Perfect Reconstruction FIR Filter Banks: Lapped Transforms, Pseudo QMF's and Paraunitary Matrices

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ABSTRACT

Perfect reconstruction FIR filter banks are analyzed both in z-transform and time-domain. A condition for equal length analysis and synthesis filters is given in terms of orthogonality constraints on overlapping parts of the filters. If the further condition that the filters themselves form an orthonormal basis set is met, one obtains a paraunitary (in z-transform domain) or unitary solution (in time-domain). For the restricted length case of $L = 2N$, solutions are shown to exist (like the lapped orthogonal transform LOT or the pseudo-QMF modulated filters) that allow perfect reconstruction and lend to a fast algorithm implementation. Therefore, there is a large class of computationally efficient perfect reconstruction FIR filter banks where analysis and synthesis have an identical frequency behavior.

I Introduction

Analysis/synthesis systems like the one depicted in fig. 1 and having the perfect reconstruction property can be regarded as generalized transforms where the "window" through which the input signal is seen is larger than the block size. Instead of processing separately adjacent blocks of the signal, such analysis/synthesis systems process overlapping blocks of the signal, thus reducing in part the problems inherent in a block transform scheme. Such analysis/synthesis systems are used in sub-band coding methods [2] for speech and image compression but, regarded as generalized transforms, their use can be much broader [6].

The initial concern on analysis/synthesis systems was focused on aliasing cancellation, since the multirate nature of such systems can lead to undesired aliased versions of the input signal in the output. Later, the attention moved on to perfect reconstruction systems [7,9], and an excellent overview of the subject can be found in [8]. In parallel to the work on filter banks and associated perfect reconstruction analysis/synthesis systems, research on extended transforms lead to the development of the lapped orthogonal transform (LOT) [1,3] which uses a window equal to twice the transform size and guarantees perfect reconstruction. Under the same constraint on the filter length, it turns out that pseudo-QMF filters [4] can lead to perfect reconstruction as well, similarly to time-domain aliasing cancellation in [5].

In the present paper, it is attempted to unify the results from various perfect reconstruction schemes. Up to now, various frameworks have been used, making comparisons and generalizations sometimes difficult. Below, the equivalence of several solutions is demonstrated, as for example between paraunitary filter banks and LOT's, allowing to carry over results and interpretations from one scheme to the other and vice versa. Conditions for equal complexity analysis and synthesis are given, and new schemes, especially in terms of low computational complexity, are indicated.

II Analysis of Filter Banks and Transform Systems

When analyzing filter banks or transform systems, one can use z-transform or time-domain methods. The former leads to matrices of polynomials (in the FIR case) while the latter gives rise to block-circulant or block-Toeplitz matrices. The two approaches are complementary and given the problem at hand, one or the other might be better suited. We assume, following fig. 1, that the analysis/synthesis systems have M channels (the number of channels is equivalent to the transform size) and that the channels are subsampled by N (the step size at which the transform window advances over the signal). The case of most interest appears when critical sampling is used, i.e. the number of channels is equal to the subsampling factor ($M = N$) and therefore the number of sample per unit of time is conserved in the system.

Since we are concerned mainly with perfect reconstruction analysis/synthesis systems where both the analysis and the synthesis filters are FIR with length L_a and L_s respectively, we will specifically look at the following problems:
- Is perfect reconstruction possible?
- Is the complexity of the synthesis equal to the complexity of the analysis?
- Are the synthesis filters identical to the analysis filters (within possible time reversal) ?

This will define classes of solutions as will be shown.

a) z-transform analysis

A filter with z-transform $H_i(z)$ followed by a subsampling by N is best described by its decomposition into polyphase components $H_{i,k}(z^N)$ [2,9].

