COLLOQUIUM MATHEMATICUM

VOL. 107 2007 NO. 1

PERFECTNESS OF THE HIGSON AND SMIRNOV COMPACTIFICATIONS

BY

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Abstract. We provide a necessary and sufficient condition for the Higson compactification to be perfect for the noncompact, locally connected, proper metric spaces. We also discuss perfectness of the Smirnov compactification.

1. Introduction. The aim of this paper is to study perfectness of the Higson and Smirnov compactifications. We follow the notation and terminology of [4].

The notion of perfect compactification was first introduced by E. G. Sklyarenko. A compactification αX of a completely regular space X is said to be *perfect* if the natural projection $f : \beta X \to \alpha X$ is monotone, where βX is the Stone–Čech compactification of X. Note that the natural projection $f : \beta X \to \alpha X$ is monotone if and only if $f|_{\beta X \setminus X} : \beta X \setminus X \to \alpha X \setminus X$ is monotone. E. G. Sklyarenko showed the following basic facts for perfect compactifications (see [8, Theorems 30.8 and 30.10] or [10, Theorems 1 and 2]).

PROPOSITION 1.1. *Let αX be a compactification of a completely regular space X. Then the following conditions are equivalent*:

(1) *αX is a perfect compactification of X.*
(2) *For every open subset U of X and every $A \subset U$, $\operatorname{Cl}_{\alpha X} A \cap \operatorname{Cl}_{\alpha X} \operatorname{Fr}_X U = \emptyset$ is equivalent to $\operatorname{Cl}_{\alpha X} A \cap \operatorname{Cl}_{\alpha X}(X \setminus U) = \emptyset$.*
(3) *For every open set U of X, $\operatorname{Cl}_{\alpha X} \operatorname{Fr}_X U = \operatorname{Fr}_{\alpha X} \operatorname{Ext}_{\alpha X} U$, where $\operatorname{Ext}_{\alpha X} U = \alpha X \setminus \operatorname{Cl}_{\alpha X}(X \setminus U)$.*

By definition, the Stone–Čech compactification of a completely regular space is perfect. Furthermore, it is known that the one-point compactifica-

2000 *Mathematics Subject Classification*: Primary 54D05, 54D35; Secondary 54D40, 54D45.

Key words and phrases: Higson compactification, Smirnov compactification, perfect compactification.

This research was done while Yuji Akaike and Naotsugu Chinen were visiting Miyakonojo National College of Technology in August, 2005. An earlier version of this paper was presented at Sūrikaisekikenkyūsho at Kyoto University, October 11, 2005.

tion of the n-dimensional Euclidean space $\mathbb{R}^n$ is perfect if and only if $n > 1$ (cf. [5]).

In the next section, we investigate perfectness of the Higson compactification. Here, the Higson compactification is defined for all proper metric spaces and is metric-dependent. A metric d on a space X is said to be *proper* if for every $r > 0$, $\mathrm{Cl}_X B_r(x,d)$ is compact, where $B_r(x,d) = \{y \in X : d(x,y) < r\}$. The Higson compactification of the proper metric space (X,d) is denoted by $\overline{X}^d$ and is the unique compactification associated with the set of all slowly oscillating bounded continuous real-valued functions on (X,d). See [2] and [7] for details. In this paper, we introduce the notion of coarse uniform connectedness at ∞, and show that $\overline{X}^d$ is perfect for every proper metric space (X,d) with this property. By virtue of this, $\overline{\mathbb{R}^n}^{d_n}$ is perfect for each n-dimensional Euclidean space $\mathbb{R}^n$ with the usual metric d_n. Moreover, we show that for every locally connected proper metric space (X,d), $\overline{X}^d$ is perfect if and only if X is coarsely uniformly connected at ∞.

In the last section, we investigate perfectness of the Smirnov compactification. The Smirnov compactification is defined for all metric spaces and is metric-dependent. The Smirnov compactification of the metric space (X,d) is denoted by $u_d X$ and is the unique compactification associated with the set of all bounded uniformly continuous real-valued functions on (X,d). See [8] or [11] for details. In [1], M. G. Charalambous showed that for every convex subset X of an arbitrary normed linear space with the subspace metric d, $u_d X$ is perfect. His result is a simple solution for a question raised in R. G. Woods' paper [11]. As a result, $u_{d_n}\mathbb{R}^n$ is a perfect compactification of $(\mathbb{R}^n, d_n)$ for each $n \in \mathbb{N}$. In this paper, we introduce the notion of uniform local connectedness at ∞, and show that $u_d X$ is perfect for every metric space (X,d) with this property. Moreover, we show that for every noncompact, locally connected, proper metric space (X,d), $u_d X$ is perfect if and only if X is uniformly locally connected at ∞.

