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Perfect powers from products of consecutive terms in arithmetic progression

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To Professor L. Lovász on his 60th birthday

Abstract

We prove that for any positive integers x, d and k with $\gcd(x, d) = 1$ and $3 < k < 35$, the product $x(x + d) \cdots (x + (k - 1)d)$ cannot be a perfect power. This yields a considerable extension of previous results of Győry *et al.* and Bennett *et al.*, which covered the cases where $k \leq 11$. We also establish more general theorems for the case where x can also be a negative integer and where the product yields an almost perfect power. As in the proofs of the earlier theorems, for fixed k we reduce the problem to systems of ternary equations. However, our results do not follow as a mere computational sharpening of the approach utilized previously; instead, they require the introduction of fundamentally new ideas. For $k > 11$, a large number of new ternary equations arise, which we solve by combining the Frey curve and Galois representation approach with local and cyclotomic considerations. Furthermore, the number of systems of equations grows so rapidly with k that, in contrast with the previous proofs, it is practically impossible to handle the various cases in the usual manner. The main novelty of this paper lies in the development of an algorithm for our proofs, which enables us to use a computer. We apply an efficient, iterated combination of our procedure for solving the new ternary equations that arise with several sieves based on the ternary equations already solved. In this way, we are able to exclude the solvability of the enormous number of systems of equations under consideration. Our general algorithm seems to work for larger values of k as well, although there is, of course, a computational time constraint.

1. Introduction and statement of new results

A classical theorem of Erdős and Selfridge [ES75] asserts that the product of consecutive positive integers is never a perfect power. A natural generalization is the diophantine equation

$$x(x + d) \cdots (x + (k - 1)d) = by^n \tag{1}$$

in non-zero integers x, d, k, b, y, n such that $\gcd(x, d) = 1$, $d \geq 1$, $k \geq 3$, $n \geq 2$ and $P(b) \leq k$. Here $P(u)$ denotes the largest prime divisor of a non-zero integer u, with the convention that $P(\pm 1) = 1$.

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Table 1. Values of the bound $P_{k,n}$.

k	$n=2$	$n=3$	$n=5$	$n\geq 7$
3	–	2	2	2
4	2	3	2	2
5	3	3	3	2
6	5	5	5	2
7	5	5	5	3
8	5	5	5	3
9	5	5	5	3
10	5	5	5	3
11	5	5	5	5

Equation (1) has an extremely rich literature. For $d=1$, it has been completely solved by Saradha [Sar97] (for $k\geq 4$) and Győry [Gyo98] (for $k<4$). Instead of attempting to review all branches of related results for $d>1$ (which would be an enormous task), we refer to the excellent survey papers by Tijdeman [Tij89] and Shorey [Sho02a, Sho02b]. Here we mention only those contributions which are closely related to the results of the present paper, that is, which provide the complete solution of (1) when the number of terms k is fixed.

If $(k,n)=(3,2)$, then (1) has infinitely many solutions even for $b=1$. Euler (see [Dic66]) showed that (1) has no solutions if $b=1$ and $(k,n)=(3,3)$ or $(4,2)$. A similar result was obtained by Obláth [Obl50, Obl51] for $(k,n)=(3,4)$, $(3,5)$ or $(5,2)$. By a conjecture of Erdős, (1) has no solutions in positive integers when $k>3$ and $b=1$; in other words, the product of k consecutive terms in a coprime positive arithmetic progression with $k>3$ can never be a perfect power. By 'coprime positive arithmetic progression' we mean an arithmetic progression of the form

$$x, x+d, \ldots, x+(k-1)d,$$

where x and d are positive integers with $\gcd(x,d)=1$.

Erdős's conjecture has recently been verified for certain values of k in a more general form. In Theorem A below, the $k=3$ case is due to Győry [Gyo99], the $k=4,5$ cases to Győry *et al.* [GHS04], and the $6\leq k\leq 11$ cases to Bennett *et al.* [BBGH06].

Theorem A. *Suppose that k and n are integers with $3\leq k\leq 11$, $n\geq 2$ prime and $(k,n)\neq(3,2)$; also suppose that x and d are coprime integers. If, further, b is a non-zero integer with $P(b)\leq P_{k,n}$ where $P_{k,n}$ is as given in Table 1, then the only solutions to (1) are with (x,d,k) from the following list:*

$$(-9,2,9), (-9,2,10), (-9,5,4), (-7,2,8), (-7,2,9),$$
$$(-6,1,6), (-6,5,4), (-5,2,6), (-4,1,4), (-4,3,3),$$
$$(-3,2,4), (-2,3,3), (1,1,4), (1,1,6).$$

It is a routine matter to extend Theorem A to arbitrary (that is, not necessarily prime) values of n. Further, we note that upon knowing the values of the variables on the left-hand side of (1), one can easily determine all the solutions (x,d,k,b,y,n) of (1).

Very recently, for $k=5$ or 6 and $n\geq 7$, the bound $P_{k,n}$ was improved to 3 by Bennett [Ben08]. Further, for $n=2$ and positive x, Theorem A was extended by Hirata-Kohno *et al.* in [HLST07]. However, they did not handle (1) for some exceptional values of $b>1$; for these values, (1) was later solved by Tengely [Ten08]. Putting together the results in [HLST07, Ten08], we have the following theorem.

THEOREM B. *Equation (1) with $n=2$, $d>1$ and $5 \le k \le 100$ has no solution in positive integers x.*

In the case of $b=1$, the assumption $k \le 100$ can be replaced by $k \le 109$ in Theorem B (see [HLST07]). When $n=3$, Hajdu *et al.* [HTT09] obtained the following extension of Theorem A.

THEOREM C. *Suppose that $n=3$ and that (x,d,k,b,y) is a solution to (1) with $k<32$ such that $P(b) \le k$ if $4 \le k \le 12$ and $P(b)<k$ if $k=3$ or $k \ge 13$. Then (x,d,k) must be from the following list:*

$$\begin{gathered}(-10,3,7),(-8,3,7),(-8,3,5),(-4,3,5),(-4,3,3),(-2,3,3),\\ (-9,5,4),(-6,5,4),(-16,7,5),(-12,7,5),\\ (x,1,k)\quad \textit{with } -30 \le x \le -4 \textit{ or } 1 \le x \le 5,\\ (x,2,k)\quad \textit{with } -29 \le x \le -3.\end{gathered}$$

Further, if $b=1$ and $k<39$, then we have

$$(x,d,k,y)=(-4,3,3,2),(-2,3,3,-2),(-9,5,4,6) \textit{ or } (-6,5,4,6).$$

Theorems A, B and C confirm the conjecture of Erdős for the corresponding values of k and n. Moreover, under the additional assumptions made on $P(b)$, they provide the complete solution to (1) for $b>1$ as well.

In this paper we give a significant extension of Theorem A, up to $k<35$. Our main result is the following theorem, which verifies Erdős's conjecture for $k<35$.

THEOREM 1.1. *If $3<k<35$, the product of k consecutive terms in a coprime positive arithmetic progression is never a perfect power.*

When $n \le 3$ or $k \le 11$, Theorem 1.1 follows from the aforementioned results. The remaining cases are covered by the following theorems.

THEOREM 1.2. *Equation (1) has no solutions with $n \ge 7$ prime, $12 \le k < 35$ and $P(b) \le P_{k,n}$, where*

$$P_{k,n}=\begin{cases}7 & \textit{if } 12 \le k \le 22,\\ \dfrac{k-1}{2} & \textit{if } 22<k<35.\end{cases}$$

THEOREM 1.3. *The only solutions to (1) with $n=5$, $8 \le k < 35$ and $P(b) \le P_{k,5}$, where*

$$P_{k,5}=\begin{cases}7 & \textit{if } 8 \le k \le 22,\\ \dfrac{k-1}{2} & \textit{if } 22<k<35,\end{cases}$$

are given by

$$\begin{gathered}(k,d)=(8,1),\quad x\in\{-10,-9,-8,1,2,3\};\ (k,d)=(8,2),\quad x\in\{-9,-7,-5\};\\ (k,d)=(9,1),\quad x\in\{-10,-9,1,2\};\ (k,d)=(9,2),\quad x\in\{-9,-7\};\\ (k,d)=(10,1),\quad x\in\{-10,1\};\ (k,d,x)=(10,2,-9).\end{gathered}$$

Note that in the $n=5$ case, Theorem 1.3 yields an extension of Theorem A already for $8 \le k \le 11$.

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Similarly to [BBGH06, GHS04], results on (1) have a simple consequence for rational solutions of equations of the form

$$u(u+1)\cdots(u+k-1)=v^n. \tag{2}$$

More precisely, we have the following.

