

# Performance Analysis of an M/G/1 Retrial Queue with Non-Persistent Calls, Two Phases of Heterogeneous Service and Different Vacation Policies

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## Abstract

*This paper studies the steady state behaviour of an M/G/1 retrial queue with non-persistent customers and two phases of heterogeneous service and different vacation policies. If the primary call, on arrival finds the server busy, it becomes impatient and leaves the system with probability  $(1 - \alpha)$  and with probability  $\alpha$ , it enters into an orbit. The server provides preliminary first essential service (FES) and followed by second essential service (SES) to primary arriving calls or calls from the retrial group. On completion of SES the server may go for  $i^{\text{th}}$  ( $i=1,2,3,\dots,M$ ) type of vacation with probability  $\beta_i$  ( $i=1,2,3,\dots,M$ ) or may remain in the system to serve the next call, if any, with probability  $\beta_0$  where  $\sum_{i=0}^M \beta_i = 1$ . The steady state queue size distribution of number of customers in retrial group, expected number of customers in the retrial group and expected waiting time of the customers in the orbit are obtained. Some special cases are also discussed. A numerical illustration is also presented.*

**Keywords:** Essential Service, non-persistent calls, vacation time, two phases of heterogeneous service

## 1 Introduction

Retrial queues have been widely used to model many problems arising in telephone switching systems, telecommunication networks, computer networks and computer systems. Retrial queueing models are characterized by the feature that arriving calls which find a server busy, do not line up or leave the system

immediately forever, but go to some virtual place called as orbit and try their luck again after some random time. During the last two decades considerable attention has been paid to the analysis of queueing system with repeated calls (also called retrial queues or queues with returning customers) See, for example, the book by Falin and Templeton [12], Artalejo[4] and the survey papers, of Artalejo [1, 2]. For many applications in telecommunications and mobile communication, Choi et al. [6,7,8] studied the single server retrial queue with priority calls and Krishnakumar et al. [15] analysed an M/G/1 retrial queue with feedback and starting failures using supplementary variable technique. Recently, there have been several contributions considering queueing system with two phases of service. SenthilKumar and Arumuganathan [19] consider a batch arrival single server retrial queue in which the server provides two phases of heterogeneous service and receives general vacation time under Bernoulli schedule. Madan [17] considers the classical M/G/1 queueing system in which the server provides the first essential service to all the arriving customers whereas some of them receive second optional service. The first essential service follows general distribution and second optional service follows exponential distribution. Medhi [18] generalizes the model by considering that the second optional service is also governed by a general distribution. Choudhury [9] investigates the queueing model with second optional service by including the waiting time distribution. Krishnakumar et al. [14] consider an M/G/1 retrial queue with additional phase of service. While at the first phase of service, the server may push-out the customer who is receiving such a service, in order to start the service of a higher priority arriving customer. The interrupted customers join a retrial queue and the customer at the head of this queue is allowed to conduct a repeated attempt in order to start again to the first phase in service after some random time. Artalejo *et al.* [3] gives a steady state analysis of the M/G/1 queueing system with repeated attempts and two phase service using embedded Markov chain method.

The first model taking into account impatience of customers was considered by Cohen [10]. He has studied an M/M/C retrial queue in which the orbital customers leave the system with some intensity. The M/M/1 model with impatient subscribers was considered by Falin [12], who obtained a solution in terms of Kummer confluent functions. Yang *et al.* [20] have presented a semi-analytic treatment for the analysis of an M/G/1 model with impatient subscribers and obtained expressions for the moments of the queue length in terms of the server utilization. Recently, Krishnamoorthy *et al.* [13] presented an analysis of the M/G/1 retrial queue with non-persistent customers and orbital search.

Queueing systems with vacation time have been found to be useful in modeling the systems in which the server has additional tasks. Doshi [11] discussed an M/G/1 system with variable vacations. In this discussion, he assumed that the

system has an infinite number of vacation types indexed by  $i=1,2,3\dots$  and after serving all customers, the server takes a vacation of type-1. Upon returning from vacation, if there are no customers, the server initiates a vacation of type -2 and so on. Krishna Reddy *et al.* [16] considered an M/G (a, b)/1 model with M different types of vacations. Recently, Arumuganathan *et al.* [5] completed a steady state analysis of a non-Markovian bulk queueing system with N-Policy and different types of vacation.

In this paper, a single server retrial queue with impatient subscribers and two phases of heterogeneous service and different vacation policies is considered. Analytical treatment of this model is obtained using the supplementary variable technique. The probability generating function of number of customers in the retrial group and various performance measures are obtained. The main motivation is from applications to Local Area Networks (LAN), client – server communication, telephone systems, electronic mail services on Internet and VoIP protocol.

*Voice-over-IP* (VoIP) is an upcoming and very promising technology that enables people to make telephone calls via an IP network such as the internet at very low, or even no cost. Voice-over-IP is the delivery of voice information over IP packet switched networks. The analogue voice information is thereby digitized, compressed and broken down into small pieces before sending it across the network, instead of having the traditional reserved (analogue) circuit. IP networks allow each packet to independently find the most efficient path to its destination utilizing the internet at its best. At the destination, the packets are reassembled in the correct order, combined, decompressed and converted to the original analogue voice signal. A VoIP call can now be established via a PC to another PC, via a PC to an ordinary telephone through a internet-PSTN gateway, via an IP-telephone handset to another handset, or combination of the methods mentioned above. Although the internet capacity grows by the minute, it is still the case that some packets may get lost due to network congestion. Packets may be dropped or be delayed for a long time, in which case, depending on the codecs used, noises in the resulting audio stream can occur.