2. Perfectness of the Higson compactification. The following characterization was proved by A. N. Dranishnikov, J. Keesling and V. V. Uspenskij in [2, Proposition 2.3].

PROPOSITION 2.1. *Let X be a noncompact metric space with a proper metric d. Then the following conditions are equivalent*:

(1) *A compactification αX of X is equivalent to $\overline{X}^d$.*
(2) *For disjoint closed sets $A, B \subset X$, $\mathrm{Cl}_{\alpha X} A \cap \mathrm{Cl}_{\alpha X} B = \emptyset$ if and only if the system $\{A, B\}$ diverges, i.e., for any $n \in \mathbb{N}$ there exists a compact subset K_n of X such that $d(x,A) + d(x,B) > n$ for all $x \in X \setminus K_n$.*

Now, we introduce the following definitions.

DEFINITION 2.2.

(1) (Cf. [3, p. 206].) A metric space (X, d) is *coarsely uniformly connected* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any two points $x, y \in X$ with $d(x, y) < \varepsilon$ there exists a connected set P in X satisfying $x, y \in P$ and $\operatorname{diam} P < \delta$.
(2) A metric space (X, d) is *coarsely uniformly connected at* ∞ if for any $\varepsilon > 0$ there exist a $\delta > 0$ and a compact set K in X such that for any two points $x, y \in X \smallsetminus K$ with $d(x, y) < \varepsilon$ there exists a connected set P in X satisfying $x, y \in P$ and $\operatorname{diam} P < \delta$.

The following theorem provides a sufficient condition for the Higson compactification $\overline{X}^d$ of the proper metric space (X, d) to be perfect.

THEOREM 2.3. *If a proper metric space* (X, d) *is coarsely uniformly connected at* ∞, *then* $\overline{X}^d$ *is a perfect compactification of* (X, d).

Proof. For brevity, let $Y = \overline{X}^d$. Let U be an open subset of X. Put $W_0 = \operatorname{Cl}_X U$, $W_1 = X \smallsetminus U$, and $V = X \smallsetminus W_0$. Since the inclusion $\operatorname{Cl}_Y \operatorname{Fr}_X U \subset \operatorname{Fr}_Y \operatorname{Ext}_Y U$ always holds, it suffices to show that $\operatorname{Cl}_Y \operatorname{Fr}_X U \supset \operatorname{Fr}_Y \operatorname{Ext}_Y U$.

If either W_0 or W_1 is compact, then $\operatorname{Fr}_Y \operatorname{Ext}_Y U = \operatorname{Fr}_X U = \operatorname{Cl}_Y \operatorname{Fr}_X U$. Thus, we may assume that both W_0 and W_1 are not compact. First, we show the following claim:

CLAIM. $\operatorname{Cl}_Y(W_0 \cap W_1) = \operatorname{Cl}_Y W_0 \cap \operatorname{Cl}_Y W_1$.

We notice that $\operatorname{Cl}_Y(W_0 \cap W_1) \subset \operatorname{Cl}_Y W_0 \cap \operatorname{Cl}_Y W_1$ always holds. Suppose that there exists a $p \in \operatorname{Cl}_Y W_0 \cap \operatorname{Cl}_Y W_1 \smallsetminus \operatorname{Cl}_Y(W_0 \cap W_1)$. Since Y is normal, there exists a closed neighborhood S of p in Y with $S \cap \operatorname{Cl}_Y(W_0 \cap W_1) = \emptyset$ and $S \cap W_i \neq \emptyset$ for $i = 0, 1$. Since $S \cap (W_0 \cap W_1) = \emptyset$ and $X \smallsetminus (W_0 \cap W_1) = U \cup V$, we obtain $S \cap X \subset U \cup V$. Thus $W_0 \cap S \subset (U \cup V) \cap \operatorname{Cl}_X U = U \cup (V \cap \operatorname{Cl}_X U)$ $= U$. Similarly, $W_1 \cap S \subset V$. Here, we note that $\operatorname{Cl}_Y(W_0 \cap S) \cap \operatorname{Cl}_Y(W_1 \cap S)$ $\neq \emptyset$ because $p \in \operatorname{Cl}_Y(W_i \cap S)$ for $i = 0, 1$. By Proposition 2.1, $\{W_0 \cap S, W_1 \cap S\}$ does not diverge. Hence, there exist an $\varepsilon > 0$, a compact cover $\{K_n\}_{n \in \mathbb{N}}$ with $K_n \subset K_{n+1}$, and an $x_n \in X \smallsetminus \operatorname{Cl}_X B_\varepsilon(K_n, d)$ such that $d(x_n, W_0 \cap S) + d(x_n, W_1 \cap S) < \varepsilon$ for each $n \in \mathbb{N}$. Since d is proper, for every $n \in \mathbb{N}$ there exists a $w_{i,n} \in W_i \cap S$ for $i = 0, 1$ such that $d(x_n, W_0 \cap S) + d(x_n, W_1 \cap S) = d(x_n, w_{0,n}) + d(x_n, w_{1,n})$. Thus, we obtain sequences $\{w_{0,n}\}_{n \in \mathbb{N}} \subset W_0 \cap S \subset U$ and $\{w_{1,n}\}_{n \in \mathbb{N}} \subset W_1 \cap S \subset V$ such that $d(w_{0,n}, w_{1,n}) < \varepsilon$ and $w_{0,n}, w_{1,n} \notin K_n$ for each $n \in \mathbb{N}$. Since (X, d) is coarsely uniformly connected at ∞, there exist a $\delta > 0$, an $l \in \mathbb{N}$, and a connected set P_n joining $w_{0,n}$ and $w_{1,n}$ in X with $\operatorname{diam} P_n < \delta$ for each $n \geq l$. Since $U \cup V$ is not connected, we see that $P_n \cap (X \smallsetminus (U \cup V)) \neq \emptyset$ for each $n \geq l$. Hence, $P_n \cap (W_0 \cap W_1) \neq \emptyset$ because $X \smallsetminus (U \cup V) = W_0 \cap W_1$. Take a $y_n \in P_n \cap (W_0 \cap W_1)$ for each $n \geq l$.