COROLLARY 1.1. *Suppose that $n \geq 2$, $1<k<35$ and $(k,n)\neq(2,2)$. Then (2) has no solutions in positive rational numbers u, v.*

For $k \leq 11$, this was proved in [BBGH06]. When $k>11$, the statement is a straightforward consequence of Theorem 1.1; see [BBGH06, GHS04] for the necessary arguments. We note that (2) was first studied by Sander [San99].

In the case of $k \leq 11$ and $n \geq 5$, Equation (1) was reduced in [Gyo99, GHS04, BBGH06, Ben08] to finitely many ternary equations of signature (n,n,n), $(n,n,2)$ or $(n,n,3)$. In our proofs, we start with the same reduction strategy. However, for $k>11$ and $n \geq 5$ prime, numerous new ternary equations arise which must be solved under certain arithmetic conditions. To solve these equations, in the $n \geq 7$ case we combine the Frey curve and modular Galois representation approach with local methods and some classical work on cyclotomic fields. Our results concerning ternary equations, which may be of independent interest, do not follow from a straightforward application of the modularity of Galois representations attached to Frey curves; it is also necessary to understand the reduction types of these curves at certain small primes.

For $n=5$, hardly any new information is available through the theory of 'general' modular forms. In this case we make use of some classical and new results concerning equations of the form $AX^5+BY^5=CZ^5$. The proof of these new results involves some cyclotomic and local considerations.

As k increases, the number of possible k-tuples $(a_0,\ldots,a_{k-1})$ introduced in (3), and hence the number of systems of ternary equations that arise, grow so rapidly that, in contrast with the $k \leq 11$ cases treated in [BBGH06, Ben08, GHS04, Gyo99], it is practically impossible to handle all the cases one by one without using a computer. The principal novelty of our paper lies in the development of an algorithm for our proof. For fixed k, we combine our algorithm for solving the new ternary equations with several sieves based on the ternary equations already solved, and we use a computer to exclude the solvability of the enormous number of systems of ternary equations. Our general method seems to work for larger k as well, and we do not see any theoretical obstacle to extending the results even further. However, the computational time consumed by the method increases rather rapidly with k, which is why we stopped at $k=34$. We give a few details of the computational endeavor here, as these may be of interest to some readers.

We used a 2.4 MHz PC with a quad processor to execute the calculations. To establish our new results for ternary equations of signature $(n,n,2)$ (see Proposition 2.2), we implemented our algorithm in Magma [BCP97]. The total running time for proving Proposition 2.2 came to about two weeks. The proof of Theorem 1.1 proceeds via proving Theorems 1.2 and 1.3; to verify these two results, we implemented our sieving procedures in Maple, separately for the $n \geq 7$ and $n=5$ cases. The program codes utilized in our computations are available from the authors upon request. In both of the cases $n \geq 7$ and $n=5$, the program took the following times to run for different values of k: a few seconds for k up to 19, a few minutes up to $k=23$, a few hours up to $k=29$, a few days for $k=30$ and 31, and about a week each for $k=32, 33$ and 34. Altogether, after having verified Proposition 2.2, the calculations to establish Theorems 1.2 and 1.3 took

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about a month each. We mention this to emphasize that, because of the huge number of cases to be looked into, if one has only the 'ternary' results, it would be hopeless to attack the problem without some additional, new 'sieving' ideas. Conversely, using only the sieving procedures with the previously known 'ternary' results, one would be left with a lot of cases which have not been dealt with. So, to prove our theorems, we need to find a balanced and efficient combination of the two techniques.

The organization of the paper is as follows. In the next section we introduce notation and summarize some old as well as establish some new results about ternary equations which will be used in the proofs of Theorems 1.1, 1.2 and 1.3. The final section is devoted to proving our theorems.

2. Notation and auxiliary results

For integers d, x, k with $k \geq 3$ and for indices $0 \leq i_1 < \cdots < i_l < k$, put

$$\Pi(i_1, \ldots, i_l) = (x + i_1 d) \cdots (x + i_l d)$$

and

$$\Pi_k = \Pi(0, 1, \ldots, k-1) = x(x+d) \cdots (x + (k-1)d).$$

Assume that (1) has a solution in non-zero integers x, d, k, b, y and n with the requested properties. Further, we may assume that n is an odd prime. From (1) one can then deduce that

$$x + id = a_i x_i^n \quad (i = 0, 1, \ldots, k-1) \tag{3}$$

where x_i is a non-zero integer and a_i is an nth-power-free positive integer with $P(a_i) \leq k$. For a given k, there are only finitely many and effectively determinable such k-tuples $(a_0, a_1, \ldots, a_{k-1})$.

For brevity, we introduce the following notation. Write

$$[i_1, i_2, i_3] : c_{i_1} a_{i_1} x_{i_1}^n + c_{i_3} a_{i_3} x_{i_3}^n = c_{i_2} a_{i_2} x_{i_2}^n, \tag{4}$$

where $0 \leq i_1 < i_2 < i_3 < k$ and $c_{i_1} = (i_3 - i_2)/D$, $c_{i_2} = (i_3 - i_1)/D$ and $c_{i_3} = (i_2 - i_1)/D$ with $D = \gcd(i_3 - i_2, i_3 - i_1, i_2 - i_1)$. Further, if $0 \leq j_1 < j_2 \leq j_3 < j_4 < k$ with $j_1 + j_4 = j_2 + j_3$, then let

$$[j_2, j_3] \times [j_1, j_4] : a_{j_2} a_{j_3} (x_{j_2} x_{j_3})^n - a_{j_1} a_{j_4} (x_{j_1} x_{j_4})^n = (j_2 j_3 - j_1 j_4) d^2.$$

Given a k-tuple $(a_0, a_1, \ldots, a_{k-1})$, we obtain in this way a complicated system of ternary equations to be solved.

In proving our theorems, we shall use several results concerning ternary equations to solve the systems of equations that arise. In this section we collect some earlier theorems and establish two new results for ternary equations which will be needed later on. We start with ternary equations of signature $(n, n, 2)$.

Proposition 2.1. *Let $n \geq 7$ be prime, let u and v be non-negative integers, and let A and B be coprime positive integers. Then the following diophantine equations have no solutions in pairwise* *coprime non-zero integers* X, Y, Z *with* $XY \neq \pm 1$:

$$X^n + 2^u Y^n = 3^v Z^2, \quad u \neq 1; \tag{5}$$

$$X^n + Y^n = CZ^2, \quad C \in \{2, 6\}; \tag{6}$$

$$X^n + 5^u Y^n = 2Z^2 \quad \text{with } n \geq 11 \text{ if } u > 0; \tag{7}$$

$$AX^n + BY^n = Z^2 \quad \text{with } AB = 2^u p^v \text{ where } u \neq 1,\ p \in \{11, 19\}. \tag{8}$$

Proof. This result is due to Bennett *et al.* [BBGH06]. □

The following result is new. For its formulation, we need some further standard notation. If m is a positive integer, let $\mathrm{rad}(m)$ denote the *radical* of m, i.e. the product of distinct prime divisors of m with the convention that $\mathrm{rad}(1) = 1$.

Set

$$\begin{aligned} I_1 = \{&(2,1), (2,3), (2,5), (2,7), (6,1), (6,5), (10,1), (10,3), (14,1), (14,3), (22,1),\\ &(26,1), (30,1), (34,1), (38,1), (42,1), (46,1), (66,1), (70,1), (78,1), (102,1),\\ &(114,1), (130,1), (138,1)\}, \end{aligned}$$

$$\begin{aligned} I_2 = \{&(3,1), (3,5), (5,1), (5,3), (7,1), (13,1), (15,1), (17,1), (21,1), (23,1), (33,1), (35,1),\\ &(39,1), (51,1), (57,1), (69,1), (165,1)\} \end{aligned}$$

and

$$\begin{aligned} I_3 = \{&(3,2), (5,6), (7,2), (11,2), (13,2), (15,2), (17,2), (19,2),\\ &(21,2), (23,2), (33,2), (35,2), (39,2)\}. \end{aligned}$$

PROPOSITION 2.2. *Let* $n > 31$ *be a prime, let* A, B *and* C *be pairwise coprime positive integers with* $(\mathrm{rad}(AB), C) \in I_1 \cup I_2 \cup I_3$, *and take* $p \in \{11, 13, 17, 19, 23, 29, 31\}$ *such that* $p \nmid AB$. *Then the equation*

$$AX^n + BY^n = CZ^2 \tag{9}$$

has no solutions in pairwise coprime non-zero integers X, Y, Z *with* $p \mid XY$ *except, possibly, in those cases listed in Table 2.*