A normal SIP session starts by locating the desired user on a registrar-server (VoIP server) containing the current IP addresses of the users. Once the user is found, an invitation is then sent to it via a proxy server which authenticates the initiator. Once a person registers with the server, that connection becomes *p2p*. Finally, call teardown is simply done by sending a message directly between peers. The following fig.1 represents the scenario of using VoIP. Assume that VoIP call arrivals follow Poisson distribution with mean arrival rate  $\lambda$ . Packets may be

dropped (non persistent calls) or be delayed for a long time (persistent calls), in which case, depending on the codecs used, noises in the resulting audio stream can occur. The service process consists mainly in transferring the information of caller to another end user. This service process of VoIP is done in three phases: (i) connection establishment (FES); (ii) transferring information (SES) and (iii) accessing differentiated services (different vacation policies). The major VoIP service technique is SIP. A normal SIP session starts by locating the desired user on a registrar-server (VoIP server) containing the current IP addresses of the users (*Connection establishment phase*). Once the user is found, an invitation is then sent to it via a proxy server which authenticates the initiator. Once a person registered with the server, that connection becomes *p2p* (*Information transferring phase*). Finally, the VoIP call can access the differentiated services (different vacation policies). This system can be modeled as an M/G/1 retrial queueing system with impatient subscribers & two phases of essential service and different vacation policies.

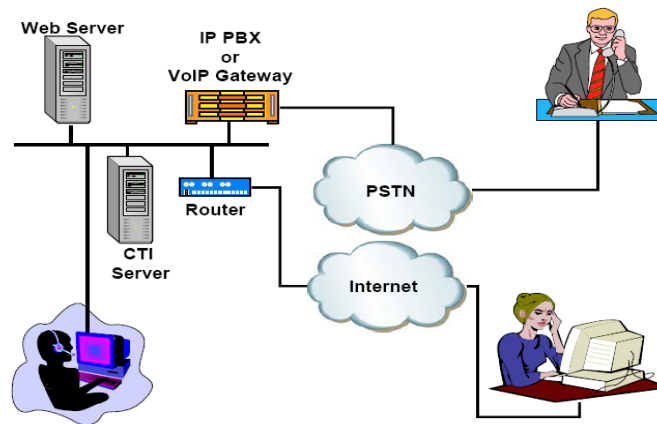


Fig 1

## 2 Mathematical Model

In this paper, a single server retrial queueing system is considered. The primary calls arrive according to Poisson process with rate  $\lambda$ . If the primary call, on arrival finds the server busy, it becomes impatient and leaves the system with probability  $(1-\alpha)$  and with probability  $\alpha$ , it enters into an orbit (retrying pool). The server provides preliminary first essential service (FES) and second essential service (SES) to all arriving customers. As soon as the SES of a call is completed, the server may go for  $i^{\text{th}}$  ( $i=1,2,3,\dots,M$ ) type of vacation with probability  $\beta_i$  or may

remain in the system to serve the next call, if any, with probability  $\beta_0$  where

$$\left( \sum_{i=0}^M \beta_i = 1 \right).$$

Primary calls finding the server free upon arrival automatically get their FES. However, if a primary call finds the server busy (attending FES or SES), then it joins the orbit with probability  $\alpha$ , in order to seek service again until it finds the server free. Otherwise, it leaves the system with probability  $(1-\alpha)$  for ever. The time between two successive repeated attempts of each call in orbit is assumed to be exponentially distributed with rate ' $\nu$ '. Let  $S_1(x)$  ( $s_1(x)$ )  $\{\tilde{s}_1(\theta)\}$  [ $S_1^0(x)$ ] be the cumulative distribution function (probability density function) {Laplace – Stieltjes transform} [remaining service time] of FES. Let  $S_2(x)$  ( $s_2(x)$ )  $\{\tilde{s}_2(\theta)\}$  [ $S_2^0(x)$ ] be cumulative distribution function (probability density function) {Laplace-Stieltjes transform} [remaining service time] of SES. Let  $V_i(x)$  ( $v_i(x)$ )  $\{\tilde{v}_i(\theta)\}$  [ $V_i^0(x)$ ] ( $i = 1, 2, 3, \dots, M$ ) be cumulative distribution function (probability density function) {Laplace-Stieltjes transform} [remaining vacation time] of  $i^{\text{th}}$  type of vacation.  $N(t)$  denotes the number of customers in the orbit at time  $t$ .

$$\text{The server state is denoted as, } C(t) = \begin{cases} 0 & \text{if the server is idle} \\ 1 & \text{if the server is doing FES} \\ 2 & \text{if the server is doing SES} \\ 3 & \text{if the server is on vacation} \end{cases}$$

Define the following probability functions:

$$P_{o,n}(t)dt = \Pr\{N(t) = n, C(t) = 0\} \quad n \geq 0 \quad \text{and}$$

$$P_{i,n}(x,t)dt = \Pr\{N(t) = n, C(t) = i, x \leq S_i^0(t) \leq x + dt, \}; n \geq 0; i = 1, 2$$

$$P_{3,n}(x,t)dt = \Pr\{N(t) = n, C(t) = 3, x \leq V_i^0(t) \leq x + dt, \}; n \geq 0; i = 1, 2, 3, \dots, M$$

### 3 Steady State System Size Distribution

The following equations are obtained for the queueing system, using the supplementary variable technique.

$$P_{o,j}(t + \Delta t) = P_{o,j}(t)(1 - \lambda\Delta t - j\nu\Delta t) + P_{3,j}(0,t)\Delta t + \beta_0 P_{2,j}(0,t)\Delta t$$

$$P_{1,j}(x - \Delta t, t + \Delta t) = P_{1,j}(x, t)(1 - \lambda\alpha\Delta t) + \lambda\Delta t P_{0,j}(t)s_1(x) \\ + (j+1)vP_{0,j+1}(t)s_1(x)\Delta t + \lambda\alpha\Delta t P_{1,j-1}(x, t)$$

$$P_{2,j}(x - \Delta t, t + \Delta t) = P_{2,j}(x, t)(1 - \lambda\alpha\Delta t) + \lambda\alpha\Delta t P_{2,j-1}(x, t) + P_{1,j}(0, t)s_2(x)\Delta t$$

$$P_{3,j}(x - \Delta t, t + \Delta t) = P_{3,j}(x, t)(1 - \lambda\alpha\Delta t) + \lambda\alpha\Delta t P_{3,j-1}(x, t) + P_{2,j}(0, t)\sum_{k=1}^M \beta_k v_k(x)\Delta t$$