Since $\operatorname{diam} P_n < \delta$ for each $n \geq l$, we see that $d(w_{i,n}, y_n) < \delta$ for $i = 0, 1$ and each $n \geq l$. Thus, $\{W_0 \cap W_1, W_i \cap S\}$ does not diverge for $i = 0, 1$. This implies that $\mathrm{Cl}_Y(W_0 \cap W_1) \cap \mathrm{Cl}_Y(W_i \cap S) \neq \emptyset$, and thus we obtain $\mathrm{Cl}_Y(W_0 \cap W_1) \cap S \neq \emptyset$. This is a contradiction which completes the proof of the claim.

Now, we note that $\mathrm{Ext}_Y U = Y \setminus \mathrm{Cl}_Y W_1$ and $\mathrm{Ext}_Y V = Y \setminus \mathrm{Cl}_Y W_0$. From the claim above, we have $Y \setminus (\mathrm{Ext}_Y U \cup \mathrm{Ext}_Y V) = \mathrm{Cl}_Y(W_0 \cap W_1) = \mathrm{Cl}_Y \mathrm{Fr}_X U$. So $Y = \mathrm{Ext}_Y U \cup \mathrm{Ext}_Y V \cup \mathrm{Cl}_Y \mathrm{Fr}_X U$ and $\mathrm{Ext}_Y U \cap \mathrm{Ext}_Y V = \emptyset$. Here, take an $x \in \mathrm{Fr}_Y \mathrm{Ext}_Y U$. Since $\mathrm{Ext}_Y U \cap \mathrm{Ext}_Y V = \emptyset$, we note that neither $x \in \mathrm{Ext}_Y U$ nor $x \in \mathrm{Ext}_Y V$. We then see that $x \in \mathrm{Cl}_Y \mathrm{Fr}_X U$. Therefore, we conclude that $\overline{X}^d$ is perfect. ■

By Theorem 2.3, we have the following corollary.

Corollary 2.4. *$\overline{\mathbb{R}^n}^{d_n}$ is a perfect compactification of the n-dimensional Euclidean space $(\mathbb{R}^n, d_n)$ for each $n \in \mathbb{N}$.*

It is known that the Stone–Čech compactification βX of a completely regular space X is a perfect compactification of X. Here, by Theorem 2.3, we provide an alternative proof of this fact for a countable discrete space.

Corollary 2.5. *Let X be a noncompact, locally compact, separable metrizable space. If the set of nonisolated points of X is compact, then there exists a proper metric d on X such that $\overline{X}^d$ is a perfect compactification of (X, d) equivalent to βX.*

Proof. By [6, Proposition 2.6], there exists a proper metric d on X such that for every $r > 0$, there exists a compact subset $K_r \subset X$ with $X \setminus K_r$ being r-discrete, i.e., $B_r(x, d) \cap (X \setminus K_r) = \{x\}$ for all $x \in X \setminus K_r$. This shows that (X, d) is coarsely uniformly connected at ∞. By Theorem 2.3, we note that $\overline{X}^d$ is perfect. Also, every bounded continuous real-valued function on (X, d) is slowly oscillating, so $\overline{X}^d$ is equivalent to βX. ■

The main result of this section is the following result.