As mentioned in the introduction, to prove our results in the $n \geq 7$ case we had to find an efficient combination of the 'modular' and 'sieving' techniques. A very large number of new ternary equations arose for each $k > 11$. We followed the strategy explained below. First, we solved a few well-chosen ternary equations (considering only a small subset I of $I_1 \cup I_2 \cup I_3$ in Proposition 2.2); then, using our sieves (which will be detailed in the next section), we tried to reduce each case $(a_0, a_1, \ldots, a_{k-1})$ to ternary equations that either had been treated already in Propositions 2.1, 2.4 or 2.5 or belonged to I. After a while (for larger values of k), exceptional cases arose for which such a reduction was unavailable. At that point we enlarged the set I in several steps, and gradually we reached the finite sets I_1, I_2 and I_3 in Proposition 2.2. By using the equations that occur in Propositions 2.1, 2.4 or 2.5 or which correspond to $I_1 \cup I_2 \cup I_3$ in Proposition 2.2, we were able to 'cover' all cases $(a_0, a_1, \ldots, a_{k-1})$, i.e. prove the insolubility of each system of equations that arose. For the details, we refer to the proof of Theorem 1.2

Proof of Proposition 2.2. To solve our equations of the form (9), we shall apply the modular approach. Specifically, to a putative non-trivial solution (X, Y, Z) of (9) one can associate a Frey

TABLE 2. The possibly exceptional cases in Proposition 2.2.

n	$(\mathrm{rad}(AB), C,\ p)$
37	(2,7,31), (3,5,31), (6,5,31), (19,2,29), (22,1,31), (46,1,29), (46,1,31), (70,1,29)
41	(2,7,11), (21,2,13), (21,2,19), (21,2,29), (22,1,31), (46,1,31), (51,1,13), (102,1,13), (165,1,13), (165,1,31)
43	(5,6,13), (6,5,23)
47	(5,6,11), (5,6,29), (6,5,31), (15,2,11), (15,2,29), (33,2,13), (33,2,23), (39,2,31)
59	(3,5,31), (6,5,31), (39,2,23), (165,1,17)
61	(5,6,13), (5,6,29),(14,3,17), (15,2,13), (15,2,29), (39,2,17), (39,2,19)
67	(165,1,29)
71	(33,2,23)
79	(5,6,17), (15,2,17), (165,1,19)
83	(165,1,29)
89	(165,1,29), (165,1,31)
97	(5,6,31), (15,2,31), (165,1,29)
107	(5,6,31), (15,2,31)
127	(33,2,31), (165,1,29)
137	(5,6,23)
193	(5,6,31), (15,2,31)
229	(33,2,31)
239	(33,2,31), (165,1,29)

curve $E/\mathbb{Q}$, with the corresponding mod n Galois representation

$$\rho_n^E : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F}_n)$$

on the n-torsion $E[n]$ of E. This representation arises from a cuspidal newform $f = \sum_{r=1}^{\infty} c_r q^r$ of weight two and trivial Nebentypus character. For details, we refer to [BS04]. As usual, for a positive integer m let $\mathrm{rad}_2(m)$ denote the *2-free radical* of m, i.e. the product of distinct odd prime divisors of m, with the convention that $\mathrm{rad}_2(1) = 1$. It can be shown that the level N of the newform considered above is contained in $\{2^\alpha \cdot \mathrm{rad}_2(AB) \cdot \mathrm{rad}_2^2(C) : \alpha = 0, 1, 2, 3, 5, 7\}$, $\{2^\alpha \cdot \mathrm{rad}_2(AB) \cdot \mathrm{rad}_2^2(C) : \alpha = 1, 5\}$ or $\{256 \cdot \mathrm{rad}_2(AB) \cdot \mathrm{rad}_2^2(C)\}$, according to whether $(\mathrm{rad}(AB), C)$ belongs to I_1, I_2 or I_3, respectively. The assumption that $p \mid XY$ for a prime $p \in \{11, 13, 17, 19, 23, 29, 31\}$ implies that if p is relatively prime to N, then

$$\mathrm{Norm}_{K_f/\mathbb{Q}}(c_p \pm (p+1)) \equiv 0 \pmod{n}, \tag{10}$$

where c_p is the pth Fourier coefficient of f and K_f is the field generated by the Fourier coefficients of f. This means that if (10) does not hold, we arrive at a contradiction. For recipes associated with this technique, see [Ben03] or [Coh07].

We illustrate our approach by considering the case of $(\mathrm{rad}(AB), C) = (38, 1)$. The corresponding levels are $19, 2 \cdot 19, 4 \cdot 19, 8 \cdot 19, 32 \cdot 19$ and $128 \cdot 19$. Suppose that X, Y, Z is a solution of the corresponding equation (9) in pairwise coprime non-zero integers such that $p \mid XY$, where p is a prime with $11 \leq p \leq 31$. Using a simple Magma program, we calculated the Fourier

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coefficients c_p of the corresponding one-dimensional newforms f at the levels considered above, obtaining

$$n \mid (c_p - (p+1))(c_p + p + 1) =: B_p. \tag{11}$$

For the corresponding higher-dimensional newforms f at the levels under consideration, we used a stronger sieve. Let

$$A_m = \mathrm{Norm}_{K_f/\mathbb{Q}}(c_m^2 - (m+1)^2) \prod_{\substack{|a|<2\sqrt{m} \\ a \text{ is even}}} \mathrm{Norm}_{K_f/\mathbb{Q}}(c_m - a)$$

for $m = 3, 5, 7$. Our method now yields that

$$n \mid \gcd(B_p, A_3, A_5, A_7). \tag{12}$$

Consequently, if for some prime p with $11 \leq p \leq 31$ the statements (11) and (12) do not hold for any f in question, then in the case of $(\mathrm{rad}(AB), C) = (38, 1)$, Equation (9) has no solution in pairwise coprime non-zero integers X, Y, Z with $p \mid XY$.

Using the same arguments for each equation considered in Proposition 2.2, we infer that (9) may have a solution with the prescribed properties only in the cases listed in Table 2.

We note that the Hasse–Weil bound implies that $B_p \neq 0$. Further, for the pairs $(\mathrm{rad}(AB), C)$ and for the higher-dimensional case, we omit A_m from the stronger sieve if $A_m = 0$ or $m \mid ABC$. □

Remark. We can choose further primes m to make a stronger sieve. For example, in the $(\mathrm{rad}(AB), C) = (165, 1)$ case we can apply the sieve $n \mid \gcd(B_p, A_7, A_{61}, A_{73})$ for higher-dimensional forms and we can exclude the cases

$$\begin{aligned}(n, p) = {} & (41, 13), (41, 31), (59, 17), (67, 29), (79, 19), \\ & (89, 31), (97, 29), (127, 29), (239, 29)\end{aligned} \tag{13}$$

as well. However, to find such appropriate primes m involves a long computation. Since, for our later purposes, Table 2 and its refinement that excludes the cases listed in (13) are already sufficient, we do not continue this procedure.

We use ternary equations of signature $(n, n, 3)$ via the following result of Bennett [Ben08]. For a prime p and non-zero integer u, $\mathrm{ord}_p(u)$ denotes, as usual, the largest integer v for which $p^v \mid u$ holds.

Proposition 2.3. *Let n be a prime with $n \geq 7$. Then*

$$x(x+d)(x+3d)(x+4d) = by^n \tag{14}$$

has only the solutions $(x, d, b, y) = (\pm 2, \mp 1, 4, 1)$ in non-zero integers x, d, b, y with $\gcd(x, d) = 1$ and $P(b) \leq 3$.

Proof. The statement is a simple consequence of a recent result of Bennett [Ben08]. However, for the sake of completeness we give the main steps of the proof.

Suppose that (x, d, b, y, n) is a solution to (14) with $by \neq 0$. If $3 \nmid x(x+d)$, then, using the notation (3), the identity $[1, 3] \times [0, 4]$ gives

$$a_1 a_3 (x_1 x_3)^n - a_0 a_4 (x_0 x_4)^n = 3d^2,$$

and we also have $\gcd(a_1 a_3 x_1 x_3, a_0 a_4 x_0 x_4) = 1$ and $P(a_0 a_1 a_3 a_4) \leq 2$. As either $\mathrm{ord}_2(a_1 a_3) = \mathrm{ord}_2(a_0 a_4) = 0$, or $\mathrm{ord}_2(a_1 a_3) = 0$ and $\mathrm{ord}_2(a_0 a_4) \geq 2$ (or vice versa), the result follows from (5) of Proposition 2.1 in this case.