Steady state equations of the above equations are,

$$(\lambda + jv) P_{0,j} = P_{3,j}(0) + \beta_0 P_{2,j}(0) \quad (1)$$

$$-\frac{d}{dx} P_{1,j}(x) = -\lambda\alpha P_{1,j}(x) + \lambda P_{0,j} s_1(x) + (j+1)v P_{0,j+1} s_1(x) + \lambda\alpha P_{1,j-1}(x) \quad (2)$$

$$-\frac{d}{dx} P_{2,j}(x) = -\lambda\alpha P_{2,j}(x) + \lambda\alpha P_{2,j-1}(x) + P_{1,j}(0) s_2(x) \quad (3)$$

$$-\frac{d}{dx} P_{3,j}(x) = -\lambda\alpha P_{3,j}(x) + \lambda\alpha P_{3,j-1}(x) + P_{2,j}(0)\sum_{k=1}^M \beta_k v_k(x) \quad (4)$$

Let  $LST\{P_{0,j}(x)\} = \tilde{P}_{0,j}(\theta)$ ;  $LST\{P_{1,j}(x)\} = \tilde{P}_{1,j}(\theta)$ ;  $LST\{P_{3,j}(x)\} = \tilde{P}_{3,j}(\theta)$

Taking LST of steady state equations (2), (3) and (4), we have,

$$\theta \tilde{P}_{1,j}(\theta) - P_{1,j}(0) = \lambda\alpha \tilde{P}_{1,j}(\theta) - \lambda P_{0,j} \tilde{S}_1(\theta) - (j+1)v P_{0,j+1} \tilde{S}_1(\theta) - \lambda\alpha \tilde{P}_{1,j-1}(\theta) \quad (5)$$

$$\theta \tilde{P}_{2,j}(\theta) - P_{2,j}(0) = \lambda\alpha \tilde{P}_{2,j}(\theta) - \lambda\alpha \tilde{P}_{2,j-1}(\theta) - P_{1,j}(0) \tilde{S}_2(\theta) \quad (6)$$

$$\theta \tilde{P}_{3,j}(\theta) - P_{3,j}(0) = \lambda\alpha \tilde{P}_{3,j}(\theta) - \lambda\alpha \tilde{P}_{3,j-1}(\theta) - P_{2,j}(0)\sum_{k=1}^M \beta_k \tilde{V}_k(\theta) \quad (7)$$

Now, we define the following probability generating functions (PGF)

$$P_0(z) = \sum_{j=0}^{\infty} P_{0,j} z^j ;$$

$$\tilde{P}_i(z, \theta) = \sum_{j=0}^{\infty} \tilde{P}_{i,j}(\theta) z^j ; \quad P_i(z, 0) = \sum_{j=0}^{\infty} P_{i,j}(0) z^j ; \quad i = 1, 2, 3$$

Using PGF equations (1), (5), (6) and (7) can be written as follows,

$$\lambda P_0(z) + v z P_0'(z) = P_3(z, 0) + \beta_0 P_2(z, 0) \quad (9)$$

$$(\theta - \lambda\alpha + \lambda\alpha z) \tilde{P}_1(z, \theta) = P_1(z, 0) - \lambda P_0(z) \tilde{S}_1(\theta) - v P_0'(z) \tilde{S}_1(\theta) \quad (10)$$

$$(\theta - \lambda\alpha + \lambda\alpha z) \tilde{P}_2(z, \theta) = P_2(z, 0) - P_1(z, 0) \tilde{S}_2(\theta) \quad (11)$$

$$(\theta - \lambda\alpha + \lambda\alpha z)\tilde{P}_3(z, \theta) = P_3(z, 0) - \sum_{k=1}^M \beta_k P_2(z, 0)\tilde{V}_k(\theta) \tag{12}$$

Substituting  $\theta = \lambda\alpha - \lambda\alpha z$  in the equations (10), (11) and (12), the equations are given as follow

$$P_1(z, 0) = \lambda P_0(z)\tilde{S}_1(\lambda\alpha - \lambda\alpha z) + \nu P_0'(z)\tilde{S}_1(\lambda\alpha - \lambda\alpha z) \tag{13}$$

$$P_2(z, 0) = P_1(z, 0)\tilde{S}_2(\lambda\alpha - \lambda\alpha z) \tag{14}$$

$$P_3(z, 0) = P_2(z, 0)\sum_{k=1}^M \beta_k V_k(\lambda\alpha - \lambda\alpha z) \tag{15}$$

Substituting (14) and (15) in the equation (9), we have,

$$\lambda P_0(z) + \nu z P_0'(z) = \left( \sum_{k=1}^M (\beta_k \tilde{V}_k(\lambda\alpha - \lambda\alpha z)) + \beta_0 \right) \tilde{S}_2(\lambda\alpha - \lambda\alpha z) \left( \lambda P_0(z)\tilde{S}_1(\lambda\alpha - \lambda\alpha z) + \nu P_0'(z)\tilde{S}_1(\lambda\alpha - \lambda\alpha z) \right) \tag{16}$$

From the equation (16), we have,

$$P_0'(z) = \frac{-\lambda}{\nu} \left( \frac{1 - [\tilde{S}_1(\lambda\alpha - \lambda\alpha z)\tilde{S}_2(\lambda\alpha - \lambda\alpha z) \left( \sum_{k=1}^M (\beta_k \tilde{V}_k(\lambda\alpha - \lambda\alpha z)) + \beta_0 \right)]}{[\tilde{S}_1(\lambda\alpha - \lambda\alpha z)\tilde{S}_2(\lambda\alpha - \lambda\alpha z) \left( \sum_{k=1}^M (\beta_k \tilde{V}_k(\lambda\alpha - \lambda\alpha z)) + \beta_0 \right)] - z} \right) P_0(z)$$