Theorem 2.6. *Let (X, d) be a noncompact, locally connected, proper metric space. Then $\overline{X}^d$ is a perfect compactification of (X, d) if and only if (X, d) is coarsely uniformly connected at ∞.*

Proof. Suppose that X is not coarsely uniformly connected at ∞. By Theorem 2.3, it suffices to show that $\overline{X}^d$ is not perfect.

By assumption there exist a strictly increasing compact cover $\{K_n\}_{n \in \mathbb{N}}$ and an $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ we have two points $x_n, y_n \in X \setminus K_n$ satisfying $d(x_n, y_n) < \varepsilon$ and $\operatorname{diam} P \geq n$ for all connected subsets P joining x_n and y_n. Furthermore, we may assume that $B_n(\{x_n, y_n\}, d) \subset K_{n+1} \setminus K_n$ and $d(K_n \cup \mathrm{Cl}_X B_n(\{x_n, y_n\}, d), \mathrm{Cl}_X(X \setminus K_{n+1})) > n$ for each $n \in \mathbb{N}$. Let C_n

be the component of $B_{n/3}(x_n, d)$ containing x_n. We see that $U = \bigcup_{n\in\mathbb{N}} C_n$ contains no y_n for all $n \in \mathbb{N}$. Since X is locally connected, C_n is open for each $n \in \mathbb{N}$. Then U is open in X and $\mathrm{Fr}_X U = \bigcup_{n\in\mathbb{N}} \mathrm{Fr}_X C_n$ because the family $\{\mathrm{Fr}_X C_n : n \in \mathbb{N}\}$ is discrete. Let $A = \{x_n : n \in \mathbb{N}\}$. Note that $\{A, X \setminus U\}$ does not diverge because $\{A, \{y_n\}_{n\in\mathbb{N}}\}$ does not diverge. By virtue of Proposition 2.1, we have $\mathrm{Cl}_{\overline{X}^d} A \cap \mathrm{Cl}_{\overline{X}^d}(X \setminus U) \neq \emptyset$.

We are going to show that $\mathrm{Cl}_{\overline{X}^d} A \cap \mathrm{Cl}_{\overline{X}^d} \mathrm{Fr}_X U = \emptyset$, thus completing the proof by Proposition 1.1. Suppose first that there exists an $l \in \mathbb{N}$ such that $\mathrm{Fr}_X C_n = \emptyset$ for each $n \geq l$. Since $\mathrm{Fr}_X U = \bigcup_{n<l} \mathrm{Fr}_X C_n$, we have $\mathrm{Cl}_{\overline{X}^d} A \cap \mathrm{Cl}_{\overline{X}^d} \mathrm{Fr}_X U = \mathrm{Cl}_{\overline{X}^d} A \cap \mathrm{Fr}_X U = \emptyset$. Hence, we may assume that $\mathrm{Fr}_X C_n \neq \emptyset$ for all $n \in \mathbb{N}$. By Propositions 1.1 and 2.1 it suffices to show that $\{A, \mathrm{Fr}_X U\}$ diverges. It follows from the local connectedness of X that $\mathrm{Fr}_X C_n \subset \{y \in X : d(x_n, y) = n/3\}$. Thus, $d(A, \mathrm{Fr}_X C_n) \geq n/3$ for each $n \in \mathbb{N}$. Now, let $x \in X \setminus K_{3n+1}$. If $x \notin \mathrm{Cl}_X \bigcup_{k\in\mathbb{N}} B_{k/3}(x_k, d)$, then

$$\begin{aligned} d(x, A) &+ d(x, \mathrm{Fr}_X U) \\ &\geq d(x, A) \\ &= \min\{\inf\{d(x, x_k) : k \geq 3n\}, \inf\{d(x, x_k) : k \leq 3n-1\}\} > n. \end{aligned}$$

If $x \in \mathrm{Cl}_X \bigcup_{k\in\mathbb{N}} B_{k/3}(x_k, d)$, then, since there exists an $m > 3n+1$ such that $x \in \mathrm{Cl}_X B_{m/3}(x_m, d)$, we have

$$d(x, A) + d(x, \mathrm{Fr}_X U) = d(x, x_m) + d(x, \mathrm{Fr}_X C_m) \geq d(x_m, \mathrm{Fr}_X C_m) \geq \frac{m}{3} > n.$$

This indicates that $\{A, \mathrm{Fr}_X U\}$ diverges. ∎

We note that every coarsely uniformly connected metric space is connected and is coarsely uniformly connected at ∞.