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Otherwise, if $3 \mid x(x+d)$, then the identity $(x+d)^2(x+4d) - x(x+3d)^2 = 4d^3$ yields

$$a_1^2 a_4 (x_1^2 x_4)^n - a_0 a_3^2 (x_0 x_3^2)^n = 4d^3.$$

After simplifying with a suitable power of 2, we get an equality that is either of the form

$$X^n + 3^v Y^n = 2^u Z^3 \quad \text{where } u \geq 1,\ v \geq 3 \quad \text{and } \gcd(X, 3Y) = 1$$

or of the form

$$AX^n + BY^n = Z^3 \quad \text{with } AB = 2^u 3^v \text{ where } u \geq 1,\ v \geq 3 \quad \text{and } \gcd(AX, BY) = 1.$$

Using results from [Ben08, BVY04] on certain ternary equations of signature $(n, n, 3)$, the statement again follows in this case. □

We will also use results on ternary equations of signature (n, n, n) which have been proved using the method involving Frey curves and modular forms; see [DM97, Kra97, Rib97, Wil95].

Proposition 2.4. *Let $n \geq 3$ and $u \geq 0$ be integers. Then the equation*

$$X^n + Y^n = 2^u Z^n$$

has no solutions in pairwise coprime non-zero integers X, Y, Z with $XYZ \neq \pm 1$.

Proof. This result is essentially due to Wiles [Wil95] (in the $n \mid u$ case), Darmon and Merel [DM97] (if $u \equiv 1 \pmod{n}$) and Ribet [Rib97] (in the remaining cases for $n \geq 5$ prime); see also Győry [Gyo99]. □

Proposition 2.5. *Let $n \geq 5$, and let A and B be coprime positive integers with $AB = 2^u 3^v$ or $2^u 5^v$, where u and v are non-negative integers with $u \geq 4$. Then the equation*

$$AX^n + BY^n = Z^n \tag{15}$$

has no solutions in pairwise coprime non-zero integers X, Y and Z.

Proof. This is [SS01, Lemma 13]. □

For $n = 5$, most of the above assertions on ternary equations cannot be applied, in which case we shall use the following results in addition.

Proposition 2.6. *Let $n \geq 3$ be an integer. All solutions of the equation*

$$x(x+1)\cdots(x+k-1) = by^n \tag{16}$$

in positive integers x, k, b, y with $k \geq 8$ and $P(b) \leq 7$ have

$$k \in \{8, 9, 10\} \quad \textit{and} \quad x \in \{1, 2, \ldots, p^{(k)} - k\}, \tag{17}$$

where $p^{(k)}$ denotes the smallest prime satisfying $p^{(k)} > k$.

Proof. This result follows from a theorem of Saradha [Sar97], which says that in (16), $P(y) \leq k$. As was seen in Győry [Gyo98], we then get $x \in \{1, 2, \ldots, p^{(k)} - k\}$, whence $p^{(k)} > x + k - 1$. Denote by $p_{(k)}$ the greatest prime with $p_{(k)} \leq k$. Then, for $k \geq 11$, we have $p_{(k)} \geq 11$. Further, by Chebyshev's theorem, $p^{(k)} < 2p_{(k)}$. In view of $p_{(k)} \leq k$, we have $p_{(k)} \mid x(x+1)\cdots(x+k-1)$. But it follows that $2p_{(k)} > x + k - 1$. Hence (16) and $P(b) \leq 7$ give $p_{(k)}^n \mid x(x+1)\cdots(x+k-1)$, which implies that $p_{(k)}^n \leq x + k - 1$. Thus we get $p_{(k)}^n \leq 2p^{(k)}$, which is a contradiction.

It remains to treat the case $k \in \{8, 9, 10\}$. Then $p^{(k)} = 11$ and it is easy to check that the values of k and x listed in (17) give the solutions of (16). □

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Lemma 2.1. *Let $n=5$. Suppose that for $k=5$, $P(b)\leq 3$, and for $6\leq k\leq 11$, $P(b)\leq 5$. Then the only solution to (1) with $d\geq 2$ is $(x,d,k)=(-5,2,6)$.*

Proof. This is a special case of [BBGH06, Theorem 1.2]. □

Lemma 2.2. *Let $n=5$. Suppose that (x,d,y,b) provides a solution to (1) with $P(b)\leq 3$ and $k=4$. Then either $(x,d)=(-3,2)$ or, up to symmetry, $(a_0,a_1,a_2,a_3)=(4,3,2,1)$ or $(9,4,1,6)$.*

Proof. This is [BBGH06, Lemma 6.3]. □

Let C be a 5th-power-free positive integer with $P(C)\leq 7$. Then we can write

$$C=2^\alpha\cdot 3^\beta\cdot 5^\gamma\cdot 7^\delta \tag{18}$$

with non-negative integers $\alpha,\beta,\gamma,\delta$ not exceeding 4.

Proposition 2.7. *If the equation*

$$X^5+Y^5=CZ^5 \tag{19}$$

has a solution in pairwise coprime non-zero integers X, Y and Z, then one of the following cases holds.

(i) *$C=2$ and $X=Y=\pm 1$.*

(ii) *$C=7^\delta$ with $1\leq\delta\leq 4$, $5\mid XY$, $5\nmid Z$ and Z is odd.*

(iii) *$C\in\{2\cdot 3^2\cdot 7^\delta, 2^2\cdot 3^4\cdot 7^\delta, 2^3\cdot 3\cdot 7^\delta, 2^4\cdot 3^3\cdot 7^\delta\}$ with $1\leq\delta\leq 4$ and $5\mid Z$.*

This implies that if, in (19), $5\nmid XYZ$, then (i) must hold. If, in particular, $P(C)\leq 5$, then Proposition 2.7 reduces to [BBGH06, Proposition 6.1].

Proof. Let (X,Y,Z) be a solution of (19) in pairwise coprime non-zero integers. By results of Dirichlet and Dénes [Den52], it suffices to deal with the case $C>2$ and $XYZ\neq\pm 1$. It follows from a theorem of Lebesgue [Dic66, p. 738, item 37] that $5\nmid C$ and

$$C\equiv\pm 1,\pm 7\pmod{5^2}. \tag{20}$$

First, assume that $5\nmid Z$. We have

$$C^4\equiv 1\pmod{5^2}\quad\text{and}\quad 2^4\not\equiv 1\pmod{5^2},$$

whence

$$C^4\not\equiv 2^4\pmod{5^2}.$$

Applying [BGP04, Lemma 6.1 and Corollary 6.2] to (19), we deduce that $5\mid XY$, CZ is odd and

$$r^4\equiv 1\pmod{5^2} \tag{21}$$

for each prime divisor r of C. In view of (18) and (21), we infer that only $r=7$ can hold, and (ii) follows.

Now suppose that $5\mid Z$. The prime 5 being regular, a theorem of Maillet (see, for example, [Dic66, p. 759, item 167]) implies that C must have at least three distinct prime factors. This means that in (18), $\gamma=0$ and $\alpha,\beta,\delta\geq 1$. It is easy to check that, together with (20), this gives (iii). □

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3. Proofs of the main theorems

First we prove Theorems 1.2 and 1.3. As was mentioned earlier, we need to consider the $n=5$ and $n \geq 7$ cases separately. The reason is that the theory of ternary equations cannot be efficiently applied in the $n=5$ case. Let us start with $n \geq 7$.

Proof of Theorem 1.2. In proving this theorem we shall eventually reduce the problem to the solution of several ternary diophantine equations. We now explain the main ideas of our proof. Suppose that under the assumptions of our theorem, (1) has a solution. Observe that, by (3), to determine all solutions to (1) with fixed k it suffices to characterize the arithmetic progressions of the form $a_0x_0^n, a_1x_1^n, \dots, a_{k-1}x_{k-1}^n$, where the x_i are non-zero integers and the a_i are positive integers such that $\gcd(a_0x_0^n, a_1x_1^n) = 1$,

$$P(a_i) \leq k \quad \text{and} \quad a_i \text{ is } n\text{th-power-free for } i = 0, 1, \dots, k-1. \tag{22}$$

Furthermore, the assumption $P(b) \leq P_{k,n}$ implies that

$$n \mid \mathrm{ord}_p\biggl(\prod_{i=0}^{k-1} a_i\biggr) \quad \text{for all primes } p > P_{k,n}. \tag{23}$$

In particular, if p is a prime and $u \geq 1$ is an integer with $p^u \mid a_ix_i^n$, then $p^u \mid a_jx_j^n$ if and only if $p^u \mid i-j$. This assertion will be used later on without any further reference.