On integration, we get

$$P_0(z) = P_0(1) \exp \left\{ \int_z^1 \frac{-\lambda}{\nu} \left( \frac{1 - [\tilde{S}_1(\lambda\alpha - \lambda\alpha u)\tilde{S}_2(\lambda\alpha - \lambda\alpha u) \left( \sum_{k=1}^M (\beta_k \tilde{V}_k(\lambda\alpha - \lambda\alpha u)) + \beta_0 \right)]}{[\tilde{S}_1(\lambda\alpha - \lambda\alpha u)\tilde{S}_2(\lambda\alpha - \lambda\alpha u) \left( \sum_{k=1}^M (\beta_k \tilde{V}_k(\lambda\alpha - \lambda\alpha u)) + \beta_0 \right)] - z} \right) du \right\} \tag{17}$$

Substituting the equation (13) in equation (10), we have,

$$(\theta - \lambda\alpha + \lambda\alpha z)\tilde{P}_1(z, \theta) = (\tilde{S}_1(\lambda\alpha - \lambda\alpha z) - \tilde{S}_1(\theta))[\lambda P_0(z) + \nu P_0'(z)] \tag{18}$$

Substituting the equation (14) in equation (11), we have,

$$(\theta - \lambda\alpha + \lambda\alpha z)\tilde{P}_2(z, \theta) = P_1(z, 0)(\tilde{S}_2(\lambda\alpha - \lambda\alpha z) - \tilde{S}_2(\theta)) \tag{19}$$

Substituting the equation (15) in equation (12), we have,

$$(\theta - \lambda\alpha + \lambda\alpha z)\tilde{P}_3(z, \theta) = P_2(z, 0) \left( \sum_{k=1}^M (\beta_k (\tilde{V}_k(\lambda\alpha - \lambda\alpha z) - \tilde{V}_k(\theta))) \right) \tag{20}$$

Using the equations (18), (19) and (20), the partial generating functions

$$\tilde{P}_i(z,0) = \sum_{j=0}^{\infty} \tilde{P}_{i,j}(0)z^j \quad ; i = 1, 2, 3 \text{ are given by,}$$

$$\tilde{P}_1(z,0) = \frac{(\tilde{S}_1(\lambda\alpha - \lambda\alpha z) - 1)[\lambda P_0(z) + vP_0'(z)]}{(-\lambda + \lambda z)\alpha} \quad (21)$$

$$\tilde{P}_2(z,0) = \frac{\tilde{S}_1(\lambda\alpha - \lambda\alpha z)(\tilde{S}_2(\lambda\alpha - \lambda\alpha z) - 1)[\lambda P_0(z) + vP_0'(z)]}{(-\lambda + \lambda z)\alpha} \quad (22)$$

$$\tilde{P}_3(z,0) = \frac{\tilde{S}_1(\lambda\alpha - \lambda\alpha z)\tilde{S}_2(\lambda\alpha - \lambda\alpha z)\left(\sum_{k=1}^M (\beta_k (\tilde{V}_k(\lambda\alpha - \lambda\alpha z) - 1))\right)[\lambda P_0(z) + vP_0'(z)]}{(-\lambda + \lambda z)\alpha} \quad (23)$$

It should be noted that the probability generating function  $P(z)$  of number of customers in orbit at an arbitrary epoch can be expressed as follows,

$$P(z) = P_0(z) + \tilde{P}_1(z,0) + \tilde{P}_2(z,0) + \tilde{P}_3(z,0)$$

Using the equations (21), (22) and (23), we have

$$P(z) = \frac{\{1 - R(z) + \alpha(R(z) - z)\}P_0(z)}{[R(z) - z]\alpha} \quad (24)$$

where

$$R(z) = \left( \tilde{S}_1(\lambda\alpha - \lambda\alpha z)\tilde{S}_2(\lambda\alpha - \lambda\alpha z)\left(\sum_{k=1}^M (\beta_k \tilde{V}_k(\lambda\alpha - \lambda\alpha z)) + \beta_0\right) \right)$$

From the equation (24), we have  $\lim_{z \rightarrow 1} P(z) = 1$ . It immediately follows that the

$$\text{steady state condition is } \rho\alpha = \lambda\alpha \left( E(S_1) + E(S_2) + \sum_{k=1}^M \beta_k E(V_k) \right) < 1$$

$$\text{(i.e. } \rho = \lambda \left( E(S_1) + E(S_2) + \sum_{k=1}^M \beta_k E(V_k) \right) )$$

$$\text{Moreover, } \lim_{z \rightarrow 1} \sum_{i=1}^3 \tilde{P}_i(z,0) = \frac{\rho}{1 - \rho\alpha} P_0(1) \text{ where } P_0(1) = \frac{(1 - \rho\alpha)}{1 + \rho(1 - \alpha)}$$



## 4 Performance Measures

Some useful results of our model are listed below.

a) *The mean number of customers in the orbit*

$$L = E[N(t)] = Lt \frac{d}{dz} P(z) \\ = \frac{\lambda^2 \alpha}{1 - \rho \alpha} \left( \frac{\left( E(S_1) + E(S_2) + \sum_{k=1}^M \beta_k E(V_k) \right)}{\nu} + \frac{\gamma_2}{2(1 + \rho(1 - \alpha))} \right) \quad (25)$$

