

Perfect matchings in r -partite r -graphs

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Abstract

Let H be an r -partite r -graph, all of whose sides have the same size n . Suppose that there exist two sides of H , each satisfying the following condition: the degree of each legal $r-1$ -tuple contained in the complement of this side is strictly larger than $\frac{n}{2}$. We prove that under this condition H must have a perfect matching. This answers a question of Kühn and Osthus.

1 Introduction

Matchings in hypergraphs are notoriously evasive. There is an abundance of conjectures in the subject, and no well developed theory similar to matching theory in graphs. It seems that we do not have as yet at our disposal the right tools to study hypergraph matchings. In this paper we prove the sufficiency of a certain condition, for the existence of perfect matching in an r -partite r -graph. This is a generalization of the well known result that if in an $n \times n$ bipartite graph the degree of every vertex is at least $\frac{n}{2}$ then the graph has a perfect matching.

We will be using the terminology of Diestel [3]. An r -uniform hypergraph H (also called an r -graph) is said to be r -partite if its vertex set $V(H)$ can be partitioned into sets V_1, V_2, \dots, V_r , called the “sides” of H , so that every edge in the edge set $E(H)$ of H consists of a choice of precisely one vertex from each side. This means that $E(H) \subseteq V_1 \times V_2 \times \dots \times V_r$, in particular that the edges of H can be considered as ordered r -tuples.

The degree $d(f)$ in H of a subset f of V is the number of edges of H containing f . An r -partite hypergraph is said to be n -balanced if $|V_i| = n$ for every $1 \leq i \leq r$. A set of vertices is called *legal* if it meets each side in at most one vertex.

In [4] Kühn and Osthus proved the following:

Theorem 1. *If in an n -balanced r -partite r -graph H every legal $r-1$ -tuple has degree at least $n/2 + \sqrt{2n \log n}$ and $n \geq 1000$ then H has a perfect matching.*

Kühn and Osthus gave the following example showing that demanding that every legal $(n-1)$ -tuple has degree at least $n/2$ does not force a perfect matching:

Example 1. Suppose that r is odd, and that n is even but not divisible by 4. For every $i \leq r$ choose a subset A_i of V_i of size $\frac{n}{2}$. Let \vec{u} be the vector in \mathbb{F}_2^V which has value 1 on all vertices of $\bigcup_{i \leq r} A_i$ and 0 elsewhere. Let H be the hypergraph containing precisely those legal r -tuples that contain an even number of vertices in $\bigcup_{i \leq r} A_i$. Then $d(f) = \frac{n}{2}$ for every legal $r-1$ -tuple f . For every characteristic vector χ_e of an edge of H we have $\chi_e \cdot \vec{u} = 0 \pmod{2}$, while $\vec{1} \cdot \vec{u} \neq 0 \pmod{2}$, where $\vec{1}$ is the all 1's vector. Thus $\vec{1}$ is not the sum of characteristic vectors of edges, meaning that H has no perfect matching.

For all other values of r and n choose A_i as above such that $||A_i| - \frac{n}{2}| \leq 1$ and $\sum |A_i|$ is odd. This yields an r -partite r -graph such that $d(f) \geq \frac{n}{2} - 1$ for every legal $r-1$ -tuple e , that has no perfect matching.

Kühn and Osthus [4] posed the question whether a minimal degree greater than $\frac{n}{2}$ forces a perfect matching. It is the aim of this paper to prove this assertion, in a somewhat stronger form:

Theorem 2. *Let H be an n -balanced r -partite r -graph with partition classes V_1, \dots, V_r . If for every legal $r-1$ -tuple f contained in $V \setminus V_1$ we have $d(f) > \frac{n}{2}$ and for every legal $r-1$ -tuple g contained in $V \setminus V_r$ we have $d(g) \geq \frac{n}{2}$ then H has a perfect matching.*

Example 1 suggests the question whether a minimal degree of $\frac{n}{2}$ forces a perfect matching if r is even or $n \neq 2 \pmod{4}$. In Section 3 we propose some further problems.