The number of possible k-tuples $(a_0, a_1, \dots, a_{k-1})$ with properties (22) and (23) grows very rapidly with k, and it is impossible to look at the different cases one by one when k is relatively large. So we apply the following strategy. We exclude the possible coefficient k-tuples $(a_0, a_1, \dots, a_{k-1})$ in several steps, by using certain procedures in a well-determined order. A k-tuple will be excluded after ensuring that in the corresponding case, (1) has no solution. We start with arguments that enable us to exclude a large number of k-tuples $(a_0, a_1, \dots, a_{k-1})$. By induction, a lot of possibilities can be excluded. Specifically, if for some $\ell \geq 3$, $P(a_0 \cdots a_{\ell-1}) \leq P_{\ell,n}$ or $P(a_{k-\ell} \cdots a_{k-1}) \leq P_{\ell,n}$ holds, then the statement follows either by induction or from Theorem A. Using this observation, the number of k-tuples to be considered can be reduced drastically. Therefore, after each successive step, it will become simpler and simpler to manage and exclude the remaining k-tuples. We shall explain the details later on, when we describe the sieves. Moreover, we will provide examples to illustrate how the sieves work.

In what follows, we will always assume that k is fixed with $11 < k < 35$. We use the following convention. Let $2 = p_1 < p_2 < \cdots < p_{\pi(k-1)}$ be the primes that are less than or equal to $k-1$, where $\pi(k-1)$ denotes the number of primes not exceeding $k-1$. Observe that as $P_{k,n} < k$ for $n \geq 7$, by (23) we have $P(a_i) < k$ in (22) for all $i = 0, 1, \dots, k-1$. We indicate the distribution of the primes $p_1, \dots, p_{\pi(k-1)}$ among the $a_ix_i^n$ or the a_i (i.e. the prime divisors not exceeding $k-1$ of the $a_ix_i^n$ or the a_i) with the aid of certain $\pi(k-1)$-tuples of the form $(m_{\pi(k-1)}, \dots, m_1)$. For $3 \leq j \leq \pi(k-1)$, let

$$m_j \in \{\times, 0, 1, \dots, p_j - 1\}, \tag{24}$$

where $m_j = \times$ if $p_j \nmid \Pi_k$ (i.e. if p_j does not divide $x(x+d)\cdots(x+(k-1)d)$) and, otherwise, m_j is the integer from $\{0, 1, \dots, p_j - 1\}$ for which $p_j \mid x + m_jd$. In our proof, we first consider cases where it is not specified which terms of the progression $x, x+d, \dots, x+(k-1)d$ are divisible by 2 or 3. Then we write $m_j = *$ for $j = 1, 2$. In such a case we say that the distribution of $p_1, \dots, p_{\pi(k-1)}$ among the $a_ix_i^n$ or the a_i corresponds to the $\pi(k-1)$-tuple $(m_{\pi(k-1)}, \dots, m_1)$.

By means of these $\pi(k-1)$-tuples we shall obtain information about the location of the coefficients a_i without 'large' prime factors, which will be of crucial importance in our proof. To each of these $\pi(k-1)$-tuples there correspond a great number of k-tuples $(a_0, a_1, \dots, a_{k-1})$ under consideration. Hence the use of our tests: sieving with all $\pi(k-1)$-tuples of the form $(m_{\pi(k-1)}, \dots, m_3, *, *)$ will enable us to exclude full branches of k-tuples $(a_0, a_1, \dots, a_{k-1})$ simultaneously; this makes our algorithm very efficient. Our first three tests below seem to be especially efficient, at least for the range of k under consideration.

Later, we shall need to refine our algorithm by specifying also those terms of $x, x+d, \dots, x+(k-1)d$ which are divisible by 2 and/or 3. For $j=1$ and 2, let

$$m_j \in \{\times, 0, 1, \dots, k-1\} \tag{25}$$

such that, as in the $j \geq 3$ case, $m_j = \times$ if $p_j \nmid \Pi_k$ and, otherwise, m_j is a number from $\{0, 1, \dots, k-1\}$ for which $p_j \mid x + m_j d$ and

$$\mathrm{ord}_{p_j}(x + m_j d) = \max_{0 \leq \ell \leq k-1} \mathrm{ord}_{p_j}(x + \ell d).$$

This will enable us to calculate the exact orders of the primes $p_1 = 2$ and $p_2 = 3$ in the numbers $a_i x_i^n$. Then we shall continue our proof with further tests, sieving first with all possible $\pi(k-1)$-tuples of the form $(m_{\pi(k-1)}, \dots, m_3, m_2, *)$ and $(m_{\pi(k-1)}, \dots, m_3, *, m_1)$, and thereafter with tuples $(m_{\pi(k-1)}, \dots, m_3, m_2, m_1)$ where m_1, m_2 satisfy (25). Finally, a relatively small number of k-tuples $(a_0, a_1, \dots, a_{k-1})$ will be left, with some small exponents n which will be excluded by means of a local sieve.

In our sieves we shall use ternary equations. We shall distinguish between (n, n, n)-, $(n, n, 3)$- and $(n, n, 2)$-sieves, according to whether the ternary equations involved are of signature (n, n, n), $(n, n, 3)$ or $(n, n, 2)$, respectively.

$(\boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n})$-sieve I. Suppose that we are dealing with a $\pi(k-1)$-tuple $T = (m_{\pi(k-1)}, \dots, m_3, *, *)$. First (with the help of T) we check whether there exists an arithmetic progression i_1, i_2, i_3 with $0 \leq i_1 < i_2 < i_3 \leq k-1$ such that $P(a_{i_1} a_{i_2} a_{i_3}) \leq 3$ and $i_1 \equiv i_2 \equiv i_3 \pmod 3$. If there are such indices, then by Proposition 2.4 the identity $[i_1, i_2, i_3]$ implies that $3 \mid x + i_1 d$ (and, consequently, $3 \mid x + i_2 d, x + i_3 d$) must be valid, otherwise we are done. We then apply an exhaustive search for indices i_4 and i_5 with which some appropriately chosen identities of the form (4) will lead to a contradiction. For example, assume that $P(a_2 a_5 a_8) \leq 3$. Then from $[2, 5, 8]$ we know that $3 \mid x + 2d, x + 5d, x + 8d$. Suppose, further, that $P(a_4 a_6) \leq 3$. Then $\gcd(x, d) = 1$ shows that $P(a_4 a_6) \leq 2$. Hence, as exactly one of $\mathrm{ord}_3(x + 2d) \geq 2$, $\mathrm{ord}_3(x + 5d) \geq 2$ or $\mathrm{ord}_3(x + 8d) \geq 2$ holds, one of the identities $[2, 4, 5]$, $[5, 6, 8]$ or $[2, 6, 8]$ (again by Proposition 2.4) will lead to a contradiction.

After having checked all the possible $\pi(k-1)$-tuples T of the form $(m_{\pi(k-1)}, \dots, m_3, *, *)$ and all the possible triples (i_1, i_2, i_3) in question, we exclude the tuples T and the corresponding k-tuples $(a_0, a_1, \dots, a_{k-1})$ which lead in this way to a contradiction.

As an example, take $k = 15$ and let

$$T = (0, 3, 0, \times, *, *).$$

Then we have $P(a_2 a_4 a_5 a_6 a_8) \leq 3$ and, by the previous argument, T and the corresponding 15-tuples can be excluded.

$(n, n, 3)$-sieve. Suppose that a $\pi(k-1)$-tuple T survives the previous test. Then we try to find an index i_0 and a difference d_0 with $P(d_0) \leq 3$, $i_0 - 2d_0 \geq 0$ and $i_0 + 2d_0 \leq k-1$ such that

$$P(a_{i_0-2d_0} a_{i_0-d_0} a_{i_0+d_0} a_{i_0+2d_0}) \leq 3. \tag{26}$$

Let $g = \gcd(x + (i_0 - 2d_0)d, d_0 d)$. Obviously, $\gcd(x, d) = 1$ and $P(d_0) \leq 3$ imply that $P(g) \leq 3$. Putting $X = (x + (i_0 - 2d_0)d)/g$ and $D = d_0 d/g$ and then using (3) and (26), we infer that for these X and D the equation

$$X(X+D)(X+3D)(X+4D) = BY^n$$

has a solution in non-zero integers B, Y with $P(B) \leq 3$. However, by Proposition 2.3 this implies that $X + 2D = 0$, which is impossible. We check all the possible i_0 and d_0, and exclude again all the T and all the corresponding k-tuples that lead in this way to a contradiction.