Where

$$\gamma_2 = E[S_1^2] + E[S_2^2] + \sum_{k=1}^M (\beta_k E[V_k^2]) + 2E[S_1]E[S_2] + 2E[S_1] \sum_{k=1}^M (\beta_k E[V_k]) \\ + 2E[S_2] \sum_{k=1}^M (\beta_k E[V_k])$$

b) *Blocking probability*

Let  $b$  be the probability that an arriving customer finds the server busy. The blocking probability is given by

$$b = \frac{\rho}{1 + \rho(1 - \alpha)} \quad (26)$$

c) *Mean waiting time in retrial queue*

We have the mean waiting time in the retrial queue ( $W$ ) as follows,

$$W = \frac{L}{\lambda \alpha} \quad (27)$$

## 5 Particular Cases

**Case I :** *If there is no second phase of service (ie.  $\tilde{S}_2(\lambda\alpha - \lambda\alpha z) = 1$ ) and no vacation (ie.  $\tilde{V}_k(\lambda\alpha - \lambda\alpha z) = 1, k = 1, 2, \dots, M$ ), then, the PGF of the customers in orbit is given by*

$$P(z) = \frac{\{1 - R(z) + \alpha(R(z) - z)\}P_0(z)}{[R(z) - z]\alpha} \quad (28)$$

where  $R(z) = \tilde{S}_1(\lambda\alpha - \lambda\alpha z)$  and

$$P_0(z) = P_0(1) \exp\left\{ \frac{-\lambda}{v} \int_z^1 \left( \frac{1 - [\tilde{S}_1(\lambda\alpha - \lambda\alpha u)]}{[\tilde{S}_1(\lambda\alpha - \lambda\alpha u) - u]} \right) du \right\}$$

Equation (28) agrees the PGF of the number of customers in the orbit of M/G/1 retrial queue with impatient subscribers in the steady state obtained by Falin [12].

**Case II:** If  $M=1$ ,  $\alpha = 1$  and no second phase of service (ie.  $\tilde{S}_2(\lambda\alpha - \lambda\alpha z) = 1$ ), then the PGF of the customers in orbit is given by

$$P(z) = \frac{\{1 - z\}P_0(z)}{[R(z) - z]} \tag{29}$$

where

$$R(z) = \tilde{S}_1(\lambda - \lambda z) (\beta_1 \tilde{V}_1(\lambda - \lambda z) + \beta_0) \text{ and } P_0(z) = P_0(1) \exp\left\{ \frac{-\lambda}{v} \int_z^1 \left( \frac{1 - R(u)}{R(u) - u} \right) du \right\}$$

Equation (29) agrees the PGF of the number of customers in the orbit of M/G/1 retrial queue in the steady state obtained by Artalejo *et al.* [3].

Further, by specifying vacation time random variable as well as service time random variables as Deterministic, Erlang and Exponential distribution, some more particular cases of this model are discussed below.

**Case III:** Single server retrial queue with impatient subscribers, two phases of essential service and **Erlangian vacation time**

It is assumed that the vacation is an k-Erlang with probability density function,  $v_i(x) = \frac{(ku_i)^k x^{k-1} e^{-ku_i x}}{(k-1)!}$ ,  $i = 1, 2, \dots, M$ ,  $k > 0$ ; where u is the parameter.

Hence the PGF of the retrial queue size distribution is as follows when

$$\tilde{V}_i(\lambda\alpha - \lambda z\alpha) = \left( \frac{u_i k}{u_i k + \lambda\alpha(1 - z)} \right)^k ; i = 1, 2, \dots, M .$$

$$P(z) = \frac{\{1 - R(z) + \alpha(R(z) - z)\}P_0(z)}{[R(z) - z]\alpha} \tag{30}$$

where  $R(z) = \tilde{S}_1(\lambda\alpha - \lambda\alpha z) \tilde{S}_2(\lambda\alpha - \lambda\alpha z) \left( \sum_{i=1}^M \beta_i \left( \frac{u_i k}{u_i k + \lambda\alpha(1 - z)} \right)^k + \beta_0 \right)$  and

$$P_0(z) = P_0(1) \exp\left\{ \frac{-\lambda}{v} \int_z^1 \left( \frac{1 - R(u)}{R(u) - u} \right) du \right\}$$

The steady state condition is obtained as  $\rho\alpha = \lambda \left( E(S_1) + E(S_2) + \sum_{i=1}^M \frac{\beta_i}{u_i} \right) \alpha < 1$

**Case IV:** A single server retrial queue with two phases of heterogeneous services, impatient subscribers and **Hyper Exponential Vacation time**

Considering the case of Hyper Exponential vacation time random variable, the pdf of Hyper Exponential vacation time is given as follows,  $v_i(x) = cu_i e^{-u_i x} + (1-c)w_i e^{-w_i x} \quad i = 1, 2, 3, \dots, M$ .

$$\text{Then } \tilde{V}_i(\lambda\alpha - \lambda z\alpha) = \left( \frac{u_i c}{u_i + \lambda\alpha(1-z)} \right) + \left( \frac{w_i(1-c)}{w_i + \lambda\alpha(1-z)} \right)$$

Hence the PGF of the retrial queue size distribution is as follows

$$P(z) = \frac{\{1 - R(z) + \alpha(R(z) - z)\}P_0(z)}{[R(z) - z]\alpha} \quad (31)$$

where

$$R(z) = \tilde{S}_1(\lambda\alpha - \lambda z\alpha) \tilde{S}_2(\lambda\alpha - \lambda z\alpha) \left( \sum_{i=1}^M \left( \frac{u_i c}{u_i + \lambda\alpha(1-z)} \right) + \left( \frac{w_i(1-c)}{w_i + \lambda\alpha(1-z)} \right) \right) + \beta_0$$

$$\text{and } P_0(z) = P_0(1) \exp \left\{ \frac{-\lambda}{v} \int_z^1 \left( \frac{1-R(u)}{R(u)-u} \right) du \right\}$$

The steady state condition is given as  $\rho\alpha = \lambda \left( E(S_1) + E(S_2) + \sum_{i=1}^M \left[ \frac{c}{u_i} + \frac{1-c}{w_i} \right] \right) \alpha < 1$

**Case V:** Particular cases of the PGF given in (24) are also discussed when service times follow Erlangian and Exponential in a similar way. We now summarize the expressions for the PGF of orbit size.