In this paper we restricted our attention to r -partite hypergraphs. Forcing perfect matchings by large minimum degree of $r-1$ -tuples in r -uniform graphs in general has been an active field lately, see [5] for example.

2 Proof of Theorem 2

In this section we prove Theorem 2.

Proof. As noted in [4], it suffices to prove the theorem for $r = 3$. To see this, let $r > 3$ and choose a perfect matching $F = g_1, g_2, \dots, g_n$ in the complete $(r-2)$ -partite $(r-2)$ -graph with vertex partition V_2, V_3, \dots, V_{r-1} . Let H' be the 3-partite 3-graph with vertex partition V_1, F, V_r where (x, g_i, y) is an edge of H' if and only if $\{x\} \cup g_i \cup \{y\}$ is an edge of H (where $x \in V_1, y \in V_r$). Clearly, H' satisfies the conditions of the theorem, with $r = 3$. Assuming that the theorem is valid in this case, H' has a perfect matching, and “de-contracting” each g_i results in a perfect matching of H .

Thus we may assume that $r = 3$. Suppose that the theorem fails. By considering a counterexample with maximal set of edges we may assume that H has a matching M that matches all but one vertex from each class; let $x_1 \in V_1, x_2 \in V_2, x_3 \in V_3$ be the unmatched vertices.

Let U be the set of pairs (u, v) where $u \in V_2, v \in V_3$ and there is an edge of M containing both u and v . Since each pair in U has more than $\frac{n}{2}$ neighbors in V_1 , there exists a vertex $w \in V_1$ that is a neighbor of at least $\frac{n}{2}$ pairs in U . We consider three cases, in all of which we will be able to construct a perfect matching of H .

The first case is when $w = x_1$. Since the pair (x_2, x_3) has more than $\frac{n}{2}$ neighbors in V_1 , there is an edge $e = (u_1, u_2, u_3) \in M$ such that $(x_1, u_2, u_3) \in H$

and $(u_1, x_2, x_3) \in H$. Then $M - e + (x_1, u_2, u_3) + (u_1, x_2, x_3)$ (standing for $M \setminus \{e\} \cup \{(x_1, u_2, u_3), (u_1, x_2, x_3)\}$) is a perfect matching of H .

The next case is when w lies on an edge $f = (w, u_2, u_3)$ of M such that $(x_1, x_2, u_3) \in E(H)$. Since the pair (u_2, x_3) has more than $\frac{n}{2}$ neighbors, there is an edge $g = (v_1, v_2, v_3) \in M$ such that v_1 is a neighbor of the pair (u_2, x_3) and the element (v_2, v_3) of U is in an edge with w . If $v_1 = w$ (in which case $f = g$) then $M - g + (x_1, x_2, v_3) + (v_1, v_2, x_3)$ is a perfect matching of H , and if $v_1 \neq w$ then $M - f - g + (x_1, x_2, v_3) + (x_3, u_2, v_1) + (w, v_2, v_3)$ is a perfect matching.

Finally, consider the case when w lies in an edge $f = (w, u_2, u_3)$ of M such that $(x_1, x_2, u_3) \notin E(H)$. Since $d((u_2, u_3)) > n/2$ and $d((x_1, x_2)) \geq n/2$ there is an edge $g = (v_1, v_2, v_3) \in M$ such that $(v_1, u_2, u_3) \in E(H)$ and $(x_1, x_2, v_3) \in E(H)$. Let M' be the matching $M - f - g + (v_1, u_2, u_3) + (x_1, x_2, v_3)$. The only vertices not matched by M' are v_2, x_3 and w . Now we can repeat the argument of the first case with w playing the role of x_1 . But in this case we have to be more careful: as w was a neighbor of at least $\frac{n}{2}$ pairs in U , and the only element of U that is not in an edge of M' is (v_2, v_3) , there are still at least $\frac{n}{2} - 1$ elements of U neighboring w that are each in an edge of M' . On the other hand, if $(w, v_2, x_3) \in E(H)$ we are done. Hence we can assume that the pair (v_2, x_3) has at least $\frac{n+1}{2}$ neighbors in $V_1 - w$. But $\frac{n}{2} - 1 + \frac{n+1}{2} > n - 1$, thus there is an edge e of M' containing a pair neighboring w and a neighbor of (v_2, x_3) . Removing e from M' and adding the two corresponding edges yields a perfect matching of H . □