To see an example, take $k = 15$ and

$$T = (0, 3, 4, 2, *, *).$$

Note that T survives the previous test. We have $P(a_5 a_6 a_8 a_9) \leq 3$, hence we can take $i_0 = 7$ and $d_0 = 1$; then, by the above test, T and the corresponding 15-tuples can be excluded.

(n, n, n)-sieve II. Consider a $\pi(k-1)$-tuple $T = (m_{\pi(k-1)}, \ldots, m_3, *, *)$ which is not excluded by the previous tests. We let m_1 run through the set $\{\times, 0, 1, \ldots, k-1\}$ and examine all $\pi(k-1)$-tuples of the form $T' = (m_{\pi(k-1)}, \ldots, m_3, *, m_1)$. We perform an exhaustive search to find an identity of the form $[i_1, i_2, i_3]$ leading to a ternary equation of the form $AX^n + BY^n = Z^n$ such that $\gcd(A, B) = 1$ and AB is $2^u 3^v$ or $2^u 5^v$, with $u \geq 4$ in either case. If we succeed, then the corresponding $\pi(k-1)$-tuple and k-tuples can be excluded by Proposition 2.5.

As an example, choose $k = 15$ and

$$T' = (0, 3, 1, 4, *, 11).$$

Note that this $\pi(k-1)$-tuple cannot be excluded by the previous tests. However, taking the identity $[2, 10, 11]$, after cancelling an appropriate power of 3 we get a ternary equation of the form $AX^n + BY^n = Z^n$ with $\gcd(A, B) = 1$ and $AB = 2^u 3^v$, $u \geq 4$. Hence we can exclude T' and the corresponding 15-tuples.

$(n, n, 2)$-sieve I. Suppose that a $\pi(k-1)$-tuple $T' = (m_{\pi(k-1)}, \ldots, m_3, *, m_1)$ passes the previous tests. Then we consider all $\pi(k-1)$-tuples of the form $T^* = (m_{\pi(k-1)}, \ldots, m_2, m_1)$ with $m_2 \in \{\times, 0, 1, \ldots, k-1\}$. We search for an identity of the form $[j_2, j_3] \times [j_1, j_4]$ which leads to a ternary equation $AX^n + BY^n = CZ^2$ such that $\gcd(A, B, C) = 1$ and one of the following holds: $AB = 2^u$ $(u \neq 1)$ and $C = 3^v$; $AB = 1$ and $C \in \{2, 6\}$; $AB = 2^u p^v$ $(u \neq 1, p \in \{11, 19\})$ and $C = 1$. Then, upon applying Proposition 2.1, the corresponding $\pi(k-1)$-tuples T^* and k-tuples $(a_0, a_1, \ldots, a_{k-1})$ can be excluded.

For example, choose $k = 15$ again, and take

$$T^* = (0, 3, 1, 2, 0, 3).$$

Note that T^* passes all of the previous sieves. However, the identity $[5, 10] \times [4, 11]$ gives rise to a ternary equation of the form $X^n + 4Y^n = 3Z^2$, which leads to a contradiction, as explained above.

$(n, n, 2)$-sieve II. Assume that a $\pi(k-1)$-tuple T^* survives the previous tests. We again try to find an identity of the form $[j_2, j_3] \times [j_1, j_4]$ that leads to a ternary equation $AX^n + BY^n = 2Z^2$

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with $AB = 5^u$, $u \geq 1$. Then Proposition 2.1 implies that $n = 7$. We collect these $\pi(k-1)$-tuples T^* into a set S, and make a note that these tuples T^* have to be separately reconsidered later for the exponent $n = 7$.

As an example, let $k = 15$ and

$$T^* = (0, 3, 4, 1, 8, 3).$$

As one can easily see, T^* survives the previous tests. However, after cancellations, the identity $[5, 6] \times [2, 9]$ leads to a ternary equation of the form $X^n + 5^u Y^n = 2Z^2$ with $u > 0$. Then Proposition 2.1 gives that $n = 7$, and we can put T^* into S.

$(n, n, 2)$-sieve III. Assume that a $\pi(k-1)$-tuple T^* survives the previous tests. Then we search for an identity $[j_2, j_3] \times [j_1, j_4]$ such that the implied ternary equation satisfies the conditions of Proposition 2.2. Then this proposition and the subsequent remark yield that n is (explicitly) bounded for the case corresponding to T^*. We put these $\pi(k-1)$-tuples T^* into the set S, and to each of them we attach the list of corresponding 'exceptional' exponents, to be checked later.

For example, let $k = 15$ and

$$T^* = (0, 3, 1, 4, 0, 0).$$

As one can check, this $\pi(k-1)$-tuple passes each earlier sieve. However, the identity $[6, 11] \times [3, 14]$ gives (after cancellations) a ternary equation of the form $X^n + 5^u Y^n = Z^2$ with $u \geq 1$ and such that $11 \mid XY$; hence by Proposition 2.2 we get that $n \leq 31$. Then we can put T^* into S.

After accomplishing the above procedures, one can exclude (or put into S) all the $\pi(k-1)$-tuples $(m_{\pi(k-1)}, \ldots, m_1)$ and corresponding k-tuples $(a_0, a_1, \ldots, a_{k-1})$ for all values of k, with very few exceptions. In the remaining cases we proceed as follows. Let $(a_0, a_1, \ldots, a_{k-1})$ be a k-tuple which passes all of the above tests. Let $T^* = (m_{\pi(k-1)}, \ldots, m_1)$ be the corresponding $\pi(k-1)$-tuple, with m_j subject to (24) for $j \geq 3$ and (25) for $j = 1, 2$. We 'split' T^* into several $\pi(k-1)$-tuples, depending on which of the indices i, j $\mathrm{ord}_5(x + id)$ and $\mathrm{ord}_7(x + jd)$ is maximal. Then, for these 'refined' $\pi(k-1)$-tuples, we try to find identities of the form $[i_1, i_2, i_3]$ or $[j_2, j_3] \times [j_1, j_4]$ such that Proposition 2.1, 2.2, 2.4 or 2.5 yields a contradiction. Obviously, for this purpose we can use the sieves explained above. On these 'refined' $\pi(k-1)$-tuples we have more information than we have on T^*. Thus, it often happens that a sieve which did not work for T^* itself manages to exclude a 'refined' $\pi(k-1)$-tuple along with the corresponding k-tuples $(a_0, a_1, \ldots, a_{k-1})$. In fact, this is exactly what we perform in all of the remaining cases. After having gone through all the remaining $\pi(k-1)$-tuples $(m_{\pi(k-1)}, \ldots, m_1)$ and corresponding k-tuples $(a_0, a_1, \ldots, a_{k-1})$, we are left only with the $\pi(k-1)$-tuples in the set S; all the other $\pi(k-1)$-tuples (and corresponding k-tuples) have already been excluded.

Let us give an example to illustrate the above method. Take $k = 24$ (there are no exceptional k-tuples for $k \leq 23$), and let

$$T^* = (m_9, \ldots, m_1) = (0, 0, 6, 10, 3, 1, 2, 4, 7).$$

One can check that this tuple passes all of the previous sieves. We then 'split' T^* into 9-tuples of the form $(0, 0, 6, 10, 3, m_4', m_3', 4, 7)$ with

$$m_3' \in \{2, 7, 12, 17, 22\} \quad \text{and} \quad m_4' \in \{1, 8, 15, 22\}.$$

Here we assume, for fixed m_j' $(j = 3, 4)$, that

$$\mathrm{ord}_{p_j}(x + m_j' d) = \max_{0 \leq \ell \leq 23} \mathrm{ord}_{p_j}(x + \ell d).$$

We try to find an identity of the form $[i_1, i_2, i_3]$ or $[j_2, j_3] \times [j_1, j_4]$ which, by Proposition 2.1, 2.2, 2.4 or 2.5, will lead to a contradiction. In this example, noting that $\mathrm{ord}_2(a_7 x_7^n) \geq 5$, one can easily check that for $m_3' = 2$ and $m_3' \in \{12, 17, 22\}$ the identity $[5, 7, 17]$ and $[2, 5, 7]$, respectively, leads to a contradiction by Proposition 2.5, regardless of the value of m_4'. Furthermore, for $m_3' = 7$, the identity $[2, 5, 17]$ yields a contradiction by Proposition 2.4, for any m_4'.

It remains to check the $\pi(k-1)$-tuples in S and the corresponding k-tuples $(a_0, a_1, \ldots, a_{k-1})$ for the remaining small values of the exponent n. This can be done very easily using the following local argument.

