_0 be the one-element MV-algebra and let M an arbitrary GMV-algebra with $0 \neq 1$. Then $M_0 \sqcup^{\mathcal{GMV}} M = M_0$, but the $\mathcal{GMV}$-free product of M_0 and M does not exist.

A class $\mathcal{V}$ of GMV-algebras is said to satisfy the *amalgamation property* (AP) if given non-trivial GMV-algebras $M, M_1, M_2 \in \mathcal{V}$ and injective homomorphisms $\sigma_i : M \to M_i$ $(i = 1, 2)$, there exists a GMV-algebra $M' \in \mathcal{V}$ and injective homomorphisms $\tau_i : M_i \to M'$ $(i = 1, 2)$ such that $\tau_1\sigma_1 = \tau_2\sigma_2$. AP is an exceedingly rare property. Of the classes of particular interest to us, the variety of abelian ℓ-groups is the only known variety of ℓ-groups that satisfies this property ([Pie1]; see also [PoTs2] for a direct proof of this result). On the other hand, the variety of all ℓ-groups fails AP ([Pie2]; see also [Gla2, Thm 7.C], or [PoTs2]). We note further that the variety of MV-algebras has AP [Mun1, Prop. 1.1], and, in fact, a variety of MV-algebras satisfies AP iff it is generated by an MV-chain [DiLe3, Thm 13].

Lemma 8.4 *Let $\mathcal{V}$, $\mathcal{U}$ be two varieties of GMV-algebras such that $\mathcal{V} \subseteq \mathcal{U}$. Assume that every GMV-algebra $M \in \mathcal{U}$ possesses a greatest subalgebra $M_{\mathcal{V}}$ contained in $\mathcal{V}$. Then we have the following:*

(1) *For every family $(M_t : t \in T)$ of algebras in $\mathcal{V}$, $\bigsqcup_t^{\mathcal{V}} M_t = \bigsqcup_t^{\mathcal{U}} M_t$.*

(2) *If given algebras $M, M_1, M_2 \in \mathcal{V}$ and injective homomorphisms $\sigma_i : M \to M_i$ $(i = 1, 2)$, there exists an algebra $M' \in \mathcal{U}$ and injective homomorphisms $\tau_i : M_i \to M'$ such that $\tau_1\sigma_1 = \tau_2\sigma_2$, then there exists $M'' \in \mathcal{V}$ and injective homomorphisms $\tau_i' : M_i \to M''$ $(i = 1, 2)$ such that $\tau_1'\sigma_1 = \tau_2'\sigma_2$.*

Proof. (1) Let $(M_t:\ t \in T)$ be a family of GMV-algebras in $\mathcal{V}$, let $M = \bigsqcup_t^{\mathcal{U}} M_t$, and let $f_t:\ M_t \to M$ $(t \in T)$ be the associated homomorphisms. We assert $M \in \mathcal{V}$. Indeed, for every $t \in T$, $f_t(M_t) \in \mathcal{V}$ and $M = \langle \bigcup_t f_t(M_t)\rangle$. Since $f_t(M_t) \subseteq M_{\mathcal{V}}$, where $M_{\mathcal{V}}$ is the greatest subalgebra of M belonging to $\mathcal{V}$, we have $M_{\mathcal{V}} = \langle \bigcup_t f_t(M_t)\rangle = M$, which proves that $M \in \mathcal{V}$ and $\bigsqcup_t^{\mathcal{V}} M_t = \bigsqcup_t^{\mathcal{U}} M_t$.

(2) Let M_1, M_2, M σ_1, σ_2, M', and τ_1, τ_2 satisfy the assumptions of the theorem. Then $\tau_i(M_i) \in \mathcal{V}$, for $i = 1, 2$, and hence $\tau_i(M_i) \subseteq M'_{\mathcal{V}}$, i.e., $\langle \tau_1(M_1) \cup \tau_2(M_2)\rangle \subseteq M'_{\mathcal{V}}$. Thus $M'' = M'_{\mathcal{V}}$ and $\tau'_i = \tau_i|M'_{\mathcal{V}}$ satisfy the conclusion of the statement. $\square$

We note that, e.g., $\mathcal{V} = \mathcal{SBP}$ and $\mathcal{U} = \mathcal{SYM} \cap \mathcal{M}$ satisfy the conditions of Lemma 8.4, due to Proposition 8.2.