$$H_i(z) = \sum_{k=0}^{N-1} H_{i,k}(z^N) z^{-k} \tag{1a}$$

$$H_{i,k}(z^N) = \sum_{n=0}^{\infty} h_{i,k+nN} z^{-nN} \tag{1b}$$

where $h_{i,n}$ are the elements of the impulse response of the i-th filter. For example, an unit impulse at time $-k$ will generate an output in the subsampled domain equal to the k-th polyphase component, that is $H_{i,k}(z)$. We define the following polyphase component matrix for the analysis filter bank:

$$\mathbf{H}_p(z) = \begin{pmatrix} H_{0,0}(z) & \dots & H_{0,N-1}(z) \\ H_{1,0}(z) & \dots & H_{1,N-1}(z) \\ \vdots & . & \vdots \\ H_{M-1,0}(z) & \dots & H_{M-1,N-1}(z) \end{pmatrix} \quad (2a)$$

and, with an inversion of the order of the polyphase components, the polyphase matrix for the synthesis filter bank:

$$\mathbf{G}_p(z) = \begin{pmatrix} G_{0,N-1}(z) & \dots & G_{0,0}(z) \\ G_{1,N-1}(z) & \dots & G_{1,0}(z) \\ \vdots & . & \vdots \\ G_{M-1,N-1}(z) & \dots & G_{M-1,0}(z) \end{pmatrix} \quad (2b)$$

It can be verified that a sufficient condition so that the analysis/synthesis system of fig. 1 is a perfect reconstruction system is that [7,8,9]:

$$[\mathbf{G}_p(z)]^T \cdot \mathbf{H}_p(z) = z^{-l} \cdot \mathbf{I} \ . \quad (3)$$

Other solutions are obtained by cyclically permuting the polyphase components and are therefore similar within a delay to the solution in (3) [7]. Note that the delay given by z^{-l} on the right side of (3) is greater or equal to zero if all filters involved are causal. From the input to the output, there is an additional delay of $N-1$ samples due to the multirate nature of the system [9]. The design problem for perfect reconstruction systems is to find pairs of analysis and synthesis filter banks so that (3) is satisfied. Usually, the analysis bank is chosen first and then the synthesis bank is found so as to satisfy perfect reconstruction. Note that invertibility of $\mathbf{H}_p(z)$ is not sufficient, since such an inverse might lead to unstable filters and because one might expect the synthesis filters to be FIR as well (assuming as we do that the analysis filters are FIR). Three classes of perfect FIR reconstruction systems can be defined.

(i) Perfect FIR Reconstruction

The necessary and sufficient condition for FIR perfect reconstruction is that the determinant of $\mathbf{H}_p(z)$ be a monomial [9]. An equivalent statement is that the Smith form of $\mathbf{H}_p(z)$ is a diagonal matrix of increasing delays. Then $\mathbf{G}_p(z)$ can be obtained from the cofactor matrix of $\mathbf{H}_p(z)$ and will yield perfect reconstruction.

(ii)Perfect FIR reconstruction with equal length analysis and synthesis filters.

A sufficient condition for this class will be given in the next section.

(iii) Perfect FIR reconstruction with identical analysis and synthesis filters

Note that the identity is within time reversal. A necessary and sufficient condition for this class is that $\mathbf{H}_p(z)$ satisfies:

$$[\mathbf{H}_p(z^{-1})]^T \cdot \mathbf{H}_p(z) = \mathbf{I} \quad (4)$$

Obviously in this case $\mathbf{G}_p(z)$ can be chosen as:

$$\mathbf{G}_p(z) = z^{-m} \cdot \mathbf{H}_p(z^{-1}) \quad (5)$$

where m is chosen so that $\mathbf{G}_p(z)$ leads to causal synthesis filters, and therefore, (3) is satisfied with $l = m$. Conversely if $\mathbf{G}_p(z)$ satisfies (5), i.e. perfect reconstruction is achieved with identical analysis and synthesis filter, then $\mathbf{H}_p(z)$ satisfies (4). In the case of critical sampling ($M = N$) a matrix $\mathbf{H}_p(z)$ that satisfies (4) is called a paraunitary matrix [7,8] and the product in (4) is commutative since $\mathbf{H}_p(z)$ is square.

Obviously, class (i) contains (ii) which in turn contains (iii). While (iii) is most desirable, it is also most constrained. Actually, we will see that certain design problem do not have a solution in (iii). The other classes have more freedom but also present more problems. In (i), the synthesis filter can be much longer than the analysis filters and in (ii) the synthesis filters can have a somewhat exotic frequency response even if the analysis filters are a perfectly well behaved set of bandpass filters.