LEMMA 2.7. *Let (X, d) be a locally connected, proper metric space. Then (X, d) is connected and coarsely uniformly connected at ∞ if and only if it is coarsely uniformly connected.*

Proof. Assume that (X, d) is connected and coarsely uniformly connected at ∞. Let $\varepsilon > 0$. There exist a compact set K of X and a $\delta_0 > 0$ such that for any $x, y \in X \setminus K$ with $d(x, y) < \varepsilon$ there exists a connected set P satisfying $x, y \in P$ and $\operatorname{diam} P < \delta_0$. Since (X, d) is a locally connected, proper metric space, there exist continua $K_1, \ldots, K_n$ such that $\mathrm{Cl}_X B_\varepsilon(K, d) \subset \bigcup_{i=1}^n K_i$.

Fix a point $z \in X \setminus K$. Let

$$L(z) = \{p \in X : p \text{ and } z \text{ belong to a continuum in } X\}.$$

Since X is locally connected and locally compact, $L(z)$ is open and closed. Since X is connected, we have $L(z) = X$. Thus, for any $i = 1, \ldots, n$, there exists a continuum L_i in X such that $z \in L_i$ and $L_i \cap K_i \neq \emptyset$. Let $K' = \bigcup_{i=1}^n (K_i \cup L_i)$ and $\delta = \operatorname{diam} K' + \delta_0$. Note that K' is a continuum.

Let $x, y \in X$ with $d(x, y) < \varepsilon$. If $x \in K$, then $x, y \in K'$ and $\operatorname{diam} K' < \delta$. If $x, y \in X \setminus K$, then by assumption, there exists a connected set P such that $x, y \in P$ and $\operatorname{diam} P < \delta_0 < \delta$. Then X is coarsely uniformly connected, as claimed. ∎

The following corollary is an immediate consequence of Theorem 2.6 and Lemma 2.7.

COROLLARY 2.8. *Let (X, d) be a noncompact, locally connected, connected, proper metric space. Then the following statements are equivalent*:

(1) $\overline{X}^d$ *is a perfect compactification of* (X, d).
(2) (X, d) *is coarsely uniformly connected at* ∞.
(3) (X, d) *is coarsely uniformly connected.*

Each of the following examples is equipped with a subspace metric induced by the usual metric in $\mathbb{R}^n$.

EXAMPLE 2.9. (1) Let $X = [0, \infty) \times \{0, 1\} \cup \bigcup_{n\in\mathbb{N}} \{2^n\} \times [0, 1] \subset \mathbb{R}^2$ with the subspace metric d. Note that (X, d) is connected and locally connected, but not coarsely uniformly connected. By Corollary 2.8, (X, d) is not coarsely uniformly connected at ∞ and $\overline{X}^d$ is not perfect.

(2) Let $Y = [0, \infty) \times \{0, 1\} \cup \bigcup_{n\in\mathbb{N}} \{n\} \times [0, 1] \subset \mathbb{R}^2$ with the subspace metric ϱ. Note that (Y, ϱ) is connected, locally connected and coarsely uniformly connected. By Corollary 2.8, $\overline{Y}^\varrho$ is perfect.

(3) Let $Z_n = \{(x, y) \in \mathbb{R}^2 : n \le x \le n + 2^{-n}$ and $y = 2^{-m}$ for $m \in \mathbb{N}\} \cup \{n\} \times [0, 1] \subset \mathbb{R}^2$ for each $n \in \mathbb{N}$ and $Z = [0, \infty) \times \{0, 1\} \cup \bigcup_{n\in\mathbb{N}} Z_n \subset \mathbb{R}^2$ with the subspace metric σ. Note that (Z, σ) is connected and coarsely uniformly connected, but not locally connected. By Theorem 2.3, $\overline{Z}^\sigma$ is perfect.

(4) Let $N = \{2^n \in \mathbb{R} : n \in \mathbb{N}\}$ with the subspace metric μ. Note that (N, μ) is coarsely uniformly connected at ∞, but not coarsely uniformly connected. By Theorem 2.3, $\overline{N}^\mu$ is perfect and is equivalent to the Stone–Čech compactification βN.

QUESTION 2.10. *Can we omit the local connectedness in Lemma 2.7?*

3. Perfectness of the Smirnov compactification. In this section, we provide a necessary and sufficient condition for the Smirnov compactification to be perfect for noncompact, locally connected, proper metric spaces.

PROPOSITION 3.1 ([11, Theorem 2.5]). *Let X be a noncompact metric space with a metric d. Then the following conditions are equivalent*:

(1) *A compactification αX of X is equivalent to $u_d X$.*
(2) *For disjoint closed sets $A, B \subset X$, $\mathrm{Cl}_{\alpha X} A \cap \mathrm{Cl}_{\alpha X} B = \emptyset$ if and only if $d(A, B) > 0$.*

DEFINITION 3.2.