Theorem 8.5

(1) *If $M_t \in \mathcal{SBP}$ for all $t \in T$, then $\bigsqcup_t^{\mathcal{SBP}} M_t = \bigsqcup_t^{\mathcal{M}} M_t$.*

(2) *If $M_t \in \mathcal{PGMV}$ for all $t \in T$, then $\bigsqcup_t^{\mathcal{PGMV}} M_t = \bigsqcup_t^{\mathcal{M}} M_t$.*

Proof.

(1) Let $M_t \in \mathcal{SBP}$ for every $t \in T$, let $M = \bigsqcup_t^{\mathcal{M}} M_t$, and let $f_t:\ M_t \to M$ be the corresponding homomorphisms. Then $M = \langle \bigcup_t f_t(M_t)\rangle$. We assert that M is symmetric. Indeed, if we set $M' := \{x \in M:\ x^\sim = x^-\}$, then M' contains $\bigcup_t f_t(M_t)$, and if $x, y \in M'$, then $(x \oplus y)^- = y^\sim \odot x^\sim = y^- \odot x^- = (x \oplus y)^\sim$ and $x \oplus y \in M'$, which yields $M' = M$. Therefore, $M \in \mathcal{SYM} \cap \mathcal{M}$.

Define $M_{\mathcal{SBP}} = \bigcap\{\langle I\rangle :\ I \in \mathcal{M}(M)\}$. According to Proposition 8.2, $M_{\mathcal{SBP}}$ is the greatest subalgebra of M that belongs to $\mathcal{SBP}$. As in Lemma 8.4, $M_{\mathcal{SBP}} = M \in \mathcal{SBP}$.

(2) We proceed as in (1). The final conclusion follows from the fact that $\langle \mathrm{Rad}(M)\rangle$ is the greatest subalgebra of M that is a perfect GMV-algebra. $\square$

We recall that the free product of any family, $(G_t:\ t \in T)$, of ℓ-groups exists in the variety of ℓ-groups $\mathcal{L}$; we denote it by $\bigsqcup_{t\in T}^{\mathcal{L}} G_t$.

Theorem 8.6 *Let $(G_t:\ t \in T)$ be a family of ℓ-groups and let $G = \bigsqcup_t^{\mathcal{L}} G_t$. Then $\mathcal{E}(G) = \bigsqcup_t^{\mathcal{PGMV}} \mathcal{E}(G_t)$.*

Proof. By Theorem 8.5(2), the coproduct, $M = \bigsqcup_t^{\mathcal{PGMV}} \mathcal{E}(G_t)$, exists, where $\mathcal{E}$ is the functor defined in (5.1). For each $t \in T$, let $f_t:\ G_t \to G$ be the homomorphism guaranteed by the existence of $\bigsqcup_t^{\mathcal{L}} G_t$, and let $\hat{f}_t : \mathcal{E}(G_t) \to$

$\mathcal{E}(G)$ be defined by $\hat{f}_t(0, g_t) = (0, f_t(g_t))$ and $\hat{f}_t(1, -g_t) = (1, -f_t(g_t))$, for $g_t \in G_t^+$.

If $\psi_t : \mathcal{E}(G_t) \to \mathcal{E}(G')$, then we can define a homomorphisms $\psi'_t : G_t \to G'$ such that $\psi_t(0, g_t) = (0, \psi'_t(g_t))$ and $\hat{\psi}_t(1, -g_t) = (1, -\psi'_t(g_t))$. Hence, there is a unique homomorphism $h : G \to G'$ such that $hf_t = \psi'_t$. Consequently, the mapping $\hat{h} : \mathcal{E}(G) \to \mathcal{E}(G')$ defined by $\hat{h}(0, g) = (0, h(g))$ and $\hat{h}(1, -g) = (1, -h(g))$, for all $g \in G^+$, is a homomorphism such that $\hat{h}\hat{f}_t = \psi_t$, for all t. The uniqueness of $\hat{h}$ follows from Theorem 2.1, which proves $\mathcal{E}(G) = M$. $\square$