b) Time domain analysis

The operation of a subsampled analysis filter bank can be described in the time domain with block-Toeplitz or block-circulant matrices (depending on how the boundaries are treated). Assuming a M-channel filter bank with subsampling by N, then the blocks are of size M by N. The structure of one line of the block matrix is of the form ($\mathbf{A}_i$ being of size M by N):

$$[\mathbf{T}_a]_{bl} = \dots\ 0\ \mathbf{A}_0\ \mathbf{A}_1\ \dots\ \mathbf{A}_{K-1}\ 0\ \dots \quad (6)$$

and describes the transform of the inputs into the subsampled channels (the subscript a stands for analysis and bl for block-line). The reconstruction in the synthesis bank can be described by an inverse transform, again with block structure and having one line as shown below ($\mathbf{B}_i$ is of size M by N):

$$[\mathbf{T}_s]_{bl} = \dots\ 0\ \mathbf{B}_{K'-1}^T\ \dots\ \mathbf{B}_1^T\ \mathbf{B}_0^T\ 0\ \dots \quad (7)$$

where the subscript s stands for synthesis. Now, we can consider the three classes of perfect FIR reconstruction systems in the context of the time-domain analysis. We assume that $\mathbf{T}_a$ in (6) is a banded block-circulant matrix (circular extensions at boundaries). FIR perfect reconstruction means that the left inverse of $\mathbf{T}_a$ is also banded (the inverse is block-circulant), and this corresponds to class (i). Class (ii), equal length analysis and synthesis filters (K = K'), can be achieved by the following sufficient condition:

Condition C1: Orthogonality of overlapping blocks, that is:

$$\sum_{i=0}^{K-1} \mathbf{A}_{i+k}^T \mathbf{A}_i = 0, \quad k = 1, \dots, K-1 \quad (8)$$

is sufficient for perfect FIR reconstruction with equal length filters given that there are N linearly independent analysis filters; this condition is sometimes called "Orthogonality of the tails of the transform" [1,3].

Proof: the product $\mathbf{T}_a^T \cdot \mathbf{T}_a$ is block diagonal because of (8) and the diagonal element is a matrix of size N by N equal to:

$$\mathbf{T}_d = \sum_{i=0}^{K-1} \mathbf{A}_i^T \cdot \mathbf{A}_i \quad (9)$$

Since there are N independent filters, $\mathbf{T}_d$ has rank N and can thus be inverted. Then, the synthesis matrix $\mathbf{T}_s$ has block lines of the form:

$$[\mathbf{T}_s]_{bl} = \ldots\ 0\ T_d^{-1} \cdot A_{K'-1}^T\ \ldots\ T_d^{-1} \cdot A_0^T\ \ldots \qquad (10)$$

The condition of orthogonality of the tails is not necessary, because the assumption that the central term of the matrix product (corresponding to (9) above) is different from zero and all others are zero is too restrictive. When the N by N matrix $\mathbf{T}_d$ equals indentity, we have the property

$$\mathbf{T}_a^T \cdot \mathbf{T}_a = I \qquad (11)$$

and the synthesis filter are equivalent to the analysis filters, and we have a class (iii) solution. When the analysis/synthesis is critically sampled ($M = N$), it means that the filters of the bank are mutually orthogonal and of norm 1, and therefore, $\mathbf{T}_a$ is a unitary matrix.

c) Relation between z-transform and time-domain analysis.

In order to see the exact relationship between the two descriptions, it is convenient to express $\mathbf{H}_p(z)$ and $\mathbf{G}_p(z)$ as polynomials with matrix coefficients (rather than matrices with polynomial coefficients). We will call this the "sum form" of the polyphase filter matrices.

$$\mathbf{H}_p(z) = \sum_{i=0}^{K-1} \mathbf{H}_{pi} \cdot z^{-i} \qquad (12)$$

and a similar relation for $\mathbf{G}_p(z)$ with matrices $\mathbf{G}_{pi}$. The values of the matrices $\mathbf{H}_{pi}$ and $\mathbf{G}_{pi}$ follow from (2) by inspection. Because of the time-reversal inherent to the convolution, it is easy to verify that:

$$\mathbf{A}_{N-K-1} = \mathbf{H}_{pk} \cdot \mathbf{J}_N, \quad \mathbf{B}_k = \mathbf{G}_{pk} \cdot \mathbf{J}_N \qquad (13)$$