(i) The PGF of orbit size distribution of a single server retrial queue with two phases of heterogeneous **Erlangian services** (FES follows  $E_{k_1}$  distribution and SES follows  $E_{k_2}$  distribution), impatient subscribers and general vacation time is obtained as,

$$P(z) = \frac{\{1 - R(z) + \alpha(R(z) - z)\}P_0(z)}{[R(z) - z]\alpha} \quad (32)$$

where

$$R(z) = \left( \frac{\mu_1 k_1}{\mu_1 k_1 + \lambda \alpha (1-z)} \right)^{k_1} \left( \frac{\mu_2 k_2}{\mu_2 k_2 + \lambda \alpha (1-z)} \right)^{k_2} \left( \sum_{k=1}^M (\beta_k \tilde{V}_k (\lambda \alpha - \lambda \alpha z)) + \beta_0 \right) \quad \text{and}$$

$$P_0(z) = P_0(1) \exp \left\{ \frac{-\lambda}{v} \int_z^1 \left( \frac{1-R(u)}{R(u)-u} \right) du \right\}$$

(ii) The PGF of orbit size distribution of single server retrial queue with two phases of heterogeneous **Exponential services**, impatient subscribers and general vacation time, is obtained as,

$$P(z) = \frac{\{1 - R(z) + \alpha(R(z) - z)\} P_0(z)}{[R(z) - z] \alpha} \quad (33)$$

where

$$R(z) = \left( \frac{\mu_1}{\mu_1 + \lambda \alpha (1-z)} \right) \left( \frac{\mu_2}{\mu_2 + \lambda \alpha (1-z)} \right) \left( \sum_{k=1}^M (\beta_k \tilde{V}_k (\lambda \alpha - \lambda \alpha z)) + \beta_0 \right) \quad \text{and}$$

$$P_0(z) = P_0(1) \exp \left\{ \frac{-\lambda}{v} \int_z^1 \left( \frac{1-R(u)}{R(u)-u} \right) du \right\}$$

## 6 Numerical Illustration

In this section, we present some numerical results to study the effect of Bernoulli vacation probability 'p' and the effect of persistent probability  $\alpha$  on mean orbit size and the mean waiting time W.

Assume that VoIP call arrivals follow Poisson distribution with mean arrival rate  $\lambda$ . Service process is mainly to transfer the information of caller to another end user. This service process of VoIP is done in three phases: (i) connection establishment (FES); (ii) transferring information (SES) and (iii) accessing differentiated services (different vacation policies). The major VoIP service technique is SIP. A normal SIP session starts by locating the desired user on a registrar-server (VoIP server) containing the current IP addresses of the users (*Connection establishment phase*). Once the user is found, an invitation is then sent to it via a proxy server which authenticates the initiator. Once a person registered with the server, that connection becomes *p2p (Information transferring phase)*. Finally, the VoIP call can access the differentiated services (different vacation policies). This system can be modeled as an M/G/1 retrial queueing system with impatient subscribers & two phases of essential service and different vacation policies.

In table 1(a), 1(b) and 1(c), for vacation time parameters  $v_2 = 20$ ,  $v_3 = 30$  and  $\beta_2 = 1/3$ , the mean orbit size is compared with varying values of the vacation probability  $\beta_1$  and with varying retrial rate when FES, SES and Vacation time follow Exponential, Erlangian of order two and Hyper – Exponential distributions, respectively. It is observed that the buffer size ‘L’ is decreased if the retrial rate ‘v’ increases. Moreover, the buffer size ‘L’ is increased if the Bernoulli vacation probability  $\beta_1$  is increased.

In table 2(a) for vacation time parameter  $v_1 = 20$  &  $v_2 = 30$ , the mean waiting time is compared with varying values of  $\alpha$  when FES, SES and Vacation time follow Exponential, Erlangian of order two and Hyper – Exponential distributions, respectively. Also, as  $\alpha$  increases, mean waiting time increases. (ie., as probability of impatience decreases, mean orbit size increases).

In fig.2(a), 2(b) and 2(c), for vacation time parameter  $v_2 = 20$ ,  $v_3 = 30$  and  $\beta_2 = 1/3$ , the mean orbit size is compared with varying values of the Bernoulli vacation probability  $\beta_1$  and with varying retrial rate when FES, SES and Vacation time follow Exponential, Erlangian of order two and Hyper – Exponential distributions, respectively. Fig. 3(b) depicts the effect of persistence probability  $\alpha$  compared to the mean waiting time. As  $\alpha$  increases, the size of retrial group increases and the mean waiting time W also increases.

Table 1(a)  
**Mean Orbit size (L)**

(retrial rate v, two types of vacation with probability  $\beta_1$  and  $\beta_2 = 1/3$ )

$v \backslash \beta_1$	2	4	6	8	10
0	0.2540764	0.1523815	0.1184832	0.101534	0.0913645
0.1	0.2546194	0.1529245	0.1190262	0.1020771	0.0919076
0.2	0.2551625	0.1534676	0.1195693	0.1026201	0.0924507
0.3	0.2557056	0.1540107	0.1201124	0.1031632	0.0929937
0.4	0.2562487	0.1545537	0.1206554	0.1037063	0.0935368
0.5	0.2567917	0.1550968	0.1211985	0.1042494	0.0940799
0.6	0.2573348	0.1556399	0.1217416	0.1047924	0.0946229
0.7	0.2578779	0.156183	0.1222846	0.1053355	0.095166
0.8	0.2584209	0.156726	0.1228277	0.1058786	0.0957091
0.9	0.258964	0.1572691	0.1233708	0.1064216	0.0962521
1	0.2595071	0.1578122	0.1239139	0.1069647	0.0967952

(FES, SES and two types of Vacation time follow Exponential distribution with service rate of FES  $s_1 = 10$ , service rate of SES  $s_2 = 15$  and two types of vacation rates  $v_1 = 20$  and  $v_2 = 30$ )

Table 1(b)  
**Mean Orbit size (L)**

(retrial rate  $v$ , two types of vacation with probability  $\beta_1$  and  $\beta_2=1/3$ )