(1) ([9, Exercise 8.42]) A metric space (X, d) is *uniformly locally connected* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any two points $x, y \in X$ with $d(x, y) < \delta$ there exists a connected subset P in X satisfying $x, y \in P$ and $\operatorname{diam} P < \varepsilon$.
(2) A metric space (X, d) is *uniformly locally connected at* ∞ if for any $\varepsilon > 0$ there exist a $\delta > 0$ and a compact set K in X such that for any two points $x, y \in X \setminus K$ with $d(x, y) < \delta$ there exists a connected subset P in X satisfying $x, y \in P$ and $\operatorname{diam} P < \varepsilon$.

THEOREM 3.3. *If a metric space (X, d) is uniformly locally connected at ∞, then $u_d X$ is a perfect compactification of (X, d).*

Proof. The proof is the same as that of Theorem 2.3 except for some modifications. Let $Y = u_d X$. Also let U, W_0, W_1, and V be as in the proof of Theorem 2.3. Here, we may assume that both W_0 and W_1 are noncompact. It suffices to show $\operatorname{Cl}_Y(W_0 \cap W_1) \supset \operatorname{Cl}_Y W_0 \cap \operatorname{Cl}_Y W_1$. Suppose that there exists a $p \in \operatorname{Cl}_Y W_0 \cap \operatorname{Cl}_Y W_1 \setminus \operatorname{Cl}_Y(W_0 \cap W_1)$ and a closed neighborhood S of p in Y such that $S \cap \operatorname{Cl}_Y(W_0 \cap W_1) = \emptyset$ and $S \cap W_i \neq \emptyset$ for $i = 0, 1$. Notice that $\operatorname{Cl}_Y(W_0 \cap S) \cap \operatorname{Cl}_Y(W_1 \cap S) \neq \emptyset$. By Proposition 3.1, $d(W_0 \cap S, W_1 \cap S) = 0$. Then there exist sequences $\{x_{0,n}\}_{n\in\mathbb{N}} \subset W_0 \cap S \subset U$ and $\{x_{1,n}\}_{n\in\mathbb{N}} \subset W_1 \cap S \subset V$ such that $d(x_{0,n}, x_{1,n}) < 1/n$ for each $n \in \mathbb{N}$. Without loss of generality, we may assume that for every compact set K in X, there exists an $l \in \mathbb{N}$ such that $\{x_{0,n}, x_{1,n}\} \cap K = \emptyset$ for each $n \geq l$. Furthermore, since X is uniformly locally connected at ∞, we may assume that for every $n \in \mathbb{N}$ there exists a connected set P_n joining $x_{0,n}$ and $x_{1,n}$ in X such that $\operatorname{diam} P_n \to 0$ if $n \to \infty$. Since $U \cup V$ is not connected, we see that $P_n \cap (W_0 \cap W_1) \neq \emptyset$ for each $n \in \mathbb{N}$. Take an element $y_n \in P_n \cap (W_0 \cap W_1)$ for each $n \in \mathbb{N}$. It follows that $d(x_{i,n}, y_n) \to 0$ as $n \to \infty$ for $i = 0, 1$. By Proposition 3.1, we have $d(W_0 \cap W_1, W_i \cap S) = 0$ for $i = 0, 1$. This gives $\operatorname{Cl}_Y(W_0 \cap W_1) \cap \operatorname{Cl}_Y(W_i \cap S) \neq \emptyset$, thus $\operatorname{Cl}_Y(W_0 \cap W_1) \cap S \neq \emptyset$, a contradiction. ∎

Since all noncompact convex subsets of normed linear spaces are uniformly locally connected at ∞, we have the following result of Charalambous as a corollary.

COROLLARY 3.4 ([1]). *For every noncompact convex subset X of an arbitrary normed linear space with the subspace metric d, the Smirnov compactification $u_d X$ is a perfect compactification of (X, d). In particular, $u_{d_n}\mathbb{R}^n$ is a perfect compactification of the n-dimensional Euclidean space $(\mathbb{R}^n, d_n)$ for each $n \in \mathbb{N}$.*

From Corollary 2.5 combined with the fact that $u_d X \geq \overline{X}^d$, we get the following corollary.

Corollary 3.5. *Let X be a noncompact, locally compact, separable metrizable space. If the set of nonisolated points of X is compact, then there exists a proper metric d on X such that u_dX is a perfect compactification of (X,d) equivalent to βX.*

Now we are ready to prove the main result of this section.