where $\mathbf{J}_N$ is the antidiagonal matrix of size N by N. As a simple example, take the case of the block transform of size N by N that advances by N samples at a time over the signal. The matrix $\mathbf{T}_a$ in (6) is therefore block diagonal with block $\mathbf{A}_0$ of size N by N. Assume that $\mathbf{A}_0$ is a unitary transform, then perfect reconstruction is achieved with $\mathbf{A}_0 = \mathbf{B}_0$ and we have $\mathbf{T}_a^{-1} = \mathbf{T}_a^T$. In terms of z-transform analysis, we have (from (13)) and since $\mathbf{J}_N^2 = \mathbf{I}$:

$$\mathbf{H}_p(z) = \mathbf{A}_0 \cdot \mathbf{J}_N, \quad \mathbf{G}_p(z) = \mathbf{A}_0 \cdot \mathbf{J}_N \qquad (14)$$

Therefore replacing (14) in (3), we get

$$\mathbf{J}_N \cdot \mathbf{A}_0^T \cdot \mathbf{A}_0 \cdot \mathbf{J}_N = \mathbf{I} \qquad (15)$$

and perfect reconstruction is verified. Note that this simple example shows also that block transform methods are a particular case of analysis/synthesis systems with filter length, number of channels and subsampling factors all equal to N (see fig. 2). Now that the relationship between z-transform and time-domain analysis has been shown, we can draw some parallels between the two representations. First, what is the meaning of condition C1 (orthogonality of the tails) in the z-transform domain. Let us consider the product:

$$[\mathbf{H}_p(z^{-1})]^T \cdot \mathbf{H}_p(z) = \sum_{i=0}^{K-1} \mathbf{H}_{pi}^T \cdot \mathbf{H}_{pi} = \mathbf{T}'_d \qquad (16)$$

This is because of (8), the orthogonality of the tails. Note that $\mathbf{T}'_d = \mathbf{J}_N \cdot \mathbf{T}_d \cdot \mathbf{J}_N$ following (9) and (13). Therefore, choosing $\mathbf{G}_p(z)$ as:

$$\mathbf{G}_p(z) = z^{-K+1} \cdot \mathbf{H}_p(z^{-1}) \cdot [\mathbf{T}'_d]^{-1} \qquad (17)$$

will yield perfect reconstruction and defines thus the class (ii) solutions in the z-transform domain. Assume now that $\mathbf{T}'_d = \mathbf{T}_d = \mathbf{I}$, then (16) means that $\mathbf{H}_p(z)$ is paraunitary (see (4)). Therefore, the condition C1 together with $\mathbf{T}_d = \mathbf{I}$ is equivalent with the fact that $\mathbf{H}_p(z)$ is paraunitary. From (11) and (16), we have the equivalence:

$$\mathbf{T}_a^T \cdot \mathbf{T}_a = \mathbf{I} \iff [\mathbf{H}_p(z^{-1})]^T \cdot \mathbf{H}_p(z) = \mathbf{I} \qquad (18)$$

Note that in the critically sampled case ($M = N$), all matrices are square and left inverses are also right inverses. Then, paraunitariness of $\mathbf{H}_p(z)$ is equivalent to $\mathbf{T}_a$ being a unitary matrix.

III Solutions in the restricted length case $L_a = 2N$

When the analysis filters are restricted to $L_a = 2N$ (that is, the analysis window is twice the step size), the filter design problem is simplified. Furthermore, this case is simple enough and should therefore illustrate some of the results from the previous section. Only the critically sampled case ($M = N$) will be considered, and same length analysis and synthesis filters are desired ($L_a = L_b = 2N$).

a) Analysis of the case $L_a = 2N$

From (6) and (7), we see that $\mathbf{T}_s \cdot \mathbf{T}_a = \mathbf{I}$ can be met by satisfying:

$$\mathbf{B}_0^T \cdot \mathbf{A}_0 + \mathbf{B}_1^T \cdot \mathbf{A}_1 = \mathbf{I} \qquad (19a)$$

$$\mathbf{B}_0^T \cdot \mathbf{A}_1 = \mathbf{B}_1^T \cdot \mathbf{A}_0 = 0 \qquad (19b)$$