$v \backslash \beta_1$	2	4	6	8	10
0	0.780891	0.5018212	0.408798	0.3622864	0.3343794
0.1	0.7830187	0.5039489	0.4109257	0.3644141	0.3365071
0.2	0.7851464	0.5060766	0.4130534	0.3665417	0.3386348
0.3	0.7872741	0.5082043	0.4151811	0.3686694	0.3407625
0.4	0.7894018	0.510332	0.4173088	0.3707971	0.3428902
0.5	0.7915295	0.5124597	0.4194365	0.3729248	0.3450179
0.6	0.7936572	0.5145874	0.4215642	0.3750525	0.3471456
0.7	0.7957849	0.5167151	0.4236919	0.3771802	0.3492733
0.8	0.7979126	0.5188428	0.4258196	0.3793079	0.351401
0.9	0.8000403	0.5209705	0.4279473	0.3814356	0.3535286
1	0.802168	0.5230982	0.430075	0.3835633	0.3556563

(FES, SES and two types of Vacation time follow Erlang -2 distribution with service rate of FES  $s_1=10$ , service rate of SES  $s_2=15$  and two types of vacation rates  $v_1=20$  and  $v_2=30$ )

Table 1(c)  
**Mean Orbit size (L)**

$v \backslash p$	2	4	6	8	10
0	0.0946163	0.0547933	0.0415189	0.0348817	0.0308994
0.1	0.0975355	0.0565252	0.0428551	0.03602	0.031919
0.2	0.1004708	0.0582664	0.0441983	0.0371642	0.0329438
0.3	0.1034225	0.0600171	0.0455486	0.0383144	0.0339739
0.4	0.1063905	0.0617772	0.0469061	0.0394706	0.0350092
0.5	0.1093751	0.0635469	0.0482708	0.0406328	0.03605
0.6	0.1123763	0.0653262	0.0496428	0.0418011	0.0370961
0.7	0.1153944	0.0671152	0.0510221	0.0429756	0.0381477
0.8	0.1184296	0.0689141	0.0524089	0.0441563	0.0392048
0.9	0.1214818	0.0707228	0.0538032	0.0453433	0.0402674
1	0.1245514	0.0725416	0.055205	0.0465367	0.0413357

(FES, SES and two types of Vacation time follow Hyper Exponential distribution with service rate of FES  $s_1=10$ , service rate of SES  $s_2=15$  and two types of vacation rates  $v_1=20$  and  $v_2=30$ )

Table 2(a)  
**Mean Waiting Time (W)**

$\alpha$	W		W		W	
	Exponential dist.		Erlangian of order 2		Hyper Exponential	
	$v_1=20$	$v_2=30$	$v_1=20$	$v_2=30$	$v_1=20$	$v_2=30$
0.1	0.1670211		0.3684296		0.0778995	
0.2	0.1724046		0.3938616		0.0790853	
0.3	0.1781316		0.4228069		0.0803067	
0.4	0.1842351		0.4560167		0.0815653	
0.5	0.1907523		0.4944677		0.082863	
0.6	0.1977254		0.5394514		0.0842014	
0.7	0.2052026		0.5927106		0.0855825	
0.8	0.2132388		0.6566528		0.0870083	
0.9	0.2218971		0.7346945		0.088481	
1	0.23125		0.8318452		0.0900028	

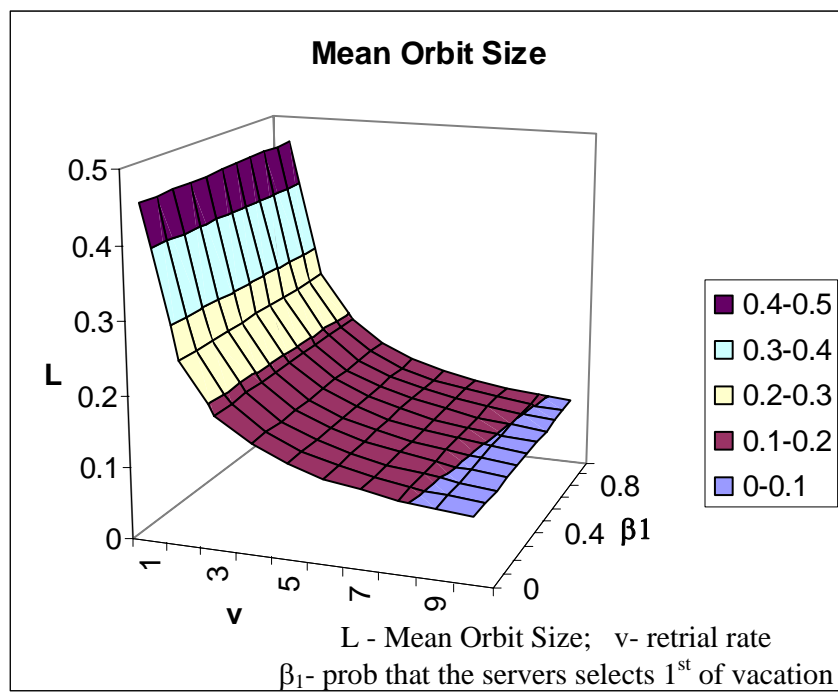


Fig 2(a)