Theorem 8.7 *The category of perfect GMV-algebras fails AP. In particular, the variety $\mathcal{SBP}$ fails this property.*

Proof. As was noted earlier, the class of ℓ-groups fails AP. Hence, there exist ℓ-groups G, G_1, G_2 and ℓ-embeddings $h_i : G \to G_i$, $i = 1, 2$, such that there is no ℓ-group G' with ℓ-embeddings $f_i : G_i \to G'$ such that $f_1 h_1 = f_2 h_2$. Let now $M = \mathcal{E}(G)$ and, for $i = 1, 2$, let $M_i = \mathcal{E}(G_i)$ and let $\sigma_i : M \to M_i$ be the injective GMV-homomorphisms induced by h_i via (5.2). That is, $\sigma_i(0, g) = (0, h_i(g))$ and $\sigma_i(1, -g) = (1, -h_i(g))$, $g \in G^+$. Hence, by 5.5, there is no perfect GMV-algebra $M' = \mathcal{E}(G')$ with embeddings $\tau_i : M_i \to M'$ such that $\tau_1\sigma_1 = \tau_2\sigma_2$.

If M' is a GMV-algebra in $\mathcal{SBP}$, then $\langle \mathrm{Rad}(M') \rangle$ is the largest perfect subalgebra of M'. Applying (2) of Lemma 8.4 and the first part of the present statement, we have that $\mathcal{SBP}$ fails AP. $\square$

Proposition 8.8 *Let $\mathcal{G}$ be a variety of ℓ-groups. Then $\mathcal{T}(\Phi(\mathcal{G})) = \mathcal{BP}$.*

Proof. Since $\mathcal{G} \subseteq \mathcal{L}$, we have, by Theorem 6.1, that $\mathcal{T}(\Phi(\mathcal{G})) \subseteq \mathcal{T}(\Phi(\mathcal{L})) = \mathcal{T}(\mathcal{BP}) = \mathcal{BP}$. If now $M \in \mathcal{T}(\Phi(\mathcal{G}))$ and I is a maximal ideal of M, then $\Gamma(\mathbb{Z}, 1) \cong M/I \in \mathcal{T}(\mathcal{G})$ and $\Gamma(\mathbb{Z}, 1) \in \mathcal{T}(\mathcal{G})$. Suppose now that $M \in \mathcal{BP}$ and let I be a maximal ideal of M. Then $M/I \cong \Gamma(\mathbb{Z}, 1)$ and that gives $M/I \in \Phi(\mathcal{G})$. Hence $\mathcal{BP} \subseteq \mathcal{T}(\Phi(\mathcal{G}))$. $\square$

Before we state the next result, we note that if $\mathcal{G}$ of ℓ-groups and $(G_t : t \in T)$ a family of ℓ-groups in $\mathcal{G}$, then the $\mathcal{G}$-free product $\coprod_{t\in T}^{\mathcal{G}}(G_t)$ exists.

Theorem 8.9 *Let $\mathcal{G}$ be a variety of ℓ-groups.*

(i) *The category $\mathcal{E}(\mathcal{G})$ has AP if and only if $\mathcal{G}$ does.*

(ii) *If $\mathcal{G}$ fails AP, then so does the variety $\Phi(\mathcal{G})$.*

(iii) *If $G_t \in \mathcal{G}$ for all $t \in T$, and $G = \bigsqcup_{t\in T}^{\mathcal{G}} G_t$ is the $\mathcal{G}$-free product of the family $(G_t :\ t \in T)$, then the free product of the family $(\mathcal{E}(G_t) :\ t \in T)$ exists the the category $\mathcal{E}(\mathcal{G})$, and $\bigsqcup_{t\in T}^{\mathcal{E}(\mathcal{G})} \mathcal{E}(G_t) = \mathcal{E}(G)$.*

Proof.

(i) Due to the categorical equivalence of $\mathcal{G}$ and $\mathcal{E}(\mathcal{G})$, one of the categories satisfies AP iff the other does.