Orthogonality of the overlapping blocks of the analysis filters, that is, $\mathbf{A}_0^T \cdot \mathbf{A}_1 = \mathbf{0}$, is sufficient to satisfy (19b) since one can choose $\mathbf{B}_i = \mathbf{A}_i \cdot [\mathbf{A}_0^T\mathbf{A}_0 + \mathbf{A}_1^T\mathbf{A}_1]^{-T}$ and achieve perfect reconstruction. Now, if the normalization equation is also satisfied, that is:

$$\mathbf{A}_0^T \cdot \mathbf{A}_0 + \mathbf{A}_1^T \cdot \mathbf{A}_1 = \mathbf{I} \qquad (20)$$

then obviously, $\mathbf{B}_i$ has to be chosen equal to $\mathbf{A}_i$ and we have a paraunitary solution. In that case, the matrix product in (4) is commutative (recall that the system is critically sampled and that therefore $\mathbf{H}_p(z)$ is of size N by N):

$$[\mathbf{H}_p(z^{-1})]^T \cdot \mathbf{H}_p(z) = \mathbf{H}_p(z) \cdot [\mathbf{H}_p(z^{-1})]^T = \mathbf{I} \qquad (21)$$

From (21) it follows that the following two relations hold simultaneously:

$$\mathbf{A}_0^T \cdot \mathbf{A}_0 + \mathbf{A}_1^T \cdot \mathbf{A}_1 = \mathbf{I}, \quad \mathbf{A}_0^T \cdot \mathbf{A}_1 = \mathbf{0} \qquad (22)$$

$$\mathbf{A}_0 \cdot \mathbf{A}_0^T + \mathbf{A}_1 \cdot \mathbf{A}_1^T = \mathbf{I}, \quad \mathbf{A}_1 \cdot \mathbf{A}_0^T = \mathbf{0} \qquad (23)$$

Note that (22) and (23) are absolutely equivalent. The orthogonality condition in (22) means that the N columns of $\mathbf{A}_0$ are each orthogonal to the N columns of $\mathbf{A}_1$, while (23) means the same for the rows of $\mathbf{A}_0$ and $\mathbf{A}_1$. The normalization in (23) means that the N filters form a size N orthogonal basis set.

b) Lapped orthogonal transforms (LOT)

Lapped orthogonal transforms have been introduced in [1] and further investigated in [3]. They are essentially obtained by condition (23) and the additional constraints that the number of channels is even and that the filters have linear phase ($N/2$ symmetric and $N/2$ antisymmetric filters). In that sense, LOT's are a sub-class of paraunitary solutions (linear phase and length constraint). Note that condition C1 together with $\mathbf{T}_d = \mathbf{I}$ is the extension of the LOT concept to arbitrary filter lengths.

Design techniques have been developed for the LOT in the case $L = 2N$ [1,3], but they do not generalize well for lengths greater than $2N$ (for example, some dependencies which are linear for $L_a = 2N$ become non-linear for longer filters). From a computational point of view, the LOT can be based on a fast transform (a DCT typically) [3], thus making the LOT computationally very efficient (two fast transforms and $N/2$ rotations as a typical case). It is interesting to note that two techniques that have been developed independently, namely perfect reconstruction FIR filter banks and lapped orthogonal transforms, lead to the same solution characterized by paraunitary (in z-transform domain) or unitary (in time domain) matrices.

c)Modulated filter bank with perfect reconstruction.

Pseudo-QMF filters (PQMF) have been proposed as an extension to N channels of the classical two-band QMF filters (e.g. [4]). PQMF analysis/synthesis systems achieve in general only the cancellation of the main aliasing term. However, when the filter length is restricted to $L_a = 2N$, they can achieve perfect reconstruction under certain conditions. The main advantages of PQMF filters are their low computational complexity as well as the fact that the window function can be tuned to satisfy additional design constraints while maintaining the exact reconstruction property.

Assume that N is even and that critical subsampling is used. Then the i-th analysis and synthesis filters are obtained by modulating a real prototype filter evenly over the frequency spectrum. While the modulating frequencies are easily obtained from the fact that the whole spectrum has to be covered with real filters, the selection of a phase term for the analysis and the synthesis is more delicate. A family of PQMF filter bank that achieves main aliasing cancellation has been designed in [4] and is of the form:

$$h_k(n) = h_{pr}(n) \cdot cos\left(\frac{2\pi(2k+1)}{4N}\left(n - \left(\frac{L-1}{2}\right)\right) + \phi_k\right) \tag{24}$$

for the analysis filter ($h_{pr}(n)$ is the impulse response of the prototype filter). In the general case the main aliasing term is canceled for the value of the phase:

$$\phi_k = \frac{\pi}{4} + k\frac{\pi}{2} \tag{25}$$

and it can be shown that this property also holds for any "non singular" value of the prototype filter. The synthesis filters have the same modulation but with a negative phase term equal to $-\frac{\pi}{4} - k\frac{\pi}{2}$.