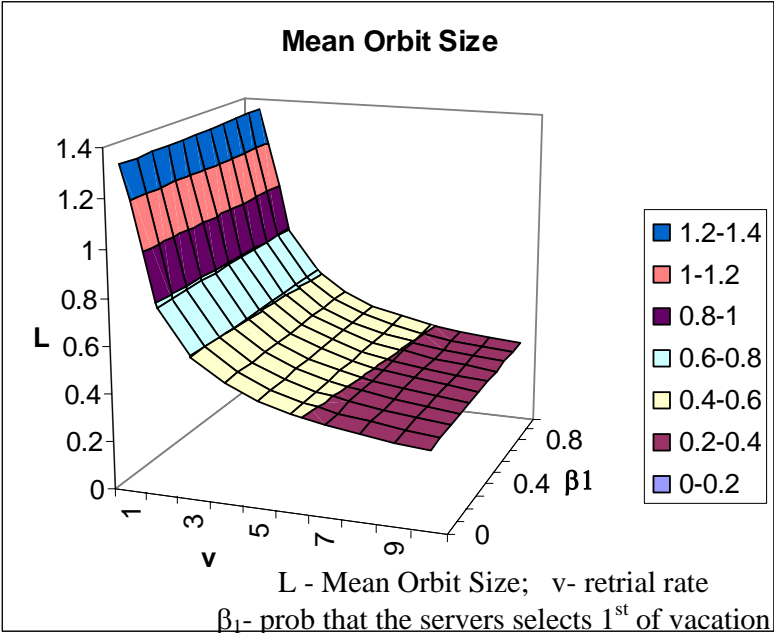


Fig 2(b)

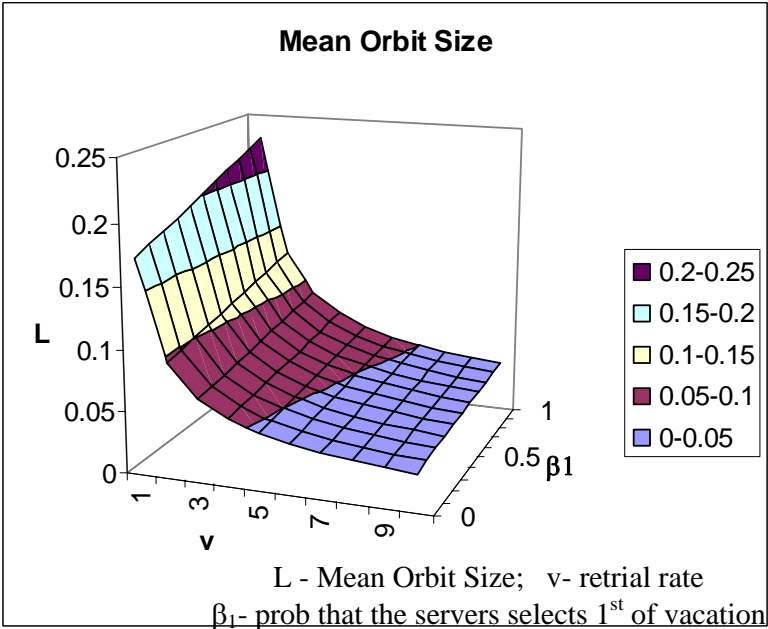


Fig 2(c)



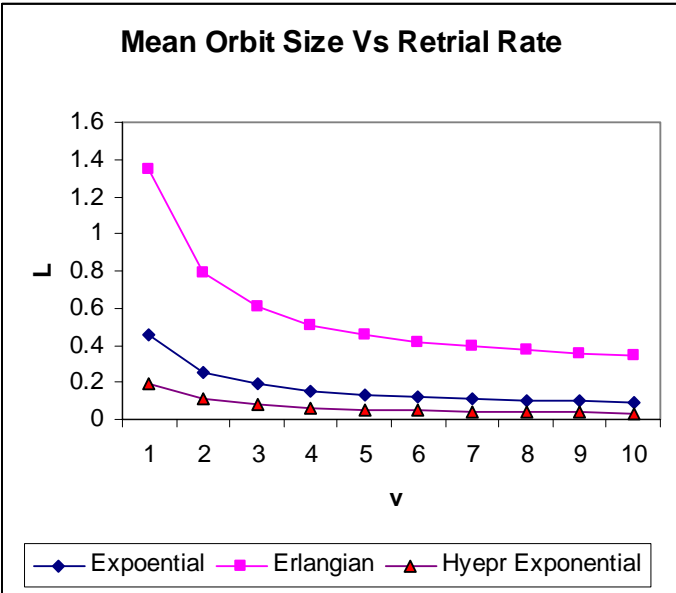


Fig 3(a)

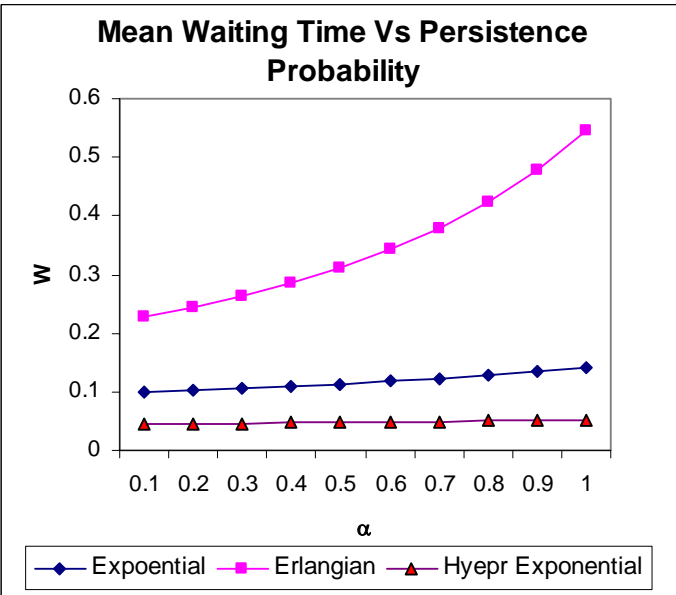


Fig 3(b)

## 7 Conclusion

In this paper, a single server retrial queue with impatient subscribers and two phases of essential service and different types of vacation policies is analyzed under the condition of stability. Several system performance measures are computed in steady state. Some numerical illustrations are also presented with potential applications.

## 8 Open Problem

A natural extension of the foregoing model consists in considering general inter-retrial times independently of the server state. It would be very interesting to examine the discrete time counterpart of our continuous time queueing system, due to the usefulness of the discrete time queueing theory to model many practical problems. The cost analysis of the proposed model is quite interesting problem to study the managerial aspects of the proposed model.

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