Theorem 3.6. *Let (X,d) be a noncompact, locally connected, proper metric space. Then u_dX is a perfect compactification of (X,d) if and only if (X,d) is uniformly locally connected at ∞.*

Proof. If X is not uniformly locally connected at ∞, then, by Theorem 3.3, it suffices to show that u_dX is not perfect.

The proof is similar to that of Theorem 2.6. Since X is not uniformly locally connected at ∞, there exist a strictly increasing compact cover $\{K_n\}_{n\in\mathbb{N}}$ and an $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exist two points $x_n, y_n \in X \setminus K_n$ with $d(x_n, y_n) < 1/n$ and $\operatorname{diam} P \geq \varepsilon$ for all connected subsets P joining x_n and y_n.

Without loss of generality, we may assume $B_\varepsilon(\{x_n, y_n\}, d) \subset K_{n+1} \setminus K_n$ and $d(K_n \cup \operatorname{Cl}_X B_\varepsilon(\{x_n, y_n\}, d), \operatorname{Cl}_X(X \setminus K_{n+1})) > \varepsilon$ for each $n \in \mathbb{N}$. Denote by C_n the component of $B_{\varepsilon/3}(x_n, d)$ containing x_n. We see that $U = \bigcup_{n\in\mathbb{N}} C_n$ is open and contains no y_n for all $n \in \mathbb{N}$, and $\operatorname{Fr}_X U = \bigcup_{n\in\mathbb{N}} \operatorname{Fr}_X C_n$ because the family $\{\operatorname{Fr}_X C_n : n \in \mathbb{N}\}$ is discrete. Let $A = \{x_n : n \in \mathbb{N}\}$. Note that $d(A, X \setminus U) \leq d(A, \{y_n : n \in \mathbb{N}\}) = 0$. By the local connectedness of X, $\operatorname{Fr}_X C_n \subset \{y \in X : d(x_n, y) = \varepsilon/3\}$. Consequently, $d(A, \operatorname{Fr}_X U) \geq \varepsilon/3$. By Proposition 1.1, this implies that u_dX is not perfect. ∎

It is clear that every uniformly locally connected metric space is locally connected and is uniformly locally connected at ∞.

Lemma 3.7. *Let (X,d) be a locally compact metric space. Then (X,d) is locally connected and uniformly locally connected at ∞ if and only if it is uniformly locally connected.*

Proof. Let $\varepsilon > 0$. Since X is uniformly locally connected at ∞, there exist a compact subset K of X and a $\delta_0 > 0$ such that for any $x, y \in X \setminus K$ with $d(x,y) < \delta_0$ there exists a connected set P satisfying $x, y \in P$ and $\operatorname{diam} P < \varepsilon$. Since X is locally compact, there is a $\delta_1 > 0$ such that $K_1 = \operatorname{Cl}_X B_{\delta_1}(K, d)$ is compact. Since X is locally connected, there exist connected open subsets $U_1, \ldots, U_n$ such that $K_1 \subset \bigcup_{i=1}^n U_i$ and $\max\{\operatorname{diam} U_i : 1 \leq i \leq n\} < \varepsilon$. Let $\delta_2 > 0$ be a Lebesgue number of the cover $\{U_i : i = 1, \ldots, n\}$, i.e., for any $x, y \in K_1$ with $d(x,y) < \delta_2$ there exists an i such that $x, y \in U_i$.

Let $\delta = \min\{\delta_0, \delta_1, \delta_2\}$. It is easy to check that if $x, y \in X$ with $d(x,y) < \delta$, then x and y belong to a connected set of diameter less than ε. ∎

The following corollary is an immediate consequence of Theorem 3.6 and Lemma 3.7.

COROLLARY 3.8. *Let (X,d) be a noncompact, locally connected, proper metric space. Then the following statements are equivalent:*

(1) *u_dX is a perfect compactification of (X,d).*
(2) *(X,d) is uniformly locally connected at ∞.*
(3) *(X,d) is uniformly locally connected.*

The following examples are equipped with the subspace metric induced by the Euclidean metric in $\mathbb{R}^n$.

EXAMPLE 3.9. (1) Fix an $n \in \mathbb{N}$. Let

$$\begin{aligned} X'_0 &= \{(x_1,\dots,x_n,0) \in \mathbb{R}^{n+1} : x_1 \geq 0\},\\ X'_1 &= \{(0,x_2,\dots,x_n,x_{n+1}) \in \mathbb{R}^{n+1} : 0 \leq x_{n+1} \leq 1\},\\ X'_{2,0} &= \{(x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} : x_1 \geq 0,\ x_{n+1} = 1\},\\ X'_{2,1} &= \{(x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} : x_1 \geq 0,\ x_{n+1} = 2^{-x_1}\}, \end{aligned}$$

and $X_i = X'_0 \cup X'_1 \cup X'_{2,i}$ with the subspace metric ϱ_i for $i = 0,1$. Note that each (X_i,ϱ_i) is locally connected and connected because it is homeomorphic to $\mathbb{R}^n$. Also, (X_0,ϱ_0) is uniformly locally connected, but (X_0,ϱ_0) is not coarsely uniformly connected and (X_1,ϱ_1) is not uniformly locally connected. By Corollaries 2.8 and 3.8, $u_{\varrho_0}X_0$ is perfect but neither $u_{\varrho_1}X_1$ nor $\overline{X_0}^{\varrho_0}$ is perfect.