Local sieve. For each element in S and for the corresponding values of n that remain (obtained by Propositions 2.1, 2.2 and the subsequent Remark) we consider the problem locally. For each such n, we choose a prime q of the form $q = tn + 1$, with t as small as possible. For example, in the cases where $n = 11, 13, 17, 19, 23$ we take $q = 23, 53, 103, 191, 47$, respectively. Then we check the putative arithmetic progressions modulo q in the following way. By the choice of the corresponding modulus, the use of the Euler–Fermat theorem guarantees that x_i^n may assume only very few values modulo q. Checking all the cases one by one and using the fact that the numbers $a_i x_i^n$ $(i = 0, 1, \ldots, k-1)$ should be consecutive terms of an arithmetic progression, we get a contradiction in each case.

To illustrate the local argument, choose $k = 15$, $n = 23$ and take the $\pi(k-1)$-tuple

$$(0, 3, 1, 4, 0, 0)$$

from S. Observe that the 23rd powers modulo 47 are exactly $-1, 0$ and 1. Hence, in this case, the putative progression $a_i x_i^{23}$ $(i = 0, 1, \ldots, 14)$ should be of the form

$$\pm 2^{\alpha_0} 3^{\beta_0} 13^{\nu_0}, \pm 7^{\delta_1}, \pm 2, \pm 3 \cdot 11^{\varepsilon_3}, \pm 2^2 \cdot 5^{\gamma_4}, \pm 1, \pm 2 \cdot 3, \pm 1, \pm 2^3 7^{\delta_8}, \pm 3^2 5^{\gamma_9},$$
$$\pm 2, \pm 1, \pm 2^2 3, \pm 13^{\nu_{13}}, \pm 2 \cdot 5^{\gamma_{14}} 11^{\varepsilon_{14}}$$

modulo 47, with non-negative exponents smaller than 23 and with the possible extra condition that at most one of the terms can be equal to 0. However, as one can easily check even by hand, such an arithmetic progression does not exist. In all other cases a similar argument works, and this completes the proof. $\square$

Proof of Theorem 1.3. Let (x, d, k, b, y) be a solution of (1) with $n = 5$. For $d = 1$, each factor $x + id$ in (1) must be positive or negative. Then we can reduce (1) to the case of $x > 0$, and Proposition 2.6 applies to give the solutions listed in the theorem.

In what follows, we assume that $d \geq 2$. Further, if $k \leq 11$, by virtue of Lemma 2.1 we can restrict ourselves to the case where $7 \mid a_0 \cdots a_{k-1}$.

For $8 \leq k \leq 13$, most of our work in proving Theorem 1.3 is concentrated on the $k = 8$ case. For the values $9 \leq k \leq 13$, we can then proceed by induction on k. We note that the above sieves can be utilized to prove our theorem for larger values of k only. For $k \leq 13$, too many exceptions would remain after using our sieves. Hence, for these values of k, we shall handle the k-tuples $(a_0, a_1, \ldots, a_{k-1})$ that arise without using sieves, tests or the computer.

The $k = 8$ case. If $7 \mid a_0, a_7$, then by omitting x and $x + 7d$ from (1) we arrive at the case of $k = 6$. By Lemma 2.1, we then get $x + d = -5$, $d = 2$. This yields the solution $(x, d) = (-7, 2)$. If $7 \mid a_1$ or $7 \mid a_6$, then we omit, respectively, the factors x and $x + d$ or $x + 6d$ and $x + 7d$ and obtain in a similar manner the solutions $(x, d) = (-9, 2), (-5, 2)$.

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The case where $7 \mid a_2 \cdots a_5$ remains. By symmetry, it suffices to consider $7 \mid a_2a_3$.

First, suppose that $7 \mid a_2$. If $5 \nmid a_0 \cdots a_7$, then Lemma 2.1 applied to $\Pi(3, 4, 5, 6, 7)$ shows that there is no solution. If $5 \mid x + 2d$, then $5 \mid x + 7d$ and for $(i_1, i_2, i_3, i_4) = (3, 4, 5, 6)$ we get

$$\Pi(i_1, i_2, i_3, i_4) = b_1 y_1^5, \tag{27}$$

where b_1 and y_1 are non-zero integers with $P(b_1) \le 3$. Then Lemma 2.2 gives that either $(x + 3d, d) = (-3, 2)$, which leads to the solution $(x, d) = (-9, 2)$, or, up to symmetry,

$$(a_3, a_4, a_5, a_6) = (4, 3, 2, 1) \text{ or } (9, 4, 1, 6).$$

If (a_3, a_4, a_5, a_6) equals $(4, 3, 2, 1)$ or $(1, 2, 3, 4)$, then by applying Proposition 2.7 to $[0, 3, 6]$ we arrive at a contradiction. In the remaining cases, Proposition 2.7 can be applied to $[1, 3, 5]$ or $[0, 1, 3]$ and we again get a contradiction.

Next, assume that $5 \mid x$. If $3 \nmid \Pi_8$ or $3 \mid x$, we can apply Proposition 2.7 to $[1, 4, 7]$. Otherwise, to obtain a contradiction, Proposition 2.7 can be applied to $[1, 3, 7]$, $[4, 6, 7]$ or $[1, 3, 4]$ if $3 \mid x + d$, and to $[1, 4, 7]$ if $3 \mid x + 2d$.

Let $5 \mid x + d$. If $3 \nmid \Pi_8$ or $3 \mid x$, then one of the equations $[3, 4, 5]$, $[0, 3, 4]$, $[5, 6, 7]$ or $[4, 5, 7]$ leads to a contradiction by Proposition 2.7. In the remaining cases, at least one of the equations $[3, 4, 5]$, $[0, 1, 3]$, $[4, 5, 7]$, $[0, 3, 6]$, $[3, 5, 7]$ or $[0, 2, 4]$ is not solvable by Proposition 2.7.

Now let $5 \mid x + 3d$. If $3 \nmid \Pi_8$ or $3 \mid x(x + 2d)$, then, by using Proposition 2.7, $[1, 4, 7]$ leads to a contradiction. If $3 \mid x + d$, we get (27) with $(i_1, i_2, i_3, i_4) = (4, 5, 6, 7)$. Then Lemma 2.2 gives that either $(x + 4d, d) = (-3, 2)$, which does not yield any solution of (1), or, up to symmetry,

$$(a_4, a_5, a_6, a_7) = (4, 3, 2, 1) \text{ or } (9, 4, 1, 6).$$

It is easy to verify that only the second option can occur. Then $[0, 3, 6]$ or $[1, 4, 5]$ has no solution, according to whether (a_4, a_5, a_6, a_7) equals $(9, 4, 1, 6)$ or $(6, 1, 4, 9)$, respectively.

Finally, assume that $5 \mid x + 4d$. Then, applying Lemma 2.2 to (27) with $(i_1, i_2, i_3, i_4) = (1, 3, 5, 7)$, we get that either $(x + d, d) = (-3, 2)$, which yields the solution $(x, d) = (-5, 2)$ of (1), or, up to symmetry,

$$(a_1, a_3, a_5, a_7) = (4, 3, 2, 1) \text{ or } (9, 4, 1, 6).$$

It follows that in each case, $x + d, x + 3d, x + 5d$ and $x + 7d$ are all divisible by 4, which contradicts the assumption that $\gcd(x, d) = 1$.

Next, consider the case where $7 \mid x + 3d$. If $5 \nmid a_0 \cdots a_7$ or if $5 \mid x + 3d$, then we have (27) with $(i_1, i_2, i_3, i_4) = (4, 5, 6, 7)$. Hence, by Lemma 2.2, (a_4, a_5, a_6, a_7) equals $(4, 3, 2, 1)$, $(1, 2, 3, 4)$, $(9, 4, 1, 6)$ or $(6, 1, 4, 9)$. Now Proposition 2.7 shows that in these four cases the following equations are not solvable: $[1, 4, 7]$; $[2, 3, 4]$ or $[1, 3, 5]$; $[0, 1, 2]$ or $[1, 4, 5]$; and $[1, 4, 5]$, respectively.

Now let $5 \mid x$. If $3 \nmid x + d$, then Proposition 2.7 applies to $[1, 4, 7]$, leading to a contradiction. If $3 \mid x + d$, then by Proposition 2.7 at least one of the equations $[2, 4, 6]$, $[1, 4, 7]$, $[4, 6, 7]$ and $[1, 2, 4]$ has no solution.