In the case $L_a = 2N$ and assuming that $h_{pr}(n) = 1, n = 0, \ldots, 2N-1$, it can be verified that eq. (19) and (22) hold, leading to a paraunitary solution (class(iii)). The matrices $\mathbf{A}_0$ and $\mathbf{A}_1$ of the unwindowed filter bank satisfy the relation

$$\mathbf{A}_0^T \cdot \mathbf{A}_1 = 0, \quad \mathbf{A}_0^T \cdot \mathbf{A}_0 + \mathbf{A}_1^T \cdot \mathbf{A}_1 = \mathbf{I} \tag{26a}$$

$$\mathbf{A}_0^T \cdot \mathbf{A}_0 = 1/2 \cdot (\mathbf{I} + \mathbf{J}), \quad \mathbf{A}_1^T \cdot \mathbf{A}_1 = 1/2 \cdot (\mathbf{I} - \mathbf{J}) \tag{26b}$$

While the two first relation are common to any $L_a = 2N$ class (iii) solution, the relation (26b) is particular to the modulated filter bank. The property that a filter bank derived by a modulation of a window function can lead to exact reconstruction in the case $L_a = 2N$ had appeared in a in a slightly different context [5]. What condition has the window $h_{pr}(n)$ to satisfy for exact reconstruction and paraunitariness to be preserved?. Assume that a symmetric window function or prototype filter is used. The new matrices $\mathbf{A}_0'$ and $\mathbf{A}_1'$ are given by:

$$\mathbf{A}_0' = \mathbf{A}_0 \cdot \mathbf{W}, \quad \mathbf{A}_1' = \mathbf{A}_1 \cdot \mathbf{W}' \tag{27}$$

Where $\mathbf{W}$ is a diagonal windowing matrix given by:

$$\mathbf{W} = Diag[h_{pr}(0),\ h_{pr}(1),\ \ldots h_{pr}(N/2-1)] \tag{28}$$

and $\mathbf{W}' = \mathbf{J}_N \cdot \mathbf{W} \cdot \mathbf{J}_N$ has the diagonal element of $\mathbf{W}$ in reverse order. We can readily verify that the condition $\mathbf{A}_0'^T \cdot \mathbf{A}_1' = 0$ is verified since

$$\mathbf{A}_0'^T \cdot \mathbf{A}_1' = \mathbf{W} \cdot \mathbf{A}_0^T \cdot \mathbf{A}_1 \cdot \mathbf{J}_N \cdot \mathbf{W} \cdot \mathbf{J}_N = 0. \tag{29}$$

follows from (26a). If we compute the product $\mathbf{A}_0'^T \cdot \mathbf{A}_0' + \mathbf{A}_1'^T \cdot \mathbf{A}_1'$ and take into account eq.(26b) we find that:

$$\mathbf{A}_0'^T \cdot \mathbf{A}_0' + \mathbf{A}_1'^T \cdot \mathbf{A}_1' = \mathbf{W}^2 + \mathbf{W}'^2. \tag{30}$$

The matrix $\mathbf{W}^2 + \mathbf{W}'^2$ is a diagonal matrix with the i-th diagonal element of the form

$$h_{pr}^2(i) + h_{pr}^2(N/2 - 1 - i) \tag{31}$$

This allows us to state the following result:

Result: The windowed modulated filter bank allows exact reconstruction in the case $L_a = 2N$ if and only if the matrix $\mathbf{W}^2 + \mathbf{W}'^2$ is non singular, furthermore if the window meets the condition

$$h_{pr}^2(i) + h_{pr}^2(N/2 - 1 - i) = 1 \tag{32}$$

then the solution is paraunitary. Some discussion of the windowed PQMF scheme seems appropriate; first, we note that windowing will never destroy the orthogonality of overlapping blocks (the columns of $\mathbf{A}_0$ and $\mathbf{A}_1$ are simply weighted by the window function, thus conserving their mutual orthogonality). Therefore exact reconstruction with filters of the same length is conserved over windowing. The degree of freedom introduced by the window allows to trade off frequency domain and time domain properties of the filter bank, thus optimizing the design for a given application.