(ii) Let $G, G_1, G_2 \in \mathcal{G}$, and let $h_i :\ G \to G_i$ be injections for which there is no amalgam in $\mathcal{G}$. Then $\mathcal{E}(G), \mathcal{E}(G_1), \mathcal{E}(G) \in \Phi(\mathcal{G})$, and let $\sigma_i :\ \mathcal{E}(G) \to \mathcal{E}(G_i)$ be induced by h_i, $i = 1, 2$. Suppose there exist $M \in \Phi(\mathcal{G})$ and GMV-embeddings $\tau_i : \mathcal{E}(G_i) \to M$ such that $\tau_1\sigma_1 = \tau_2\sigma_2$. Then $M' := \langle \mathrm{Rad}(M)\rangle$ is the largest perfect subalgebra of M, and $M' \in \Phi(\mathcal{G})$. Therefore, there exists $G' \in \mathcal{G}$ such that $M' = \mathcal{E}(G')$. Since $\tau_i(\mathcal{E}(G_i))$ are perfect subalgebras of M, $\tau_i(\mathcal{E}(G_i)) \subseteq M' = \mathcal{E}(G)$. The categorical equivalence of $\mathcal{E}(\mathcal{G})$ and $\mathcal{G}$ gives that G, G_1, G_2 with the injections h_1, h_2 have an amalgam in $\mathcal{G}$, which is a contradiction.

(iii) It follows as in the proof of Theorem 7.6. □

Question: Does $\Phi(\mathcal{G})$ has AP whenever $\mathcal{G}$ does?

The results below further confirm the scarcity of AP for varieties of GMV-algebras.

Theorem 8.10 *There exist uncountably many varieties of symmetric GMV-algebras that fail AP.*

Proof. There is an uncountable interval in the subvariety lattice of representable ℓ-groups each variety of which fails the amalgamation property (refer to the comments following [PoTs1, Thm 4]). Thus the note prior to Theorem 8.9 produces the desired result. □

Now we show that the above mentioned result of [DiLe3, Thm 13], which states that a variety of MV-algebras satisfies AP iff it is generated by an MV-chain, does not have an analogue for varieties of GMV-algebras.

Theorem 8.11 *There exist varieties of symmetric GMV-algebras generated by a single GMV-chain that fail AP.*

Proof. Let $\mathbb{Z}$ be the ℓ-group of natural numbers, and let $\mathbb{Z}\,\mathrm{wr}\,\mathbb{Z}$ be the small (or restricted) wreath product. The elements of $\mathbb{Z}\,\mathrm{wr}\,\mathbb{Z}$ are written in the form $((a_i), m) := ((\ldots, a_i, \ldots), m)$, where the support, $\mathrm{supp}(a_i) = \{i \in \mathbb{Z} : a_i \neq 0\}$, of the vector (a_i) is finite, and the group operation, $*$, on $\mathbb{Z}\,\mathrm{wr}\,\mathbb{Z}$ is given by $((a_i), m) * ((b_i), n) := ((c_i), m+n)$, where $c_i = a_i + b_{i+m}$. This group admits two natural linear orders. The positive cone of one of these orders is defined by $((a_i), m) \geq ((0), 0)$ if $m > 0$ or $m = 0$ and $a_j > 0$, where j is the largest index such that $a_j \neq 0$. The second order is defined by $((a_i), m) \geq ((0), 0)$ if $m > 0$ or $m = 0$ and $a_j > 0$, where j is the smallest index such that $a_j \neq 0$.

Let $W^+ := \mathbb{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbb{Z}$ and $W^- := \mathbb{Z}\,\overrightarrow{\mathrm{wr}}\,\mathbb{Z}$ be the small wreath products endowed with these two linear orders.

Let $\mathcal{M}^+$ and $\mathcal{M}^-$ be the varieties of ℓ-groups generated by W^+ and W^-, respectively. They are subvarieties of the variety of representable ℓ-groups that cover the variety, $\mathcal{A}$, of abelian ℓ-groups, and according to [PoTs1, Thm 4], they fail AP. Now $\mathcal{E}(W^+)$ and $\mathcal{E}(W^-)$ symmetric GMV-chains, and generate the varieties $\mathcal{V}(\mathcal{E}(W^+)) = \mathcal{V}(\mathcal{E}(\mathcal{M}^+))$ and $\mathcal{V}(\mathcal{E}(W^-)) = \mathcal{V}(\mathcal{E}(\mathcal{M}^-))$. According to the note preceding Theorem 8.9, these two varieties fail AP. $\square$

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