3 Open problems

The condition in Theorem 2, although sharp, is very strong. It is likely that it can be weakened, in more than one way. We offer some conjectures as possible weakenings of the condition. Let H be an n -balanced r -partite r -graph fixed throughout this section. For a subset I of $[r] := \{1, 2, \dots, r\}$ an I -tuple is an element of $\times_{i \in I} V_i$. Let $I^c := [r] \setminus I$.

Conjecture 1. *Let I be a subset of $[r]$. If $d(f) > \frac{n^{r-|I|}}{2}$ for every I -tuple f and $d(g) \geq \frac{n^{|I|}}{2}$ for every I^c -tuple g (i.e. each I -tuple has degree larger than half its degree in the complete r -partite hypergraph and each I^c -tuple has degree at least half its degree in the complete r -partite hypergraph) then H has a perfect matching.*

A stronger version of Conjecture 1 is that it suffices to assume that for every legal r -tuple z not belonging to $E(H)$ there holds:

$$\frac{d(z \cap I)}{n^{r-|I|}} + \frac{d(z \cap I^c)}{n^{|I|}} > 1.$$

We shall prove a fractional version of this conjecture. A *fractional matching* of H is a function $h : E(H) \rightarrow \mathbb{R}^+$ such that for every vertex x in H there holds $\sum \{h(e) \mid x \in e\} \leq 1$. We say that h is *perfect* if $\sum \{h(e) \mid x \in e\} = 1$ for every vertex x .

Theorem 3. *Let I be a subset of $[r]$. If $\frac{d(z \cap I)}{n^{r-|I|}} + \frac{d(z \cap I^c)}{n^{|I|}} \geq 1$ for every legal r -tuple z not belonging to $E(H)$ then there exists a perfect fractional matching.*

Proof. For a real valued function f and a set S contained in its domain, we write $f[S]$ for $\sum\{f(s) \mid s \in S\}$. A *fractional cover* is a function $g : V(H) \rightarrow \mathbb{R}_{\geq 0}$ such that $g[e] \geq 1$ for every $e \in E(H)$.

We have to show that $\nu^*(H) = n$ where $\nu^*(H)$, the *fractional matching number of H* , is the maximum value of $h[E(H)]$ over all fractional matchings h of H . By linear programming duality (see [7] for an introduction to the subject), this is equivalent to showing that $\tau^*(H) = n$, namely that $g[V] \geq n$ holds for every fractional cover ($\tau^*(H)$ is the minimum value of $g[V]$ over all fractional covers g of H).

So let g be fractional cover. For every $j \in [r]$ let $\alpha(j)$ be the minimal value of g on V_j , and let v_j be a vertex of V_j with $g(v_j) = \alpha(j)$. Also let $\beta = \alpha[I]$ and $\gamma = \alpha[I^c]$.

Consider the r -tuple $z = (v_j)_{j \in [r]}$. By the minimality of the $\alpha(j)$'s, we have $g[V] \geq ng[z]$. Hence we may assume that $g[z] = \beta + \gamma < 1$. In particular, we have $z \notin E(H)$.