It is the particular form of the products $\mathbf{A}_i^T \cdot \mathbf{A}_i$ in (26b) that provides the "window independence". For the sake of illustration, we indicate an example on how to generate matrices $\mathbf{A}_i$ that will satisfy (26b). Assume an orthogonal basis of size N given by the vectors $V_0\ V_1\ \cdots\ V_{N-1}$, then choose:

$$\mathbf{A}_0 = [V_0\ V_1\ \cdots\ V_{\frac{N}{2}-1}\ V_{\frac{N}{2}-1} \cdots\ V_1\ V_0] \tag{33a}$$

$$\mathbf{A}_1 = [V_{\frac{N}{2}}\ V_{\frac{N}{2}+1}\ \cdots\ V_{N-1}\ -V_{N-1} \cdots -V_{\frac{N}{2}+1}\ -V_{\frac{N}{2}}] \tag{33b}$$

Then (26) is automatically satisfied. The PQMF filter bank is a particular case of (33). Unfortunately such a technique to design the $L_a = 2N$ filter bank excludes linear phase solutions. The condition on the window given by eq.(32) is quite important, if it does not hold, the synthesis filter bank will be given by:

$$\mathbf{B}_i = \mathbf{A}_i \cdot (\mathbf{W}^2 + \mathbf{W}'^2)^{-1} \qquad (34)$$

The inverse which has to be applied at the synthesis might deemphasize any benefits introduced by the window.

IV Conclusion

After having set up the analysis framework both in z-transform and time-domain, we have shown conditions for perfect FIR reconstruction that guarantee also certain constraints on the synthesis filters (equal complexity, same amplitude frequency response as the analysis filters). The concept of orthogonal overlaps has been shown to be powerful in that context, and the equivalence of unitary time-domain matrices and paraunitary z-transform domain matrices has been demonstrated. In the restricted length case ($L = 2N$), it has been shown that several schemes exist that achieve both perfect reconstruction and low computational complexity, as for example the modulated PQMF filter banks. Note that the focus here was on "sum" forms of polyphase matrices (see equ.(12)). Another approach uses "product" forms instead [8] and a companion paper [10] describes results on perfect reconstruction FIR filter banks where the filters are constrained to some symmetry conditions.

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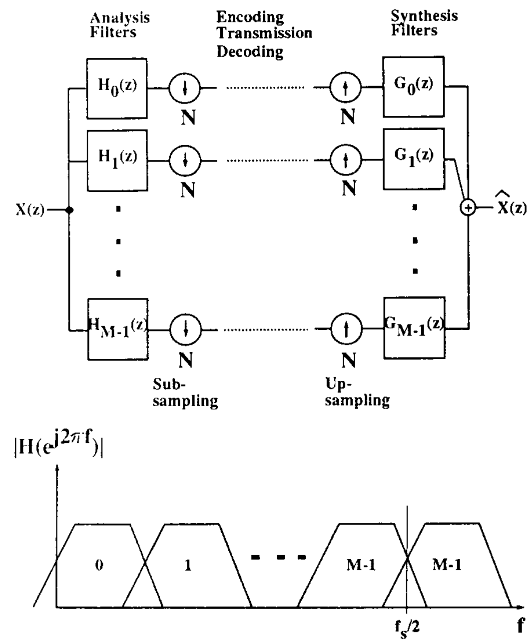

Fig. 1: Analysis/synthesis system with M channels and subsampling by N, as well as typical frequency responses of the filters.

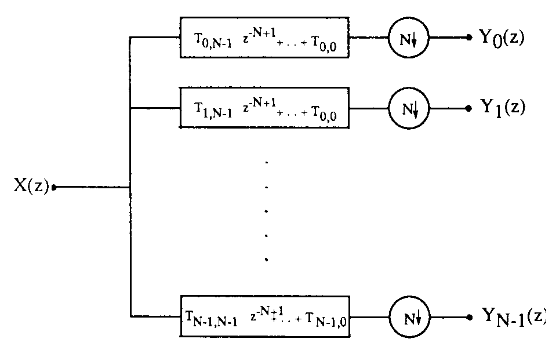

Fig. 2: Size-N block transform interpretation as a filter bank with N filters of length-N subsampled by N.