Assume now that $5 \mid x + d$. If $3 \nmid \Pi_8$ or $3 \mid x$, then by Proposition 2.7 at least one of the equations $[0, 2, 4]$, $[2, 3, 4]$ and $[5, 6, 7]$ has no such solution which would yield a solution of (1). Suppose that $3 \mid x + d$. If x is odd, then the equation $[0, 1, 2]$ is not solvable by Proposition 2.7. Otherwise, if x is even, then since $\gcd(x, d) = 1$ we have that d is odd, whence $2^2 \mid x$ or $2^2 \mid x + 2d$. If $3^2 \nmid x + 7d$ or $3^2 \mid x + 7d$ and $2^2 \mid x$, then Proposition 2.7 shows, respectively, that $[4, 5, 7]$ or $[2, 4, 5]$ is not solvable. If $3^2 \mid x + 7d$ and $2^2 \mid x + 2d$, then using the fact that

$$X^5 \equiv 0, \pm 1 \pmod{11} \tag{28}$$

for any integer X, we deduce that $x_1 \equiv x_4 \equiv x_5 \equiv 0$ is the only solution of [1, 4, 5] (mod 11), and this leads to a contradiction.

Next, let $3 \mid x+2d$. If x is odd or $\mathrm{ord}_2(x) = \mathrm{ord}_2(x+4d)$, then in view of Proposition 2.7, [0, 2, 4] has no solution. As $\gcd(x, d) = 1$, it remains to consider the case where $2^3 \mid x$ or $2^3 \mid x+4d$. If $3^2 \nmid x+2d$ and $3^2 \nmid x+5d$, then [2, 4, 5] is not solvable by Proposition 2.7.

Assume that $3^2 \mid x+2d$. If $2^3 \mid x$, then [4, 5, 7] yields only the solution

$$x_4^5 \equiv x_5^5 \equiv x_7^5 \equiv \pm 1 \pmod{11}.$$

Together with (3), this gives $d \equiv \mp 1 \pmod{11}$ and $x \equiv \pm 8 \pmod{11}$. Then $x + d \equiv 5x_1^5 \pmod{11}$ with $5 \nmid x_1$ cannot hold. Thus $5^2 \mid x+d$, whence $\mathrm{ord}_5(x+6d) = 1$, and [5, 6, 7] yields a contradiction (mod 11). If $2^3 \mid x+4d$, then [0, 5, 7] is not solvable (mod 11). Finally, consider the case where $3^2 \mid x+5d$. If $2^3 \mid x+4d$, then Proposition 2.5 shows that the equation [1, 4, 7] is not solvable. By assumption, we have $5 \mid x+6d$. If $2^3 \mid x$, then [1, 4, 7] or [2, 4, 6] is not solvable (mod 11), according to whether $5^2 \mid x+6d$ or not.

Now let $5 \mid x+2d$. If $3 \nmid \Pi_8$, then upon solving [4, 5, 6] by means of Proposition 2.7 we do not get any solution for (1). First assume that $3 \mid x+d$. Then, by Proposition 2.7 and the fact that $5 \nmid x(x+6d)$, we find that either [0, 3, 6] or [4, 5, 6] has no solution, according to whether $2^2 \nmid x$ or $2^2 \mid x$, respectively. Next, let $3 \mid x+2d$. Then Proposition 2.7 implies that [0, 1, 4], [0, 4, 6] or [1, 2, 4] is not solvable, depending on whether $2^3 \mid x$, $2^3 \mid x+4d$ or $\mathrm{ord}_2(x) = \mathrm{ord}_2(x+4d)$, respectively. Assume now that $3 \mid x$. If $\mathrm{ord}_2(x+d) = \mathrm{ord}_2(x+5d)$, then [1, 3, 5] is not solvable by virtue of Proposition 2.7. It remains to deal with the cases $2^3 \mid x+d$ and $2^3 \mid x+5d$. Then Proposition 2.5 shows that [0, 1, 4] has no solution.

Finally, assume that $5 \mid x+4d$. If $3 \nmid \Pi_8$ or $3 \mid x+2d$, then at least one of the equations [0, 3, 6] and [1, 4, 7] is not solvable by Proposition 2.7. If $3 \mid x$, then, by Proposition 2.7, [1, 2, 5], [1, 5, 7] or [1, 3, 5] is not solvable, according to whether $2^3 \mid x+d$, $2^3 \mid x+5d$ or $\mathrm{ord}_2(x+d) = \mathrm{ord}_2(x+5d)$, respectively. If $3 \mid x+d$, then [0, 2, 6], [2, 5, 6] or [2, 4, 6] has no solution, according to whether $2^3 \mid x+2d$, $2^3 \mid x+6d$ or $\mathrm{ord}_2(x+2d) = \mathrm{ord}_2(x+6d)$, respectively. This completes the proof of the $k = 8$ case.

The cases of $k = 9, 10, 11$. In view of $P(b) \leq 7$, (1) implies (3) with $P(a_i) \leq 7$ for each i. Hence we deduce from (1) that

$$\Pi(0, 1, \ldots, k-2) = b_2 y_2^5, \tag{29}$$

where b_2 and y_2 are non-zero integers with $P(b_2) \leq 7$. We can now proceed by induction on k. For $k = 9$, we apply to (29) the results we proved above in the $k = 8$ case and infer that all the solutions of (1) with $d \geq 2$ are given by $d = 2$, $x \in \{-9, -7\}$. For $k = 10$ we obtain in a similar fashion that $d = 2$ and $x = -9$, while for $k = 11$ we do not get any solution for (1).

The cases of $k = 12, 13$. First we suppose that at most one factor, say $x + id$, is divisible by 11. Then $11 \nmid a_i$, and we get (29). Using induction on k again, we infer that in these cases (1) has no solution. If two factors, say $x + id$ and $x + jd$ with $i < j$, are divisible by 11, then we deduce from (1) that

$$\Pi(i+1, \ldots, j-1) = b_3 y_3^5, \tag{30}$$

where $j = i + 11$ and b_3 and y_3 are non-zero integers with $P(b_3) \leq 7$. We can now apply the results we obtained for $k = 10$, and it follows that no new solutions of (1) arise.

The cases of $k \geq 14$. From this point on, it is definitely worth algorithmizing the proof and using a computer. We execute the following tests; as they are rather similar to those employed in case of $n \geq 7$, we use the same notation.

(5, 5, 5)-sieve I–II. We apply the sieves $(\boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n})$**-sieve I** and $(\boldsymbol{n}, \boldsymbol{n}, \boldsymbol{n})$**-sieve II** as in case of $n \geq 7$, but consecutively. As the underlying Propositions 2.4 and 2.5 are valid also for $n = 5$, this can be done without any restrictions.

(5, 5, 5)-sieve III. This is a new sieve. From this point on we work with $\pi(k-1)$-tuples T^* of the same type $(m_{\pi(k-1)}, \ldots, m_1)$ as in '$(\boldsymbol{n}, \boldsymbol{n}, \mathbf{2})$**-sieve I**' in the proof of Theorem 1.2; that is, m_j satisfies (24) for $j \geq 3$ and (25) for $j = 1, 2$. For each such $\pi(k-1)$-tuple T^*, we check whether it is possible to find three terms of the arithmetic progressions under consideration such that their linear combination leads to an equation of the form

$$X^5 + Y^5 = CZ^5$$

with $P(C) \leq 5$. If we can find such terms, then the corresponding $\pi(k-1)$-tuple T^* and the k-tuples $(a_0, a_1, \ldots, a_{k-1})$ can be excluded by Proposition 2.7. (We can easily take care of the cases corresponding to part (i) of the proposition.) If a $\pi(k-1)$-tuple T^* cannot be excluded in this way, we put it into a set S.

Sieve modulo 11. Similarly as in '**Local sieve**', we test all elements of S locally. In this case, we can obviously use the prime 11 because of (28). With the aid of the same method as in the proof of Theorem 1.2, all $\pi(k-1)$-tuples in S and hence all the k-tuples $(a_0, a_1, \ldots, a_{k-1})$ can be excluded; thus the proof is complete. $\square$

Proof of Theorem 1.1. We must prove that for $3 < k < 35$ and $b = 1$, (1) has no solution in positive integers x, d, y and n. Suppose that such a solution exists. By the result of Erdős and Selfridge, we have $d > 1$. Further, as was mentioned earlier, we may assume without loss of generality that n is prime. If $n = 2$ or $n = 3$, then the statement immediately follows from Theorem B or Theorem C, respectively. In case of $n = 5$, Theorem 1.1 is a consequence of Theorem A and Theorem 1.3. Finally, for any prime $n \geq 7$, Theorem A and Theorem 1.2 together imply the assertion. $\square$

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