Write $\frac{d(z \cap I)}{n^{|I|}} = \theta$ and $\frac{d(z \cap I^c)}{n^{|I|}} = \zeta$. Call an I -tuple y *good* if $y \cup (z \cap I^c) \in E(H)$. Consider the complete $|I|$ -partite graph on $\bigcup_{j \in I} V_j$. It is a well known fact (easily proved by induction) that its edge set can be partitioned into $n^{|I|-1}$ perfect matchings. Since there are $\zeta n^{|I|}$ good I -tuples, one of those perfect matchings contains at least ζn good I -tuples; we thus have a set Y of at least ζn disjoint good I -tuples. For each $j \in I$, denote by A_j the set of vertices in V_j that are contained in an I -tuple in Y . Since $g[y] \geq 1 - \gamma$ for each good I -tuple y , we have $g[\bigcup_{j \in I} A_j] \geq |Y|(1 - \gamma)$. This yields $g[\bigcup_{j \in I} V_j] = g[\bigcup_{j \in I} A_j] + g[\bigcup_{j \in I} (V_j \setminus A_j)] \geq |Y|(1 - \gamma) + (n - |Y|)\beta$. Since $\beta < 1 - \gamma$ and $|Y| \geq \zeta n$, we obtain $g[\bigcup_{j \in I} V_j] \geq n\beta + |Y|(1 - \gamma - \beta) \geq n\beta + n\zeta(1 - \gamma - \beta) \geq \zeta n(1 - \gamma) + (1 - \zeta)n\beta$. Similarly, we have $g[\bigcup_{j \in I^c} V_j] \geq \theta n(1 - \beta) + (1 - \theta)n\gamma$ and thus

$$\begin{aligned} g[V] &\geq \zeta n(1 - \gamma) + (1 - \zeta)n\beta + \theta n(1 - \beta) + (1 - \theta)n\gamma \\ &= n(\zeta + \theta) + n\beta(1 - \zeta - \theta) + n\gamma(1 - \zeta - \theta) \\ &= n(1 + (\beta + \gamma - 1)(1 - \zeta - \theta)) \\ &\geq n, \end{aligned}$$

since $\beta + \gamma - 1 < 0$ and $1 - \zeta - \theta \leq 0$. □

Next we ask what condition on the degrees of vertices, rather than I -tuples, suffices for the existence of a perfect matching in an n -balanced r -partite hypergraph.

Problem 2. *Is it true that if $d(x) \geq (1 - 1/e)n^{r-1}$ for every vertex x of H then there is a perfect matching?*

Taking a subset X_i of V_i of size a bit less than $\frac{n}{r}$ for each $i \in [r]$, and letting H be the hypergraph consisting of all edges meeting $\bigcup_{i \in [r]} X_i$, shows that if the assertion of Problem 2 is true then it is asymptotically tight.

Some of the most intriguing conjectures on 3-partite hypergraphs were originally formulated in terms of Latin squares. Here is one of the best known of those, the Brualdi-Ryser conjecture ([2, 6]):

Conjecture 3. *Let H be an n -balanced 3-partite hypergraph in which every legal 2-tuple participates in precisely one edge. If n is odd then there exists a perfect matching and if n is even there is a matching of size $n - 1$.*

As Stein pointed out in [8], the condition of the Brualdi-Ryser conjecture is probably way too strong, and the conclusion is probably valid assuming much less than that. Here is a rather bold conjecture of this type:

Conjecture 4. *Let H be an n -balanced r -partite r -graph, and let I be a subset of $[r]$. If $d(e) = d(f)$ for every two I -tuples e, f and $d(g) = d(z)$ for every two I^c -tuples g, z then there is a perfect matching unless r is odd and n even.*

Let us mention a result in this direction, in which the assumptions are again probably way too strong:

Theorem 4 ([1]). *Let H be a 3-partite hypergraph, with sides $V_i, i = 1, 2, 3$, where $|V_1| = n$ and $|V_2| \geq 2n - 1$. Suppose, furthermore, that the degree of every pair in $(V_1 \times V_2)$ is 1 and the degree of every pair in $(V_1 \times V_3)$ is at most 1. Then there exists in H a matching of size n .*

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