(2) Let $Y_n = \{n + kn^{-1} \in \mathbb{R} : k = 0,1,\dots,n-1\}$ and $Y = \bigcup_{n\in\mathbb{N}} Y_n$ with the subspace metric s. Note that (Y,s) is locally connected, but not uniformly locally connected at ∞. By Corollary 3.8, (Y,s) is not uniformly locally connected and u_sY is not perfect.

(3) Let (Z,σ) be as in Example 2.9(3). Note that (Z,σ) is uniformly locally connected at ∞, but neither locally connected nor uniformly locally connected. By Theorem 3.3, $u_\sigma Z$ is perfect.

(4) Consider $\mathbb{N}$ as a subset of $\mathbb{R}$ with the subspace metric μ. Note that $(\mathbb{N},\mu)$ is uniformly locally connected. By Theorem 3.3, $u_\mu\mathbb{N}$ is perfect and is equivalent to the Stone–Čech compactification $\beta\mathbb{N}$.

(5) Let $X_n = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 1 \text{ and } x = n + 2^{-m} \text{ for } m \in \mathbb{N}\} \cup \{n\} \times [0,1] \subset \mathbb{R}^2$ for each $n \in \mathbb{N}$ and $X = [0,\infty) \times \{0,1\} \cup \bigcup_{n\in\mathbb{N}} X_n \subset \mathbb{R}^2$ with the subspace metric d. Then (X,d) is coarsely uniformly connected at ∞, but neither locally connected nor uniformly locally connected at ∞. By Theorem 2.3, $\overline{X}^d$ is perfect. Let $W_n = \{(x,y) \in X : x = n + 2^{-m}$ for $m \in \mathbb{N}$ with $m \geq n\} \cup \{n\} \times [0,1] \cup [n, n+2^{-n}] \times \{0,1\}$ for each $n \in \mathbb{N}$, $W = \bigcup_{n\in\mathbb{N}} W_n$, $U = \operatorname{Int}_X W$, and $A = \{(n,2^{-1}) : n \in \mathbb{N}\}$. We see that $d(A, \operatorname{Fr}_X U) \geq 2^{-1}$ and $d(A, X \setminus U) = 0$, therefore, u_dX is not perfect by Propositions 1.1 and 3.1.

REFERENCES

[1] M. G. Charalambous, *A remark on R. G. Woods' paper "The minimum uniform compactification of a metric space"*, Fund. Math. 149 (1996), 287–288.

[2] A. N. Dranishnikov, J. Keesling and V. V. Uspenskij, *On the Higson corona of uniformly contractible spaces*, Topology 37 (1998), 791–803.

[3] A. N. Dranishnikov and M. Zarichnyi, *Universal spaces for asymptotic dimension*, Topology Appl. 140 (2004), 203–225.

[4] R. Engelking, *General Topology*, Helderman, Berlin, 1989.

[5] C. F. K. Jung, *Locally compact spaces whose Alexandroff one-point compactifications are perfect*, Colloq. Math. 27 (1973), 247–249.

[6] K. Kawamura and K. Tomoyasu, *Approximations of Stone–Čech compactifications by Higson compactifications*, ibid. 88 (2001), 75–92.

[7] J. Keesling, *The one-dimensional Čech cohomology of the Higson compactification and its corona*, Topology Proc. 19 (1994), 129–148.

[8] Y. Kodama and K. Nagami, *General Topology*, Iwanami, Tokyo, 1974 (in Japanese).

[9] S. B. Nadler, Jr. *Continuum Theory. An Introduction*, Monogr. Textbooks Pure Appl. Math. 158, Dekker, New York, 1992.

[10] E. G. Sklyarenko, *On perfect bicompact extensions*, Dokl. Akad. Nauk SSSR 137 (1961), 39–41 (in Russian); English transl.: Soviet Math. Dokl. 2 (1961), 238–240.

[11] R. G. Woods, *The minimum uniform compactification of a metric space*, Fund. Math. 147 (1995), 39–59.

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Received 1 April 2006;
revised 5 May 2006 (4767)